Discrete de Rham methods

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NEMESIS

New generation methods
for numerical simulations

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Outline

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1. From finite element to fully discrete complexes
2. The three-dimensional Discrete de Rham complex
3. The Discrete de Rham complex of differential forms
4. Restoring function spaces

Setting

- Let $\Omega \subset \mathbb{R}^3$ be a connected polyhedral domain with Betti numbers b_i
- $b_0 = 1$ (number of connected components) and $b_3 = 0$ (since $d = 3$)
- b_1 and b_2 respectively account for the number of tunnels and voids

$(b_0, b_1, b_2, b_3) = (1, 1, 0, 0)$

$(b_0, b_1, b_2, b_3) = (1, 0, 1, 0)$

A unified tool for well-posedness: The de Rham complex

$$H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl};\Omega) \xrightarrow{\text{curl}} H(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow \{0\}$$

- Key properties:

$$\begin{aligned} \operatorname{Im}\text{grad} &\subset \operatorname{Ker}\text{curl}, \\ \operatorname{Im}\text{curl} &\subset \operatorname{Ker}\text{div}, \\ \boldsymbol{\Omega} \subset \mathbb{R}^3 \ \big(b_3 = 0\big) &\implies \operatorname{Im}\text{div} = L^2(\boldsymbol{\Omega}) \end{aligned}$$

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- Key properties:

$$\text{no tunnels crossing } \Omega \ (b_1 = 0) \implies \mathrm{Im}\,\mathrm{grad} = \mathrm{Ker}\,\mathrm{curl}$$
$$\text{no voids contained in } \Omega \ (b_2 = 0) \implies \mathrm{Im}\,\mathrm{curl} = \mathrm{Ker}\,\mathrm{div}$$
$$\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega)$$

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- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathcal{H}_1 \coloneqq \operatorname{Ker}\text{curl}/\operatorname{Im}\text{grad} \quad \text{and} \quad \mathcal{H}_2 \coloneqq \operatorname{Ker}\text{div}/\operatorname{Im}\text{curl}$$

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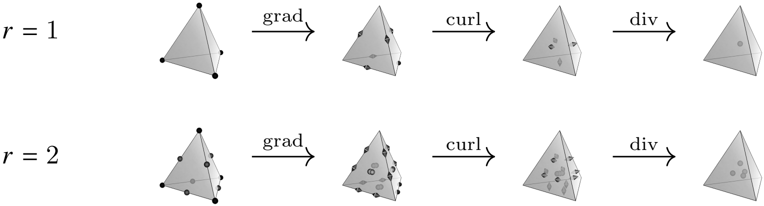
- **Emulating these properties is key for stable discretizations!**

The Finite Element way

- Trimmed FE complexes on a tetrahedron T[1]: For any $r \geq 1$



- On a conforming tetrahedral mesh $\mathcal{T}_h$, local spaces can be glued together

$$\begin{tikzcd}
H^1(\Omega) \arrow[r, "\mathrm{grad}"] & H(\mathrm{curl};\Omega) \arrow[r, "\mathrm{curl}"] & H(\mathrm{div};\Omega) \arrow[r, "\mathrm{div}"] & L^2(\Omega) \\
\mathcal{P}^r_{\mathrm{c}}(\mathcal{T}_h) \arrow[r, "\mathrm{grad}"] \arrow[u, hook] & \mathcal{N}^r(\mathcal{T}_h) \arrow[r, "\mathrm{curl}"] \arrow[u, hook] & \mathcal{RT}^r(\mathcal{T}_h) \arrow[r, "\mathrm{div}"] \arrow[u, hook] & \mathcal{P}^{r-1}(\mathcal{T}_h) \arrow[u, hook]
\end{tikzcd}$$

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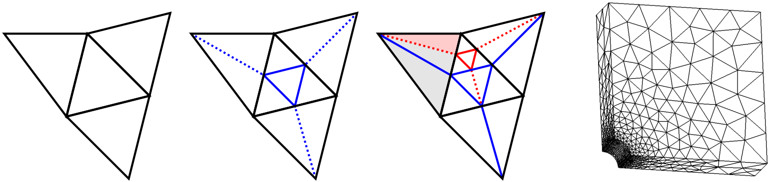
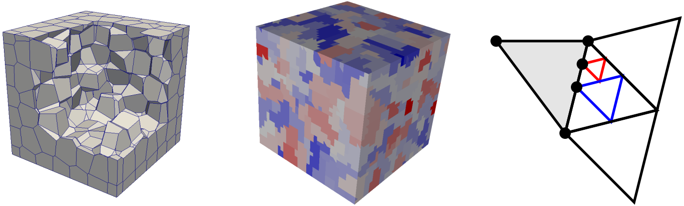
[1] [Raviart and Thomas, 1977, Nédélec, 1980]

Limitations



- Approach limited to conforming meshes with standard elements
  - $\Longrightarrow$ Local refinement requires to trade mesh size for quality
  - $\Longrightarrow$ Complex geometries may require a large number of elements
  - $\Longrightarrow$ The element shape cannot be adapted to the solution
- The extension to advanced complexes is also not straightforward

Polytopal approaches



- Key idea: replace spaces and, possibly, operators by discrete counterparts
- Support of polyhedral meshes and high-order
- Higher-level point of view, possibly resulting in leaner constructions
- Several strategies to reduce the number of unknowns on general shapes
- Agglomeration-based techniques[2] for adaptivity and h-multigrid

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[2] [Bassi et al., 2012], [Antonietti et al., 2013]

Mesh-related notations I

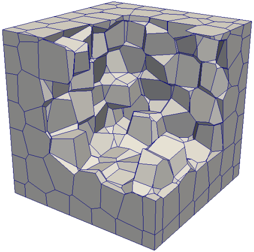

Figure: Example of polyhedral mesh $\mathcal{M}_h = \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$

- $\mathcal{T}_h = \Delta_3(\mathcal{M}_h)$ set of polyhedral elements (3-cells)
- $\mathcal{F}_h = \Delta_2(\mathcal{M}_h)$ set of polygonal (flat) faces (2-cells)
- $\mathcal{E}_h = \Delta_1(\mathcal{M}_h)$ set of edges (1-cells)
- $\mathcal{V}_h = \Delta_0(\mathcal{M}_h)$ set of vertices (0-cells)
- Cochain spaces are denoted with an asterisk, e.g., $\mathcal{T}_h^*$ or $\Delta_2(\mathcal{M}_h)^*$

Mesh-related notations II

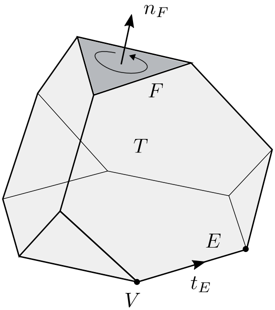

Figure: Notations for a polyhedral element. The orientations of a face F relative to T and of an edge E relative to F are respectively denoted by ω_{TF} and ω_{FE}.

