Polynomial Chaos and Hybrid High-Order methods for poroelasticity with random coefficients

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Abstract

We propose a novel numerical method for the Biot problem with uncertain poroelastic coefficients. The uncertainty is modelled using a finite set of parameters with prescribed distribution. We present the variational formulation of the stochastic partial differential system and establish its well-posedness. The approximation is based on sparse spectral projection methods, which essentially amount to performing an ensemble of deterministic model simulations to estimate the Polynomial Chaos expansion coefficients. The deterministic solver is based on the Hybrid High-Order discretization of [1] supporting general polyhedral meshes and arbitrary approximation orders. We numerically investigate the convergence of the Polynomial Chaos approximations with respect to the level of the sparse grid. Finally, we assess the propagation of the input uncertainty onto the solution considering an injection-extraction problem.

1. The Biot problem with random coefficients

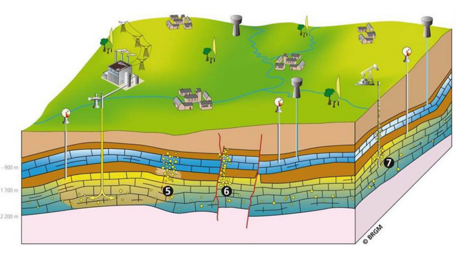
Let $\mu, \lambda, \alpha, c_0, \kappa : \Theta \to \mathbb{R}$ be random variables defined on the probability space $(\Theta, \mathcal{B}, \mathcal{P})$. For a given domain $D \subset \mathbb{R}^d$, final time $t_F > 0$, load $\boldsymbol{f}$, source g, and initial fluid content ϕ_0; find the displacement $\boldsymbol{u}$ and pressure p solution of

$$-\nabla\cdot\boldsymbol{\sigma}(\theta) + \nabla(\alpha(\theta)p(\theta)) = \boldsymbol{f}, \quad \text{in } D\times\Theta\times(0,t_F],$$
$$\mathrm{d}_t\phi(\theta) - \nabla\cdot(\kappa(\theta)\nabla p(\theta)) = g, \quad \text{in } D\times\Theta\times(0,t_F],$$
$$\phi(\theta, t=0) = \phi_0, \quad \text{in } D\times\Theta, \ (+\text{ BCs }).$$

- Stress tensor: $\boldsymbol{\sigma}(\theta) = 2\mu(\theta)\nabla_s\boldsymbol{u}(\theta) + \lambda(\theta)(\nabla\cdot\boldsymbol{u}(\theta))\boldsymbol{I}_d$
- Fluid content: $\phi(\theta) = c_0(\theta)p(\theta) + \alpha(\theta)\nabla\cdot\boldsymbol{u}(\theta)$

Applications

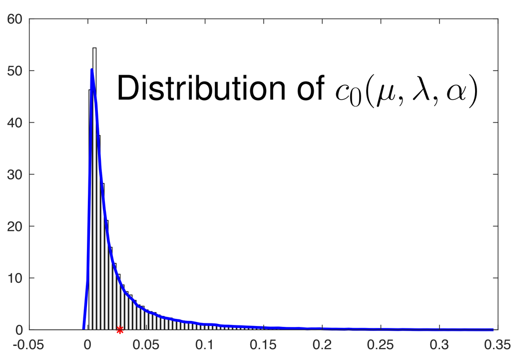
- Groundwater flow,
- Reservoir modelling,
- Earthquake engineering,
- CO_2 capture and storage...



Probabilistic model

We use a set of uniformly *iid* canonical random variables, collected into a random vector $\boldsymbol{\xi} : \Theta \to [-1,1]^4$, to describe the uncertainty of the poroelastic coefficients.

$$\mu(\boldsymbol{\xi}) = 10^{(\xi_1+1)}\ \mathrm{kPa},$$
$$\lambda(\boldsymbol{\xi}) = 2\cdot 10^{(\xi_2+1)}\ \mathrm{kPa},$$
$$\alpha(\boldsymbol{\xi}) = \frac{1+\alpha_{\min}}{2} + \xi_3\frac{1-\alpha_{\min}}{2},$$
$$\kappa(\boldsymbol{\xi}) = 10^{(\xi_4-1)}\ \mathrm{m^2kPa^{-1}s^{-1}}.$$



