A Hybrid High-Order method for the incompressible Navier-Stokes problem robust for large irrotational body forces

Daniel Castanon Quiroz[†‡] and Daniele A. Di Pietro[†]

[†]Institut Montpelliérain Alexander Grothendieck, University of Montpellier, Montpellier, France
[‡]danielcq.mathematics@gmail.com

Introduction

Let $\Omega \subset \mathbb{R}^3$ denote an open, bounded, simply connected polyhedral domain with Lipschitz boundary $\partial\Omega$. Let $\nu > 0$ be a real number representing the kinematic viscosity of the fluid, and let $\boldsymbol{f} \in L^2(\Omega)^3$ be a given vector field representing a body force. Setting $\boldsymbol{U} := H_0^1(\Omega)^3$ and $P := \left\{q \in L^2(\Omega) : \int_\Omega q = 0\right\}$, we consider the steady incompressible Navier–Stokes problem: Find $(\boldsymbol{u}, p) \in \boldsymbol{U} \times P$ such that

$$\nu a(\boldsymbol{u}, \boldsymbol{v}) + t(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \ell(\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{U}, \tag{1a}$$

$$-b(\boldsymbol{u}, q) = 0 \qquad \forall q \in L^2(\Omega), \tag{1b}$$

with bilinear forms $a : \boldsymbol{U} \times \boldsymbol{U} \to \mathbb{R}$, $b : \boldsymbol{U} \times L^2(\Omega) \to \mathbb{R}$, and $\ell : L^2(\Omega)^d \times \boldsymbol{U} \to \mathbb{R}$ defined by

$$a(\boldsymbol{w}, \boldsymbol{v}) := \int_\Omega \nabla \boldsymbol{w} : \nabla \boldsymbol{v}, \quad b(\boldsymbol{v}, q) := -\int_\Omega (\nabla \cdot \boldsymbol{v}) q, \quad \ell(\boldsymbol{f}, \boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v},$$

and trilinear form $t : \boldsymbol{U} \times \boldsymbol{U} \times \boldsymbol{U} \to \mathbb{R}$ such that

$$t(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{z}) := \int_\Omega (\nabla \times \boldsymbol{w}) \times \boldsymbol{v} \cdot \boldsymbol{z}. \tag{2}$$

Above, $\nabla \cdot$ and $\nabla \times$ denote, respectively, the divergence and curl operators, while $\times$ is the cross product of two vectors. The convective term in (2) is expressed in rotational form, so p is here the so-called Bernoulli pressure, which is related to the kinematic pressure p_{kin} by the equation $p = p_{\text{kin}} + \frac{1}{2}|\boldsymbol{u}|^2$.
The domain Ω being simply connected, we have the following Hodge decomposition of the body force:

$$\boldsymbol{f} = \boldsymbol{g} + \lambda \nabla \psi, \tag{3}$$

where $\boldsymbol{g} \in \boldsymbol{H}_{\boldsymbol{0}}(\text{curl}; \Omega) := \left\{\boldsymbol{v} \in L^2(\Omega)^3 : \boldsymbol{\gamma_\tau v} = \boldsymbol{0} \text{ on } \partial\Omega\right\}$ with $\boldsymbol{\gamma_\tau}$ denoting the tangent trace operator on $\partial\Omega$, $\psi \in H^1(\Omega)$ is such that $\|\nabla\psi\|_{L^2(\Omega)^3} = 1$, and $\lambda \in \mathbb{R}^+$.

Objective

To design an HHO discretization method for problem (1) such that the velocity error estimates are **uniform** in λ and **independent** of the pressure.

The HHO Space

Let a polynomial degree $k \geq 0$ be fixed. We define the following global space of discrete velocity unknowns:

$$\underline{\boldsymbol{U}}_h^k := \{\underline{\boldsymbol{v}}_h = ((\boldsymbol{v}_T)_{T \in \mathcal{T}_h}, (\boldsymbol{v}_F)_{F \in \mathcal{F}_h}) : \boldsymbol{v}_T \in \mathbb{P}^k(T)^3 \quad \forall T \in \mathcal{T}_h, \text{ and } \quad \boldsymbol{v}_F \in \mathbb{P}^k(F)^3 \quad \forall F \in \mathcal{F}_h\}.$$

