Some New Estimates for Virtual Element Methods

Susanne C. Brenner

Department of Mathematics
and
Center for Computation & Technology

Louisiana State University

USA

POEMS 2019

Some New Estimates
for
Virtual Element Methods

Joint work with

Li-yeng Sung

A Model Poisson Problem

Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \qquad \forall\, v \in H_0^1(\Omega)$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \text{and} \quad (f,v) = \int_\Omega f v \, dx$$

Ω is a polygonal/polyhedral domain in $\mathbb{R}^2$/$\mathbb{R}^3$.

f belongs to $L_2(\Omega)$.

A Model Poisson Problem

Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \qquad \forall\, v \in H_0^1(\Omega)$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \text{and} \quad (f,v) = \int_\Omega f v \, dx$$

Ω is a polygonal/polyhedral domain in $\mathbb{R}^2/\mathbb{R}^3$.

f belongs to $L_2(\Omega)$.

For simplicity we assume that Ω is convex so that $u \in H^2(\Omega)$.

Outlne

- Virtual Element Methods in 2D

Outlne

- Virtual Element Methods in 2D
  - Shape Regularity Assumptions
  - Estimates for Computable Projections
  - Inverse Estimates
  - Estimates for an Interpolation Operator
  - Stabilization Estimates
  - Error Estimates in H^1 and L_2
  - Error Estimates in L_∞

Outlne

- Virtual Element Methods in 2D
- Extensions to 3D
- Concluding Remarks

References

Basic principles of virtual element methods (2013)

Beirão da Veiga-Brezzi-Cangiani-Manzini-Marini-Russo

Equivalent projectors for virtual element methods (2013)

Ahmad-Alsaedi-Brezzi-Marini-Russo

Virtual element method for general second-order elliptic problems on polygonal meshes (2016)

Beirão da Veiga-Brezzi-Marini-Russo

High-order virtual element method on polyhedral meshes (2017)

Beirão da Veiga-Dassi–Russo

Stability analysis for the virtual element method (2017)

Beirão da Veiga-Lovadina-Russo

Virtual element methods on meshes with small edges or faces (2018)

B.-Sung

Virtual Element Methods
in 2D

Local Virtual Element Spaces

D is a bounded polygon.

$\mathcal{E}_D$ is the set of the edges of D.

$\mathbb{P}_k$ is the space of polynomials of total degree $\leq k$.

$\mathbb{P}_{-1} = \{0\}$

$\mathbb{P}_k(D)$ is the restriction of $\mathbb{P}_k$ to D.

$\mathbb{P}_k(e)$ is the restriction of $\mathbb{P}_k$ to the edge e.

$\mathbb{P}_k(\partial D) = \{v \in C(\partial D) : v\big|_e \in \mathbb{P}_k(e)$ for all $e \in \mathcal{E}_D\}$

Local Virtual Element Spaces

$\Pi^{\nabla}_{k,D}$ is the projection from $H^1(D)$ onto $\mathbb{P}_k(D)$ with respect to the inner product

$$(\!(\zeta,\eta)\!) = \int_D \nabla\zeta\cdot\nabla\eta\,dx + \Big(\int_{\partial D}\zeta\,ds\Big)\Big(\int_{\partial D}\eta\,ds\Big)$$

i.e., $\Pi^{\nabla}_{k,D}\zeta \in \mathbb{P}_k(D)$ satisfies

$$(\!(\Pi^{\nabla}_{k,D}\zeta, q)\!) = (\!(\zeta, q)\!) \qquad \forall\, q \in \mathbb{P}_k(D).$$

Local Virtual Element Spaces

$\Pi^{\nabla}_{k,D}$ is the projection from $H^1(D)$ onto $\mathbb{P}_k(D)$ with respect to the inner product

$$(\!(\zeta,\eta)\!) = \int_D \nabla\zeta\cdot\nabla\eta\,dx + \Big(\int_{\partial D}\zeta\,ds\Big)\Big(\int_{\partial D}\eta\,ds\Big)$$

i.e., $\Pi^{\nabla}_{k,D}\zeta \in \mathbb{P}_k(D)$ satisfies

$$(\!(\Pi^{\nabla}_{k,D}\zeta, q)\!) = (\!(\zeta, q)\!) \qquad \forall\, q\in\mathbb{P}_k(D).$$

Equivalently,

$$\int_D \nabla(\Pi^{\nabla}_{k,D}\zeta)\cdot\nabla q\,dx = \int_D \nabla\zeta\cdot\nabla q\,dx \qquad \forall\, q\in\mathbb{P}_k(D)$$

$$\int_{\partial D}\Pi^{\nabla}_{k,D}\zeta\,ds = \int_{\partial D}\zeta\,ds$$

Local Virtual Element Spaces

$\Pi^{\nabla}_{k,D}$ is the projection from $H^1(D)$ onto $\mathbb{P}_k(D)$ with respect to the inner product

$$(\!(\zeta,\eta)\!) = \int_D \nabla\zeta\cdot\nabla\eta\,dx + \Big(\int_{\partial D}\zeta\,ds\Big)\Big(\int_{\partial D}\eta\,ds\Big)$$

$\Pi^{0}_{k,D}$ is the projection from $L_2(D)$ onto $\mathbb{P}_k(D)$.

Local Virtual Element Spaces

$\Pi^{\nabla}_{k,D}$ is the projection from $H^1(D)$ onto $\mathbb{P}_k(D)$ with respect to the inner product

$$((\zeta,\eta)) = \int_D \nabla\zeta\cdot\nabla\eta\,dx + \Big(\int_{\partial D}\zeta\,ds\Big)\Big(\int_{\partial D}\eta\,ds\Big)$$

$\Pi^{0}_{k,D}$ is the projection from $L_2(D)$ onto $\mathbb{P}_k(D)$.

Virtual Element Space $\mathcal{Q}^k(D)$ $(k \geq 1)$

$v \in H^1(D)$ belongs to $\mathcal{Q}^k(D)$ if and only if

- The trace of v on ∂D belongs to $\mathbb{P}_k(\partial D)$.
- The distribution Δv belongs to $\mathbb{P}_k(D)$.
- $\Pi^0_{k,D} v - \Pi^{\nabla}_{k,D} v \in \mathbb{P}_{k-2}(D)$

Ahmad-Alsaedi-Brezzi-Marini-Russo (2013)

Local Virtual Element Spaces

Virtual Element Space $\mathcal{Q}^k(D)$ $(k \geq 1)$

$v \in H^1(D)$ belongs to $\mathcal{Q}^k(D)$ if and only if

- The trace of v on ∂D belongs to $\mathbb{P}_k(\partial D)$.
- The distribution Δv belongs to $\mathbb{P}_k(D)$.
- $\Pi^0_{k,D} v - \Pi^\nabla_{k,D} v \in \mathbb{P}_{k-2}(D)$

Ahmad-Alsaedi-Brezzi-Marini-Russo (2013)

Properties of $v \in \mathcal{Q}^k(D)$

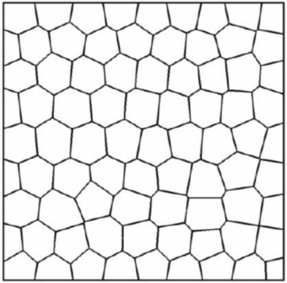
- v is uniquely determined by $v\big|_{\partial D}$ and $\Pi^0_{k-2,D} v$.

$$\dim \mathcal{Q}^k(D) = \dim \mathbb{P}_k(\partial D) + \dim \mathbb{P}_{k-2}(D)$$

- $\Pi^\nabla_{k,D} v$ and $\Pi^0_{k,D} v$ are computable.
- v is continuous on $\bar{D}$.