2. Stochastic discretization

Polynomial chaos expansions

Let $\rho : [-1,1]^N \to \mathbb{R}^+$ a pdf and $\{\phi_{\boldsymbol{k}}(\boldsymbol{\xi}) : \boldsymbol{k} \in \mathbb{N}^N\}$ an Hilbertian basis of orthogonal multivariate polynomials in $\boldsymbol{\xi}$:

$$\langle \phi_{\boldsymbol{k}}, \phi_{\boldsymbol{l}} \rangle = \int_{\Xi} \phi_{\boldsymbol{k}}(\boldsymbol{\xi})\phi_{\boldsymbol{l}}(\boldsymbol{\xi})\rho(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = \delta_{\boldsymbol{k},\boldsymbol{l}}.$$

The PC expansion of a second-order random variable X is

$$X(\boldsymbol{\xi}) = \sum_{\boldsymbol{k}\in\mathbb{N}^N} X_{\boldsymbol{k}}\phi_{\boldsymbol{k}}(\boldsymbol{\xi}).$$

The PC approximation $X_{\mathcal{K}}(\boldsymbol{\xi})$ of $X(\boldsymbol{\xi})$ is obtained by truncating the expansion to a finite set of multi-indices $\mathcal{K} \subset \mathbb{N}^N$.

Sparse Pseudo-Spectral Projection

In the spectral projection method the modes $X_{\boldsymbol{k}}$ of the PC expansion are computed using a numerical quadrature rule

$$X_{\boldsymbol{k}} = \langle X, \phi_{\boldsymbol{k}} \rangle \simeq \sum_{q=1}^{N_q} w^{(q)} X(\boldsymbol{\xi}^{(q)})\phi_{\boldsymbol{k}}(\boldsymbol{\xi}^{(q)}),$$

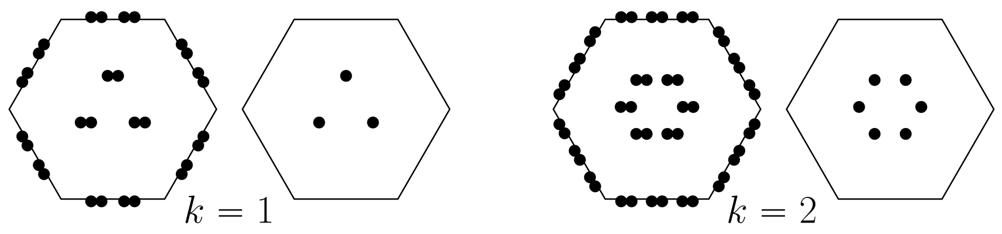
where the N_q nodes $\boldsymbol{\xi}^{(q)}$ and weights $w^{(q)}$ are constructed by tensorization of one-dimensional quadrature rules. The key-idea of PSP (cf. [3]) is to apply the Smolyak's formula on the projection operator, yielding, for the same sparse grid, a larger set $\mathcal{K}$ of basis functions $\phi_{\boldsymbol{k}}$ without internal aliasing:

$$\forall \boldsymbol{k}, \boldsymbol{l} \in \mathcal{K}, \ \sum_{q=1}^{N_q} w^{(q)}\phi_{\boldsymbol{k}}(\boldsymbol{\xi}^{(q)})\phi_{\boldsymbol{l}}(\boldsymbol{\xi}^{(q)}) = \delta_{\boldsymbol{kl}}.$$

3. HHO method for poroelasticity

Let $\mathcal{T}_h$ be an admissible mesh (cf. [2,4]), $\mathcal{F}_h$ the set collecting the mesh faces, and $k \geqslant 1$ a polynomial degree.

DOFs: $\underline{\boldsymbol{U}}_T^k \times \mathbb{P}_d^k(T)$, with $\underline{\boldsymbol{U}}_T^k := \mathbb{P}_d^k(T)^d \times \left\{ \bigtimes_{F\in\mathcal{F}_h} \mathbb{P}_{d-1}^k(F)^d \right\}$



The discretization of the **elasticity operator** is realized by

$$a_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) := \sum_{T\in\mathcal{T}_h}\left(\int_T \boldsymbol{\sigma}(\cdot, \boldsymbol{G}_{s,T}^k\underline{\boldsymbol{u}}_T) : \boldsymbol{G}_{s,T}^k\underline{\boldsymbol{v}}_T + s_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T)\right),$$
$$s_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T) := \sum_{F\in\mathcal{F}_T} h_F^{-1}\int_F \Delta_{TF}^k\underline{\boldsymbol{u}}_T \cdot \Delta_{TF}^k\underline{\boldsymbol{v}}_T.$$

In s_T we penalize in a least-square sense the face-based residual $\Delta_{TF}^k\underline{\boldsymbol{v}}_T := \boldsymbol{\pi}_F^k(\boldsymbol{r}_T^{k+1}\underline{\boldsymbol{v}}_T - \boldsymbol{v}_F) - \boldsymbol{\pi}_T^k(\boldsymbol{r}_T^{k+1}\underline{\boldsymbol{v}}_T - \boldsymbol{v}_T)$.