We define the global interpolation operator on a smooth function over Ω by $\underline{\boldsymbol{I}}_h^k : H^1(\Omega)^3 \to \underline{\boldsymbol{U}}_h^k$ such that,

$$\underline{\boldsymbol{I}}_h^k \boldsymbol{v} := ((\boldsymbol{\pi}_T^k \boldsymbol{v}_{|T})_{T \in \mathcal{T}_h}, (\boldsymbol{\pi}_F^k \boldsymbol{v}_{|F})_{F \in \mathcal{F}_h}) \qquad \forall \boldsymbol{v} \in H^1(\Omega)^3,$$

where $\boldsymbol{\pi}_T^k$, and $\boldsymbol{\pi}_F^k$ are the L^2-orthogonal projectors over cells and faces, respectively. We furnish $\underline{\boldsymbol{U}}_h^k$ with the discrete H^1-like seminorm such that, for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$,

$$\|\underline{\boldsymbol{v}}_h\|_{1,h}^2 := \sum_{T \in \mathcal{T}_h} \|\underline{\boldsymbol{v}}_T\|_{1,T}^2,$$

where, for all $T \in \mathcal{T}_h$,

$$\|\underline{\boldsymbol{v}}_T\|_{1,T}^2 := \|\nabla \boldsymbol{v}_T\|_{L^2(T)^{3\times 3}}^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|\boldsymbol{v}_F - \boldsymbol{v}_T\|_{L^2(F)^3}^2.$$

The global spaces of discrete unknowns for the velocity and the pressure, respectively accounting for the wall boundary condition and the zero-average condition, are

$$\underline{\boldsymbol{U}}_{h,0}^k := \left\{\underline{\boldsymbol{v}}_h = ((\boldsymbol{v}_T)_{T \in \mathcal{T}_h}, (\boldsymbol{v}_F)_{F \in \mathcal{F}_h}) \in \underline{\boldsymbol{U}}_h^k : \boldsymbol{v}_F = 0 \quad \forall F \in \mathcal{F}_h^{\text{b}}\right\}, \qquad P_h^k := \mathbb{P}^k(\mathcal{T}_h) \cap P.$$

Velocity Reconstruction

Let an element $T \in \mathcal{T}_h$ be fixed, and denote by $\mathbb{RTN}^k(T) := \mathbb{P}^k(T)^3 + \boldsymbol{x}\mathbb{P}^k(T)$ the local Raviart–Thomas–Nédélec space of degree k. We define the local velocity reconstruction operator $\boldsymbol{R}_T^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{RTN}^k(T)$ such that, for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$,

$$\int_T \boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T \cdot \boldsymbol{w} = \int_T \boldsymbol{v}_T \cdot \boldsymbol{w}, \qquad \forall \boldsymbol{w} \in \mathbb{P}^{k-1}(T)^3, \tag{4a}$$

$$\boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T \cdot \boldsymbol{n}_{TF} = \boldsymbol{v}_F \cdot \boldsymbol{n}_{TF} \qquad \forall F \in \mathcal{F}_T. \tag{4b}$$

Proposition

It holds for all $r \in [1, 6]$ and all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$,

$$\|\boldsymbol{R}_h^k \underline{\boldsymbol{v}}_h\|_{L^r(\Omega)^3} \leq C \|\underline{\boldsymbol{v}}_h\|_{1,h}. \tag{5}$$

To discretize ℓ in (1) we introduce $\ell_h : L^2(\Omega)^3 \times \underline{\boldsymbol{U}}_h^k \to \mathbb{R}$ such that, for any $\boldsymbol{f} \in L^2(\Omega)^3$ and any $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$,

$$\ell_h(\boldsymbol{f}, \underline{\boldsymbol{v}}_h) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{R}_h^k \underline{\boldsymbol{v}}_h. \tag{6}$$

To discretize t in (1) we introduce the global trilinear form t_h on $\underline{\boldsymbol{U}}_h^k \times \underline{\boldsymbol{U}}_h^k \times \underline{\boldsymbol{U}}_h^k \to \mathbb{R}$ such that

$$\begin{aligned} t_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h, \underline{\boldsymbol{z}}_h) := & \sum_{T \in \mathcal{T}_h} \left[\int_T \nabla \boldsymbol{w}_T \boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T \cdot \boldsymbol{R}_T^k \underline{\boldsymbol{z}}_T - \int_T \nabla \boldsymbol{w}_T \boldsymbol{R}_T^k \underline{\boldsymbol{z}}_T \cdot \boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T\right] \\ & + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_F (\boldsymbol{w}_F - \boldsymbol{w}_T) \cdot \boldsymbol{R}_T^k \underline{\boldsymbol{z}}_T \left(\boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T \cdot \boldsymbol{n}_{TF}\right) \\ & - \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_F (\boldsymbol{w}_F - \boldsymbol{w}_T) \cdot \boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T \left(\boldsymbol{R}_T^k \underline{\boldsymbol{z}}_T \cdot \boldsymbol{n}_{TF}\right). \end{aligned}$$

