DDFV method for Navier-Stokes problem with outflow boundary conditions

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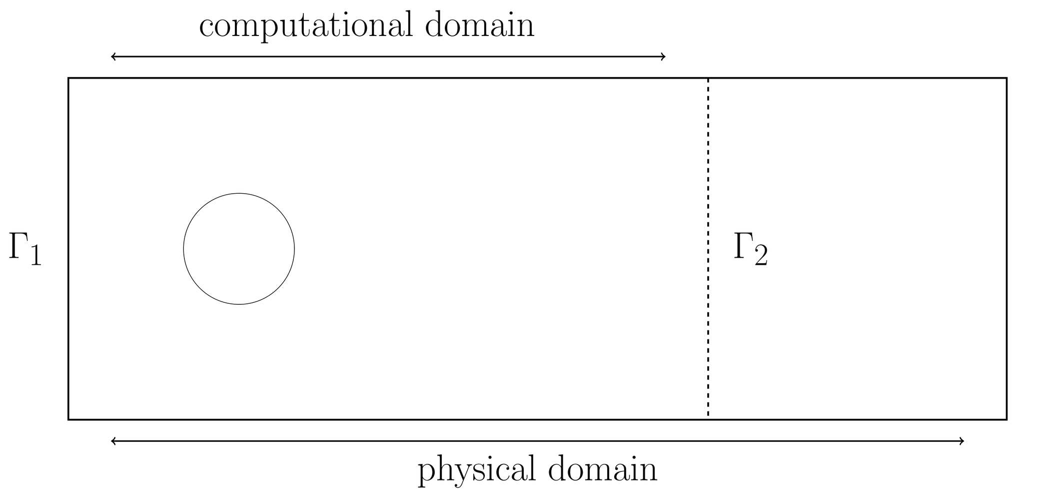
1. The problem

We consider a computational domain Ω that is strictly smaller than the physical domain:



We propose a DDFV (Discrete Duality Finite Volume) scheme for the following incompressible Navier Stokes problem on Ω:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}(\sigma(\mathbf{u},\mathrm{p})) = 0 & \text{in } \Omega_T = \Omega\times[0,T[,\\ \operatorname{div}(\mathbf{u}) = 0 & \text{in } \Omega_T,\\ \mathbf{u} = \mathbf{g}_1 & \text{on } \Gamma_1\times(0,T),\\ \text{outflow boundary conditions} & \text{on } \Gamma_2\times(0,T),\\ \mathbf{u}(0) = \mathbf{u}_{init} & \text{in } \Omega \end{cases} \qquad (\mathcal{P})$$

- Ω is a polygonal bounded open set of $\mathbb{R}^2$, $\partial\Omega = \Gamma_1 \cup \Gamma_2$
- $T > 0$, $\mathbf{u}_{init} \in (L^\infty(\Omega))^2$ and $\vec{\mathbf{n}}$ the outer normal
- $\mathbf{g}_1 \in (H^{\frac12}(\partial\Omega))^2$
- $\sigma(\mathbf{u},\mathrm{p}) = \frac{2}{\mathrm{Re}}\mathrm{D}\mathbf{u} - \mathrm{pId}$ and $\mathrm{D}\mathbf{u} = \frac12(\nabla\mathbf{u} + {}^t\nabla\mathbf{u})$.

where we suppose that the velocity is prescribed upstream and we impose the following artificial boundary condition, introduced in [BF94] and studied in [BF12] on the non-physical part of the boundary Γ_2:

$$\sigma(\mathbf{u},\mathrm{p})\cdot\vec{\mathbf{n}} + \frac12(\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}-\mathbf{u}_{ref}) = \sigma_{ref}\cdot\vec{\mathbf{n}}$$

with $\mathbf{u}_{ref} \in (H^1(\Omega))^2$, $\sigma_{ref}\cdot\vec{\mathbf{n}} \in (H^{-\frac12}(\Omega))^2$.