Isomorphism in cohomology

Theorem (Complexes with isomorphic cohomologies[3])

$$\begin{tikzcd}
(V,d): & \cdots \arrow[r] & V_i \arrow[r, "d_i"] \arrow[d, bend right, "E_i"', leftarrow] & V_{i+1} \arrow[r] \arrow[d, bend right, "E_{i+1}"', leftarrow] & \cdots \\
(W,\partial): & \cdots \arrow[r] & W_i \arrow[r, "\partial_i"] \arrow[u, bend right, "R_i"'] & W_{i+1} \arrow[r] \arrow[u, bend right, "R_{i+1}"'] & \cdots
\end{tikzcd}$$

Let the reduction R and extension E maps be s.t., for all i,

- $\partial_i E_i = E_{i+1} d_i$ *and* $d_i R_i = R_{i+1} \partial_i$;
- $R_i E_i = \mathrm{Id}_{W_i}$;
- $(E_{i+1} R_{i+1} - \mathrm{Id}_{V_{i+1}}) \operatorname{Ker} d_{i+1} \subset \operatorname{Im} d_i$.

Then, (V,d) and (W,∂) are complexes with isomorphic cohomologies.

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[3] [DP, Droniou, Pitassi, 2023]

Cohomology of the trimmed FE complex

- If $\mathcal{M}_h$ is a simplicial complex (FE mesh), we have

$$\begin{tikzcd}
\mathcal{V}_h^* \arrow[r, "\partial_0"] & \mathcal{E}_h^* \arrow[r, "\partial_1"] & \mathcal{F}_h^* \arrow[r, "\partial_2"] & \mathcal{T}_h^* \\
\mathcal{P}_{\mathrm{c}}^1(\mathcal{T}_h) \arrow[u, "\overset{\kappa_{0,h}}{\cong}"] \arrow[r, "\mathrm{grad}"] \arrow[d, bend right] & \mathcal{N}^1(\mathcal{T}_h) \arrow[u, "\overset{\kappa_{1,h}}{\cong}"] \arrow[r, "\mathrm{curl}"] \arrow[d, bend right] & \mathcal{RT}^1(\mathcal{T}_h) \arrow[u, "\overset{\kappa_{2,h}}{\cong}"] \arrow[r, "\mathrm{div}"] \arrow[d, bend right] & \mathcal{P}^0(\mathcal{T}_h) \arrow[u, "\overset{\kappa_{3,h}}{\cong}"] \arrow[d, bend right] \\
\mathcal{P}_{\mathrm{c}}^r(\mathcal{T}_h) \arrow[r, "\mathrm{grad}"] \arrow[u, bend right, "I^1_{\mathrm{grad},h}"'] & \mathcal{N}^r(\mathcal{T}_h) \arrow[r, "\mathrm{curl}"] \arrow[u, bend right, "I^1_{\mathrm{curl},h}"'] & \mathcal{RT}^r(\mathcal{T}_h) \arrow[r, "\mathrm{div}"] \arrow[u, bend right, "I^1_{\mathrm{div},h}"'] & \mathcal{P}^{r-1}(\mathcal{T}_h) \arrow[u, bend right, "\pi_h^0"']
\end{tikzcd}$$

with κ_h de Rham map, $I^1_{\bullet,h}$ interpolator, and π_h^0 L^2-orthogonal projector

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- By de Rham's Theorem, the two top rows have isomorphic cohomologies

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\end{tikzcd}$$

  with κ_h de Rham map, $I^1_{\bullet,h}$ interpolator, and π_h^0 L^2-orthogonal projector
- By de Rham's Theorem, the two top rows have isomorphic cohomologies
- The two bottom rows fulfill the assumptions of the theorem with

$$R = \text{interpolators and } E = \text{injections}$$

Shifting point of view I

- Denote by “dofs” the standard FE degrees of freedom
- By unisolvency, we have

$$\begin{tikzcd}
\mathcal{V}_h^* \arrow[r, "\partial_0"] & \mathcal{E}_h^* \arrow[r, "\partial_1"] & \mathcal{F}_h^* \arrow[r, "\partial_2"] & \mathcal{T}_h^* \\
\mathcal{P}^1_{\mathrm{c}}(\mathcal{T}_h) \arrow[r, "\mathrm{grad}"] \arrow[u, "\kappa_{0,h}", "\cong"'] & \mathcal{N}^1(\mathcal{T}_h) \arrow[r, "\mathrm{curl}"] \arrow[u, "\kappa_{1,h}", "\cong"'] & \mathcal{RT}^1(\mathcal{T}_h) \arrow[r, "\mathrm{div}"] \arrow[u, "\kappa_{2,h}", "\cong"'] & \mathcal{P}^0(\mathcal{T}_h) \arrow[u, "\kappa_{3,h}", "\cong"'] \\
\mathbb{R}^{\mathcal{V}_h} \arrow[u, "\mathsf{dofs}^{-1}", "\cong"'] & \mathbb{R}^{\mathcal{E}_h} \arrow[u, "\mathsf{dofs}^{-1}", "\cong"'] & \mathbb{R}^{\mathcal{F}_h} \arrow[u, "\mathsf{dofs}^{-1}", "\cong"'] & \mathbb{R}^{\mathcal{T}_h} \arrow[u, "\mathsf{dofs}^{-1}", "\cong"']
\end{tikzcd}$$

- In the previous diagram, we can erase the middle row

Shifting point of view II

- Set $K_h := \kappa_h \circ \text{dofs}^{-1}$, i.e., $\forall(\underline{q}_h, \underline{v}_h, \underline{w}_h, \underline{r}_h) \in \mathbb{R}^{\mathcal{V}_h} \times \mathbb{R}^{\mathcal{E}_h} \times \mathbb{R}^{\mathcal{F}_h} \times \mathbb{R}^{\mathcal{T}_h}$,

$$\begin{array}{llrl} K_{0,h}\underline{q}_h(V) := q_V & \forall V \in \mathcal{V}_h, & K_{1,h}\underline{v}_h(E) := |E| v_E & \forall E \in \mathcal{E}_h, \\ K_{2,h}\underline{w}_h(F) := |F| v_F & \forall F \in \mathcal{F}_h, & K_{3,h} r_h := |T| r_T & \forall T \in \mathcal{T}_h \end{array}$$