Global Virtual Element Spaces

$\mathcal{T}_h$ is a partition of Ω into polygonal subdomains.



$$\mathcal{Q}_h^k = \{v \in H_0^1(\Omega) : v\big|_D \in \mathcal{Q}^k(D) \quad \forall\, D \in \mathcal{T}_h\}$$

Virtual Element Methods

For $v \in \mathcal{Q}_h^k$, we have

$$a(v,v) = \sum_{D\in\mathcal{T}_h} a^D(v,v) \qquad\qquad \Big(a^D(w,v) = \int_D \nabla w\cdot\nabla v\,dx\Big)$$

$$= \sum_{D\in\mathcal{T}_h} \big[a^D(\Pi^\nabla_{k,D} v, \Pi^\nabla_{k,D} v) + a^D(v - \Pi^\nabla_{k,D} v, v - \Pi^\nabla_{k,D} v)\big]$$

because $\Pi^\nabla_{k,D} v \in \mathbb{P}_k(D)$ and hence

$$a^D(v-\Pi^\nabla_{k,D} v, \Pi^\nabla_{k,D} v) = \int_D \nabla(v - \Pi^\nabla_{k,D} v)\cdot\nabla\Pi^\nabla_{k,D} v\,dx = 0$$

by the definition of $\Pi^\nabla_{k,D}$.

$$\int_D \nabla(\Pi^\nabla_{k,D}\zeta)\cdot\nabla q\,dx = \int_D \nabla\zeta\cdot\nabla q\,dx \qquad \forall\, q\in\mathbb{P}_k(D)$$

Virtual Element Methods

For $v \in \mathcal{Q}_h^k$, we have

$$
\begin{aligned}
a(v,v) &= \sum_{D\in\mathcal{T}_h} a^D(v,v) \qquad\qquad \Big(a^D(w,v) = \int_D \nabla w\cdot\nabla v\,dx\Big)\\
&= \sum_{D\in\mathcal{T}_h} \big[a^D(\Pi^\nabla_{k,D}v, \Pi^\nabla_{k,D}v) + a^D(v-\Pi^\nabla_{k,D}v, v-\Pi^\nabla_{k,D}v)\big]
\end{aligned}
$$

The first term on the right-hand side is at our disposal because $\Pi^\nabla_{k,D}v$ is computable.

The second term is not available because v is not known in D, and it needs to be replaced by a stabilization term.

Virtual Element Methods

Find $u_h \in \mathcal{Q}_h^k$ such that

$$a_h(u_h, v) = (f, \Xi_h v) \qquad \forall\, v \in \mathcal{Q}_h^k$$

where

$$a_h(w, v) = \sum_{D \in \mathcal{T}_h} \Big[a^D(\Pi^{\nabla}_{k,D} w, \Pi^{\nabla}_{k,D} v) + S^D(w - \Pi^{\nabla}_{k,D} w, v - \Pi^{\nabla}_{k,D} v) \Big]$$

$$a^D(w, v) = \int_D \nabla w \cdot \nabla v \, dx$$

and

$$\Xi_h = \begin{cases} \Pi^0_{1,h} & \text{if } k = 1, 2, \\ \Pi^0_{k-2,h} & \text{if } k \geq 3. \end{cases}$$

Virtual Element Methods

First stabilization bilinear form

$$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$$

Wriggers-Rust-Reddy (2016)

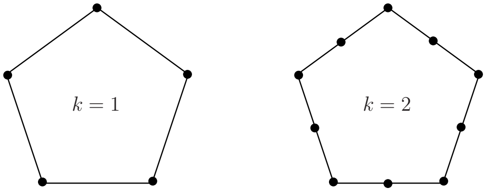
Virtual Element Methods

Second stabilization bilinear form

$$S^D(w, v) = \sum_{p \in \mathcal{N}_{\partial D}} w(p)v(p)$$

where $\mathcal{N}_{\partial D}$ is the set of the nodes on ∂D that determines functions in $\mathbb{P}_k(\partial D)$.

Beirão da Veiga-Lovadina-Russo (2017)



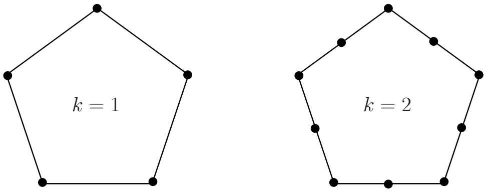
Virtual Element Methods

Second stabilization bilinear form

$$S^D(w,v) = \sum_{p \in \mathcal{N}_{\partial D}} w(p)v(p)$$

where $\mathcal{N}_{\partial D}$ is the set of the nodes on ∂D that determines functions in $\mathbb{P}_k(\partial D)$.

Beirão da Veiga-Lovadina-Russo (2017)



We can take these nodal values and the moments of v on D up to order $k-2$ as the dofs of $\mathcal{Q}^k(D)$.

Virtual Element Methods

- Derive error estimates for u_h, $\Pi^{\nabla}_{k,D} u_h$ and $\Pi^{0}_{k,D} u_h$ in the H^1 norm and the L_2 norm.
- Derive error estimate for u_h in the L_∞ norm over the skeleton of $\mathcal{T}_h$ where u_h is known.
- Derive error estimates for $\Pi^{\nabla}_{k,D} u_h$ and $\Pi^{0}_{k,D} u_h$ in the L_∞ norm over Ω.

Virtual Element Methods

- Derive error estimates for u_h, $\Pi^{\nabla}_{k,D}u_h$ and $\Pi^{0}_{k,D}u_h$ in the H^1 norm and the L_2 norm.
- Derive error estimate for u_h in the L_∞ norm over the skeleton of $\mathcal{T}_h$ where u_h is known.
- Derive error estimates for $\Pi^{\nabla}_{k,D}u_h$ and $\Pi^{0}_{k,D}u_h$ in the L_∞ norm over Ω.

Ingredients

- Stability estimates for $a_h(\cdot,\cdot)$.
- Estimates for $\Pi^{\nabla}_{k,D}$ and $\Pi^{0}_{k,D}$.
- Interpolation error estimates.

Virtual Element Methods

- Derive error estimates for u_h, $\Pi^{\nabla}_{k,D} u_h$ and $\Pi^{0}_{k,D} u_h$ in the H^1 norm and the L_2 norm.
- Derive error estimate for u_h in the L_∞ norm over the skeleton of $\mathcal{T}_h$ where u_h is known.
- Derive error estimates for $\Pi^{\nabla}_{k,D} u_h$ and $\Pi^{0}_{k,D} u_h$ in the L_∞ norm over Ω.

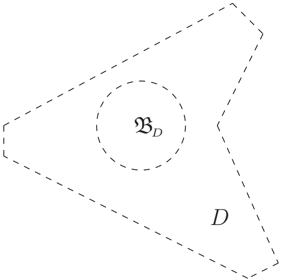
Challenge

Control the constants in all the estimates in terms of the shape regularity of the subdomains, especially in the presence of small edges or faces.

Shape Regularity Assumptions

Local Shape Regularity Assumptions

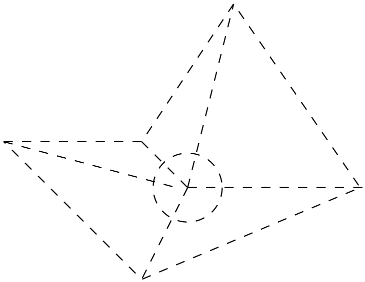
The (open) polygon D is star-shaped with respect to a disc $\mathfrak{B}_D \subset D$ with radius $\rho_D\, h_D$, where h_D is the diameter of D.



Local Shape Regularity Assumptions

The (open) polygon D is star-shaped with respect to a disc $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

Under the additional assumption that the lengths of the edges of D are comparable, we can generate a background mesh that satisfies a minimum angle condition by connecting the center of $\mathfrak{B}_D$ and the vertices of D.



Local Shape Regularity Assumptions

The (open) polygon D is star-shaped with respect to a disc $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

Under the additional assumption that the lengths of the edges of D are comparable, we can generate a background mesh that satisfies a minimum angle condition by connecting the center of $\mathfrak{B}_D$ and the vertices of D.

In the presence of small edges such a background mesh only satisfies a maximum angle condition.

Local Shape Regularity Assumptions

The (open) polygon D is star-shaped with respect to a disc $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

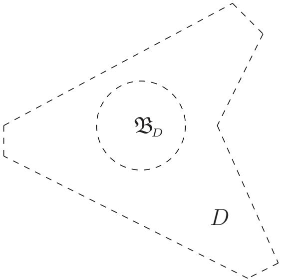
Under the additional assumption that the lengths of the edges of D are comparable, we can generate a background mesh that satisfies a minimum angle condition by connecting the center of $\mathfrak{B}_D$ and the vertices of D.

