Divergence-free virtual element method for Stokes and Navier-Stokes problems

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a joint work with

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Divergence-free virtual element method

for Stokes and Navier-Stokes problems

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Divergence-free condition

- standard Finite Element do not have it!
- pressure-robust method *only* in a weak sense, for instance A. Linke *et al.* (2017).

$$||\nabla(\boldsymbol{u}-\boldsymbol{u}_h)||_{L^2(\Omega)} \lesssim \inf_{\boldsymbol{w}_h\in\mathbf{V}_h} ||\nabla(\boldsymbol{u}-\boldsymbol{w}_h)||_{L^2(\Omega)} + \frac{1}{\nu}\inf_{q_h\in Q_h}||p-q_h||_{L^2(\Omega)}$$

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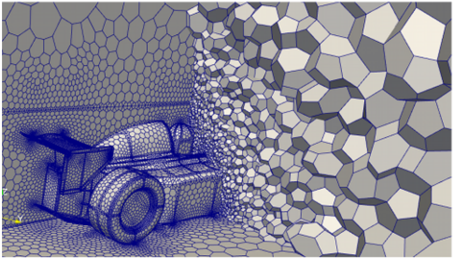
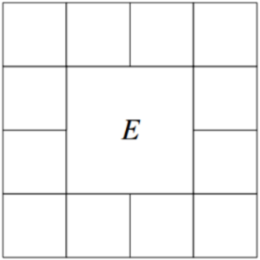
Divergence-free **virtual element method**

for Stokes and Navier-Stokes problems

What is the Virtual Element Method (VEM)?

A generalization of the Finite Element Method introduced in 2013





- general **polygonal** and **polyhedral** meshes (also non convex)
- additional interesting **features** and **properties**

"Basic principles of virtual element method"
L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. D. Marini and A. Russo (2013)

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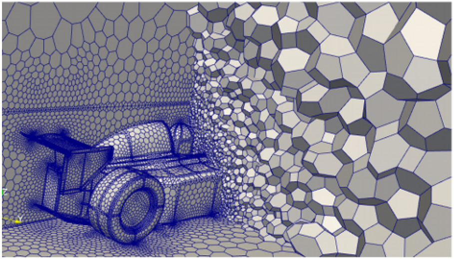

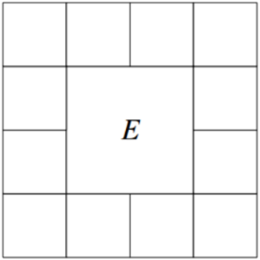

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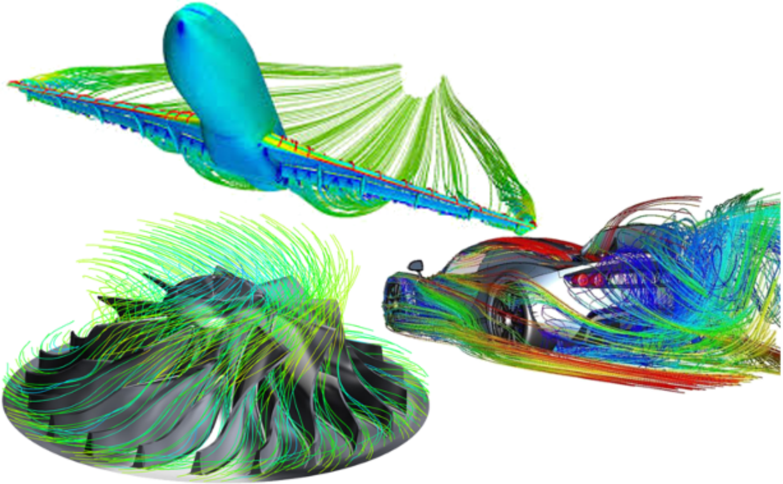
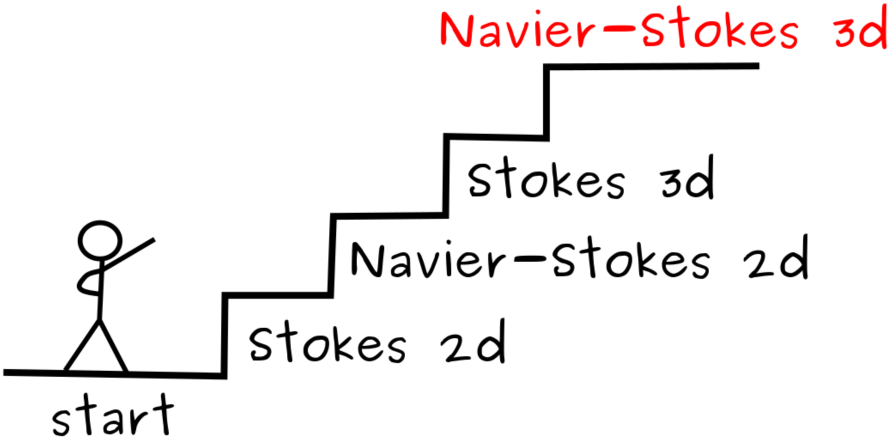
Divergence-free virtual element method

for Stokes and Navier-Stokes problems

Stokes and Navier-Stokes problems

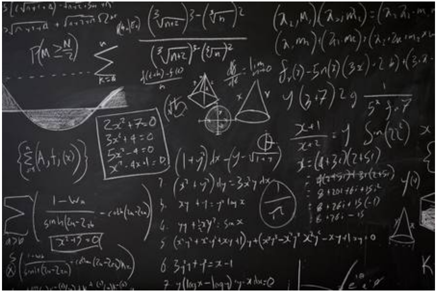



Navier-Stokes 3d
Stokes 3d
Navier-Stokes 2d
Stokes 2d
start

Talk outline

1. Problem definition
2. VEM spaces
   - Velocity field virtual space
   - Pressure virtual space
3. Problem discretization
4. Numerical examples
5. Conclusions

Problem definition



Stokes problem - continuous formulation

We search for a velocity field $\boldsymbol{u}$ and pressure p, such that

$$\begin{cases} -\nu\,\boldsymbol{\Delta u} - \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \operatorname{div}(\boldsymbol{u}) &= 0 \quad \text{in } \Omega \\ \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega \end{cases}$$

where

- Ω be a simply connected domain in $\mathbb{R}^2$
- $\mathbf{f} \in [L^2(\Omega)]^2$