- This graded map induces the following isomorphisms:

$$\begin{tikzcd} \mathcal{V}_h^* \arrow[r, "\partial_0"] & \mathcal{E}_h^* \arrow[r, "\partial_1"] & \mathcal{F}_h^* \arrow[r, "\partial_2"] & \mathcal{T}_h^* \\ \mathbb{R}^{\mathcal{V}_h} \arrow[u, "\overset{K_{0,h}}{\cong}"] & \mathbb{R}^{\mathcal{E}_h} \arrow[u, "\overset{K_{1,h}}{\cong}"] & \mathbb{R}^{\mathcal{F}_h} \arrow[u, "\overset{K_{2,h}}{\cong}"] & \mathbb{R}^{\mathcal{T}_h} \arrow[u, "\overset{K_{3,h}}{\cong}"] \end{tikzcd}$$

- **Can we complete the bottom row to form a complex?**

Shifting point of view III

- Define the following discrete gradient, curl, and divergence operators:

$$\underline{G}_h^0 \underline{q}_h \coloneqq K_{1,h}^{-1} \partial_0 K_{0,h}, \quad \underline{C}_h^0 \underline{v}_h \coloneqq K_{2,h}^{-1} \partial_1 K_{1,h}, \quad \underline{D}_h^0 \underline{w}_h \coloneqq K_{3,h}^{-1} \partial_2 K_{2,h},$$

- Notice that, by construction,

$$\underline{C}_h^0 \circ \underline{G}_h^0 = \underline{0} \text{ and } \underline{D}_h^0 \circ \underline{C}_h^0 = \underline{0}$$

- Hence, we have two complexes with isomorphic cohomologies:

$$\begin{tikzcd}
\mathcal{V}_h^* \arrow[r, "\partial_0"] & \mathcal{E}_h^* \arrow[r, "\partial_1"] & \mathcal{F}_h^* \arrow[r, "\partial_2"] & \mathcal{T}_h^* \\
\mathbb{R}^{\mathcal{V}_h} \arrow[u, "\overset{K_{0,h}}{\cong}"] \arrow[r, "\underline{G}_h^0"] & \mathbb{R}^{\mathcal{E}_h} \arrow[u, "\overset{K_{1,h}}{\cong}"] \arrow[r, "\underline{C}_h^0"] & \mathbb{R}^{\mathcal{F}_h} \arrow[u, "\overset{K_{2,h}}{\cong}"] \arrow[r, "\underline{D}_h^0"] & \mathbb{R}^{\mathcal{T}_h} \arrow[u, "\overset{K_{3,h}}{\cong}"]
\end{tikzcd}$$

- **Still true with $\mathcal{M}_h$ CW complex associated to a polyhedral mesh!**

Moral of the story so far

By getting rid of FE spaces, we can now handle polyhedral meshes!

A closer look at the discrete operators

- $\underline{G}_h^0$, $\underline{C}_h^0$, and $\underline{D}_h^0$ are actually the mimetic operators[4]:

$$\underline{G}_h^0 \underline{q}_h := \left(G_E^0 \underline{q}_E = \frac{q_{V_2} - q_{V_1}}{|E|} \right)_{E \in \mathcal{E}_h}$$

$$\underline{C}_h^0 \underline{v}_h := \left(C_F^0 \underline{v}_F = -\frac{1}{|F|} \sum_{E \in \mathcal{E}_F} \omega_{FE} |E| v_E \right)_{F \in \mathcal{F}_h}$$

$$\underline{D}_h^0 \underline{w}_h := \left(D_T^0 \underline{w}_T = \frac{1}{|T|} \sum_{F \in \mathcal{F}_T} \omega_{TF} |F| w_F \right)_{T \in \mathcal{T}_h}$$

- These operators are polynomially exact and commute with the interpolators (component-wise L^2-orthogonal projectors)

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[4] See, e.g., [Beirão da Veiga et al., 2014] and [Bonelle and Ern, 2014]

Outline

The arbitrary-order case $r \geq 0$

$$\begin{tikzcd}
\underline{X}^r_{\text{grad},h} \arrow[r, "\underline{G}^r_h"] & \underline{X}^r_{\text{curl},h} \arrow[r, "\underline{C}^r_h"] & \underline{X}^r_{\text{div},h} \arrow[r, "D^r_h"] & \mathcal{P}^r(\mathcal{T}_h)
\end{tikzcd}$$

	V	E	F	T
$\underline{X}^r_{\text{grad},h}$	$\mathbb{R}$	$\mathcal{P}^{r-1}(E)$	$\mathcal{P}^{r-1}(F)$	$\mathcal{P}^{r-1}(T)$
$\underline{X}^r_{\text{curl},h}$	–	$\mathcal{P}^r(E)$	$\mathcal{RT}^r(F)$	$\mathcal{RT}^r(T)$
$\underline{X}^r_{\text{div},h}$	–	–	$\mathcal{P}^r(F)$	$\mathcal{N}^r(T)$
$\mathcal{P}^r(\mathcal{T}_h)$	–	–	–	$\mathcal{P}^r(T)$

- Discrete de Rham (DDR) [DP, Droniou, Rapetti, 2020]
- Version with Koszul complements [DP and Droniou, 2023a]
- Cohomology on domains of general topology [DP and Droniou, 2023b]
- Serendipity version [DP and Droniou, 2023b]

An example: The arbitrary-order curl space I

- The DDR curl space reads:

$$\underline{X}^r_{\mathrm{curl},h} \coloneqq \Big\{ \underline{v}_h = \big((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_h}, (v_E)_{E\in\mathcal{E}_h} \big) \ : \\ v_T \in \mathcal{RT}^r(T) \text{ for all } T \in \mathcal{T}_h, \\ v_F \in \mathcal{RT}^r(F) \text{ for all } F \in \mathcal{F}_h, \\ v_E \in \mathcal{P}^r(E) \text{ for all } E \in \mathcal{E}_h \Big\}$$

- The components are interpreted through the interpolator s.t., for all $v : \Omega \to \mathbb{R}^3$ smooth enough,

$$\underline{I}^r_{\mathrm{curl},h} v \coloneqq \Big((\pi_{\mathcal{RT}^r(T)} v)_{T\in\mathcal{T}_h}, (\pi_{\mathcal{RT}^r(F)} v_{t,F})_{F\in\mathcal{F}_h}, (\pi_{\mathcal{P}^r(E)} (v\cdot t_E))_{E\in\mathcal{E}_h} \Big),$$

with $v_{t,F} \coloneqq n_F \times (v_{|F} \times n_F)$ and $\pi_{\mathcal{X}}$ L^2-orthogonal projector onto $\mathcal{X}$

