Nonlinear free energy diminishing schemes for convection-diffusion equations : convergence and long time behaviour

Claire Chainais-Hillairet

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Joint work with
C. Cancès (Lille), M. Herda (Lille) and S. Krell (Nice)

Laboratoire Paul Painlevé

Université de Lille 1 Sciences et Technologies

Outline of the talk

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1. Motivation
2. Nonlinear TPFA schemes
3. Nonlinear DDFV schemes
4. Some numerical results

About convection-diffusion equations

Model problem : Fokker-Planck equation

$$\begin{cases} \partial_t u + \text{div}\, \mathbf{J} = 0, \quad \mathbf{J} = \mathbf{\Lambda}(-\nabla u - u\nabla V), \text{ in } \Omega \times (0,T) \\ + \text{ boundary conditions (Dirichlet on } \Gamma^D\text{/no-flux on } \Gamma^N) \\ u(\cdot, 0) = u_0 \geq 0 \end{cases}$$

Examples

- Semiconductor models, corrosion models
  - ➠ $\mathbf{\Lambda} = \mathbf{I}$
  - ➠ coupling with a Poisson equation for V
- Porous media flow
  - ➠ $\mathbf{\Lambda}$ bounded, symmetric and uniformly elliptic
  - ➠ $V = gz$

Assumptions : $V \in C^1(\Omega, \mathbb{R}^+)$, $\quad \int_\Omega u_0 > 0.$

Structural properties

$$\begin{cases} \partial_t u + \text{div}\, \mathbf{J} = 0, \quad \mathbf{J} = \mathbf{\Lambda}(-\nabla u - u\nabla V), \\ u(\cdot, 0) = u_0 \geq 0 \;\; + \text{boundary conditions} \end{cases}$$

- Existence and uniqueness of the solution
- Nonnegativity of u, mass conservation if $\Gamma^D = \emptyset$
- Existence of a thermal equilibrium :

$$u_\infty = \lambda e^{-V} (\Longrightarrow \mathbf{J} = 0)$$

  - if $\Gamma^D = \emptyset$,

$$\lambda = \frac{\int_\Omega u_0}{\int_\Omega e^{-V}}, \quad \text{so that} \quad \int_\Omega u_\infty = \int_\Omega u_0$$

  - if $\Gamma^D \neq \emptyset$ and the boundary data u^D satisfy a compatibility assumption

$$u^D = \lambda e^{-V} \text{ on } \Gamma^D.$$

Long time behaviour of the Fokker-Planck equation

$$\begin{cases} \partial_t u + \text{div } \mathbf{J} = 0, \quad \mathbf{J} = -\nabla u - u\nabla V, \\ \mathbf{J} \cdot \mathbf{n} = 0 \text{ on } \Gamma \text{ and } \ u(\cdot, 0) = u_0 \geq 0 \end{cases} \qquad u^\infty = \lambda e^{-V}$$

Main result : exponential decay towards the steady-state

$$\|u(t) - u^\infty\|_1^2 \leq C(u_0, V)e^{-\kappa t}$$

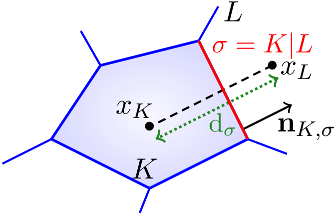
- Energy dissipation, with $E = \int_\Omega u \log(u/u^\infty)$

$$\frac{d}{dt}E = -\int_\Omega u|\nabla \log(\frac{u}{u^\infty})|^2 = -4\int_\Omega u^\infty \left|\nabla\sqrt{\frac{u}{u^\infty}}\right|^2$$

as $\mathbf{J} = -u\nabla(\log u + V) = -u\nabla \log \frac{u}{u^\infty}$

- Logarithmic Sobolev + Csiszar-Kullback inequalities

$\mathbf{\Lambda} = \mathbf{I}$, TPFA B-schemes



Numerical fluxes

$$\mathbf{J} = -\nabla u - u\nabla V \Longrightarrow \mathcal{F}_{K,\sigma} \approx \int_\sigma (-\nabla u - u\nabla V) \cdot \mathbf{n}_{K,\sigma}$$

$$\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B\big(V_L - V_K\big)u_K - B\big(-V_L + V_K\big)u_L\Big)$$