Stokes problem - variational formulation

$$
\begin{cases}
\text{find } (\boldsymbol{u},\, p) \in \mathbf{V}_0(\Omega) \times Q(\Omega) \text{ such that:} \\
\displaystyle\int_\Omega \nu\, \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\Omega + \int_\Omega \mathrm{div}(\boldsymbol{v})\, p \,\mathrm{d}\Omega = \int_\Omega \mathbf{f} \cdot \boldsymbol{v} \,\mathrm{d}\Omega & \forall \boldsymbol{v} \in \mathbf{V}_0(\Omega) \\
\displaystyle\int_\Omega \mathrm{div}(\boldsymbol{u})\, q \,\mathrm{d}\Omega = 0 & \forall q \in Q(\Omega)
\end{cases}
$$

$$
\begin{aligned}
\mathbf{V}_0(\Omega) &:= \left\{ \boldsymbol{v} \in [H^1(\Omega)]^2 \,:\, \boldsymbol{v} = 0 \text{ on } \partial\Omega \right\} \\
Q(\Omega) &:= \left\{ q \in L^2(\Omega) \,:\, \int_\Omega q \,\mathrm{d}\Omega = 0 \right\}
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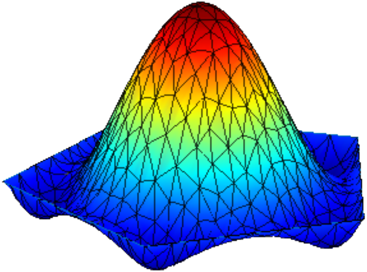
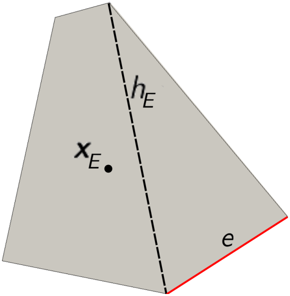
VEM spaces



Notation for polygons

polygon E



$|E| = \text{area}$

Notation for 2d monomials

Let $E \subset \mathbb{R}^2$, $k \in \mathbb{N}\backslash\{0\}$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ be a multi-index, we define the **scaled monomials**

$$m_{\boldsymbol{\alpha}} := \left(\frac{x - x_E}{h_E}\right)^{\alpha_1} \left(\frac{y - y_E}{h_E}\right)^{\alpha_2}$$

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$$\boldsymbol{m}^1_{\boldsymbol{\alpha}} = \begin{bmatrix} m_{\boldsymbol{\alpha}} \\ 0 \end{bmatrix}, \quad \boldsymbol{m}^2_{\boldsymbol{\alpha}} = \begin{bmatrix} 0 \\ m_{\boldsymbol{\alpha}} \end{bmatrix} \quad \text{and} \quad \boldsymbol{m}^{\perp} = \begin{bmatrix} m_{(0,1)} \\ -m_{(1,0)} \end{bmatrix}$$

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- $\{\boldsymbol{m}_{\boldsymbol{\alpha}}^i\}$ is a basis of $[\mathbb{P}_k(E)]^2$

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3d case is analogous

Polynomial decomposition

It is **essential** the following property

$$[\mathbb{P}_k(E)]^2 = \nabla\mathbb{P}_{k+1}(E) \oplus \mathbf{x}^{\perp}\mathbb{P}_{k-1}(E)$$

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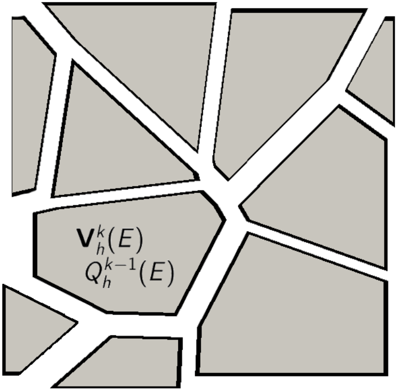
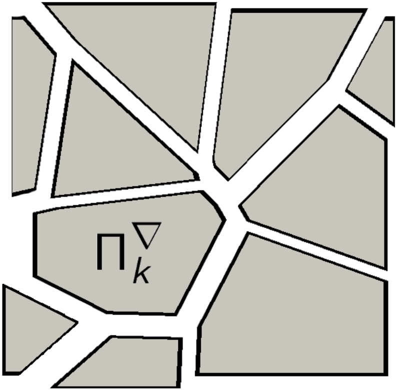
VEM space definition - the plan

- VEM local spaces



VEM space definition - the plan

- VEM local spaces
- local projection operators



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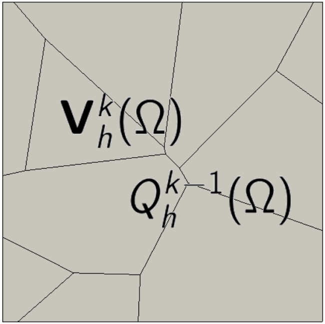
- glue spaces



Velocity field virtual space

$$\mathbf{V}_h^k(E) := \Big\{ \boldsymbol{v}_h \in [H^1(E) \cap C^0(E)]^2 \,:\, \boldsymbol{v}_h|_e \in [\mathbb{P}_k(e)]^2 \,\forall e \in \partial E\,, \\ -\boldsymbol{\Delta} \boldsymbol{v}_h + \nabla s \in [\mathbb{P}_{k-2}(E)]^2,\ s \in L_0^2(E)\,, \\ \mathrm{div}(\boldsymbol{v}_h) \in \mathbb{P}_{k-1}(E) \Big\}$$

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Useful to count space dimension

$k \geq 2$, for $k = 1$ P. F. Antonietti *et al.* (2014)

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Velocity field virtual space - d.o.f.

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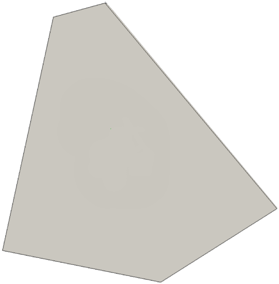

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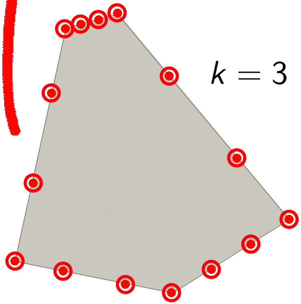
- vectorial values at the vertices and $k-1$ internal nodes



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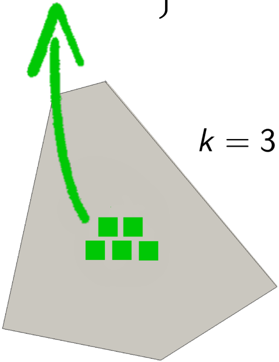
- vectorial values at the vertices and $k-1$ internal nodes
- $k(k+1)/2 - 1$ divergence moments

$$\int_E \mathrm{div}(\boldsymbol{v}_h) \, m_{\boldsymbol{\alpha}} \, \mathrm{d}E$$



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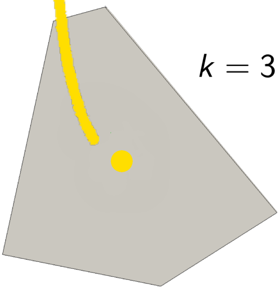
- vectorial values at the vertices and $k-1$ internal nodes
- $k(k+1)/2 - 1$ divergence moments
$$\int_E \mathrm{div}(\boldsymbol{v}_h)\, m_{\boldsymbol{\alpha}} \, \mathrm{d}E$$
- $(k-1)(k-2)/2$ perp moments
$$\int_E (\boldsymbol{v}_h \cdot \boldsymbol{m}^\perp)\, m_{\boldsymbol{\beta}} \, \mathrm{d}E$$