Those conditions are derived from a weak formulation of the Navier Stokes equations that ensures an **energy estimate**. If Ψ is a test function in the space $V = \{\psi \in (H^1(\Omega))^2,\ \psi|_{\Gamma_1} = 0,\ \operatorname{div}(\psi) = 0\}$, we get:

$$\int_\Omega \partial_t\mathbf{u}\cdot\Psi + \frac{2}{\mathrm{Re}}\int_\Omega \mathrm{D}(\mathbf{u}):\mathrm{D}(\Psi) + \frac12\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac12\int_\Omega(\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u}$$
$$= -\frac12\int_{\Gamma_2}(\mathbf{u}\cdot\vec{\mathbf{n}})(\mathbf{u}\cdot\Psi) + \int_{\Gamma_2}\sigma(\mathbf{u},\mathrm{p})\cdot\vec{\mathbf{n}}\cdot\Psi$$

that thanks to the boundary conditions becomes:

$$\int_\Omega \partial_t\mathbf{u}\cdot\Psi + \frac{2}{\mathrm{Re}}\int_\Omega \mathrm{D}(\mathbf{u}):\mathrm{D}(\Psi) + \frac12\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac12\int_\Omega(\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u}$$
$$= -\frac12\int_{\Gamma_2}(\mathbf{u}\cdot\vec{\mathbf{n}})^+(\mathbf{u}\cdot\Psi) + \frac12\int_{\Gamma_2}(\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}_{ref}\cdot\Psi) + \int_{\Gamma_2}\sigma_{ref}\cdot\vec{\mathbf{n}}\cdot\Psi. \qquad (\mathcal{WF})$$

Properties:

- Existence and uniqueness of $\mathbf{u} \in L^\infty(]0,T[,\overline{V}) \cap L^2(]0,T[,V)$, $\mathrm{p} \in W^{-1,\infty}(]0,T[,L^2(\Omega))$, weak solution of $(\mathcal{P})$
- Inf-sup condition, Korn's inequality, Trace theorem
- Energy inequality

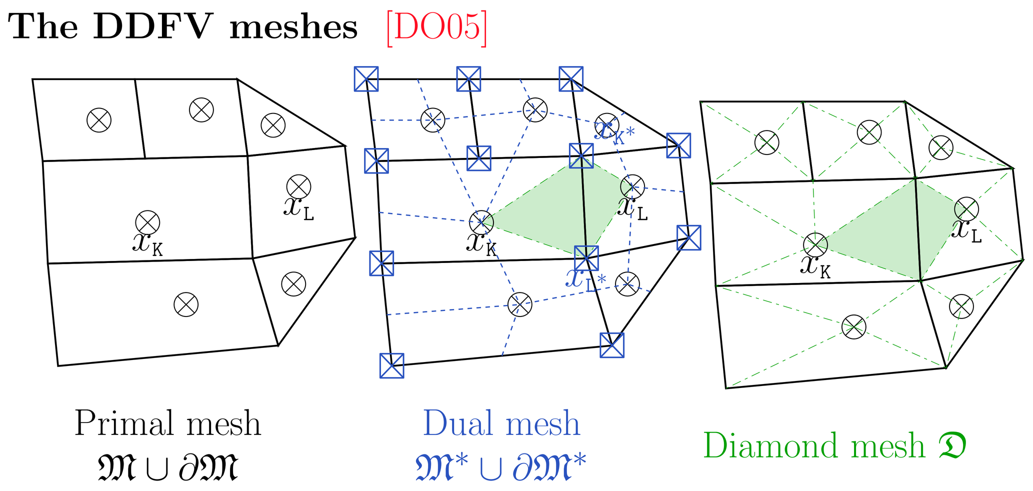
→ **Goal:** Reproduce the results at a discrete level.