Discrete Problem

The HHO discretization of problem (1) then reads: Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ such that

$$\nu a_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + t_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + b_h(\underline{\boldsymbol{v}}_h, p_h) = \ell_h(\boldsymbol{f}, \underline{\boldsymbol{v}}_h) \quad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}, \tag{7a}$$

$$-b_h(\underline{\boldsymbol{u}}_h, q_h) = 0 \qquad \forall q_h \in \mathbb{P}^k(\mathcal{T}_h). \tag{7b}$$

Theorem

Recalling the decomposition (3) of $\boldsymbol{f}$, we assume that it holds, for some $\alpha \in (0, 1)$,

$$\|\boldsymbol{g}\|_{L^2(\Omega)^3} \leq C\alpha\nu^2. \tag{8}$$

Let $(\boldsymbol{u}, p) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$ be a solution to the Navier–Stokes equations (1), and $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_h^k \times P_h^k$ be a solution to the HHO scheme (7). Then, it holds:

$$\begin{aligned} & \|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_k^h \boldsymbol{u}\|_{1,h} + \nu^{-1} \|p_h - I_k^h p\|_{L^2(\Omega)} \\ & \leq C h^{k+1} (1 - \alpha)^{-1} \left(|\boldsymbol{u}|_{H^{k+2}(\mathcal{T}_h)^3} + \nu^{-1} \|\boldsymbol{u}\|_{W^{1,4}(\Omega)^3} |\boldsymbol{u}|_{W^{k+1,4}(\mathcal{T}_h)^3}\right) \end{aligned} \tag{9}$$

Remark

Observe that the right hand side of the inequality (9) is **independent** of λ and p. For more details see [2].

Numerical test: 2D lid-driven cavity flow

The domain is the unit square $\Omega = (0, 1)^2$ and we set $\boldsymbol{f} = \boldsymbol{0}$. Homogeneous (wall) boundary conditions are enforced at all but the top horizontal wall (at $x_2 = 1$), where we enforce a unit tangential velocity $\boldsymbol{u} = (1, 0)$. In Figure 1, we report the horizontal component u_1 of the velocity along the vertical centerline $x_1 = \frac{1}{2}$ and the vertical component u_2 of the velocity along the horizontal centerline $x_2 = \frac{1}{2}$ for the two dimensional flow at global Reynolds numbers $\text{Re} := \frac{1}{\nu}$. References solutions from the literature [4, 3] are also included for the sake of comparison. To check the robustness of the method we run the same test case but with $\boldsymbol{f} = \lambda \nabla \psi$, where $\psi = \frac{1}{3}(x^3 + y^3)$. In Figure 2 we report the results. As expected, the velocity profiles are not affected by the value of λ. The same plot also contains the results obtained with the original HHO formulation of [1].