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$$\ldots + \nabla s \in \mathbf{x}^{\perp} \mathbb{P}_{k-3}(E),\ s \in L^2_0(E)\,,$$

$$\ldots \mathbb{P}_{k-1}(E) \Big\}$$

- vectorial values at the …
  and $k-1$ internal n…
- $k(k+1)/2 - 1$ di…

  $\int_E$ …

- $(k-1)(k-$ …

  $\int_E$ …

Not *only* polynomials are in $\mathbf{V}_h^k(E)$

Projection operator Π_k^∇

$$
\left\{
\begin{array}{rcl}
\displaystyle\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) : \boldsymbol{\nabla}\boldsymbol{p}_k \,\mathrm{d}E & = & 0 \quad \forall \boldsymbol{p}_k \in [\mathbb{P}_k(E)]^2 \\
\displaystyle\int_{\partial E} (\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) \cdot \boldsymbol{p}_0 \,\mathrm{d}e & = & 0 \quad \forall \boldsymbol{p}_0 \in [\mathbb{P}_0(E)]^2
\end{array}
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$$

Projection operator Π_k^{∇}

$$\begin{cases} \displaystyle\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^{\nabla}\boldsymbol{v}_h) : \boldsymbol{\nabla}\boldsymbol{p}_k \,\mathrm{d}E = 0 & \forall \boldsymbol{p}_k \in [\mathbb{P}_k(E)]^2 \\ \displaystyle\int_{\partial E} (\boldsymbol{v}_h - \Pi_k^{\nabla}\boldsymbol{v}_h) \cdot \boldsymbol{p}_0 \,\mathrm{d}e = 0 & \forall \boldsymbol{p}_0 \in [\mathbb{P}_0(E)]^2 \end{cases}$$

- $\mathbb{M}_k(E) := \{\boldsymbol{m}_i\}_{i=1}^n$
- $\mathbb{M}_0(E) := \{\boldsymbol{m}_1, \boldsymbol{m}_2\}$

Projection operator Π_k^∇

$$\begin{cases} \displaystyle\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) : \boldsymbol{\nabla}\boldsymbol{p}_k \,\mathrm{d}E = 0 & \forall \boldsymbol{p}_k \in [\mathbb{P}_k(E)]^2 \\ \displaystyle\int_{\partial E} (\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) \cdot \boldsymbol{p}_0 \,\mathrm{d}e = 0 & \forall \boldsymbol{p}_0 \in [\mathbb{P}_0(E)]^2 \end{cases}$$

- $\mathbb{M}_k(E) := \{\boldsymbol{m}_i\}_{i=1}^n$
- $\mathbb{M}_0(E) := \{\boldsymbol{m}_1,\, \boldsymbol{m}_2\}$
- $\Pi_k^\nabla \boldsymbol{v}_h := c_0\boldsymbol{m}_1 + c_1\boldsymbol{m}_1 + \ldots + c_n\boldsymbol{m}_n$

Projection operator Π_k^∇

$$\begin{cases} \displaystyle\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) : \boldsymbol{\nabla} \boldsymbol{p}_k \, \mathrm{d}E = 0 & \forall \boldsymbol{p}_k \in [\mathbb{P}_k(E)]^2 \\ \displaystyle\int_{\partial E} (\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) \cdot \boldsymbol{p}_0 \, \mathrm{d}e = 0 & \forall \boldsymbol{p}_0 \in [\mathbb{P}_0(E)]^2 \end{cases}$$

- $\mathbb{M}_k(E) := \{\boldsymbol{m}_i\}_{i=1}^n$
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- $\Pi_k^\nabla \boldsymbol{v}_h := c_0 \boldsymbol{m}_1 + c_1 \boldsymbol{m}_1 + \ldots + c_n \boldsymbol{m}_n$
- conditions according to $\mathbb{M}_k(E)$ and $\mathbb{M}_0(E)$

Projection operator Π_k^∇

$$\begin{cases} \displaystyle\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) \ldots \boldsymbol{\nabla} \boldsymbol{p}_k \, \mathrm{d}E = 0 & \forall \boldsymbol{p}_k \in [\mathbb{P}_k(E)]^2 \\ \displaystyle\int_{\partial E} (\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) \cdot \boldsymbol{p}_0 \, \mathrm{d}e = 0 & \forall \boldsymbol{p}_0 \in [\mathbb{P}_0(E)]^2 \end{cases}$$

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- $\Pi_k^\nabla \boldsymbol{v}_h := c_0 \boldsymbol{m}_1 + c_1 \boldsymbol{m}_1 + \ldots +$ …
- conditions according to $\mathbb{M}_k(E)$ a…

Projection operator Π_k^∇ - Yes we can compute it!

$$\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) : \boldsymbol{\nabla} \boldsymbol{m}_i \,\mathrm{d}E = 0$$

Projection operator Π_k^∇ - Yes we can compute it!

$$\begin{aligned}\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) : \boldsymbol{\nabla} \boldsymbol{m}_i \,\mathrm{d}E &= 0 \\ \sum_{j=1}^{n} c_j \int_E \boldsymbol{\nabla} \boldsymbol{m}_j : \boldsymbol{\nabla} \boldsymbol{m}_i \,\mathrm{d}E &= \int_E \boldsymbol{\nabla} \boldsymbol{v}_h : \boldsymbol{\nabla} \boldsymbol{m}_i \,\mathrm{d}E\end{aligned}$$

Projection operator Π_k^∇ - Yes we can compute it!

$$\begin{aligned}\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E &= 0\\ \sum_{j=1}^{n} c_j \int_E \boldsymbol{\nabla}\boldsymbol{m}_j : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E &= \int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E\end{aligned}$$

Focus on the virtual part

$$\int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E$$

Projection operator Π_k^∇ - Yes we can compute it!

$$\begin{aligned}
\int_E \nabla(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) : \nabla \boldsymbol{m}_i \,\mathrm{d}E &= 0 \\
\sum_{j=1}^{n} c_j \int_E \nabla \boldsymbol{m}_j : \nabla \boldsymbol{m}_i \,\mathrm{d}E &= \int_E \nabla \boldsymbol{v}_h : \nabla \boldsymbol{m}_i \,\mathrm{d}E
\end{aligned}$$

Focus on the virtual part

$$\int_E \nabla \boldsymbol{v}_h : \nabla \boldsymbol{m}_i \,\mathrm{d}E = -\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta} \boldsymbol{m}_i \,\mathrm{d}E + \int_{\partial E} \boldsymbol{v}_h \cdot (\nabla \boldsymbol{m}_i \,\mathbf{n}) \,\mathrm{d}e$$

Projection operator Π_k^∇ - Yes we can compute it!

$$\begin{aligned}\int_E \nabla(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) : \nabla \boldsymbol{m}_i \,\mathrm{d}E &= 0 \\ \sum_{j=1}^{n} c_j \int_E \nabla \boldsymbol{m}_j : \nabla \boldsymbol{m}_i \,\mathrm{d}E &= \int_E \nabla \boldsymbol{v}_h : \nabla \boldsymbol{m}_i \,\mathrm{d}E\end{aligned}$$