Examples of B functions

$$B_{up}(s) = 1 + s^-, \quad B_{ce}(s) = 1 - s/2, \quad B_{sg}(s) = \frac{s}{e^s - 1}$$

Hypotheses on B

- $B(0) = 1$ and $B(s) > 0 \quad \forall s \in \mathbb{R}$,
- $B(s) - B(-s) = -s \quad \forall s \in \mathbb{R}$.

$\mathbf{\Lambda} = \mathbf{I}$, TPFA B-schemes : long time behaviour

$$\begin{cases} \mathrm{m}(K)\dfrac{u_K^{n+1}-u_K^n}{\Delta t} + \displaystyle\sum_{\sigma\in\mathcal{E}_K^{int}} \mathcal{F}_{K,\sigma}^{n+1} = 0, \\ \mathcal{F}_{K,\sigma} = \dfrac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B\big(V_L - V_K\big)u_K - B\big(-V_L + V_K\big)u_L\Big). \end{cases}$$

Properties

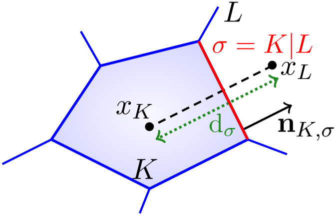
- Existence, uniqueness of the solution to the scheme
- Preservation of positivity, conservation of mass
- Existence of a steady-state $(u_K^\infty)_{K\in\mathcal{T}}$.
- The scheme preserves the thermal equilibrium iff $B = B_{sg}$:

$$u_K^\infty = \lambda \exp(-V_K) \Longrightarrow \mathcal{F}_{K,\sigma} = 0.$$

Exponential decay towards the discrete steady-state

❑ C.-H., Herda, submitted

Motivation



Main drawbacks of the TPFA scheme

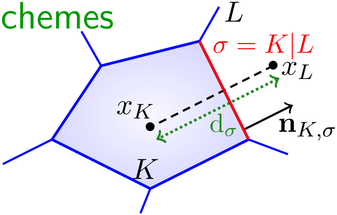
- Admissibility of the mesh
- $\mathbf{\Lambda} = \mathbf{I}$

Requirements wanted for a new scheme

- To be applicable on almost-general meshes
- To be applicable for anisotropic equations
- To preserve thermal equilibrium
- To be energy-diminishing
- To ensure the positivity

Outline of the talk

Principle of the nonlinear TPFA schemes



Numerical fluxes

$$\mathbf{J} = -\nabla u - u\nabla V = -u\nabla(\log u + V)$$

$$\mathcal{F}_{K,\sigma} \approx \int_\sigma -u\nabla(\log u + V) \cdot \mathbf{n}_{K,\sigma}$$

$$\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma} r(u_K, u_L)\Big(\log u_K + V_K - \log u_L - V_L\Big)$$

Examples of r functions

$$r(x,y) = \frac{x+y}{2}, \quad r(x,y) = \frac{x-y}{\log x - \log y},$$

or other choices of mean value.

Principle of the nonlinear TPFA schemes

Some references about nonlinear schemes for linear equations

- Burman, Ern, 2004
- Le Potier, 2005, Cancès, Cathala, Le Potier, 2013
- Cancès, Guichard, 2016

The nonlinear schemes for Fokker-Planck equations

$$\begin{cases} \mathrm{m}(K)\dfrac{u_K^{n+1}-u_K^n}{\Delta t} + \displaystyle\sum_{\sigma\in\mathcal{E}_K^{int}} \mathcal{F}_{K,\sigma}^{n+1} = 0, \\ \mathcal{F}_{K,\sigma} = \dfrac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma} r(u_K, u_L)\Big(\log u_K + V_K - \log u_L - V_L\Big) \end{cases}$$

The schemes preserve the thermal equilibrium

- $u_K^\infty = \lambda e^{-V_K}$ is a steady-state,
- λ is fixed by the conservation of mass.