In the presence of small edges such a background mesh only satisfies a maximum angle condition.

We do not use any background mesh in our approach.

Local Shape Regularity Assumptions

The (open) polygon D is star-shaped with respect to a disc $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.



There exists a Lipschitz isomorphism $\Phi : \mathfrak{B}_D \longrightarrow D$ such that both $|\Phi|_{W^{1,\infty}(\mathfrak{B}_D)}$ and $|\Phi^{-1}|_{W^{1,\infty}(D)}$ are bounded by a constant that only depends on ρ_D.

Maz'ya, *Sobolev Spaces*

Local Shape Regularity Assumptions

The (open) polygon D is star-shaped with respect to a disc $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

Under the star-shaped assumption we can use ρ_D to control the constants in many estimates.

B.-Scott

The Mathematical Theory of Finite Element Methods

Local Shape Regularity Assumptions

The (open) polygon D is star-shaped with respect to a disc $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

A Sobolev Inequality

$$\|\zeta\|_{L_\infty(D)} \lesssim h_D^{-(1/2)} \|\zeta\|_{L_2(D)} + |\zeta|_{H^1(D)} + h_D |\zeta|_{H^2(D)}$$

for all $\zeta \in H^2(D)$

(The hidden constant only depends on ρ_D.)

Local Shape Regularity Assumptions

The (open) polygon D is star-shaped with respect to a disc $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

A Sobolev Inequality

$$\|\zeta\|_{L_\infty(D)} \lesssim h_D^{-(1/2)} \|\zeta\|_{L_2(D)} + |\zeta|_{H^1(D)} + h_D |\zeta|_{H^2(D)}$$

for all $\zeta \in H^2(D)$

Bramble-Hilbert Estimates

$$\inf_{q \in \mathbb{P}_\ell} |\zeta - q|_{H^m(D)} \lesssim h_D^{\ell+1-m} |\zeta|_{H^{\ell+1}(D)} \qquad \forall\, \zeta \in H^{\ell+1}(D)$$

where $\ell = 0, \ldots, k$ and $m \leq \ell$

(The hidden constants only depend on ρ_D and k.)

Local Shape Regularity Assumptions

The (open) polygon D is star-shaped with respect to a disc $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

Poincaré-Friedrichs Inequalities

$$h_D^{-(1/2)} \|\zeta\|_{L_2(D)} \lesssim h_D^{-2} \Big| \int_D \zeta \, dx \Big| + |\zeta|_{H^1(D)} \qquad \forall\, \zeta \in H^1(D)$$

$$h_D^{-(1/2)} \|\zeta\|_{L_2(D)} \lesssim h_D^{-1} \Big| \int_{\partial D} \zeta \, ds \Big| + |\zeta|_{H^1(D)} \qquad \forall\, \zeta \in H^1(D)$$

(The hidden constants only depend on ρ_D.)

Local Shape Regularity Assumptions

The (open) polygon D is star-shaped with respect to a disc $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

Poincaré-Friedrichs Inequalities

$$h_D^{-(1/2)}\|\zeta\|_{L_2(D)} \lesssim h_D^{-2}\left|\int_D \zeta\,dx\right| + |\zeta|_{H^1(D)} \qquad \forall\,\zeta \in H^1(D)$$

$$h_D^{-(1/2)}\|\zeta\|_{L_2(D)} \lesssim h_D^{-1}\left|\int_{\partial D} \zeta\,ds\right| + |\zeta|_{H^1(D)} \qquad \forall\,\zeta \in H^1(D)$$

A Trace Inequality

$$\|\zeta\|^2_{L_2(\partial D)} \lesssim h_D^{-1}\|\zeta\|^2_{L_2(D)} + h_D|\zeta|^2_{H^1(D)} \qquad \forall\,\zeta \in H^1(D)$$

(The hidden constants only depend on ρ_D.)

Local Shape Regularity Assumptions

The (open) polygon D is star-shaped with respect to a disc $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

Estimates for $|\cdot|_{H^{1/2}(\partial D)}$

$$|\zeta|_{H^{1/2}(\partial D)} \lesssim h_D^{1/2} |\zeta|_{H^1(\partial D)} \qquad \forall \zeta \in H^1(\partial D)$$

$$|\zeta|_{H^{1/2}(\partial D)} \lesssim |\zeta|_{H^1(D)} \qquad \forall \zeta \in H^1(D)$$

(The hidden constants only depend on ρ_D.)

There exists a Lipschitz isomorphism $\Phi : \mathfrak{B}_D \longrightarrow D$ such that both $|\Phi|_{W^{1,\infty}(\mathfrak{B}_D)}$ and $|\Phi^{-1}|_{W^{1,\infty}(D)}$ are bounded by a constant that only depends on ρ_D.

Local Shape Regularity Assumptions

The (open) polygon D is star-shaped with respect to a disc $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

Scaling Estimates for Polynomials

$$\|q\|^2_{L_2(\partial D)} \lesssim h_D^{-1}\|q\|^2_{L_2(D)} \qquad \forall\, q \in \mathbb{P}_k$$

$$|q|_{H^1(D)} \lesssim h_D^{-1}\|q\|_{L_2(D)} \qquad \forall\, q \in \mathbb{P}_k$$

$$\|q\|_{L_\infty(D)} \lesssim |\bar{q}_{\partial D}| + h_D^{1-(d/2)}|q|_{H^1(D)} \qquad \forall\, q \in \mathbb{P}_k$$

$$\|q\|_{L_\infty(D)} \lesssim |\bar{q}_D| + h_D^{1-(d/2)}|q|_{H^1(D)} \qquad \forall\, q \in \mathbb{P}_k$$

where

$$\bar{q}_{\partial D} = \frac{1}{|\partial D|}\int_{\partial D} q\, ds \quad \text{and} \quad \bar{q}_D = \frac{1}{|D|}\int_D q\, dx$$

(The hidden constants only depend on ρ_D and k.)

Global Shape Regularity Assumptions

The (open) polygon D is star-shaped with respect to a disc $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

Assumption 1 There exists a positive number $\rho \in (0,1)$, independent of h, such that

$$\rho_D \geq \rho \qquad \forall\, D \in \mathcal{T}_h$$

This is the only assumption we need for the first stabilization

$$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$$

Global Shape Regularity Assumptions

The (open) polygon D is star-shaped with respect to a disc $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

Assumption 1 There exists a positive number $\rho \in (0,1)$, independent of h, such that

$$\rho_D \geq \rho \qquad \forall\, D \in \mathcal{T}_h$$

Assumption 2 There exists a positive integer N, independent of h, such that

$$|\mathcal{E}_D| \leq N \qquad \forall\, D \in \mathcal{T}_h$$

We need both assumptions for the second stabilization

$$S^D(w,v) = \sum_{p \in \mathcal{N}_{\partial D}} w(p)v(p)$$

A Maximum Principle

Lemma 1

There exists a linear operator $\Delta^\dagger : \mathbb{P}_k(D) \longrightarrow \mathbb{P}_{k+2}(D)$ such that

$$\Delta(\Delta^\dagger q) = q \qquad \forall\, q \in \mathbb{P}_k(D)$$

$$\|\Delta^\dagger q\|_{L_\infty(D)} \lesssim h_D^2 \|q\|_{L_\infty(D)} \qquad \forall\, q \in \mathbb{P}_k(D)$$

where the hidden constant depends only on k and ρ_D

Beirão da Veiga-Lovadina-Russo (2017)

Lemma 1

There exists a linear operator $\Delta^\dagger : \mathbb{P}_k(D) \longrightarrow \mathbb{P}_{k+2}(D)$ such that

$$\Delta(\Delta^\dagger q) = q \qquad \forall\, q \in \mathbb{P}_k(D)$$

$$\|\Delta^\dagger q\|_{L_\infty(D)} \lesssim h_D^2 \|q\|_{L_\infty(D)} \qquad \forall\, q \in \mathbb{P}_k(D)$$

where the hidden constant depends only on k and ρ_D

Beirão da Veiga-Lovadina-Russo (2017)