Focus on the virtual part

$$\begin{aligned}\int_E \nabla \boldsymbol{v}_h : \nabla \boldsymbol{m}_i \,\mathrm{d}E &= -\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta} \boldsymbol{m}_i \,\mathrm{d}E + \int_{\partial E} \boldsymbol{v}_h \cdot (\nabla \boldsymbol{m}_i \, \mathbf{n}) \,\mathrm{d}e \\ &= -\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta} \boldsymbol{m}_i \,\mathrm{d}E + \sum_{e \in \partial E} \int_e \boldsymbol{v}_h \cdot (\nabla \boldsymbol{m}_i \, \mathbf{n}_e) \,\mathrm{d}e\end{aligned}$$

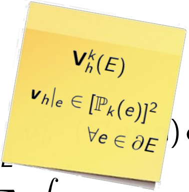
Projection operator Π_k^∇ - Yes we can compute it!

$$\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E = 0$$

$$\sum_{j=1}^{n} c_j \int_E \boldsymbol{\nabla}\boldsymbol{m}_j : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E = \int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E$$

Focus on the virtual part

$$\int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E = -\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta}\boldsymbol{m}_i \,\mathrm{d}E + \int_{\partial E} \ldots) \,\mathrm{d}e$$



$$= -\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta}\boldsymbol{m}_i \,\mathrm{d}E + \sum_{e\in\partial E} \int_e \boldsymbol{v}_h \cdot (\boldsymbol{\nabla}\boldsymbol{m}_i \,\mathbf{n}_e) \,\mathrm{d}e$$

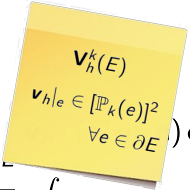
Projection operator Π_k^{∇} - Yes we can compute it!

$$\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^{\nabla}\boldsymbol{v}_h) : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E = 0$$

$$\sum_{j=1}^{n} c_j \int_E \boldsymbol{\nabla}\boldsymbol{m}_j : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E = \int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E$$

Focus on the virtual part

$$\int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E = -\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta}\boldsymbol{m}_i \,\mathrm{d}E + \int_{\partial E} \ldots) \,\mathrm{d}e$$



$$= -\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta}\boldsymbol{m}_i \,\mathrm{d}E + \sum_{e \in \partial E} \int_e \boldsymbol{v}_h \cdot (\boldsymbol{\nabla}\boldsymbol{m}_i\, \mathbf{n}_e) \,\mathrm{d}e$$

Projection operator Π_k^{∇} - Yes we can compute it!

$$-\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta m}_i \,\mathrm{d}E$$

Projection operator Π_k^{∇} - Yes we can compute it!

$$-\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta m}_i \,\mathrm{d}E$$

$$\boldsymbol{\Delta m}_i = c_\alpha \boldsymbol{m}_\alpha + c_\beta \boldsymbol{m}_\beta + c_\delta \boldsymbol{m}_\delta + c_\gamma \boldsymbol{m}_\gamma$$

Projection operator Π_k^∇ - Yes we can compute it!

$$-\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta m}_i \,\mathrm{d}E$$

$$\boldsymbol{\Delta m}_i = c_\alpha \boldsymbol{m}_\alpha + c_\beta \boldsymbol{m}_\beta + c_\delta \boldsymbol{m}_\delta + c_\gamma \boldsymbol{m}_\gamma$$

$$\boldsymbol{m}_\alpha, \boldsymbol{m}_\beta, \boldsymbol{m}_\delta, \boldsymbol{m}_\gamma \in \mathbb{P}_{k-2}(E)$$

Projection operator Π_k^∇ - Yes we can compute it!

$$-\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta m}_i \,\mathrm{d}E$$

$$\boldsymbol{\Delta m}_i = c_\alpha \boldsymbol{m}_\alpha + c_\beta \boldsymbol{m}_\beta + c_\delta \boldsymbol{m}_\delta + c_\gamma \boldsymbol{m}_\gamma$$

$$\boldsymbol{m}_\alpha, \boldsymbol{m}_\beta, \boldsymbol{m}_\delta, \boldsymbol{m}_\gamma \in \mathbb{P}_{k-2}(E)$$

$$\boldsymbol{m}_\alpha = c_\zeta^\alpha \nabla m_\zeta + c_\eta^\alpha \boldsymbol{m}^\perp m_\eta$$

$m_\zeta \in \mathbb{P}_{k-1}(E)$ and $m_\eta \in \mathbb{P}_{k-3}(E)$

Projection operator Π_k^{∇} - Yes we can compute it!

$$\int_E \boldsymbol{v}_h \cdot \boldsymbol{m}_\alpha \,\mathrm{d}E$$

Projection operator Π_k^∇ - Yes we can compute it!

$$\int_E \boldsymbol{v}_h \cdot \boldsymbol{m}_\alpha \, \mathrm{d}E = \int_E \boldsymbol{v}_h \cdot (c_\zeta^{\boldsymbol{\alpha}} \nabla m_\zeta + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \boldsymbol{m}^\perp m_{\boldsymbol{\eta}}) \, \mathrm{d}E$$

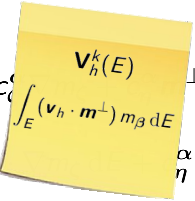
Projection operator Π_k^∇ - Yes we can compute it!

$$
\begin{aligned}
\int_E \boldsymbol{v}_h \cdot \boldsymbol{m}_\alpha \,\mathrm{d}E &= \int_E \boldsymbol{v}_h \cdot (c_\zeta^{\boldsymbol{\alpha}} \nabla m_\zeta + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \boldsymbol{m}^\perp m_{\boldsymbol{\eta}}) \,\mathrm{d}E \\
&= c_\zeta^{\boldsymbol{\alpha}} \int_E \boldsymbol{v}_h \cdot \nabla m_\zeta \,\mathrm{d}E + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \int_E (\boldsymbol{v}_h \cdot \boldsymbol{m}^\perp) m_{\boldsymbol{\eta}} \,\mathrm{d}E
\end{aligned}
$$

Projection operator Π_k^∇ - Yes we can compute it!

$$\int_E \boldsymbol{v}_h \cdot \boldsymbol{m}_\alpha \,\mathrm{d}E = \int_E \boldsymbol{v}_h \cdot (c_\zeta^\alpha \text{ [illegible] } {}^\perp m_\eta)\,\mathrm{d}E$$

$$= c_\zeta^\alpha \int_E \boldsymbol{v}_h \text{ [illegible] } {}_\eta^\alpha \int_E (\boldsymbol{v}_h \cdot \boldsymbol{m}^\perp) m_\eta \,\mathrm{d}E$$



Projection operator Π_k^∇ - Yes we can compute it!