Dissipativity of the nonlinear TPFA schemes

$$\begin{cases} \mathsf{m}(K)\dfrac{u_K^{n+1}-u_K^n}{\Delta t} + \displaystyle\sum_{\sigma\in\mathcal{E}_K^{int}} \mathcal{F}_{K,\sigma}^{n+1} = 0, \\ \mathcal{F}_{K,\sigma} = \dfrac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma} r(u_K,u_L)\Big(\log u_K + V_K - \log u_L - V_L\Big) \end{cases}$$

Dissipation of some discrete entropies

- Φ, regular convex function,
- Discrete relative entropy : $\mathbb{E}_\Phi^n = \displaystyle\sum_{K\in\mathcal{T}} u_K^\infty \Phi(\frac{u_K^n}{u_K^\infty})$

$$\frac{\mathbb{E}_\Phi^{n+1}-\mathbb{E}_\Phi^n}{\Delta t} + \mathbb{I}_\Phi^{n+1} \leq 0$$

with

$$\mathbb{I}_\Phi = \sum_{\sigma\in\mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma} r(u_K,u_L)\big(\log\frac{u_K}{u_K^\infty} - \log\frac{u_L}{u_L^\infty}\big)\big(\Phi'(\frac{u_K}{u_K^\infty}) - \Phi'(\frac{u_L}{u_L^\infty})\big)$$

Main results for the nonlinear TPFA schemes

A priori estimates and existence of a solution to the scheme

- Uniform bounds : $\Phi(s) = (s - M)^+$ and $\Phi(s) = (s - m)^-$,

  with $M = \max(1, \max \frac{u_K^0}{u_K^\infty})$, $m = \min(1, \min \frac{u_K^0}{u_K^\infty})$
- Existence via a topological degree argument.

Towards the exponential decay

$$\Phi(s) = s \log s, \quad \mathbb{E}_\Phi^n = \sum_{K \in \mathcal{T}} u_K^n \log \frac{u_K^n}{u_K^\infty}$$

$$\mathbb{I}_\Phi = \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma} r(u_K, u_L) \Big(\log \frac{u_K}{u_K^\infty} - \log \frac{u_L}{u_L^\infty}\Big)^2 \quad \geq \widehat{\mathbb{I}}_\Phi$$

with $\widehat{\mathbb{I}}_\Phi = 4 \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma} \min(u_K^\infty, u_L^\infty) \Big(\sqrt{\frac{u_K}{u_K^\infty}} - \sqrt{\frac{u_L}{u_L^\infty}}\Big)^2$

➤ Discrete Logarithmic-Sobolev inequality : $\mathbb{E}_\Phi \leq C \, \widehat{\mathbb{I}}_\Phi$

Outline of the talk

Introduction to DDFV schemes

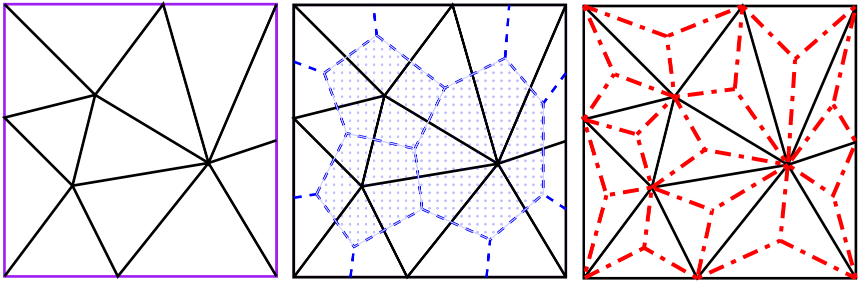
Some (partial) references

- Domelevo, Omnes, 2005
- Coudière, Vila, Villedieu, 1999
- Andreianov, Boyer, Hubert, 2007
- Andreianov, Bendahmane, Karlsen, 2010

Principles (for diffusion equations)

- Unknowns located at the centers and the vertices of the mesh
- Discrete gradient defined on a diamond mesh
- Discrete divergence defined on primal and dual meshes
- Integration of the equation on primal cells and dual cells
- Discrete-duality formula