An example: The arbitrary-order curl space II

- Tangential traces on edges are the components $(v_E)_{E\in\mathcal{E}_h}$
- A face rotor and tangential trace are built mimicking the following IBP:
  For all $v : F \to \mathbb{R}^2$ and all $q : F \to \mathbb{R}$ smooth enough,

$$\int_F \mathrm{rot}_F\, v\, q = \int_F v \cdot \mathrm{curl}_F\, q - \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E (v \cdot t_E)\, q$$

- Similarly, element curl and potential result from mimicking:
  For all $v, w : T \to \mathbb{R}^3$ smooth enough,

$$\int_T \mathrm{curl}\, v \cdot w = \int_T v \cdot \mathrm{curl}\, w + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F v_{t,F} \cdot (w \times n_F)$$

An example: The arbitrary-order curl space III

- The face rotor $R_F^r : \underline{X}_{\mathrm{curl},F}^r \to \mathcal{P}^r(F)$ is s.t.

$$\int_F R_F^r \underline{v}_F\, q = \int_F v_F \cdot \mathrm{curl}_F\, q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E\, q \quad \forall q \in \mathcal{P}^r(F)$$

- Let $\mathcal{R}^{\mathrm{c},r}(F) := (x - x_F)\mathcal{P}^{r-1}(F)$, complement of $\mathrm{curl}_F\, \mathcal{P}^{r+1}(F)^2$
- The trace $\gamma_F^r : \underline{X}_{\mathrm{curl},F}^r \to \mathcal{P}^r(F)^2$ is s.t., $\forall (r, w) \in \mathcal{P}^{r+1}(F) \times \mathcal{R}^{\mathrm{c},r}(F)$,

$$\int_F \gamma_F^r \underline{v}_F \cdot (\mathrm{curl}_F\, r + w) = \int_F R_F^r \underline{v}_F\, r + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E\, r + \int_F v_F \cdot w$$

- We build the element curl $C_T^r : \underline{X}_{\mathrm{curl},h}^r \to \mathcal{P}^r(T)^3$ similarly to R_F^r and set

$$\begin{aligned} \underline{C}_h^r : \underline{X}_{\mathrm{curl},h}^r &\to \underline{X}_{\mathrm{div},h}^r \\ \underline{v}_h &\mapsto \Big((\pi_{\mathcal{N}^r(T)} C_T^r \underline{v}_T)_{T \in \mathcal{T}_h}, (R_F^r \underline{v}_F)_{F \in \mathcal{F}_h} \Big) \end{aligned}$$

An example: The arbitrary-order curl space IV

- From C_T^r and γ_F^r, we build an element potential

$$P_{\mathrm{curl},T}^r : \underline{X}_{\mathrm{curl},T}^r \to \mathcal{P}^r(T)^3$$

- The local L^2-product in $\underline{X}_{\mathrm{curl},T}^r$ is

$$(\underline{w}_T, \underline{v}_T)_{\mathrm{curl},T} \coloneqq \int_T P_{\mathrm{curl},T}^r \underline{w}_T \cdot P_{\mathrm{curl},T}^r \underline{v}_T + \text{stab.}$$

  where stab. penalizes $\underline{I}_{\mathrm{curl},T}^r P_{\mathrm{curl},T}^r \underline{v}_T - \underline{v}_T$ in a least-square sense
- The global discrete L^2-product is obtained assemblying element-wise:

$$(\underline{w}_h, \underline{v}_h)_{\mathrm{curl},h} \coloneqq \sum_{T \in \mathcal{T}_h} (\underline{w}_T, \underline{v}_T)_{\mathrm{curl},T}$$

An example of numerical scheme

- Let us consider the magnetostatics problem:
  Find $(H, A) \in H(\mathrm{curl};\Omega) \times H(\mathrm{div};\Omega)$ s.t.

$$
\begin{aligned}
\mu \int_\Omega H \cdot \tau - \int_\Omega A \cdot \mathrm{curl}\,\tau &= 0 && \forall \tau \in H(\mathrm{curl};\Omega),\\
\int_\Omega \mathrm{curl}\,H \cdot v + \int_\Omega \mathrm{div}\,A\,\mathrm{div}\,v &= \int_\Omega J \cdot v && \forall v \in H(\mathrm{div};\Omega)
\end{aligned}
$$

- A DDR scheme for this problem is obtained with obvious substitutions:
  Find $(\underline{H}_h, \underline{A}_h) \in \underline{X}^r_{\mathrm{curl},h} \times \underline{X}^r_{\mathrm{div},h}$ s.t.

$$
\begin{aligned}
\mu(\underline{H}_h, \underline{\tau}_h)_{\mathrm{curl},h} - (\underline{A}_h, \underline{C}^r_h \underline{\tau}_h)_{\mathrm{div},h} &= 0 && \forall \underline{\tau}_h \in \underline{X}^r_{\mathrm{curl},h},\\
(\underline{C}^r_h \underline{H}_h, \underline{v}_h)_{\mathrm{div},h} + \int_\Omega D^r_h \underline{A}_h\, D^r_h \underline{v}_h &= (\underline{I}^r_{\mathrm{div},h} J, \underline{v}_h)_{\mathrm{div},h} && \forall \underline{v}_h \in \underline{X}^r_{\mathrm{div},h}
\end{aligned}
$$

- **Stability mimics the continuous argument for well-posedness**

Outline

1 From finite element to fully discrete complexes

2 The three-dimensional Discrete de Rham complex

3 The Discrete de Rham complex of differential forms

4 Restoring function spaces

The DDR complex of differential forms

- The construction extends to the de Rham complex of differential forms[5]
- Denote by $\Lambda^k(M)$ the space of k-forms

$$\omega = \sum_{1 \le \sigma_1 < \cdots < \sigma_k \le n} a_\sigma \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}$$

- When regularity on the a_σ is required, we prepend it to $\Lambda^k(M)$, e.g.,

$L^2\Lambda^k(M)$ = space of k-forms with coefficients a_σ square-integrable on M

$\mathcal{P}^r\Lambda^k(M)$ = space of k-forms with coefficients a_σ in $\mathcal{P}^r(M)$

---

[5] [Bonaldi, DP, Droniou, Hu, 2025]

Exterior derivative

- The exterior derivative d is the (unbounded) graded operator s.t.