The inf-sup stability condition reads:

$$\inf_{p\in L_0^2(\Omega)}\left(\sup_{\mathbf{u}\in(H_0^1(\Omega))^2}\frac{\int_\Omega p\,(\operatorname{div}\mathbf{u})}{\|\mathbf{u}\|_{H^1}\|p\|_{L^2}}\right) > 0.$$

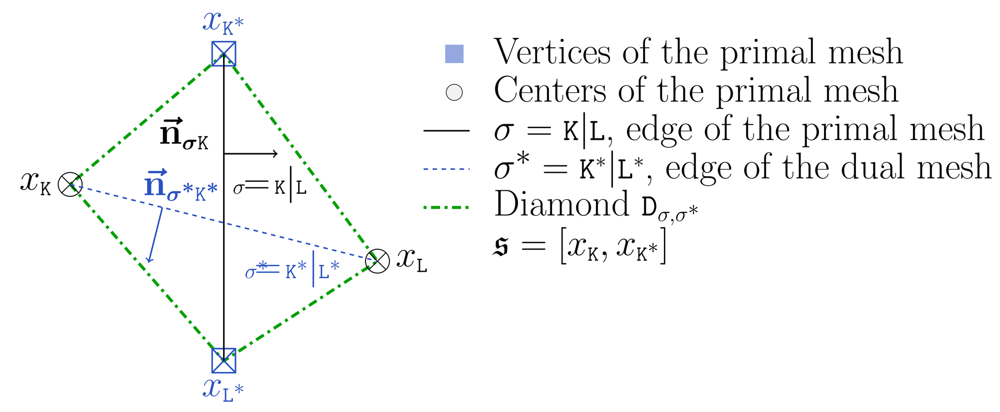
2. DDFV discretization

- **Previous works on 2D DDFV for Navier-Stokes problem** in the case of variable viscosity and Dirichlet boundary conditions: [K11]
- **The DDFV meshes** [DO05]



and $\mathfrak{T} = \mathfrak{M}\cup\partial\mathfrak{M}\cup\mathfrak{M}^*\cup\partial\mathfrak{M}^*$.

- **Zoom on the diamond cells**



- **The discrete unknowns**:

$$\mathrm{p}^{\mathfrak{D}} = \left(\mathrm{p}^{\mathrm{D}}\right)_{\mathrm{D}\in\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}},$$
$$\mathbf{u}^{\mathfrak{T}} = \left((\mathbf{u}_K)_{K\in\mathfrak{M}\cup\partial\mathfrak{M}}, (\mathbf{u}_{K^*})_{K^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*}\right) \in (\mathbb{R}^2)^{\mathfrak{T}}$$

- **The discrete gradient**: $\nabla^{\mathfrak{D}}$ constant on each diamond cell

$$\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}}\cdot(x_L - x_K) = \mathbf{u}_L - \mathbf{u}_K,$$
$$\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}}\cdot(x_{L^*} - x_{K^*}) = \mathbf{u}_{L^*} - \mathbf{u}_{K^*},$$

$$\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} = \frac{1}{2m_{\mathrm{D}}}\left[m_\sigma(\mathbf{u}_L - \mathbf{u}_K)\otimes\vec{\mathbf{n}}_{\sigma K} + m_{\sigma^*}(\mathbf{u}_{L^*} - \mathbf{u}_{K^*})\otimes\vec{\mathbf{n}}_{\sigma^* K^*}\right],\ \forall \mathrm{D}\in\mathfrak{D}.$$

- **The discrete strain rate tensor**: $\mathrm{D}^{\mathfrak{D}}$ constant on each diamond cell

$$\mathrm{D}^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} = \frac{\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} + {}^t(\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}})}{2}.$$

- **The discrete divergences**: $\mathbf{div}^{\mathfrak{T}}$ constant on each primal and dual cell. For $\xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$:

$$\forall K\in\mathfrak{M},\quad \mathrm{div}^K\xi^{\mathfrak{D}} = \frac{1}{m_K}\sum_{\sigma\subset\partial K} m_\sigma\xi^{\mathrm{D}}\vec{\mathbf{n}}_{\sigma K}$$
$$\forall K^*\in\mathfrak{M}^*,\quad \mathrm{div}^{K^*}\xi^{\mathfrak{D}} = \frac{1}{m_{K^*}}\sum_{\sigma^*\subset\partial K^*} m_{\sigma^*}\xi^{\mathrm{D}}\vec{\mathbf{n}}_{\sigma^* K^*}$$

and $\mathrm{div}^{\mathfrak{D}}$ constant on each diamond cell

$$\mathrm{div}^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} = \mathrm{Tr}(\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}})$$

- **Trace operators**:

– On the boundary of the domain $\gamma^{\mathfrak{T}}$: $\gamma_\sigma(\mathbf{u}^{\mathfrak{T}}) = \frac{\mathbf{u}_{K^*} + 2\mathbf{u}_L + \mathbf{u}_{L^*}}{4}\quad \forall\sigma\in\partial\mathfrak{M}$,

– On the boundary diamond mesh $\gamma^{\mathfrak{D}}$: $\gamma^{\mathfrak{D}}(\Phi^{\mathfrak{D}}) = (\Phi^{\mathrm{D}})_{\mathrm{D}\in\mathfrak{D}\cap\partial\Omega}$.