Symmetric gradient reconstruction operator

$$\boldsymbol{G}_{s,T}^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}_d^k(T)_{\text{sym}}^{d\times d} \ \text{ s.t. } \ \forall\boldsymbol{\tau} \in \mathbb{P}_d^k(T)_{\text{sym}}^{d\times d}$$
$$\int_T \boldsymbol{G}_{s,T}^k\underline{\boldsymbol{v}}_T : \boldsymbol{\tau} = -\int_T \boldsymbol{v}_T\cdot(\nabla\cdot\boldsymbol{\tau}) + \sum_{F\in\mathcal{F}_T}\int_F \boldsymbol{v}_F\cdot(\boldsymbol{\tau}\boldsymbol{n}_{TF})$$

Lemma 1. *The following commuting property holds for* $\boldsymbol{G}_{s,T}^k$*:*

$$\boldsymbol{G}_{s,T}^k\underline{\boldsymbol{I}}_T^k\boldsymbol{v} = \boldsymbol{\pi}_T^k(\nabla_s\boldsymbol{v})$$

Displacement reconstruction operator

$$\boldsymbol{r}_T^{k+1} : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}_d^{k+1}(T)^d \ \text{ s.t. } \ \forall\boldsymbol{w} \in \mathbb{P}_d^{k+1}(T)^d$$
$$\int_T(\nabla_s\boldsymbol{r}_T^{k+1}\underline{\boldsymbol{v}}_T - \boldsymbol{G}_{s,T}^k\underline{\boldsymbol{v}}_T) : \nabla_s\boldsymbol{w} = 0 \quad + \text{rigid-body motions}.$$

The **hydro-mechanical coupling** is realized by means of

$$b_h(\underline{\boldsymbol{v}}_h, q_h) := -\sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{G}_{s,T}^k\underline{\boldsymbol{v}}_T : q_h\boldsymbol{I}_d, \ \forall\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k, \ \forall q_h \in \mathbb{P}_d^k(\mathcal{T}_h).$$

Lemma 2 (Inf-sup condition for b_h)**.**

$$\exists\beta > 0 \ \textit{s.t.} \ \|q_h\| \leqslant \beta \sup_{\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{U}}_{h,0}^k\setminus\{\underline{\boldsymbol{0}}\}} \frac{b_h(\underline{\boldsymbol{v}}_h, q_h)}{\|\underline{\boldsymbol{v}}_h\|_{a,h}}, \quad \forall q_h \in \mathbb{P}_d^k(\mathcal{T}_h)\cap L_0^2(\Omega)$$

The discrete counterpart of the **Darcy operator** is given by

$$c_h(r_h, q_h) := \int_\Omega \boldsymbol{\kappa}\nabla_h r_h\nabla_h q_h + \sum_{F\in\mathcal{F}_h}\frac{\varsigma\lambda_{\kappa,F}}{h_F}\int_F [\![r_h]\!]_F[\![q_h]\!]_F +$$
$$-\sum_{F\in\mathcal{F}_h}\int_F\left([\![q_h]\!]_F\{\!\{\boldsymbol{\kappa}\nabla_h r_h\}\!\}_{\omega,F} + [\![r_h]\!]_F\{\!\{\boldsymbol{\kappa}\nabla_h q_h\}\!\}_{\omega,F}\right)\cdot\boldsymbol{n}_F,$$

where $\{\!\{\cdot\}\!\}_{\omega,F}$ and $[\![\cdot]\!]_F$ are the average and jump operators.