$$\mathrm{d}^k : L^2\Lambda^k(M) \to L^2\Lambda^{k+1}(M)$$

$$\omega \mapsto \sum_{1\le\sigma_1<\cdots<\sigma_k\le n} \sum_{i=1}^{n} \frac{\partial a_\sigma}{\partial x_i}\, \mathrm{d}x^i \wedge \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}$$

  which satisfies

$$\mathrm{d}^k \circ \mathrm{d}^{k-1} = 0$$

- When $M = \Omega$ domain of $\mathbb{R}^3$, through vector proxies we can identify

$$\mathrm{d}^0 \cong \mathrm{grad}, \quad \mathrm{d}^1 \cong \mathrm{curl}, \quad \mathrm{d}^2 \cong \mathrm{div}$$

- In what follows, we define the domain of the exterior derivative

$$H\Lambda^k(M) := \left\{\omega \in L^2\Lambda^k(M) \;:\; \mathrm{d}\omega \in L^2\Lambda^{k+1}(M)\right\}$$

The continuous de Rham complex

- The de Rham complex for a domain Ω of $\mathbb{R}^n$ reads

$$H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}^0} \cdots \xrightarrow{\mathrm{d}^{k-1}} H\Lambda^k(\Omega) \xrightarrow{\mathrm{d}^k} \cdots \xrightarrow{\mathrm{d}^{n-1}} H\Lambda^n(\Omega) \longrightarrow \{0\}$$

- For $n = 3$, the following isomorphisms are provided by vector proxies:

$$\begin{tikzcd}
H\Lambda^0(\Omega) \arrow[r, "\mathrm{d}^0"] \arrow[d, leftrightarrow, "\cong"] & H\Lambda^1(\Omega) \arrow[r, "\mathrm{d}^1"] \arrow[d, leftrightarrow, "\cong"] & H\Lambda^2(\Omega) \arrow[r, "\mathrm{d}^2"] \arrow[d, leftrightarrow, "\cong"] & H\Lambda^3(\Omega) \arrow[r] \arrow[d, leftrightarrow, "\cong"] & \{0\} \\
H^1(\Omega) \arrow[r, "\mathrm{grad}"] & H(\mathrm{curl};\Omega) \arrow[r, "\mathrm{curl}"] & H(\mathrm{div};\Omega) \arrow[r, "\mathrm{div}"] & L^2(\Omega) \arrow[r] & \{0\}
\end{tikzcd}$$

Local Koszul differential and complements I

- Given $d \in [0, n]$, let $f \in \Delta_d(\mathcal{M}_h)$ denote a d-face of $\mathcal{M}_h$ and fix $x_f \in f$
- The Koszul differential $\kappa_f : \Lambda^{\ell+1}(f) \to \Lambda^{\ell}(f)$ is s.t.

$$(\kappa_f \omega)_x(v_1, \ldots, v_\ell) = \omega_x(x - x_f, v_1, \ldots, v_\ell)$$

for all $x \in f$ and $v_1, \ldots, v_\ell$ tangent vectors to f

- We define the Koszul complement space

$$\mathcal{K}^{r,\ell}(f) \coloneqq \kappa_f \mathcal{P}^{r-1}\Lambda^{\ell+1}(f)$$

- Assuming $d \geq 1$, for any integers $\ell \in [0, d]$ and $r \geq 0$, we have

$$\begin{aligned}
\mathcal{P}^r\Lambda^0(f) &= \mathcal{P}^0\Lambda^0(f) \oplus \mathcal{K}^{r,0}(f), \\
\mathcal{P}^r\Lambda^\ell(f) &= \mathrm{d}\mathcal{P}^{r+1}\Lambda^{\ell-1}(f) \oplus \mathcal{K}^{r,\ell}(f) \qquad \text{if } \ell \geq 1
\end{aligned}$$

Local Koszul differential and complements II

- Lowering the first polynomial degree yields trimmed polynomial spaces

$$\begin{aligned}\mathcal{P}^r_-\Lambda^0(f) &:= \mathcal{P}^r\Lambda^0(f),\\ \mathcal{P}^r_-\Lambda^\ell(f) &:= \mathrm{d}\mathcal{P}^r\Lambda^{\ell-1}(f) \oplus \mathcal{K}^{r,\ell}(f) \quad \text{if } \ell \geq 1\end{aligned}$$

- The L^2-orthogonal projector onto $\mathcal{P}^r_-\Lambda^k(f)$ is s.t.

$$\forall \omega \in L^2\Lambda^k(f), \quad \int_f \pi_{\mathcal{P}^r_-\Lambda^k(f)}\omega \wedge \star\mu = \int_f \omega \wedge \star\mu \quad \forall \mu \in \mathcal{P}^r_-\Lambda^k(f)$$

Local Koszul differential and complements III

Example (Trimmed spaces in dimensions 2 and 3)

Let $n = 3$. For $T = f_3 \in \Delta_3(\mathcal{M}_h) = \mathcal{T}_h$, the vector proxies for trimmed spaces are the Nédélec and Raviart–Thomas spaces:

$$\begin{aligned}
\mathcal{P}^r_-\Lambda^1(f_3) &\cong \operatorname{grad}\mathcal{P}^r(T) \oplus (x - x_T) \times \mathcal{P}^{r-1}(T)^3 =: \mathcal{N}^r(T),\\
\mathcal{P}^r_-\Lambda^2(f_3) &\cong \operatorname{curl}\mathcal{P}^r(T)^3 \oplus (x - x_T)\mathcal{P}^{r-1}(T) =: \mathcal{RT}^r(T).
\end{aligned}$$

For $F = f_2 \in \Delta_2(\mathcal{M}_h)$, we have

$$\mathcal{P}^r_-\Lambda^1(f_2) \cong \operatorname{curl}_F \mathcal{P}^r(F) \oplus (x - x_F)\mathcal{P}^{r-1}(F) =: \mathcal{RT}^r(F).$$

Discrete spaces and interpolators

- The discrete $H\Lambda^k(\Omega)$ space, $0 \le k \le n$, is

$$\underline{X}_h^{r,k} := \bigtimes_{d=k}^{n} \bigtimes_{f \in \Delta_d(\mathcal{M}_h)} \mathcal{P}_-^r \Lambda^{d-k}(f)$$

- Its restrictions to $f \in \Delta_d(\mathcal{M}_h)$, $k \le d \le n$, and ∂f are $\underline{X}_f^{r,k}$ and $\underline{X}_{\partial f}^{r,k}$
- The components are interpreted as L^2-orthogonal projections of traces:

$$\begin{aligned} \underline{I}_f^{r,k} : C^0\Lambda^k(\overline{f}) &\to \underline{X}_f^{r,k} \\ \omega &\mapsto \left(\pi_{\mathcal{P}_-^r \Lambda^{d'-k}(f')} (\star \operatorname{tr}_{f'} \omega) \right)_{f' \in \Delta_{d'}(f),\, d' \in [k,d]} \end{aligned}$$

with trace operator $\operatorname{tr}_{f'}$ pullback of the inclusion $f' \hookrightarrow f$