- $\Delta : \mathbb{P}_{k+2} \longrightarrow \mathbb{P}_k$ is a surjection.
- scaling arguments for polynomials

Lemma 2

$$\|\Delta v\|_{L_2(D)} \lesssim h_D^{-1}|v|_{H^1(\Omega)} \qquad \forall\, v \in \mathcal{Q}^k(D).$$

where the hidden constant depends only on k and ρ_D

Beirão da Veiga-Lovadina-Russo (2017)

Lemma 2

$$\|\Delta v\|_{L_2(D)} \lesssim h_D^{-1}|v|_{H^1(\Omega)} \qquad \forall\, v \in \mathcal{Q}^k(D).$$

where the hidden constant depends only on k and ρ_D

Beirão da Veiga-Lovadina-Russo (2017)

- $\Delta v \in \mathbb{P}_k(D)$
- scaling arguments for polynomials

A Maximum Principle

There exists a positive constant C depending only on k and ρ_D such that

$$\|v\|_{L_\infty(D)} \le \|v\|_{L_\infty(\partial D)} + C|v|_{H^1(D)} \qquad \forall\, v \in \mathcal{Q}^k(D)$$

A Maximum Principle

There exists a positive constant C depending only on k and ρ_D such that

$$\|v\|_{L_\infty(D)} \leq \|v\|_{L_\infty(\partial D)} + C|v|_{H^1(D)} \qquad \forall\, v \in \mathcal{Q}^k(D)$$

$$\|v\|_{L_\infty(D)} \leq \|v - \Delta^\dagger \Delta v\|_{L_\infty(D)} + \|\Delta^\dagger \Delta v\|_{L_\infty(D)}$$

A Maximum Principle

There exists a positive constant C depending only on k and ρ_D such that

$$\|v\|_{L_\infty(D)} \leq \|v\|_{L_\infty(\partial D)} + C|v|_{H^1(D)} \qquad \forall\, v \in \mathcal{Q}^k(D)$$

$$\begin{aligned} \|v\|_{L_\infty(D)} &\leq \|v - \Delta^\dagger \Delta v\|_{L_\infty(D)} + \|\Delta^\dagger \Delta v\|_{L_\infty(D)} \\ &= \|v - \Delta^\dagger \Delta v\|_{L_\infty(\partial D)} + \|\Delta^\dagger \Delta v\|_{L_\infty(D)} \end{aligned}$$

Maximum Principle for Harmonic Functions

$$\Delta(v - \Delta^\dagger \Delta v) = \Delta v - \Delta v = 0$$

A Maximum Principle

There exists a positive constant C depending only on k and ρ_D such that

$$\|v\|_{L_\infty(D)} \leq \|v\|_{L_\infty(\partial D)} + C|v|_{H^1(D)} \qquad \forall\, v \in \mathcal{Q}^k(D)$$

$$\begin{aligned}
\|v\|_{L_\infty(D)} &\leq \|v - \Delta^\dagger \Delta v\|_{L_\infty(D)} + \|\Delta^\dagger \Delta v\|_{L_\infty(D)} \\
&= \|v - \Delta^\dagger \Delta v\|_{L_\infty(\partial D)} + \|\Delta^\dagger \Delta v\|_{L_\infty(D)} \\
&\leq \|v\|_{L_\infty(\partial D)} + 2\|\Delta^\dagger \Delta v\|_{L_\infty(D)}
\end{aligned}$$

A Maximum Principle

There exists a positive constant C depending only on k and ρ_D such that

$$\|v\|_{L_\infty(D)} \le \|v\|_{L_\infty(\partial D)} + C|v|_{H^1(D)} \qquad \forall\, v \in \mathcal{Q}^k(D)$$

$$\begin{aligned}
\|v\|_{L_\infty(D)} &\le \|v - \Delta^\dagger \Delta v\|_{L_\infty(D)} + \|\Delta^\dagger \Delta v\|_{L_\infty(D)} \\
&= \|v - \Delta^\dagger \Delta v\|_{L_\infty(\partial D)} + \|\Delta^\dagger \Delta v\|_{L_\infty(D)} \\
&\le \|v\|_{L_\infty(\partial D)} + 2\|\Delta^\dagger \Delta v\|_{L_\infty(D)} \\
&\le \|v\|_{L_\infty(\partial D)} + C\, h_D^2 \|\Delta v\|_{L_\infty(D)}
\end{aligned}$$

$$\|\Delta^\dagger q\|_{L_\infty(D)} \lesssim h_D^2 \|q\|_{L_\infty(D)}$$

A Maximum Principle

There exists a positive constant C depending only on k and ρ_D such that

$$\|v\|_{L_\infty(D)} \leq \|v\|_{L_\infty(\partial D)} + C|v|_{H^1(D)} \qquad \forall\, v \in \mathcal{Q}^k(D)$$

$$\begin{aligned}
\|v\|_{L_\infty(D)} &\leq \|v - \Delta^\dagger \Delta v\|_{L_\infty(D)} + \|\Delta^\dagger \Delta v\|_{L_\infty(D)} \\
&= \|v - \Delta^\dagger \Delta v\|_{L_\infty(\partial D)} + \|\Delta^\dagger \Delta v\|_{L_\infty(D)} \\
&\leq \|v\|_{L_\infty(\partial D)} + 2\|\Delta^\dagger \Delta v\|_{L_\infty(D)} \\
&\leq \|v\|_{L_\infty(\partial D)} + C h_D^2 \|\Delta v\|_{L_\infty(D)} \\
&\leq \|v\|_{L_\infty(\partial D)} + C h_D \|\Delta v\|_{L_2(D)}
\end{aligned}$$

$$\Delta v \in \mathbb{P}_k(D)$$

A Maximum Principle

There exists a positive constant C depending only on k and ρ_D such that

$$\|v\|_{L_\infty(D)} \le \|v\|_{L_\infty(\partial D)} + C|v|_{H^1(D)} \qquad \forall\, v \in \mathcal{Q}^k(D)$$

$$\begin{aligned}
\|v\|_{L_\infty(D)} &\le \|v - \Delta^\dagger \Delta v\|_{L_\infty(D)} + \|\Delta^\dagger \Delta v\|_{L_\infty(D)} \\
&= \|v - \Delta^\dagger \Delta v\|_{L_\infty(\partial D)} + \|\Delta^\dagger \Delta v\|_{L_\infty(D)} \\
&\le \|v\|_{L_\infty(\partial D)} + 2\|\Delta^\dagger \Delta v\|_{L_\infty(D)} \\
&\le \|v\|_{L_\infty(\partial D)} + C h_D^2 \|\Delta v\|_{L_\infty(D)} \\
&\le \|v\|_{L_\infty(\partial D)} + C h_D \|\Delta v\|_{L_2(D)} \\
&\le \|v\|_{L_\infty(\partial D)} + C|v|_{H^1(D)}
\end{aligned}$$

$$\|\Delta v\|_{L_2(D)} \lesssim h_D^{-1} |v|_{H^1(\Omega)}$$

Estimates for Computable Projections

Estimates for $\Pi^{\nabla}_{k,D} : H^1(D) \longrightarrow \mathbb{P}_k(D)$

We have an obvious stability estimate

$$|\Pi^{\nabla}_{k,D}\zeta|_{H^1(D)} \leq |\zeta|_{H^1(D)} \qquad \forall\,\zeta \in H^1(D)$$

that follows from

$$\int_D \nabla(\Pi^{\nabla}_{k,D}\zeta) \cdot \nabla q\,dx = \int_D \nabla\zeta \cdot \nabla q\,dx \qquad \forall\,q \in \mathbb{P}_k(D)$$

Estimates for $\Pi^{\nabla}_{k,D} : H^1(D) \longrightarrow \mathbb{P}_k(D)$

We have an obvious stability estimate

$$|\Pi^{\nabla}_{k,D}\zeta|_{H^1(D)} \leq |\zeta|_{H^1(D)} \qquad \forall\,\zeta \in H^1(D)$$

which implies (Bramble-Hilbert)

$$|\zeta - \Pi^{\nabla}_{k,D}\zeta|_{H^1(D)} \lesssim h_D^{\ell}|\zeta|_{H^{\ell+1}(D)} \qquad 1 \leq \ell \leq k$$

since $\Pi^{\nabla}_{k,D}q = q$ for all $q \in \mathbb{P}_k(D)$.