- **Inner products**:

$$[[\mathbf{u}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}}]]_{\mathfrak{T}} = \frac12\left(\sum_{K\in\mathfrak{M}} m_K\,\mathbf{u}_K\cdot\mathbf{v}_K + \sum_{K^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} m_{K^*}\,\mathbf{u}_{K^*}\cdot\mathbf{v}_{K^*}\right)$$
$$(\Phi^{\mathfrak{D}},\mathbf{v}^{\mathfrak{T}})_{\partial\Omega} = \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}\cap\partial\Omega} m_\sigma\Phi^{\mathrm{D}}\cdot\mathbf{v}_\sigma$$
$$(\xi^{\mathfrak{D}}:\Phi^{\mathfrak{D}})_{\mathfrak{D}} = \sum_{\mathrm{D}\in\mathfrak{D}} m_{\mathrm{D}}\,(\xi^{\mathrm{D}}:\Phi^{\mathrm{D}}),$$

to which we can associate norms, e.g.

$$\|\mathbf{u}^{\mathfrak{T}}\|_2 = [[\mathbf{u}^{\mathfrak{T}},\mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}}^{\frac12},\qquad \|\xi^{\mathfrak{D}}\|_2 = (\xi^{\mathfrak{D}}:\xi^{\mathfrak{D}})_{\mathfrak{D}}^{\frac12}$$

Theorem. (Discrete Green's formula) For all $\xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$, $\mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$:

$$[[\mathbf{div}^{\mathfrak{T}}\xi^{\mathfrak{D}},\mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} = -(\xi^{\mathfrak{D}}:\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}} + (\gamma^{\mathfrak{D}}(\xi^{\mathfrak{D}})\vec{\mathbf{n}},\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}))_{\partial\Omega}.$$

- **Convection term:** $\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}})$ constant on each primal and dual cell. For instance, on the primal mesh we define $\forall K\in\mathfrak{M}$:

$$m_K\mathbf{b}_K(\mathbf{u}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}}) = \sum_{\substack{\sigma\subset\partial K,\\ \sigma\notin\partial\Omega}} F_{K,\sigma}(\mathbf{u}^{\mathfrak{T}})\mathbf{v}_{\sigma^+} + \sum_{\substack{\sigma\subset\partial K,\\ \sigma\in\partial\Omega}} F_{K,\sigma}(\mathbf{u}^{\mathfrak{T}})\gamma^\sigma(\mathbf{v}^{\mathfrak{T}})$$

$$\text{where } F_{K,\sigma}(\mathbf{u}^{\mathfrak{T}}) = \begin{cases} -\displaystyle\sum_{\mathfrak{s}\in D_{\sigma,\sigma^*}\cap K} m_{\mathfrak{s}}\frac{\mathbf{u}_K + \mathbf{u}_{K^*}}{2}\cdot\vec{\mathbf{n}}_{\mathfrak{s}D} & \text{if } \sigma\notin\partial\Omega\\ m_\sigma\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})\cdot\vec{\mathbf{n}}_{\sigma K} & \text{if } \sigma\in\partial\Omega \end{cases}$$

$$\text{and } \mathbf{v}_{\sigma^+} = \begin{cases}\mathbf{v}_K \text{ if } F_{K,\sigma}\ge 0\\ \mathbf{v}_L \text{ otherwise}\end{cases}.$$