$$
\begin{aligned}
\int_E \boldsymbol{v}_h \cdot \boldsymbol{m}_\alpha \,\mathrm{d}E &= \int_E \boldsymbol{v}_h \cdot (c_\zeta^{\boldsymbol{\alpha}} \nabla m_\zeta + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \boldsymbol{m}^\perp m_{\boldsymbol{\eta}}) \,\mathrm{d}E \\
&= c_\zeta^{\boldsymbol{\alpha}} \int_E \boldsymbol{v}_h \cdot \nabla m_\zeta \,\mathrm{d}E + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \int_E (\boldsymbol{v}_h \cdot \boldsymbol{m}^\perp) m_{\boldsymbol{\eta}} \,\mathrm{d}E \\
&= -c_\zeta^{\boldsymbol{\alpha}} \int_E \mathrm{div}(\boldsymbol{v}_h) m_\zeta \,\mathrm{d}E + c_\zeta^{\boldsymbol{\alpha}} \int_{\partial E} (\boldsymbol{v}_h \cdot \mathbf{n})\, m_\zeta \,\mathrm{d}e + \ldots
\end{aligned}
$$

Projection operator Π_k^∇ - Yes we can compute it!

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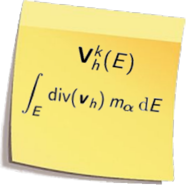
\end{aligned}
$$



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$$
\begin{aligned}
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&= c_\zeta^{\boldsymbol{\alpha}} \int_E \boldsymbol{v}_h \cdot \nabla m_\zeta \, \mathrm{d}E + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \int_E (\boldsymbol{v}_h \cdot \boldsymbol{m}^\perp) m_{\boldsymbol{\eta}} \, \mathrm{d}E \\
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&= -c_\zeta^{\boldsymbol{\alpha}} \int_E \mathrm{div}(\boldsymbol{v}_h) m_\zeta \, \mathrm{d}E + c_\zeta^{\boldsymbol{\alpha}} \sum_{e \in \partial E} (\boldsymbol{v}_h \cdot \mathbf{n}_e) \, m_\zeta \, \mathrm{d}e + \dots
\end{aligned}
$$

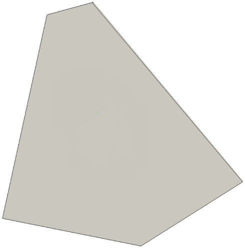
$\mathbf{V}_h^k(E)$

$\boldsymbol{v}_h|_e \in [\mathbb{P}_k(e)]^2$

$\forall e \in \partial E$

Pressure virtual space and d.o.f.

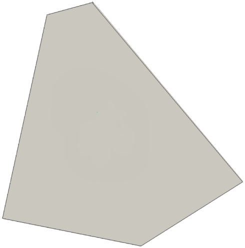
$$Q_h^{k-1}(E) := \{q_h \ : \ q_h \in \mathbb{P}_{k-1}(E)\}$$



Pressure virtual space and d.o.f.

$$Q_h^{k-1}(E) := \{q_h \,:\, q_h \in \mathbb{P}_{k-1}(E)\}$$

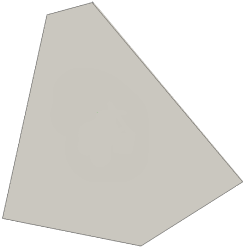
- no VEM approximation



Pressure virtual space and d.o.f.

$$Q_h^{k-1}(E) := \{q_h \ : \ q_h \in \mathbb{P}_{k-1}(E)\}$$

- no VEM approximation
- no projection

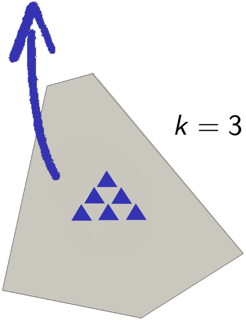


Pressure virtual space and d.o.f.

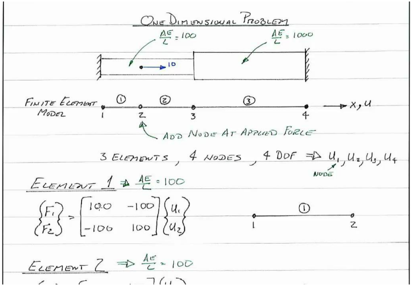
$$Q_h^{k-1}(E) := \{q_h \,:\, q_h \in \mathbb{P}_{k-1}(E)\}$$

- no VEM approximation
- no projection
- $k(k+1)/2$ moments

$$\int_E q_h \, m_\alpha \, \mathrm{d}E$$



Problem discretization



Problem discretization

Consider a polyhedral decomposition Ω_h of Ω, then we solve:

$$\begin{cases} \text{find } (\boldsymbol{u}_h, p_h) \in \mathbf{V}_h^k \times Q_h^{k-1} \text{ such that} \\ a_h(\boldsymbol{u}_h\,, \boldsymbol{v}_h) + \displaystyle\int_{\Omega_h} \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h = \int_{\Omega_h} \mathbf{f}_h \cdot \boldsymbol{v}_h \,\mathrm{d}\Omega_h & \forall \boldsymbol{v}_h \in \mathbf{V}_{h,0}^k \\ \displaystyle\int_{\Omega_h} \mathrm{div}(\boldsymbol{u}_h)\, q_h \,\mathrm{d}\Omega_h = 0 & \forall q_h \in Q_h^{k-1} \end{cases}$$

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where

- we define from Π_k^∇ and dofs

$$a_h(\cdot,\cdot) \approx \int_\Omega \nu \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\Omega$$

Problem discretization

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where

- we define from Π_k^∇ and dofs

$$a_h(\cdot,\cdot) \approx \int_\Omega \nu \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \, \mathrm{d}\Omega$$

- $\mathbf{f}_h$ is a proper L^2 projection of $\mathbf{f}$

Defintion of $a_h(\cdot, \cdot)$

Follow a standard VEM approach

$$a_h(\boldsymbol{v}_h, \boldsymbol{w}_h) = \sum_{E \in \Omega_h} a_{h,E}(\boldsymbol{v}_h, \boldsymbol{w}_h)$$

Defintion of $a_h(\cdot,\cdot)$

Follow a standard VEM approach

$$a_h(\boldsymbol{v}_h,\, \boldsymbol{w}_h) = \sum_{E\in\Omega_h} a_{h,E}(\boldsymbol{v}_h,\, \boldsymbol{w}_h)$$

where

$$\begin{aligned} a_{h,E}(\boldsymbol{v}_h,\, \boldsymbol{w}_h) \quad &:= \quad \int_E \boldsymbol{\nabla}(\Pi_k^\nabla \boldsymbol{v}_h) \,:\, \boldsymbol{\nabla}(\Pi_k^\nabla \boldsymbol{w}_h)\, \mathrm{d}E \\ &\qquad + s_E(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h,\, \boldsymbol{w}_h - \Pi_k^\nabla \boldsymbol{w}_h) \end{aligned}$$