(The hidden constants only depend on ρ_D and k.)

Estimates for $\Pi^{\nabla}_{k,D} : H^1(D) \longrightarrow \mathbb{P}_k(D)$

We have an obvious stability estimate

$$|\Pi^{\nabla}_{k,D}\zeta|_{H^1(D)} \leq |\zeta|_{H^1(D)} \qquad \forall\, \zeta \in H^1(D)$$

which implies (Bramble-Hilbert)

$$|\zeta - \Pi^{\nabla}_{k,D}\zeta|_{H^1(D)} \lesssim h_D^{\ell}|\zeta|_{H^{\ell+1}(D)} \qquad 1 \leq \ell \leq k$$

and hence (Poincaré-Friedrichs)

$$\|\zeta - \Pi^{\nabla}_{k,D}\zeta\|_{L_2(D)} \lesssim \int_{\partial D}(\zeta - \Pi^{\nabla}_{k,D}\zeta)ds + h_D|\zeta - \Pi^{\nabla}_{k,D}\zeta|_{H^1(D)}$$

Estimates for $\Pi^{\nabla}_{k,D} : H^1(D) \longrightarrow \mathbb{P}_k(D)$

We have an obvious stability estimate

$$|\Pi^{\nabla}_{k,D}\zeta|_{H^1(D)} \leq |\zeta|_{H^1(D)} \qquad \forall\, \zeta \in H^1(D)$$

which implies (Bramble-Hilbert)

$$|\zeta - \Pi^{\nabla}_{k,D}\zeta|_{H^1(D)} \lesssim h_D^{\ell}|\zeta|_{H^{\ell+1}(D)} \qquad 1 \leq \ell \leq k$$

and hence (Poincaré-Friedrichs)

$$\begin{aligned}\|\zeta - \Pi^{\nabla}_{k,D}\zeta\|_{L_2(D)} &\lesssim \int_{\partial D}(\zeta - \Pi^{\nabla}_{k,D}\zeta)ds + h_D|\zeta - \Pi^{\nabla}_{k,D}\zeta|_{H^1(D)} \\ &= h_D|\zeta - \Pi^{\nabla}_{k,D}\zeta|_{H^1(D)}\end{aligned}$$

$$\int_{\partial D}\Pi^{\nabla}_{k,D}\zeta\, ds = \int_{\partial D}\zeta\, ds$$

Estimates for $\Pi^{\nabla}_{k,D} : H^1(D) \longrightarrow \mathbb{P}_k(D)$

We have an obvious stability estimate

$$|\Pi^{\nabla}_{k,D}\zeta|_{H^1(D)} \leq |\zeta|_{H^1(D)} \qquad \forall\,\zeta \in H^1(D)$$

which implies (Bramble-Hilbert)

$$|\zeta - \Pi^{\nabla}_{k,D}\zeta|_{H^1(D)} \lesssim h_D^{\ell}|\zeta|_{H^{\ell+1}(D)} \qquad 1 \leq \ell \leq k$$

and hence (Poincaré-Friedrichs)

$$\begin{aligned}\|\zeta - \Pi^{\nabla}_{k,D}\zeta\|_{L_2(D)} &\lesssim \int_{\partial D}(\zeta - \Pi^{\nabla}_{k,D}\zeta)ds + h_D|\zeta - \Pi^{\nabla}_{k,D}\zeta|_{H^1(D)} \\ &= h_D|\zeta - \Pi^{\nabla}_{k,D}\zeta|_{H^1(D)} \\ &\lesssim h_D^{\ell+1}|\zeta|_{H^{\ell+1}(D)} \qquad 0 \leq \ell \leq k\end{aligned}$$

(The hidden constants only depend on ρ_D and k.)

Estimates for $\Pi^{\nabla}_{k,D} : H^1(D) \longrightarrow \mathbb{P}_k(D)$

There is also a stability estimate for $\Pi^{\nabla}_{k,D}$ in the L_2 norm.

Estimates for $\Pi^{\nabla}_{k,D} : H^1(D) \longrightarrow \mathbb{P}_k(D)$

There is also a stability estimate for $\Pi^{\nabla}_{k,D}$ in the L_2 norm.

Computation of $\Pi^{\nabla}_{k,D}\zeta$ ($\zeta \in H^1(D)$)

$$\begin{aligned}\int_D \nabla\Pi^{\nabla}_{k,D}\zeta \cdot \nabla q\,dx &= \int_D \nabla\zeta\cdot\nabla q\,dx\\ &= \int_{\partial D}\zeta\frac{\partial q}{\partial n}ds - \int_D \zeta(\Delta q)dx \quad \forall\, q\in\mathbb{P}_k(D)\end{aligned}$$

$$\int_{\partial D}\Pi^{\nabla}_{k,D}\zeta\,ds = \int_{\partial D}\zeta\,ds$$

Estimates for $\Pi^{\nabla}_{k,D} : H^1(D) \longrightarrow \mathbb{P}_k(D)$

There is also a stability estimate for $\Pi^{\nabla}_{k,D}$ in the L_2 norm.

Computation of $\Pi^{\nabla}_{k,D}\zeta$ $\quad(\zeta \in H^1(D))$

$$\int_D \nabla \Pi^{\nabla}_{k,D}\zeta \cdot \nabla q\, dx = \int_{\partial D} \zeta \frac{\partial q}{\partial n} ds - \int_D \zeta(\Delta q) dx \quad \forall\, q \in \mathbb{P}_k(D)$$

$$\int_{\partial D} \Pi^{\nabla}_{k,D}\zeta\, ds = \int_{\partial D} \zeta\, ds$$

This only requires the moments of ζ up to order $(k-1)$ on the edges of D and the moments of ζ up to order $(k-2)$ on D.

$$\partial q/\partial n \in \mathbb{P}_{k-1}(e) \qquad \forall\, e \in \mathcal{E}_D$$

$$\Delta q \in \mathbb{P}_{k-2}(D)$$

Estimates for $\Pi^{\nabla}_{k,D} : H^1(D) \longrightarrow \mathbb{P}_k(D)$

There is also a stability estimate for $\Pi^{\nabla}_{k,D}$ in the L_2 norm.

Computation of $\Pi^{\nabla}_{k,D}\zeta$ $\quad (\zeta \in H^1(D))$

$$\int_D \nabla \Pi^{\nabla}_{k,D}\zeta \cdot \nabla q\,dx = \int_{\partial D} \zeta \frac{\partial q}{\partial n} ds - \int_D \zeta(\Delta q)dx \quad \forall\, q \in \mathbb{P}_k(D)$$

$$\int_{\partial D} \Pi^{\nabla}_{k,D}\zeta\,ds = \int_{\partial D} \zeta\,ds$$

This only requires the moments of ζ up to order $(k-1)$ on the edges of D and the moments of ζ up to order $(k-2)$ on D.

$$\|\Pi^{\nabla}_{k,D}\zeta\|^2_{L_2(D)} \lesssim \|\Pi^0_{k-2,D}\zeta\|^2_{L_2(D)} + h_D \sum_{e\in\mathcal{E}_D} \|\Pi^0_{k-1,e}\zeta\|^2_{L_2(e)}$$

for all $\zeta \in H^1(D)$

(The hidden constants only depend on ρ_D and k.)

Estimates for $\Pi^0_{k,D} : L_2(D) \longrightarrow \mathbb{P}_k(D)$

Estimates for $\Pi^0_{k,D} : L_2(D) \longrightarrow \mathbb{P}_k(D)$

It follows from the obvious stability estimate

$$\|\Pi^0_{k,D}\zeta\|_{L_2(D)} \le \|\zeta\|_{L_2(D)} \qquad \forall\,\zeta \in L_2(D)$$

that (Bramble-Hilbert)

$$\|\zeta - \Pi^0_{k,D}\zeta\|_{L_2(D)} \lesssim h_D^{\ell+1}|\zeta|_{H^{\ell+1}(D)} \qquad \forall\,\zeta \in H^{\ell+1}(D)$$

where $0 \le \ell \le k$.