Discrete potential and exterior derivative I

- Let $d \in \mathbb{N}$ be s.t. $0 \leq d \leq n$, $f \in \Delta_d(\mathcal{M}_h)$, and notice that

$$\mathrm{tr}_{\partial f} : \Lambda^k(f) \to \Lambda^k(\partial f)$$

- Stokes formula: For $\omega \in \Lambda^k(f)$ and $\mu \in \Lambda^{d-k-1}(f)$ smooth enough,

$$\int_f \mathrm{d}\omega \wedge \mu = (-1)^{k+1} \int_f \omega \wedge \mathrm{d}\mu + \int_{\partial f} \mathrm{tr}_{\partial f}\, \omega \wedge \mathrm{tr}_{\partial f}\, \mu$$

- Local reconstructions are obtained emulating this formula

Discrete potential and exterior derivative II

- For $d = k$, we set

$$P_f^{r,k} \underline{\omega}_f := \star^{-1} \omega_f \in \mathcal{P}^r \Lambda^d(f)$$

- For $d = k + 1 \dots n$, we first let, for all $\underline{\omega}_f \in \underline{X}_f^{r,k}$ and $\mu \in \mathcal{P}^r \Lambda^{d-k-1}(f)$,

$$\int_f \mathrm{d}_f^{r,k} \underline{\omega}_f \wedge \mu = (-1)^{k+1} \int_f \star^{-1} \omega_f \wedge \mathrm{d}\mu + \int_{\partial f} P_{\partial f}^{r,k} \underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f} \mu$$

then, for all $(\mu, \nu) \in \mathcal{K}^{r+1,d-k-1}(f) \times \mathcal{K}^{r,d-k}(f)$,

$$(-1)^{k+1} \int_f P_f^{r,k} \underline{\omega}_f \wedge (\mathrm{d}\mu + \nu) = \int_f \mathrm{d}_f^{r,k} \underline{\omega}_f \wedge \mu - \int_{\partial f} P_{\partial f}^{r,k} \underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f} \mu + (-1)^{k+1} \int_f \star^{-1} \omega_f \wedge \nu$$

Consistency of the local reconstructions I

Lemma (Polynomial consistency)

For all integers $0 \le k \le d \le n$ *and all* $f \in \Delta_d(\mathcal{M}_h)$*, it holds*

$$P_f^{r,k}(\underline{I}_f^{r,k}\omega) = \omega \qquad \forall \omega \in \mathcal{P}^r\Lambda^k(f),$$

and, if $d \ge k+1$*,*

$$\mathrm{d}_f^{r,k}(\underline{I}_f^{r,k}\omega) = \mathrm{d}\omega \qquad \forall \omega \in \mathcal{P}_-^{r+1}\Lambda^k(f).$$

There also exists $P_f^{r+1,0}$ *consistent up to degree* $r+1$*.*

Consistency of the local reconstructions II

Example (The case $(n, d) = (3, 3)$)

The above properties translate as follows for $(n, d) = (3, 3)$:

- For $k = 0$,

$$\begin{aligned} P^r_{\mathrm{grad},T} \underline{I}^r_{\mathrm{grad},T} q &= q && \forall q \in \mathcal{P}^r(T), \\ G^r_T \underline{I}^r_{\mathrm{grad},T} q &= \operatorname{grad} q && \forall q \in \mathcal{P}^{r+1}(T); \end{aligned}$$

- For $k = 1$,

$$\begin{aligned} P^r_{\mathrm{curl},T} \underline{I}^r_{\mathrm{curl},T} v &= v && \forall v \in \mathcal{P}^r(T)^3, \\ C^r_T \underline{I}^r_{\mathrm{curl},T} v &= \operatorname{curl} v && \forall v \in \mathcal{N}^{r+1}(T); \end{aligned}$$

- For $k = 2$,

$$\begin{aligned} P^r_{\mathrm{div},T} \underline{I}^r_{\mathrm{div},T} w &= w && \forall w \in \mathcal{P}^r(T)^3, \\ D^r_T \underline{I}^r_{\mathrm{curl},T} w &= \operatorname{div} w && \forall w \in \mathcal{RT}^{r+1}(T). \end{aligned}$$

Consistency of the local reconstructions III

Theorem (Consistency for smooth functions)

Let $s \geq \frac{d}{2}$ *and set,* $\forall \omega \in H^{\max\{r+1,s\}}\Lambda^k(f)$,

$$|\omega|_{H^{(r+1,s)}\Lambda^k(f)} := \begin{cases} |\omega|_{H^{r+1}\Lambda^k(f)} & \text{if } s \leq r+1, \\ \sum_{t=r+1}^{s} h_f^{t-r-1} |\omega|_{H^t\Lambda^k(f)} & \text{otherwise.} \end{cases}$$

Then, for any $0 \leq m \leq r+1$, *it holds, for all* $\omega \in H^{\max\{r+1,s\}}\Lambda^k(f)$,

$$|P_f^{r,k} \underline{I}_f^{r,k} \omega - \omega|_{H^m\Lambda^k(f)} \lesssim h_f^{r+1-m} |\omega|_{H^{(r+1,s)}\Lambda^k(f)}.$$

If, moreover, $\mathrm{d}\omega \in H^{\max\{r+1,s\}}\Lambda^{k+1}(f)$,

$$|\mathrm{d}_f^{r,k} \underline{I}_f^{r,k} \omega - \mathrm{d}\omega|_{H^m\Lambda^{k+1}(f)} \lesssim h_f^{r+1-m} |\mathrm{d}\omega|_{H^{(r+1,s)}\Lambda^{k+1}(f)}.$$

Global discrete exterior derivative and DDR complex

- The spaces $\underline{X}_h^{r,k}$ are connected by the global discrete exterior derivative

$$\begin{aligned}\underline{\mathrm{d}}_h^{r,k} : \underline{X}_h^{r,k} &\to \underline{X}_h^{r,k+1}\\ \underline{\omega}_h &\mapsto \left(\pi_{\mathcal{P}^r_-\Lambda^{d-k-1}(f)}(\star \mathrm{d}_f^{r,k}\underline{\omega}_f)\right)_{f\in\Delta_d(\mathcal{M}_h),\, d\in[k+1,n]}\end{aligned}$$

- The DDR sequence then reads

$$\underline{X}_h^{r,0} \xrightarrow{\underline{\mathrm{d}}_h^{r,0}} \underline{X}_h^{r,1} \longrightarrow \cdots \longrightarrow \underline{X}_h^{r,n-1} \xrightarrow{\underline{\mathrm{d}}_h^{r,n-1}} \underline{X}_h^{r,n} \longrightarrow \{0\}$$

- For $n = 3$, we recover the DDR complex discussed in the first part

Algebraic properties I

Theorem (Complex property)

For all $0 \le k \le d \le n$ *and all* $f \in \Delta_d(\mathcal{M}_h)$*, it holds,*

$$P_f^{r,k} \circ \underline{\mathrm{d}}_f^{r,k-1} = \mathrm{d}_f^{r,k-1},$$

and, if $d \ge k+1$*,*

$$\underline{\mathrm{d}}_f^{r,k} \circ \underline{\mathrm{d}}_f^{r,k-1} = \underline{0},$$

so that the DDR sequence defines a complex.