Defintion of $a_h(\cdot,\cdot)$

Follow a standard VEM approach

$$a_h(\boldsymbol{v}_h,\, \boldsymbol{w}_h) = \sum_{E\in\Omega_h} a_{h,E}(\boldsymbol{v}_h,\, \boldsymbol{w}_h)$$

where

consistency

$$a_{h,E}(\boldsymbol{v}_h,\, \boldsymbol{w}_h) := \int_E \boldsymbol{\nabla}(\Pi_k^\nabla \boldsymbol{v}_h) : \boldsymbol{\nabla}(\Pi_k^\nabla \boldsymbol{w}_h)\,\mathrm{d}E$$
$$+ s_E(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h,\, \boldsymbol{w}_h - \Pi_k^\nabla \boldsymbol{w}_h)$$

Defintion of $a_h(\cdot,\cdot)$

Follow a standard VEM approach

$$a_h(\boldsymbol{v}_h,\,\boldsymbol{w}_h) = \sum_{E\in\Omega_h} a_{h,E}(\boldsymbol{v}_h,\,\boldsymbol{w}_h)$$

where

$$a_{h,E}(\boldsymbol{v}_h,\,\boldsymbol{w}_h) \;:=\; \int_E \boldsymbol{\nabla}(\Pi_k^{\nabla}\boldsymbol{v}_h) : \boldsymbol{\nabla}(\Pi_k^{\nabla}\boldsymbol{w}_h)\,\mathrm{d}E$$

$$+s_E(\boldsymbol{v}_h - \Pi_k^{\nabla}\boldsymbol{v}_h,\,\boldsymbol{w}_h - \Pi_k^{\nabla}\boldsymbol{w}_h)$$

stability

Defintion of $a_h(\cdot, \cdot)$

Follow a standard VEM approach

$$a_h(\boldsymbol{v}_h,\, \boldsymbol{w}_h) = \sum_{E \in \Omega_h} a_{h,E}(\boldsymbol{v}_h,\, \boldsymbol{w}_h)$$

where

$$a_{h,E}(\boldsymbol{v}_h,\, \boldsymbol{w}_h) \quad := \quad \int_E \boldsymbol{\nabla}(\Pi_k^\nabla \boldsymbol{v}_h) : \boldsymbol{\nabla}(\Pi_k^\nabla \boldsymbol{w}_h)\,\mathrm{d}E$$

$$+ s_E(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h,\, \boldsymbol{w}_h - \Pi_k^\nabla \boldsymbol{w}_h)$$

stability

Defintion of $a_h(\cdot,\cdot)$ - stability

We substitute with a computable approximation,
L. Beirão *et al.* 2013

$$\int_E \nabla(I-\Pi_E^\nabla)\phi_i\cdot\nabla(I-\Pi_E^\nabla)\phi_j$$
$$\approx \sum_{r=1}^{N_{\text{dofs}}^E} \text{dof}_r\left(\left(I-\Pi_E^\nabla\right)\phi_i\right)\text{dof}_r\left(\left(I-\Pi_E^\nabla\right)\phi_j\right)$$

where N_{dofs}^E is the number of dofs of E.

Mixed term

$$\int_{\Omega_h} \operatorname{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h$$

Mixed term

$$\int_{\Omega_h} \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h = \sum_{E\in\Omega_h} \int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E$$

Mixed term

$$\int_{\Omega_h} \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h = \sum_{E\in\Omega_h} \int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E$$

there is no approximation

Mixed term

$$\int_{\Omega_h} \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h = \sum_{E\in\Omega_h} \int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E$$

if $p_h \in \mathbb{R}$

there is no approximation

$$\begin{aligned}\int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E &= p_h \int_E \mathrm{div}(\boldsymbol{v}_h)\,\mathrm{d}E \\ &= p_h \int_{\partial E} (\boldsymbol{v}_h \cdot \mathbf{n})\,\mathrm{d}e \\ &= p_h \sum_{e\in\partial E} \int_e (\boldsymbol{v}_h \cdot \mathbf{n}_e)\,\mathrm{d}e\end{aligned}$$

Mixed term

$$\int_{\Omega_h} \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h = \sum_{E\in\Omega_h} \int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E$$

if $p_h \in \mathbb{R}$

there is no approximation

$$\begin{aligned}\int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E &= p_h \int_E \mathrm{div}(\boldsymbol{v}_h) \,\mathrm{d}E \\ &= p_h \int_{\partial E} (\boldsymbol{v}_h \cdot \mathbf{n}) \,\mathrm{d}e \\ &= p_h \sum_{e\in\partial E} \int_e (\boldsymbol{v}_h \cdot \mathbf{n}_e) \,\mathrm{d}e\end{aligned}$$

$\mathbf{V}_h^k(E)$

$\boldsymbol{v}_h|_e \in [\mathbb{P}_k(e)]^2 \quad \forall e \in \partial E$

Mixed term

$$\int_{\Omega_h} \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h = \sum_{E\in\Omega_h} \int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E$$

if $p_h \in \mathbb{P}_{k-1}(E)\backslash\mathbb{R}$

there is no approximation

$$\int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E = \sum_{s=1}^{n} c_s^{p_h} \int_E \mathrm{div}(\boldsymbol{v}_h)\, m_s \,\mathrm{d}E$$

Mixed term

$$\int_{\Omega_h} \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h = \sum_{E\in\Omega_h} \int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E$$

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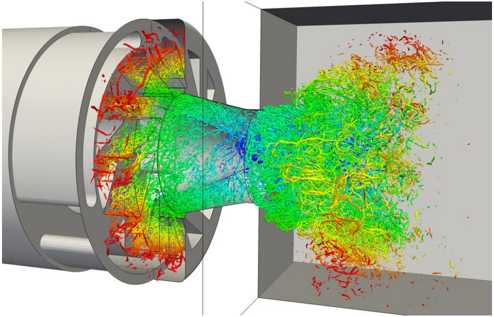
$\mathbf{V}_h^k(E)$

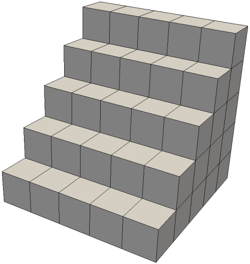
$$\int_E \mathrm{div}(\boldsymbol{v}_h)\, m_\alpha \,\mathrm{d}E$$

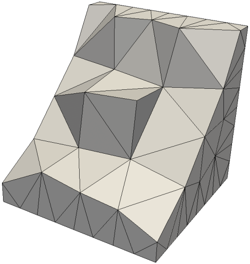
Numerical examples



Mesh types

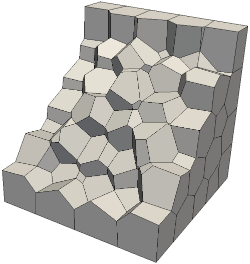
Cube

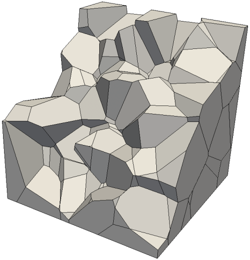




Tetra

CVT





Random

Error norms

- H^1**–velocity error**:

$$e_{H^1}^{\boldsymbol{u}} := \sqrt{\sum_{E\in\Omega_h} \left|\left|\nabla \boldsymbol{u} - \boldsymbol{\Pi}_{k-1}^0 \nabla \boldsymbol{u}_h\right|\right|_{L^2(E)}^2} \sim h^k$$