(The hidden constants only depend on ρ_D and k.)

Estimates for $\Pi^0_{k,D} : L_2(D) \longrightarrow \mathbb{P}_k(D)$

It follows from the obvious stability estimate

$$\|\Pi^0_{k,D}\zeta\|_{L_2(D)} \leq \|\zeta\|_{L_2(D)} \qquad \forall\, \zeta \in L_2(D)$$

that (Bramble-Hilbert)

$$\|\zeta - \Pi^0_{k,D}\zeta\|_{L_2(D)} \lesssim h_D^{\ell+1}|\zeta|_{H^{\ell+1}(D)} \qquad \forall\, \zeta \in H^{\ell+1}(D)$$

Stability Estimate in the H^1 Norm

$$\begin{aligned}
|\Pi^0_{k,D}\zeta|_{H^1(D)} &\leq |\Pi^0_{k,D}\zeta - \Pi^\nabla_{k,D}\zeta|_{H^1(D)} + |\Pi^\nabla_{k,D}\zeta|_{H^1(D)} \\
&\lesssim h_D^{-1}\|\Pi^0_{k,D}\zeta - \Pi^\nabla_{k,D}\zeta\|_{L_2(D)} + |\zeta|_{H^1(D)} \\
&\lesssim h_D^{-1}\big(\|\Pi^0_{k,D}\zeta - \zeta\|_{L_2(D)} + \|\zeta - \Pi^\nabla_{k,D}\zeta\|_{L_2(D)}\big) \\
&\qquad\quad + |\zeta|_{H^1(D)} \\
&\lesssim |\zeta|_{H^1(D)}
\end{aligned}$$

(The hidden constants only depend on ρ_D and k.)

Estimates for $\Pi^0_{k,D} : L_2(D) \longrightarrow \mathbb{P}_k(D)$

It follows from the obvious stability estimate

$$\|\Pi^0_{k,D}\zeta\|_{L_2(D)} \leq \|\zeta\|_{L_2(D)} \qquad \forall\,\zeta \in L_2(D)$$

that (Bramble-Hilbert)

$$\|\zeta - \Pi^0_{k,D}\zeta\|_{L_2(D)} \lesssim h_D^{\ell+1}|\zeta|_{H^{\ell+1}(D)} \qquad \forall\,\zeta \in H^{\ell+1}(D)$$

Stability Estimate in the H^1 Norm

$$|\Pi^0_{k,D}\zeta|_{H^1(D)} \lesssim |\zeta|_{H^1(D)} \qquad \forall\,\zeta \in H^1(D)$$

implies (Bramble-Hilbert)

$$|\zeta - \Pi^0_{k,D}\zeta|_{H^1(D)} \lesssim h_D^{\ell}|\zeta|_{H^{\ell+1}(D)} \qquad \forall\,\zeta \in H^{\ell+1}(D)$$

where $1 \leq \ell \leq k$ (The hidden constants only depend on ρ_D and k.)

Estimates for $\Pi^0_{k,D} : L_2(D) \longrightarrow \mathbb{P}_k(D)$

Given $v \in \mathcal{Q}^k(D)$, we can compute $\Pi^0_{k,D}v$ from

$$\begin{aligned}\Pi^0_{k,D}v &= \Pi^0_{k-2,D}v + (\Pi^0_{k,D} - \Pi^0_{k-2,D})\Pi^0_{k,D}v \\ &= \Pi^0_{k-2,D}v + (\Pi^0_{k,D} - \Pi^0_{k-2,D})\Pi^\nabla_{k,D}v\end{aligned}$$

$\Pi^0_{k,D}v - \Pi^\nabla_{k,D}z \in \mathbb{P}_{k-2}(D)$

Estimates for $\Pi^0_{k,D} : L_2(D) \longrightarrow \mathbb{P}_k(D)$

Given $v \in \mathcal{Q}^k(D)$, we can compute $\Pi^0_{k,D}\, v$ from

$$\begin{aligned}\Pi^0_{k,D}\, v &= \Pi^0_{k-2,D} v + (\Pi^0_{k,D} - \Pi^0_{k-2,D})\Pi^0_{k,D}\, v \\ &= \Pi^0_{k-2,D} v + (\Pi^0_{k,D} - \Pi^0_{k-2,D})\Pi^\nabla_{k,D}\, v\end{aligned}$$

$\Pi^0_{k,D}\, v - \Pi^\nabla_{k,D}\, z \in \mathbb{P}_{k-2}(D)$

It follows that   Pythagoras' Theorem

$$\|\Pi^0_{k,D}\, v\|^2_{L_2(D)} = \|\Pi^0_{k-2,D} v\|^2_{L_2(D)} + \|(\Pi^0_{k,D} - \Pi^0_{k-2,D})\Pi^\nabla_{k,D}\, v\|^2_{L_2(D)}$$

Estimates for $\Pi^0_{k,D} : L_2(D) \longrightarrow \mathbb{P}_k(D)$

Given $v \in \mathcal{Q}^k(D)$, we can compute $\Pi^0_{k,D}v$ from

$$\begin{aligned}\Pi^0_{k,D}v &= \Pi^0_{k-2,D}v + (\Pi^0_{k,D} - \Pi^0_{k-2,D})\Pi^0_{k,D}v \\ &= \Pi^0_{k-2,D}v + (\Pi^0_{k,D} - \Pi^0_{k-2,D})\Pi^\nabla_{k,D}v\end{aligned}$$

$\Pi^0_{k,D}v - \Pi^\nabla_{k,D}z \in \mathbb{P}_{k-2}(D)$

It follows that Pythagoras' Theorem

$$\begin{aligned}\|\Pi^0_{k,D}v\|^2_{L_2(D)} &= \|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + \|(\Pi^0_{k,D} - \Pi^0_{k-2,D})\Pi^\nabla_{k,D}v\|^2_{L_2(D)} \\ &\le \|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + \|\Pi^\nabla_{k,D}v\|^2_{L_2(D)}\end{aligned}$$

Estimates for $\Pi^0_{k,D} : L_2(D) \longrightarrow \mathbb{P}_k(D)$

Given $v \in \mathcal{Q}^k(D)$, we can compute $\Pi^0_{k,D} v$ from

$$\Pi^0_{k,D} v = \Pi^0_{k-2,D} v + (\Pi^0_{k,D} - \Pi^0_{k-2,D})\Pi^0_{k,D} v$$
$$= \Pi^0_{k-2,D} v + (\Pi^0_{k,D} - \Pi^0_{k-2,D})\Pi^\nabla_{k,D} v$$

$\Pi^0_{k,D} v - \Pi^\nabla_{k,D} z \in \mathbb{P}_{k-2}(D)$

It follows that Pythagoras' Theorem

$$\|\Pi^0_{k,D} v\|^2_{L_2(D)} = \|\Pi^0_{k-2,D} v\|^2_{L_2(D)} + \|(\Pi^0_{k,D} - \Pi^0_{k-2,D})\Pi^\nabla_{k,D} v\|^2_{L_2(D)}$$
$$\leq \|\Pi^0_{k-2,D} v\|^2_{L_2(D)} + \|\Pi^\nabla_{k,D} v\|^2_{L_2(D)}$$
$$\lesssim \|\Pi^0_{k-2,D} v\|^2_{L_2(D)} + h_D \sum_{e \in \mathcal{E}_D} \|\Pi^0_{k-1,e} v\|^2_{L_2(e)}$$

$\|\Pi^\nabla_{k,D} \zeta\|^2_{L_2(D)} \lesssim \|\Pi^0_{k-2,D} \zeta\|^2_{L_2(D)} + h_D \sum_{e \in \mathcal{E}_D} \|\Pi^0_{k-1,e} \zeta\|^2_{L_2(e)}$

Estimates for $\Pi^0_{k,D} : L_2(D) \longrightarrow \mathbb{P}_k(D)$

We have the estimate

$$\|\Pi^0_{k,D} v\|^2_{L_2(D)} \lesssim \|\Pi^0_{k-2,D} v\|^2_{L_2(D)} + h_D \sum_{e \in \mathcal{E}_D} \|\Pi^0_{k-1,e} v\|^2_{L_2(e)}$$

for all $v \in \mathcal{Q}^k(D)$

(The hidden constants only depend on ρ_D and k.)