Meshes : primal, dual and diamond meshes



$\mathfrak{M}$, $\partial\mathfrak{M}$ $\qquad$ $\mathfrak{M}^*$, $\partial\mathfrak{M}^*$ $\qquad$ $\mathfrak{D}$

Space of discrete unknowns $\quad \mathbb{R}^{\mathcal{T}}$

$$u_{\mathcal{T}} = \left((u_K)_{K \in \mathfrak{M} \cup \partial\mathfrak{M}}, (u_{K^*})_{K^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*} \right)$$

Space of discrete gradients $\quad (\mathbb{R}^2)^{\mathfrak{D}}$

$$\boldsymbol{\xi}_{\mathfrak{D}} = \left((\boldsymbol{\xi}_{\mathcal{D}})_{\mathcal{D} \in \mathfrak{D}} \right)$$

Discrete operators and discrete duality property

Discrete gradient

$$\nabla^{\mathfrak{D}} \; : \; u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}} \mapsto (\nabla^{\mathcal{D}} u_{\mathcal{T}})_{\mathcal{D}\in\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}$$

Discrete divergence

$$\mathrm{div}^{\mathcal{T}} : \; \boldsymbol{\xi}_{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}} \mapsto \mathrm{div}^{\mathcal{T}}(\boldsymbol{\xi}_{\mathfrak{D}}) \in \mathbb{R}^{\mathcal{T}}$$

Scalar products and norms

$$\begin{aligned} [\![v_{\mathcal{T}}, u_{\mathcal{T}}]\!]_{\mathcal{T}} &= \frac{1}{2}\Big(\sum_{K\in\mathfrak{M}} \mathrm{m}_K u_K v_K + \sum_{K^*\in\overline{\mathfrak{M}^*}} \mathrm{m}_{K^*} u_{K^*} v_{K^*} \Big), \\ (\boldsymbol{\xi}_{\mathfrak{D}}, \boldsymbol{\varphi}_{\mathfrak{D}})_{\mathfrak{D}} &= \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}} \, \boldsymbol{\xi}_{\mathcal{D}} \cdot \boldsymbol{\varphi}_{\mathcal{D}}. \end{aligned}$$

Discrete duality formula

$$[\![\mathrm{div}^{\mathcal{T}} \boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}]\!]_{\mathcal{T}} = -(\boldsymbol{\xi}_{\mathfrak{D}}, \nabla^{\mathfrak{D}} v_{\mathcal{T}})_{\mathfrak{D}} + \langle \gamma^{\mathfrak{D}}(\boldsymbol{\xi}_{\mathfrak{D}}) \cdot \mathbf{n}, \gamma^{\mathcal{T}}(v_{\mathcal{T}}) \rangle_{\partial\Omega}$$

DDFV scheme for an anisotropic diffusion equation

$$\begin{cases} -\operatorname{div}\boldsymbol{\Lambda}\nabla u = f \\ + \text{ boundary conditions} \end{cases}$$

The DDFV scheme

$$\begin{cases} -\operatorname{div}^{\mathcal{T}}\Big(\boldsymbol{\Lambda}_{\mathfrak{D}}\nabla^{\mathfrak{D}}u_{\mathcal{T}}) = f_{\mathcal{T}} \\ + \text{ boundary conditions} \end{cases}$$

"Variational" formulation

$$(\boldsymbol{\Lambda}_{\mathfrak{D}}\nabla^{\mathfrak{D}}u_{\mathcal{T}}, \nabla^{\mathfrak{D}}v_{\mathcal{T}})_{\mathfrak{D}} = [\![f_{\mathcal{T}}, v_{\mathcal{T}}]\!]_{\mathcal{T}} \quad \forall v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$$

DDFV scheme for an anisotropic diffusion equation

$$\begin{cases} -\operatorname{div}\mathbf{\Lambda}\nabla u = f \\ + \text{ boundary conditions} \end{cases}$$