- L^2**–pressure error**:

$$e_{L^2}^{p} := \sqrt{\sum_{E\in\Omega_h} \left|\left|p - p_h\right|\right|_{L^2(E)}^2} \sim h^k$$

"Divergence free Virtual Elements for the Stokes problem on polygonal meshes"
L. Beirão da Veiga, C. Lovadina, and G.Vacca (2017)

Example 1: Convergence analysis for Stokes

Let us consider a Stokes problem

$$\begin{cases} -\nu\, \boldsymbol{\Delta u} - \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \operatorname{div}(\boldsymbol{u}) &= 0 \quad \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{r} \quad \text{on } \partial\Omega \end{cases}$$

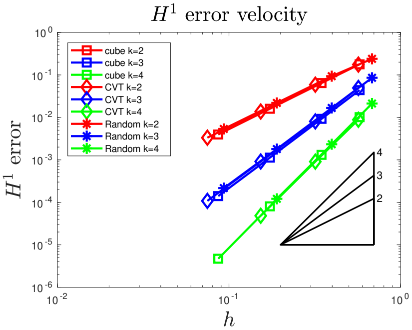
where the exact solution is

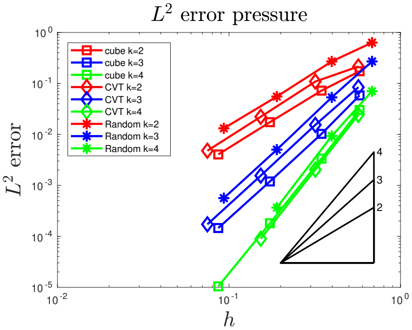
$$\boldsymbol{u}(x,\, y,\, z) := \begin{pmatrix} \sin(\pi x)\, \cos(\pi y)\, \cos(\pi z) \\ \cos(\pi x)\, \sin(\pi y)\, \cos(\pi z) \\ -2\cos(\pi x)\, \cos(\pi y)\, \sin(\pi z) \end{pmatrix}$$

and

$$p(x,\, y,\, z) := -\pi\, \cos(\pi x)\, \cos(\pi y)\, \cos(\pi z)\,.$$

Example 1: Convergence analysis for Stokes





"The Stokes complex for Virtual Elements in three dimensions"
L. Beirão da Veiga, F, Dassi, and G.Vacca submitted

Example 2: Convergence analysis for Navier-Stokes

Let us consider a Navier-Stokes problem

$$\begin{cases} -\nu\,\boldsymbol{\Delta u} + \boldsymbol{u}\nabla\boldsymbol{u} - \nabla p & = & \mathbf{f} & \text{in } \Omega \\ \text{div}(\boldsymbol{u}) & = & 0 & \text{in } \Omega \\ \boldsymbol{u} & = & \boldsymbol{r} & \text{on } \partial\Omega \end{cases}$$

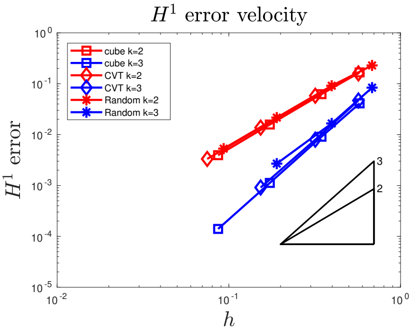
where the exact solution is

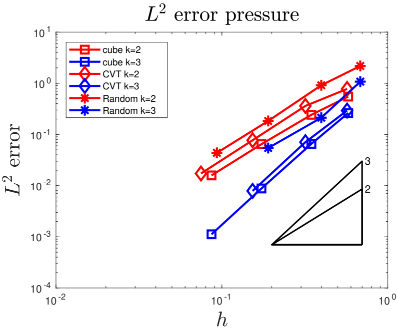
$$\boldsymbol{u}(x,\,y,\,z) := \begin{pmatrix} \sin(\pi x)\,\cos(\pi y)\,\cos(\pi z) \\ \cos(\pi x)\,\sin(\pi y)\,\cos(\pi z) \\ -2\cos(\pi x)\,\cos(\pi y)\,\sin(\pi z) \end{pmatrix}$$

and

$$p(x,\,y,\,z) := \sin(2\pi x)\,\sin(2\pi y)\,\sin(2\pi z)\,.$$

Example 2: Convergence analysis for Navier-Stokes





"The Stokes complex for Virtual Elements in three dimensions"
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Example 3: Benchmark problems

Let us consider a Stokes problem, we have the following estimate

$$|\boldsymbol{u} - \boldsymbol{u}_h|_1 \lesssim h^s \, \mathcal{F}(\boldsymbol{u};\, \nu, \gamma) + h^{s+2} \, \mathcal{H}(\mathbf{f}; \nu)$$

for suitable functions $\mathcal{F}$, $\mathcal{H}$, $\mathcal{K}$ independent of h.

Example 3: Benchmark problems

Let us consider a Stokes problem, we have the following estimate

$$|\boldsymbol{u} - \boldsymbol{u}_h|_1 \lesssim h^s \, \mathcal{F}(\boldsymbol{u};\, \nu, \gamma) + h^{s+2} \, \mathcal{H}(\mathbf{f}; \nu)$$

for suitable functions $\mathcal{F}$, $\mathcal{H}$, $\mathcal{K}$ independent of

divergece-free property

Example 3: Benchmark problems

We consider two problems

$$\boldsymbol{u}(x,\, y,\, z) := \begin{pmatrix} k\, x\, z^{k-1} \\ k\, y\, z^{k-1} \\ (2-k)\, x^k + (2-k)\, y^k - 2\, z^k \end{pmatrix},$$

and

$$p_1(x,\, y,\, z) := x^k\, y + y^k\, z + z^k\, x - \frac{3}{2(k+1)},$$

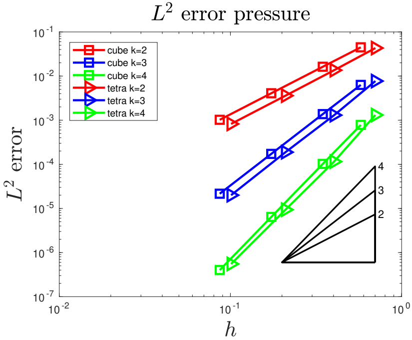
or

$$p_2(x,\, y,\, z) := \sin(2\pi x)\, \sin(2\pi y)\, \sin(2\pi z)\,.$$

Example 3: Benchmark problem, case p_1

H^1 error velocity

k	**Cube**	**Tetra**
2	1.0576e-13	7.2075e-13
3	2.7333e-13	1.1927e-12
4	1.5266e-12	2.2718e-10