Inverse Estimates

Inverse Estimates

These are estimates that bound the H^1 norm of a virtual element function $v \in \mathcal{Q}^k(D)$ in terms of $\|\Pi^0_{k-2,D} v\|_{L_2(D)}$ and (semi-) norms that only involve the boundary data of v.

Inverse Estimates

These are estimates that bound the H^1 norm of a virtual element function $v \in \mathcal{Q}^k(D)$ in terms of $\|\Pi^0_{k-2,D} v\|_{L_2(D)}$ and (semi-) norms that only involve the boundary data of v.

A Minimum Energy Principle

The inequality

$$|v|_{H^1(D)} \leq |\zeta|_{H^1(D)}$$

holds for any $v \in \mathcal{Q}^k(D)$ and $\zeta \in H^1(D)$ such that

$$(\zeta - v)\big|_{\partial D} = 0 \quad \text{and} \quad \Pi^0_{k,D}(\zeta - v) = 0$$

Inverse Estimates

These are estimates that bound the H^1 norm of a virtual element function $v \in \mathcal{Q}^k(D)$ in terms of $\|\Pi^0_{k-2,D} v\|_{L_2(D)}$ and (semi-) norms that only involve the boundary data of v.

A Minimum Energy Principle

The inequality

$$|v|_{H^1(D)} \leq |\zeta|_{H^1(D)}$$

holds for any $v \in \mathcal{Q}^k(D)$ and $\zeta \in H^1(D)$ such that

$$(\zeta - v)\big|_{\partial D} = 0 \quad \text{and} \quad \Pi^0_{k,D}(\zeta - v) = 0$$

$$\int_D \nabla v \cdot \nabla(\zeta - v)\, dx = \int_D (-\Delta v)(\zeta - v) dx = 0$$

$\Delta v \in \mathbb{P}_k(D)$

and hence

$$|\zeta|^2_{H^1(D)} = |\zeta - v|^2_{H^1(D)} + |v|^2_{H^1(D)}$$

Inverse Estimates

There exists a positive constant C, depending only on ρ_D and k, such that

$$|v|^2_{H^1(D)} \le C\Big[h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + |v|^2_{H^{1/2}(\partial D)}\Big]$$

for all $v \in \mathcal{Q}^k(D)$.

Inverse Estimates

There exists a positive constant C, depending only on ρ_D and k, such that

$$|v|^2_{H^1(D)} \le C\Big[h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + |v|^2_{H^{1/2}(\partial D)}\Big]$$

for all $v \in \mathcal{Q}^k(D)$.

By the inverse trace theorem, there exists $w \in H^1(\Omega)$ such that

$$w = v \ \text{ on } \ \partial D \quad \text{and} \quad \|w\|_{H^1(D)} \lesssim |v|_{H^{1/2}(\partial D)}$$

There exists a Lipschitz isomorphism $\Phi : \mathfrak{B}_D \longrightarrow D$ such that both $|\Phi|_{W^{1,\infty}(\mathfrak{B}_D)}$ and $|\Phi^{-1}|_{W^{1,\infty}(D)}$ are bounded by a constant that only depends on ρ_D.

Inverse Estimates

Let

$$\zeta = w + p\phi$$

where $\phi \geq 0$ is a smooth (bump) function supported on a compact subset of the disc $\mathfrak{B}_D \subset D$ in the star-shaped assumption, and $p \in \mathbb{P}_k(D)$ is chosen such that

$$\Pi^0_{k,D}\zeta = \Pi^0_{k,D}v$$

Inverse Estimates

Let

$$\zeta = w + p\phi$$

where $\phi \geq 0$ is a smooth (bump) function supported on a compact subset of the disc $\mathfrak{B}_D \subset D$ in the star-shaped assumption, and $p \in \mathbb{P}_k(D)$ is chosen such that

$$\Pi^0_{k,D}\zeta = \Pi^0_{k,D}v$$

By construction

$$(\zeta - v)\big|_{\partial D} = 0 \quad \text{and} \quad \Pi^0_{k,D}(\zeta - v) = 0$$

and hence

$$|v|_{H^1(D)} \leq |\zeta|_{H^1(D)}$$

by the minimum energy principle.

Inverse Estimates

Let

$$\zeta = w + p\phi$$

where $\phi \geq 0$ is a smooth (bump) function supported on a compact subset of the disc $\mathfrak{B}_D \subset D$ in the star-shaped assumption, and $p \in \mathbb{P}_k(D)$ is chosen such that

$$\Pi^0_{k,D}\zeta = \Pi^0_{k,D}v$$

On the other hand we have

$$|\zeta|^2_{H^1(D)} \lesssim h_D^{-2}\|\Pi^0_{k,D}v\|^2_{L_2(D)} + \|w\|^2_{H^1(D)}$$

Inverse Estimates

Let

$$\zeta = w + p\phi$$

where $\phi \geq 0$ is a smooth (bump) function supported on a compact subset of the disc $\mathfrak{B}_D \subset D$ in the star-shaped assumption, and $p \in \mathbb{P}_k(D)$ is chosen such that

$$\Pi^0_{k,D}\zeta = \Pi^0_{k,D}v$$

On the other hand we have

$$\begin{aligned}|\zeta|^2_{H^1(D)} &\lesssim h_D^{-2}\|\Pi^0_{k,D}v\|^2_{L_2(D)} + \|w\|^2_{H^1(D)}\\ &\lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)}\\ &\qquad + |v|^2_{H^{1/2}(\partial D)}\end{aligned}$$

$$\|\Pi^0_{k,D}v\|^2_{L_2(D)} \lesssim \|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)}$$

$$\|w\|_{H^1(D)} \lesssim |v|_{H^{1/2}(\partial D)}$$

Inverse Estimates

$$|v|^2_{H^1(D)} \lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + |v|^2_{H^{1/2}(\partial D)}$$

for all $v \in \mathcal{Q}^k(D)$ (The hidden constant only depends on k and ρ_D.)

Inverse Estimates

$$|v|^2_{H^1(D)} \lesssim h_D^{-2} \|\Pi^0_{k-2,D} v\|^2_{L_2(D)} + h_D^{-1} \sum_{e \in \mathcal{E}_D} \|\Pi^0_{k-1,e} v\|^2_{L_2(e)} + |v|^2_{H^{1/2}(\partial D)}$$

for all $v \in \mathcal{Q}^k(D)$ (The hidden constant only depends on k and ρ_D.)

Corollary 1

$$|v|^2_{H^1(D)} \lesssim h_D^{-2} \|\Pi^0_{k-2,D} v\|^2_{L_2(D)} + h_D^{-1} \sum_{e \in \mathcal{E}_D} \|\Pi^0_{k-1,e} v\|^2_{L_2(e)} + h_D \|\partial v / \partial s\|^2_{L_2(\partial D)}$$

for all $v \in \mathcal{Q}^k(D)$

$$|\zeta|_{H^{1/2}(\partial D)} \lesssim h_D^{1/2} |\zeta|_{H^1(\partial D)} \qquad \forall \, \zeta \in H^1(D)$$

Inverse Estimates

$$|v|^2_{H^1(D)} \lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + |v|^2_{H^{1/2}(\partial D)}$$

for all $v \in \mathcal{Q}^k(D)$ (The hidden constant only depends on k and ρ_D.)

Corollary 1

$$|v|^2_{H^1(D)} \lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + h_D\|\partial v/\partial s\|^2_{L_2(\partial D)}$$

This is relevant for the first stabilization

$$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$$

Inverse Estimates

$$|v|^2_{H^1(D)} \lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + |v|^2_{H^{1/2}(\partial D)}$$

for all $v \in \mathcal{Q}^k(D)$ (The hidden constant only depends on k and ρ_D.)