The DDFV scheme

$$\begin{cases} -\operatorname{div}^{\mathcal{T}}\Big(\mathbf{\Lambda}_{\mathfrak{D}}\nabla^{\mathfrak{D}}u_{\mathcal{T}}) = f_{\mathcal{T}} \\ + \text{ boundary conditions} \end{cases}$$

"Variational" formulation

$$\underbrace{(\mathbf{\Lambda}_{\mathfrak{D}}\nabla^{\mathfrak{D}}u_{\mathcal{T}}, \nabla^{\mathfrak{D}}v_{\mathcal{T}})_{\mathfrak{D}}}_{(\nabla^{\mathfrak{D}}u_{\mathcal{T}}, \nabla^{\mathfrak{D}}v_{\mathcal{T}})_{\mathbf{\Lambda},\mathfrak{D}}} = [\![f_{\mathcal{T}}, v_{\mathcal{T}}]\!]_{\mathcal{T}} \quad \forall v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$$

with $(\boldsymbol{\xi}_{\mathfrak{D}}, \boldsymbol{\varphi}_{\mathfrak{D}})_{\mathbf{\Lambda},\mathfrak{D}} = \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}\, \boldsymbol{\xi}_{\mathcal{D}} \cdot \mathbf{\Lambda}_{\mathcal{D}}\boldsymbol{\varphi}_{\mathcal{D}}, \quad \mathbf{\Lambda}_{\mathcal{D}} = \frac{1}{\mathrm{m}_{\mathcal{D}}}\int_{\mathcal{D}} \mathbf{\Lambda}.$

Structure of the scalar product of two discrete gradients

Discrete gradient

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{2\mathrm{m}_{\mathcal{D}}}\Big(\mathrm{m}_{\sigma}(u_L - u_K)\mathbf{n}_{\sigma K} + \mathrm{m}_{\sigma^*}(u_{L^*} - u_{K^*})\mathbf{n}_{\sigma^* K^*}\Big).$$

$$\delta^{\mathcal{D}} u_{\mathcal{T}} = \begin{pmatrix} u_K - u_L \\ u_{K^*} - u_{L^*} \end{pmatrix}$$

Scalar product

$$\begin{aligned}(\nabla^{\mathfrak{D}} u_{\mathcal{T}}, \nabla^{\mathfrak{D}} v_{\mathcal{T}})_{\mathbf{\Lambda},\mathfrak{D}} &= \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}\ \nabla^{\mathcal{D}} u_{\mathcal{T}} \cdot \mathbf{\Lambda}_{\mathcal{D}} \nabla^{\mathcal{D}} v_{\mathcal{T}}, \\ &= \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}\ \delta^{\mathcal{D}} u_{\mathcal{T}} \cdot \mathbb{A}_{\mathcal{D}} \delta^{\mathcal{D}} v_{\mathcal{T}}.\end{aligned}$$

Local matrices $\mathbb{A}_{\mathcal{D}}$

➟ Uniform bound on $\mathrm{Cond}_2(\mathbb{A}_{\mathcal{D}})$

Nonlinear formulation of the problem

$$\partial_t u + \text{div } \mathbf{J} = 0, \quad \mathbf{J} = -u\boldsymbol{\Lambda}\nabla(\log u + V)$$

How to approximate the current ?

- $V_{\mathcal{T}}$ given. For $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, we define $g_{\mathcal{T}} = \log u_{\mathcal{T}} + V_{\mathcal{T}}$.
- $\nabla^{\mathfrak{D}} g_{\mathcal{T}}$ has a sense.
- Reconstruction of u on the diamond mesh, $r^{\mathfrak{D}}(u_{\mathcal{T}})$

$$r^{\mathcal{D}}(u_{\mathcal{T}}) = \frac{1}{4}(u_K + u_L + u_{K^*} + u_{L^*}) \quad \forall \mathcal{D} \in \mathfrak{D}$$

- We can define a discrete current on the diamond mesh :

$$\mathbf{J}_{\mathfrak{D}} = -r^{\mathfrak{D}}(u_{\mathcal{T}})\boldsymbol{\Lambda}_{\mathfrak{D}}\nabla^{\mathfrak{D}} g_{\mathcal{T}}.$$