Theorem (Cohomology)

The cohomology of the DDR complex of differential forms is isomorphic to that of the de Rham complex.

Algebraic properties II

Proof.

$$\begin{tikzcd}
\cdots \arrow[r] & \Delta^*_k(\mathcal{M}_h) \arrow[r, "\partial_k"] & \Delta^*_{k+1}(\mathcal{M}_h) \arrow[r] & \cdots \\
\cdots \arrow[r] & \underline{X}_h^{0,k} \arrow[u, "\kappa_k"] \arrow[r, "\underline{\mathrm{d}}^k_{0,h}"] \arrow[d, bend left, "\underline{E}_h^k"] & \underline{X}_h^{0,k+1} \arrow[u, "\kappa_{k+1}"] \arrow[r] \arrow[d, bend left, "\underline{E}_h^{k+1}"] & \cdots \\
\cdots \arrow[r] & \underline{X}_h^{r,k} \arrow[u, bend left, "\underline{R}_h^k"] \arrow[r, "\underline{\mathrm{d}}^k_{r,h}"] & \underline{X}_h^{r,k+1} \arrow[u, bend left, "\underline{R}_h^{k+1}"] \arrow[r] & \cdots
\end{tikzcd}$$

Key point: design of the extension cochain map $\underline{E}_h$ □

Discrete L^2-products I

- We define the global discrete L^2-product s.t., $\forall(\underline{\omega}_h, \underline{\mu}_h) \in \underline{X}_d^{r,k} \times \underline{X}_d^{r,k}$,

$$(\underline{\omega}_h, \underline{\mu}_h)_{k,h} \coloneqq \sum_{f \in \Delta_n(\mathcal{M}_h)} (\underline{\omega}_f, \underline{\mu}_f)_{k,f},$$

$$(\underline{\omega}_f, \underline{\mu}_f)_{k,f} \coloneqq \int_f P_f^{r,k} \underline{\omega}_f \wedge \star P_f^{r,k} \underline{\mu}_f + s_{k,f}(\underline{\omega}_f, \underline{\mu}_f)$$

- $s_{k,f}$ is a stabilization ensuring positivity and polynomial consistency s.t.

$$s_{k,f}(\underline{\omega}_f, \underline{\mu}_f) = \mathcal{S}_{k,f}(\underline{I}_f^{r,k} P_f^{r,k} \underline{\omega}_f - \underline{\omega}_f, \underline{I}_f^{r,k} P_f^{r,k} \underline{\mu}_f - \underline{\mu}_f),$$

  with $\mathcal{S}_{k,f}$ scaling in h_f like the $L^2\Lambda^k(f)$-product

- For any $d \in [k, n]$ and any $f \in \Delta_d(\mathcal{M}_h)$, we define the norm s.t.

$$\|\underline{\omega}_f\|_{k,f} \coloneqq (\underline{\omega}_f, \underline{\omega}_f)_{k,f}^{\frac{1}{2}} \quad \forall \underline{\omega}_f \in \underline{X}_f^{r,k}$$

Discrete L^2-products II

Theorem (Consistency of the local discrete L^2-products)

Let $f \in \Delta_n(\mathcal{M}_h)$*. Then, for any integers* $s \geq \frac{d}{2}$ *and* $0 \leq m \leq r+1$ *and all* $(\omega, \underline{\mu}_f) \in H^{\max\{r+1,s\}}\Lambda^k(f) \times \underline{X}_f^{r,k}$*, it holds,*

$$\left| (\underline{I}_h^{r,k}\omega, \underline{\mu}_f)_{k,f} - \int_f \omega \wedge \star P_f^{r,k}\underline{\mu}_f \right| \lesssim h_f^{r+1-m} |\mathrm{d}\omega|_{H^{(r+1,s)}\Lambda^{k+1}(f)} \|\underline{\mu}_f\|_{k,f}.$$

Discrete Poincaré inequality

Theorem (Discrete Poincaré inequality[6])

Let $\|\cdot\|_{k,h} := (\cdot,\cdot)_{k,h}^{\frac{1}{2}}$. *Then, for all* $\underline{\omega}_h \in \underline{X}_h^{r,k}$, *there exists* $\underline{\mu}_h \in \underline{X}_h^{r,k}$ *s.t.*

$$\underline{\mathrm{d}}_h^{r,k}\underline{\mu}_h = \underline{\omega}_h \text{ and } \|\underline{\mu}_h\|_{k,h} \lesssim \|\underline{\mathrm{d}}_h^{r,k}\underline{\omega}_h\|_{k+1,h}.$$

Equivalently,

$$\min_{\underline{\mu}_h \in \operatorname{Ker}\underline{\mathrm{d}}_h^{r,k}} \|\underline{\omega}_h + \underline{\mu}_h\|_{k,h} \lesssim \|\underline{\mathrm{d}}_h^{r,k}\underline{\omega}_h\|_{k+1,h}.$$

Corollary (Discrete Poincaré inequality on the orthogonal complement)

For all $\underline{\omega}_h \in (\operatorname{Ker}\underline{\mathrm{d}}_h^{r,k})^\perp$ *with orthogonal taken w.r.t. to* $(\cdot,\cdot)_{k,h}$, *it holds*

$$\|\underline{\omega}\|_{k,h} \lesssim \|\underline{\mathrm{d}}_h^{r,k}\underline{\omega}\|_{k+1,h}.$$

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[6] See [DP, Droniou, Hanot, Pitassi, 2025] and the precursor work [DP and Hanot, 2024]

Adjoint consistency

The following results estimates the error on the global Stokes formula

$$\int_\Omega \omega \wedge \mathrm{d}\mu + (-1)^k \int_\Omega \mathrm{d}\omega \wedge \mu = 0 \qquad \forall (\omega, \mu) \in \Lambda^k(\Omega) \times \Lambda^{n-k-1}(\Omega)$$

Theorem (Adjoint consistency)