Using an implicit time discretization (e.g. $\delta_t\varphi^n := \frac{\varphi^n - \varphi^{n-1}}{\tau}$), we obtain the **discrete coupled problems**:

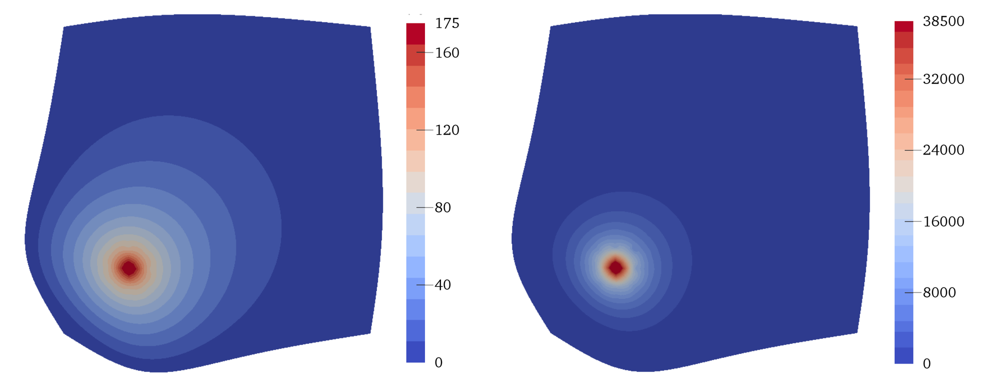
At each step $1 \leqslant n \leqslant N$, find $\underline{\boldsymbol{u}}_h^n \in \underline{\boldsymbol{U}}_h^k$ and $p_h^n \in \mathbb{P}_d^k$ s.t.

$$a_h(\underline{\boldsymbol{u}}_h^n, \underline{\boldsymbol{v}}_h) + b_h(\underline{\boldsymbol{v}}_h, p_h^n) = \int_\Omega \boldsymbol{f}^n\cdot\boldsymbol{v}_h \ \ \forall\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$$
$$(c_0\delta_t p_h^n, q_h) - b_h(\delta_t\underline{\boldsymbol{u}}_h^n, q_h) + c_h(p_h^n, q_h) = \int_\Omega g^n q_h \ \ \forall q_h \in \mathbb{P}_d^k$$

4. Point injection and poroelastic footing tests

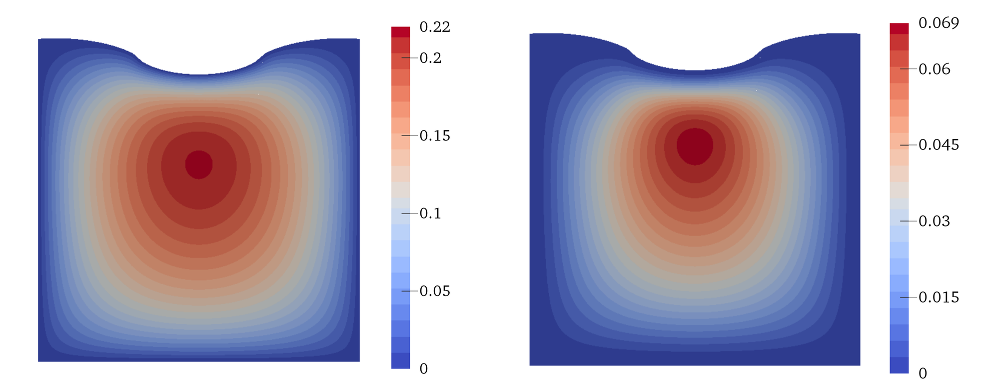
Validation tests using the PSP method with $l{=}5$ and $N_q{=}2561$

- Data: $D = [0,1]^2$, $\boldsymbol{f} = \boldsymbol{0}$, $\phi_0 = 0$, $t_F = 1\mathrm{s}$.
  Point source: $g = 10\cdot\delta(\boldsymbol{x} - \boldsymbol{x}_0)$, where $\boldsymbol{x}_0 = (0.25, 0.25)$.
  BCs on ∂D: $\boldsymbol{u}\cdot\boldsymbol{\tau} = 0$, $\nabla\boldsymbol{u}\boldsymbol{n}\cdot\boldsymbol{n} = 0$, $p = 0$.



Mean and variance pressure fields obtained using the HHO method with $k = 3$ on a Cartesian mesh with 1024 elements.