The scheme (no-flux boundary conditions)

$$\partial_t u + \text{div}\,\mathbf{J} = 0, \quad \mathbf{J} = -u\boldsymbol{\Lambda}\nabla(\log u + V)$$

"Classical" formulation

$$\frac{u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^{n}}{\Delta t} + \text{div}^{\mathcal{T}}(J_{\mathfrak{D}}^{n+1}) = 0,\ \ J_{\mathfrak{D}}^{n+1} = -r^{\mathfrak{D}}[u_{\mathcal{T}}^{n+1}]\boldsymbol{\Lambda}^{\mathfrak{D}}\nabla^{\mathfrak{D}} g_{\mathcal{T}}^{n+1},$$

$$\mathrm{m}_\sigma J_{\mathcal{D}}^{n+1} \cdot \mathbf{n} = 0, \qquad \forall\ \mathcal{D} = \mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}.$$

Compact form

$$\left[\!\left[\frac{u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^{n}}{\Delta t}, \psi_{\mathcal{T}} \right]\!\right]_{\mathcal{T}} + T_{\mathfrak{D}}(u_{\mathcal{T}}^{n+1}; g_{\mathcal{T}}^{n+1}, \psi_{\mathcal{T}}) = 0, \quad \forall \psi_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}},$$

$$T_{\mathfrak{D}}(u_{\mathcal{T}}^{n+1}; g_{\mathcal{T}}^{n+1}, \psi_{\mathcal{T}}) = \sum_{\mathcal{D}\in\mathfrak{D}} r^{\mathcal{D}}(u_{\mathcal{T}}^{n+1})\, \delta^{\mathcal{D}} g_{\mathcal{T}}^{n+1} \cdot \mathbb{A}^{\mathcal{D}} \delta^{\mathcal{D}} \psi_{\mathcal{T}},$$

$$g_{\mathcal{T}}^{n+1} = \log(u_{\mathcal{T}}^{n+1}) + V_{\mathcal{T}}.$$

Key discrete properties

$$\left[\!\left[\frac{u_{\mathcal{T}}^{n+1}-u_{\mathcal{T}}^{n}}{\Delta t},\psi_{\mathcal{T}}\right]\!\right]_{\mathcal{T}}+T_{\mathfrak{D}}(u_{\mathcal{T}}^{n+1};g_{\mathcal{T}}^{n+1},\psi_{\mathcal{T}})=0,\quad \forall\psi_{\mathcal{T}}\in\mathbb{R}^{\mathcal{T}},$$

$$T_{\mathfrak{D}}(u_{\mathcal{T}}^{n+1};g_{\mathcal{T}}^{n+1},\psi_{\mathcal{T}})=\sum_{\mathcal{D}\in\mathfrak{D}}r^{\mathcal{D}}(u_{\mathcal{T}}^{n+1})\,\delta^{\mathcal{D}}g_{\mathcal{T}}^{n+1}\cdot\mathbb{A}^{\mathcal{D}}\delta^{\mathcal{D}}\psi_{\mathcal{T}},$$

Mass conservation

$$\sum_{K\in\mathfrak{M}}\mathrm{m}_K u_K^n=\sum_{K\in\mathfrak{M}}\mathrm{m}_K u_K^0,\qquad \sum_{K^*\in\overline{\mathfrak{M}^*}}\mathrm{m}_{K^*}u_{K^*}^n=\sum_{K^*\in\overline{\mathfrak{M}^*}}\mathrm{m}_{K^*}u_{K^*}^0$$

Steady-state

$$u_K^\infty=\rho e^{-V(x_K)},\ u_{K^*}^\infty=\rho^* e^{-V(x_{K^*})}$$

$\rho,\ \rho^*$ ensuring the conservation of mass.