Let $\omega \in C^0\Lambda^k(\overline{\Omega}) \cap H^{r+2}\Lambda^k(\mathcal{T}_h)$ *be s.t.* $\mathrm{tr}_{\partial\Omega}\,\omega = 0$*. Then, for all* $\underline{\mu}_h \in \underline{X}_h^{r,n-k-1}$*, it holds*

$$\left| \int_\Omega \omega \wedge \mathrm{d}_h^{r,n-k-1} \underline{\mu}_h + (-1)^k \sum_{f \in \Delta_n(\mathcal{M}_h)} \int_f \mathrm{d}\omega \wedge P_f^{r,n-k-1} \underline{\mu}_f \right|$$
$$\lesssim h^{r+1} \left(|\omega|_{H^{r+1}\Lambda^k(\mathcal{T}_h)} \|\underline{\mathrm{d}}_h^{r,n-k-1} \underline{\mu}_h\|_{n-k,h} + |\mathrm{d}\omega|_{H^{r+1}\Lambda^{k+1}(\mathcal{T}_h)} \|\underline{\mu}_h\|_{n-k-1,h} \right).$$

See J. Droniou's presentation for the relevance and proof of this result

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Restoring function spaces

- Key difference with respect to FEEC[7]: no conforming function spaces
- Virtual function spaces can be devised[8]
- Alternative construction based on harmonic extensions in FES[9]
- Computable conforming FE liftings on a submesh[10]

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[7] [Arnold, 2018]
[8] [Beirão da Veiga, Dassi, DP, Droniou, 2020]
[9] [Christiansen and Rapetti, 2025]
[10] [DP, Droniou, Pitassi, 2025]

Virtual function spaces

- We can construct a complex of virtual spaces for which the DDR DOFs are unisolvent:

$$\begin{tikzcd}
H^1(\Omega) \arrow[r, "\mathrm{grad}"] & H(\mathrm{curl};\Omega) \arrow[r, "\mathrm{curl}"] & H(\mathrm{div};\Omega) \arrow[r, "\mathrm{div}"] & L^2(\Omega) \\
V^k_{\mathrm{grad},h} \arrow[u, hook] \arrow[r, "\mathrm{grad}"] & V^k_{\mathrm{curl},h} \arrow[u, hook] \arrow[r, "\mathrm{curl}"] & V^k_{\mathrm{div},h} \arrow[u, hook] \arrow[r, "\mathrm{div}"] & \mathcal{P}^k(\mathcal{T}_h) \arrow[u, hook] \\
\underline{X}^k_{\mathrm{grad},h} \arrow[u, "\mathsf{dofs}^{-1}" , "\cong"'] \arrow[r, "\underline{G}^k_h"] & \underline{X}^k_{\mathrm{curl},h} \arrow[u, "\mathsf{dofs}^{-1}", "\cong"'] \arrow[r, "\underline{C}^k_h"] & \underline{X}^k_{\mathrm{div},h} \arrow[u, "\mathsf{dofs}^{-1}", "\cong"'] \arrow[r, "\underline{D}^k_h"] & \mathcal{P}^k(\mathcal{T}_h) \arrow[u, leftrightarrow]
\end{tikzcd}$$

- The spaces $V^k_{\bullet,h}$ are finite-dimensional but not polynomial in general
- This is the key idea of the Virtual Element Method (VEM)[11]

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[11] See [Beirão da Veiga et al., 2016 and 2018a] and [Beirão da Veiga, Dassi, DP, and Droniou, 2022] for the present construction

An example: The virtual curl space I

- For $X \in \mathcal{T}_h \cup \mathcal{F}_h$, let $\mathcal{P}^{k-1|k+1}(X)$ be a complement of $\mathcal{P}^{k-1}(X)$ in $\mathcal{P}^{k+1}(X)$
- The curl space on a face $F \in \mathcal{F}_h$ is

$$\begin{aligned} V^k_{\mathrm{curl}}(F) \coloneqq \Big\{ & v \in L^2(F)^{d-1} \;:\; \mathrm{div}_F\, v \in \mathcal{P}^{k+1}(F),\ \mathrm{rot}_F\, v \in \mathcal{P}^k(F), \\ & v \cdot t_E \in \mathcal{P}^k(E) \text{ for all } E \in \mathcal{E}_F, \\ & \int_F (v - \pi_{\mathcal{P}^k(F)} v) \cdot (x - x_F)\, p = 0 \text{ for all } p \in \mathcal{P}^{k-1|k+1}(F) \Big\} \end{aligned}$$

- The curl space on a mesh element $T \in \mathcal{T}_h$ is

$$\begin{aligned} V^k_{\mathrm{curl}}(T) \coloneqq \Big\{ & v \in L^2(T)^d \;:\; n_{TF} \times (v \times n_{TF}) \in V^k_{\mathrm{curl}}(F) \text{ for all } F \in \mathcal{F}_T, \\ & \int_T (\mathrm{curl}\, v - \pi_{\mathcal{P}^k(T)^d}\, \mathrm{curl}\, v) \cdot (x_T \times w) = 0 \text{ for all } w \in \mathcal{P}^{k-1|k}(T)^d, \\ & \int_T (v - \pi_{\mathcal{P}^k(T)^d} v) \cdot (x - x_T)\, p = 0 \text{ for all } p \in \mathcal{P}^{k-1|k+1}(T) \Big\} \end{aligned}$$

An example: The virtual curl space II

- The global curl space is defined setting

$$V^k_{\mathrm{curl}}(\mathcal{T}_h) := \left\{ v \in H(\mathrm{curl};\Omega) \ :\ v_{|T} \in V^k_{\mathrm{curl}}(T) \text{ for all } T \in \mathcal{T}_h \right\}$$

- The degrees of freedom are:
  - For each edge $E \in \mathcal{E}_h$,

$$V^k_{\mathrm{curl}}(\mathcal{T}_h) \ni v \mapsto \int_E (v \cdot t_E)\, p \in \mathbb{R} \quad \forall p \in \mathcal{P}^k(E)$$

  - If $k \geq 1$, for each face $F \in \mathcal{F}_h$,

$$V^k_{\mathrm{curl}}(\mathcal{T}_h) \ni v \mapsto \int_F v_{t,F} \cdot w \in \mathbb{R} \quad \forall w \in \mathcal{RT}^k(F)$$

  - If $k \geq 1$, for each element $T \in \mathcal{T}_h$,

$$V^k_{\mathrm{curl}}(\mathcal{T}_h) \ni v \mapsto \int_T v \cdot w \in \mathbb{R} \quad \forall w \in \mathcal{RT}^k(T)$$

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Thank you for your attention!

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