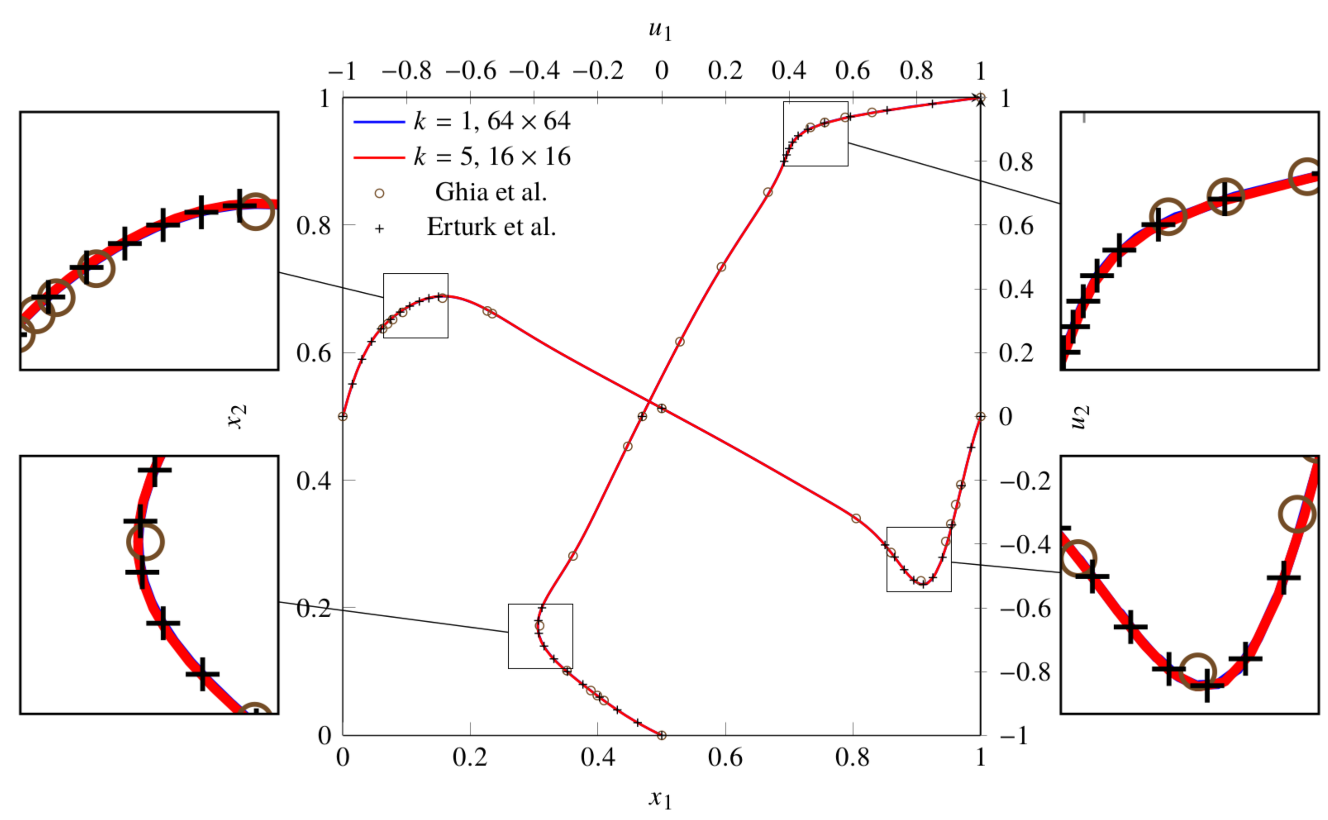

Fig. 1: Comparison for two-dimensional lid-driven cavity flow for $\text{Re} = 1{,}000$.

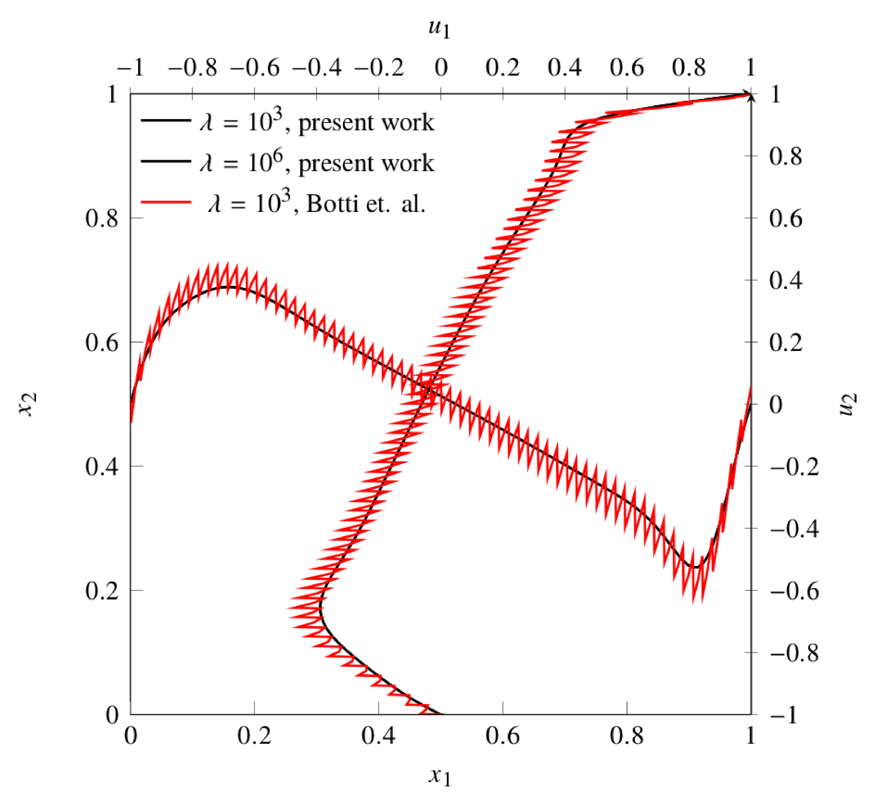

Fig. 2: 2D lid-driven cavity flow with irrotational force $\boldsymbol{f} = \lambda\nabla\psi$. Comparison between the present method [2] and the original HHO formulation of [1].

References

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