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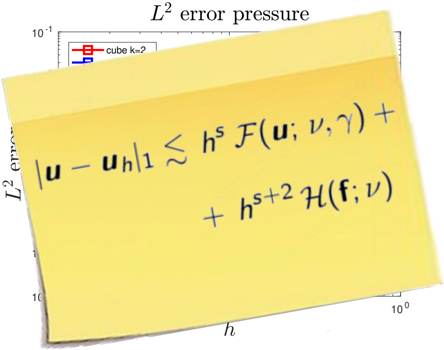

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Example 3: Benchmark problem, case p_1

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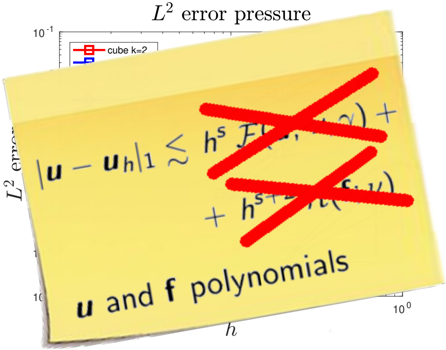

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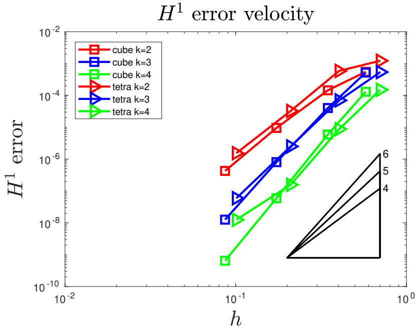
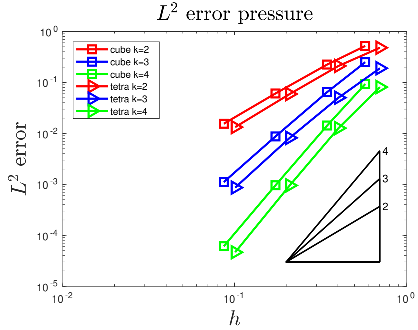
Example 3: Benchmark problem, case p_2





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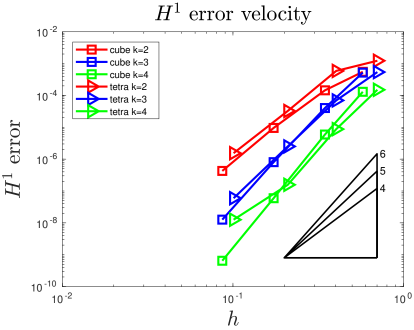
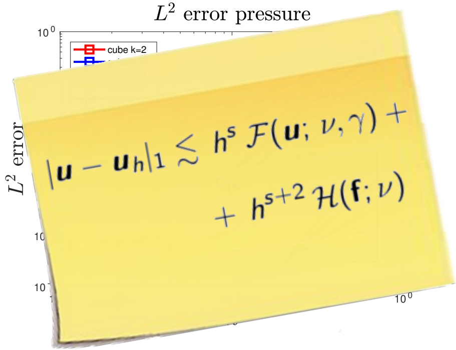
Example 3: Benchmark problem, case p_2





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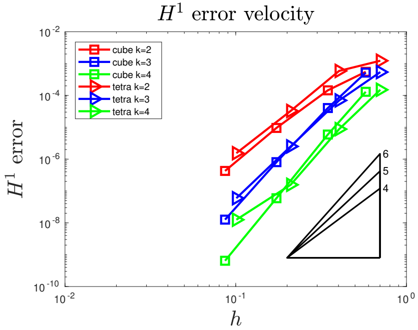
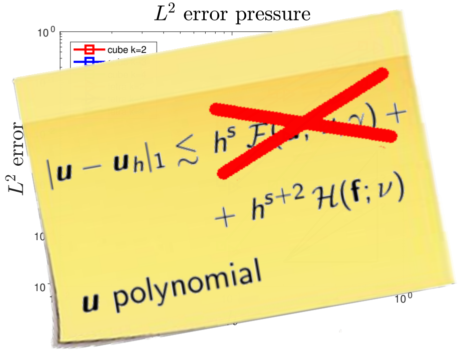
Example 3: Benchmark problem, case p_2





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Conclusions

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We presented Virtual Element approach

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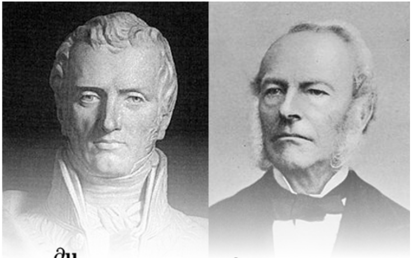
We presented Virtual Element approach

- for Stokes and Navier-Stokes problems 2d/3d



$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \nabla^2 \mathbf{u} = -\nabla p + \mathbf{g}.$$

Conclusions

We presented Virtual Element approach

- for Stokes and Navier-Stokes problems 2d/3d
- div-free property

$$|\boldsymbol{u} - \boldsymbol{u}_h|_1 \lesssim h^s \mathcal{F}(\boldsymbol{u}; \nu, \gamma) + h^{s+2} \mathcal{H}(\mathbf{f}; \nu)$$

H^1 error velocity

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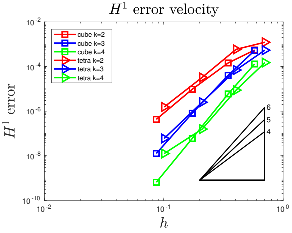

Future works

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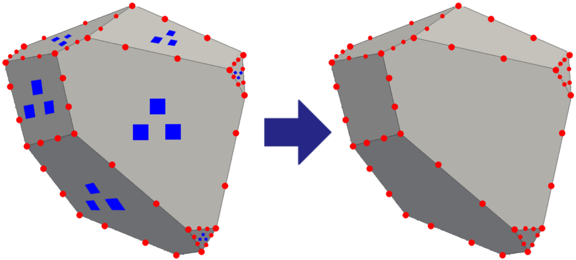
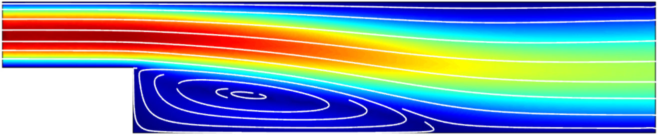
- serendipity on faces



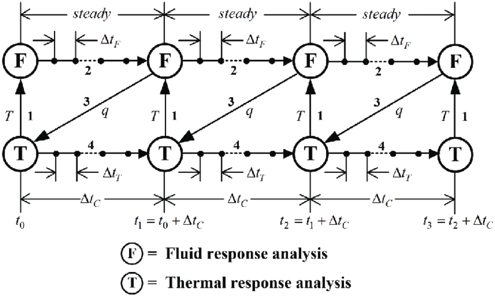
Future works

- serendipity on faces
- non static case



Future works

- serendipity on faces
- non static case
- coupling with other PDEs

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YAQHANYELAY
SUKSAMA
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GRAZIE
MAAKE
PALDIES
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