3. The scheme

Let $N\in\mathbb{N}^*$. We note $\delta t = \frac{T}{N}$. We look for $(\mathbf{u}^{n+1},\mathrm{p}^{n+1})$ by knowing the solution at the previous time step $(\mathbf{u}^n,\mathrm{p}^n)$. We can rewrite the weak formulation $(\mathcal{WF})$ in the DDFV framework as:

$$[[\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\delta t},\Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1},\mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} + \frac12[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n,\mathbf{u}^{n+1}),\Psi^{\mathfrak{T}}]]_{\mathfrak{T}}$$
$$-\frac12[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n,\Psi^{\mathfrak{T}}),\mathbf{u}^{n+1}]]_{\mathfrak{T}} = -\frac12\sum_{\mathrm{D}\in\mathfrak{D}\cap\Gamma_2}(F_{K,\sigma}(\mathbf{u}^n))^+\gamma^\sigma(\mathbf{u}^{n+1})\gamma^\sigma(\Psi^{\mathfrak{T}})$$
$$+\frac12\sum_{\mathrm{D}\in\mathfrak{D}\cap\Gamma_2}(F_{K,\sigma}(\mathbf{u}^n))^-\gamma^\sigma(\mathbf{u}_{ref})\gamma^\sigma(\Psi^{\mathfrak{T}}) + \sum_{\mathrm{D}\in\mathfrak{D}\cap\Gamma_2} m_\sigma(\sigma^{\mathrm{D}}_{\mathrm{ref}}\cdot\vec{\mathbf{n}}_{\sigma K})\cdot\gamma^\sigma(\Psi^{\mathfrak{T}}),$$

where $\Psi^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ is a test function in the discrete space that satisfies:

$$\Psi^{\mathfrak{T}} = 0 \quad \text{on } \Gamma_1,\qquad \mathrm{div}^{\mathfrak{D}}(\Psi^{\mathfrak{T}}) = 0.$$

Theorem. (Well posedness) Let $\mathfrak{T}$ be a mesh that satisfies inf-sup stability condition. Then the scheme admits a **unique** solution $(\mathbf{u}^n,\mathrm{p}^n)_{n\in\{0,\dots,N\}} \in \left((\mathbb{R}^2)^{\mathfrak{T}}\right)^{N+1}\times(\mathbb{R}^{\mathfrak{D}})^{N+1}$.

Remark: We require *inf-sup* condition because we need it in order to prove Korn's inequality and, then, the energy estimate. We could overcome this difficulty by adding a stabilization term and we would obtain existence and uniqueness for general meshes.

4. Discrete energy estimate

In order to prove the discrete energy estimate, it is necessary to prove:

Theorem. (Korn's inequality) Let $\mathfrak{T}$ be a mesh that satisfies inf-sup stability condition. Then there exists $C > 0$ such that :

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \le C\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \qquad \forall\mathbf{u}^{\mathfrak{T}}\in\mathbb{E}_0$$

Theorem. (Trace theorem) Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. For all $p > 1$, there exists a constant $C > 0$, such that $\forall\mathbf{u}^{\mathfrak{T}}\in\mathbb{E}_0$ and for all $s\ge 1$:

$$\|\gamma(\mathbf{u}^{\mathfrak{T}})\|^s_{s,\partial\Omega} \le C\|\mathbf{u}^{\mathfrak{T}}\|_{1,p}\|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{p(s-1)}{p-1}}$$

We then obtain:

Theorem. (Energy estimate) Let $\mathfrak{T}$ be a DDFV mesh associated to Ω that satisfies inf-sup stability condition. Let $(\mathbf{u}^n,\mathrm{p}^n)$, $n\ge 1$, be the solution of the DDFV scheme and $\mathbf{u}^n = \mathbf{v}^n + \mathbf{u}_{ref}$. For $N > 1$, there exists a constant $C > 0$ such that:

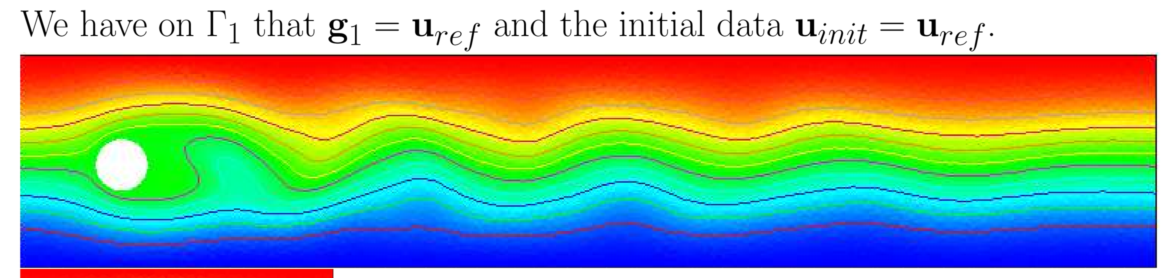
$$\sum_{j=0}^{N-1}\|\mathbf{v}^{j+1}-\mathbf{v}^j\|_2^2 \le C,\quad \|\mathbf{v}^N\|_2^2 \le C,$$
$$\sum_{j=0}^{N-1}\delta t\frac{1}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{j+1}\|_2^2 \le C,\quad \delta t\frac{1}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^N\|_2^2 \le C,$$
$$\sum_{j=0}^{N-1}\delta t\sum_{\mathrm{D}\in\mathfrak{D}\cap\Gamma_2}(F_{K,\sigma}(\mathbf{v}^j + \mathbf{u}^{\mathfrak{T}}_{ref}))^+(\gamma^\sigma(\mathbf{v}^{j+1}))^2 \le C.$$

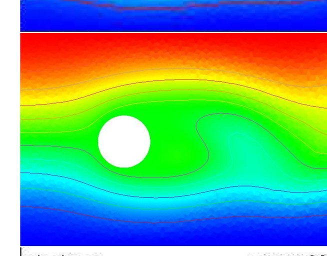
5. Numerical results

▶ **Flow around an obstacle:** The bigger computational domain is $\Omega = [0,2.2]\times[0,0.41]$, the smaller one is $\Omega' = [0,0.6]\times[0,0.41]$. The viscosity is $\eta = 10^{-3}$, $T = 5s$, $\delta t = 1.46\times10^{-3}$. The reference flow on Γ_2 is a Poiseuille flow:

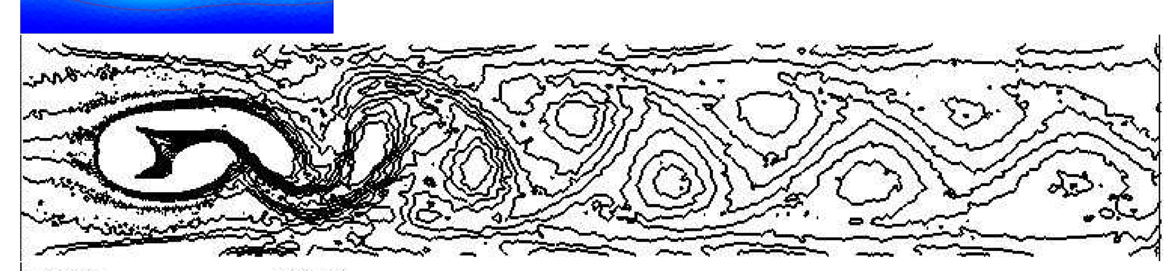
$$\mathbf{u}_{ref}(x,y) = \begin{pmatrix}\frac{1}{0.41^2}6y(0.41-y)\\ 0\end{pmatrix}$$
$$\sigma_{ref}(\mathbf{u},\mathrm{p})\cdot\vec{\mathbf{n}} = \begin{pmatrix}0\\ \frac{1}{0.41^2}6\eta(0.41-2y)\end{pmatrix}$$

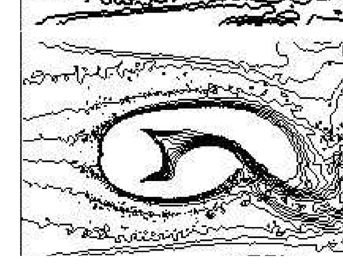
We have on Γ_1 that $\mathbf{g}_1 = \mathbf{u}_{ref}$ and the initial data $\mathbf{u}_{init} = \mathbf{u}_{ref}$.





Streamlines at Re = 100.





Vorticity lines at Re = 100.

We observe the efficiency of the condition: at the cut it does not introduce perturbations to the flow.

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