Energy-dissipation property

$$\frac{\mathbb{E}_{\mathcal{T}}^{n+1}-\mathbb{E}_{\mathcal{T}}^{n}}{\Delta t}+\mathbb{I}_{\mathcal{T}}^{n+1}\le 0.$$

$$\mathbb{E}_{\mathcal{T}}^n=\left[\!\left[u_{\mathcal{T}}^n\log\left(\frac{u_{\mathcal{T}}^n}{u_{\mathcal{T}}^\infty}\right),1_{\mathcal{T}}\right]\!\right]_{\mathcal{T}},\qquad \mathbb{I}_{\mathcal{T}}^n=T_{\mathfrak{D}}\left(u_{\mathcal{T}}^n;g_{\mathcal{T}}^n,g_{\mathcal{T}}^n\right)$$

Consequences

- Cancès, C.-H., Krell, 2018

- Decay of the free energy + bounds
- Further a priori estimates related to Fisher information
- Positivity + lower bound of the approximate solution
- Existence of a solution to the scheme
- Compactness of a sequence of approximate solutions
- Convergence (if penalization term)

Outline of the talk

Test case

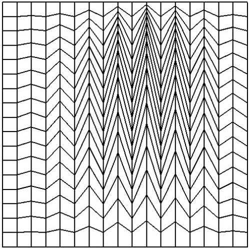
Data and solution

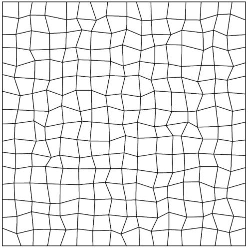
- $\Omega = (0,1)^2$, and $V(x_1, x_2) = -x_2$.
- Exact solution, with $\alpha = \pi^2 + \frac{1}{4}$,

$$u_{\text{ex}}((x_1,x_2),t) = e^{-\alpha t + \frac{x_2}{2}} \left(\pi \cos(\pi x_2) + \frac{1}{2} \sin(\pi x_2) \right) + \pi e^{(x_2 - \frac{1}{2})}$$

- Initial condition : $u_0(x) = u_{\text{ex}}(x, 0)$.

Meshes





Convergence with respect to the grid

On Kershaw meshes

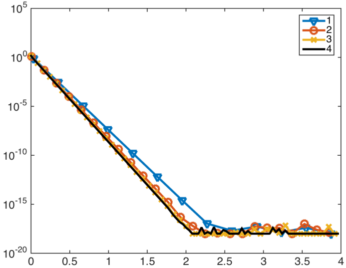
M	dt	erru	ordu	N_{max}	N_{mean}	Min u^n
1	2.0E-03	7.2E-03	—	9	2.15	1.010E-01
2	5.0E-04	1.7E-03	2.09	8	2.02	2.582E-02
3	1.2E-04	7.2E-04	2.20	7	1.49	6.488E-03
4	3.1E-05	4.0E-04	2.11	7	1.07	1.628E-03
5	3.1E-05	2.6E-04	1.98	7	1.04	1.628E-03

On quadrangle meshes

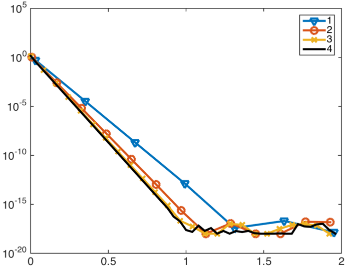
M	dt	erru	ordu	N_{max}	N_{mean}	Min u^n
1	4.0E-03	2.1E-02	—	9	2.26	1.803E-01
2	1.0E-03	5.1E-03	2.08	9	2.04	5.079E-02
3	2.5E-04	1.3E-03	2.06	8	1.96	1.352E-02
4	6.3E-05	3.3E-04	2.09	8	1.22	3.349E-03
5	1.2E-05	7.7E-05	1.70	7	1.01	8.695E-04

Long time behaviour

Exponential decay of the discrete relative energy $\mathbb{E}_{\mathcal{T}}^{n}$



Kershaw



quadrangles

Conclusion

Results obtained for the nonlinear schemes

- well-posedness of the schemes
- preservation of the positivity / bounds
- preservation of the steady-state (thermal equilibrium)
- exponential decay towards the steady-state
- TPFA/DDFV schemes

Question

Is it possible to extend these techniques to high order methods ?