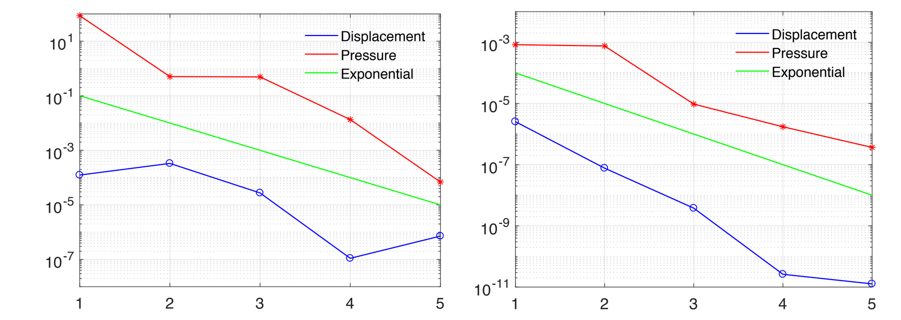
- Data: $D = [0,1]^2$, $\boldsymbol{f} = \boldsymbol{0}$, $g = 0$, $\phi_0 = 0$, $t_F = 0.2\mathrm{s}$.
  BCs: $\boldsymbol{\sigma n} = (0,-5)$ on $\Gamma_N := \{\boldsymbol{x} \mid 0.3 \leqslant x_1 \leqslant 0.7, \ x_2 = 1\}$, $\boldsymbol{\sigma n} = \boldsymbol{0}$ on $\{x_2 = 1\}\setminus\Gamma_N$, $\boldsymbol{u} = \boldsymbol{0}$ on $\partial D\setminus\{x_2 = 1\}$, $p = 0$ on ∂D.



Mean and variance pressure fields obtained using the HHO method with $k = 2$ on a triangular mesh with 3584 elements.

Convergence analysis

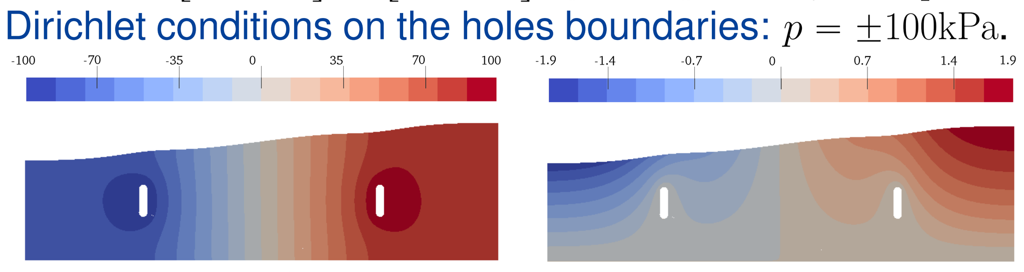
The accuracy of the PCEs is evaluated on a 500 points LHS.



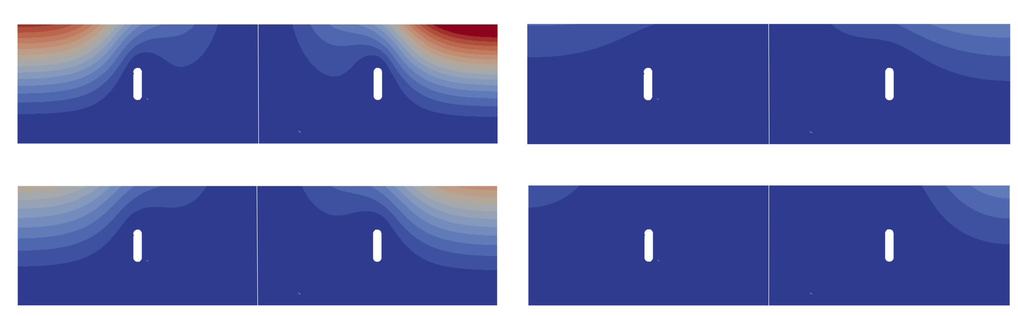
Errors $\|\mathrm{MSE}(\boldsymbol{u} - \boldsymbol{u}_{\mathcal{K}})\|$ and $\|\mathrm{MSE}(p - p_{\mathcal{K}})\|$ vs. level l of the Sparse Grid for the injection (left) and footing (right) tests.

5. Injection-extraction test and sensitivity analysis

Data: $D{=}[0,4\ \mathrm{Km}] \times [0,1\ \mathrm{Km}]$, $\boldsymbol{f}{=}\boldsymbol{0}$, $g{=}0$, $\phi_0{=}0$, $t_F{=}1\mathrm{d}$.
Dirichlet conditions on the holes boundaries: $p = \pm 100\mathrm{kPa}$.



Mean pressure field in kPa and vertical displacement in mm obtained with $k = 1$ on a Voronoi mesh with 10^4 elements.



First and total-order partial variances of the vertical displacement related to μ (top left), λ (top right), α (bottom left) and κ (bottom right). PSP method with $l = 3$ and $N_q = 209$.

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