Corollary 2 (The hidden constant only depends on k, ρ_D and $|\mathcal{E}_D|$.)

$$|v|^2_{H^1(D)} \lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + \ln(1+\tau_D)\|v\|^2_{L_\infty(\partial D)}$$

for all $v \in \mathcal{Q}^k(D)$, where

$$\tau_D = \frac{\max_{e\in\mathcal{E}_D} h_e}{\min_{e\in\mathcal{E}_D} h_e}$$

Inverse Estimates

$$|v|^2_{H^1(D)} \lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + |v|^2_{H^{1/2}(\partial D)}$$

for all $v \in \mathcal{Q}^k(D)$ (The hidden constant only depends on k and ρ_D.)

Corollary 2 (The hidden constant only depends on k, ρ_D and $|\mathcal{E}_D|$.)

$$|v|^2_{H^1(D)} \lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + \ln(1+\tau_D)\|v\|^2_{L_\infty(\partial D)}$$

Lemma (The hidden constant depends on k and $|\mathcal{E}_D|$.)

$$|v|^2_{H^{1/2}(\partial D)} \lesssim \ln(1+\tau_D)\|v\|^2_{L_\infty(\partial D)} \qquad \forall\, v \in \mathbb{P}_k(\partial D)$$

Beirão da Veiga-Lovadina-Russo (2017)

Inverse Estimates

$$|v|^2_{H^1(D)} \lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + |v|^2_{H^{1/2}(\partial D)}$$

for all $v \in \mathcal{Q}^k(D)$ (The hidden constant only depends on k and ρ_D.)

Corollary 2 (The hidden constant only depends on k, ρ_D and $|\mathcal{E}_D|$.)

$$|v|^2_{H^1(D)} \lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + \ln(1+\tau_D)\|v\|^2_{L_\infty(\partial D)}$$

This is relevant for

$$S^D(w,v) = \sum_{p\in\mathcal{N}_{\partial D}} w(p)v(p)$$

since $S^D(v,v) \approx \|v\|^2_{L_\infty(\partial D)}$ for $v \in \mathbb{P}_k(\partial D)$.

Estimates for an Interpolation Operator

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

The (local) interpolation operator $I_{k,D}$ is defined by the condition that $\zeta \in H^2(D)$ and $I_{k,D}\zeta \in \mathcal{Q}^k(D)$ share the same dofs, i.e.,

$$(I_{k,D}\zeta)(p) = \zeta(p) \qquad \forall\, p \in \mathcal{N}_{\partial D}$$

$$\Pi^0_{k-2,D}(I_{k,D}\zeta) = \Pi^0_{k-2,D}\zeta$$

In particular

$$I_{k,D}q = q \qquad \forall\, q \in \mathbb{P}_k(D)$$

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

The (local) interpolation operator $I_{k,D}$ is defined by the condition that $\zeta \in H^2(D)$ and $I_{k,D}\zeta \in \mathcal{Q}^k(D)$ share the same dofs, i.e.,

$$(I_{k,D}\zeta)(p) = \zeta(p) \qquad \forall\, p \in \mathcal{N}_{\partial D}$$

$$\Pi^0_{k-2,D}(I_{k,D}\zeta) = \Pi^0_{k-2,D}\zeta$$

In particular

$$I_{k,D}q = q \qquad \forall\, q \in \mathbb{P}_k(D)$$

Stability Estimates

$$|I_{k,D}\zeta|_{H^1(D)} \lesssim |\zeta|_{H^1(D)} + h_D|\zeta|_{H^2(D)}$$

$$\|I_{k,D}\zeta\|_{L_2(D)} \lesssim \|\zeta\|_{L_2(D)} + h_D|\zeta|_{H^1(D)} + h_D^2|\zeta|_{H^2(D)}$$

for all $\zeta \in H^2(D)$ (The hidden constants only depend on k and ρ_D.)

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

$$|I_{k,D}\zeta|^2_{H^1(D)} = |I_{k,D}(\zeta - \bar{\zeta})|^2_{H^1(D)}$$

$$\bar{\zeta} = \frac{1}{|D|}\int_D \zeta \, dx$$

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

$$\begin{aligned}
|I_{k,D}\zeta|^2_{H^1(D)} &= |I_{k,D}(\zeta - \bar{\zeta})|^2_{H^1(D)} \\
&\lesssim h_D^{-2}\|\Pi^0_{k-2,D} I_{k,D}(\zeta - \bar{\zeta})\|^2_{L_2(D)} \\
&\qquad + h_D^{-1} \sum_{e\in\mathcal{E}_D} \|\Pi^0_{k-1,e} I_{k,D}(\zeta - \bar{\zeta})\|^2_{L_2(e)} \\
&\qquad\qquad + h_D\|\partial I_{k,D}(\zeta - \bar{\zeta})/\partial s\|^2_{L_2(\partial D)}
\end{aligned}$$

$$\begin{aligned}
|v|^2_{H^1(D)} \lesssim h_D^{-2}\|\Pi^0_{k-2,D} v\|^2_{L_2(D)} + h_D^{-1} \sum_{e\in\mathcal{E}_D} \|\Pi^0_{k-1,e} v\|^2_{L_2(e)} \\
+ h_D\|\partial v/\partial s\|^2_{L_2(\partial D)} \qquad \forall\, v \in \mathcal{Q}^k(D)
\end{aligned}$$

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

$$
\begin{aligned}
|I_{k,D}\zeta|^2_{H^1(D)} &= |I_{k,D}(\zeta-\bar{\zeta})|^2_{H^1(D)} \\
&\lesssim h_D^{-2}\|\Pi^0_{k-2,D} I_{k,D}(\zeta-\bar{\zeta})\|^2_{L_2(D)} \\
&\qquad + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e} I_{k,D}(\zeta-\bar{\zeta})\|^2_{L_2(e)} \\
&\qquad\qquad + h_D\|\partial I_{k,D}(\zeta-\bar{\zeta})/\partial s\|^2_{L_2(\partial D)} \\
&= h_D^{-2}\|\Pi^0_{k-2,D}(\zeta-\bar{\zeta})\|^2_{L_2(D)} \\
&\qquad + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e} I_{k,D}(\zeta-\bar{\zeta})\|^2_{L_2(e)} \\
&\qquad\qquad + h_D\|\partial I_{k,D}\zeta/\partial s\|^2_{L_2(\partial D)}
\end{aligned}
$$

$$\Pi^0_{k-2,D}(I_{k,D}\zeta) = \Pi^0_{k-2,D}\zeta$$

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

$$
\begin{aligned}
|I_{k,D}\zeta|^2_{H^1(D)} &= |I_{k,D}(\zeta - \bar{\zeta})|^2_{H^1(D)} \\
&\lesssim h_D^{-2} \|\Pi^0_{k-2,D} I_{k,D}(\zeta - \bar{\zeta})\|^2_{L_2(D)} \\
&\qquad + h_D^{-1} \sum_{e \in \mathcal{E}_D} \|\Pi^0_{k-1,e} I_{k,D}(\zeta - \bar{\zeta})\|^2_{L_2(e)} \\
&\qquad\qquad + h_D \|\partial I_{k,D}(\zeta - \bar{\zeta})/\partial s\|^2_{L_2(\partial D)} \\
&= h_D^{-2} \|\Pi^0_{k-2,D}(\zeta - \bar{\zeta})\|^2_{L_2(D)} \\
&\qquad + h_D^{-1} \sum_{e \in \mathcal{E}_D} \|\Pi^0_{k-1,e} I_{k,D}(\zeta - \bar{\zeta})\|^2_{L_2(e)} \\
&\qquad\qquad + h_D \|\partial I_{k,D}\zeta/\partial s\|^2_{L_2(\partial D)}
\end{aligned}
$$

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

$$\begin{aligned} |I_{k,D}\zeta|^2_{H^1(D)} &= |I_{k,D}(\zeta-\bar{\zeta})|^2_{H^1(D)} \\ &\lesssim h_D^{-2}\|\Pi^0_{k-2,D} I_{k,D}(\zeta-\bar{\zeta})\|^2_{L_2(D)} \\ &\qquad + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e} I_{k,D}(\zeta-\bar{\zeta})\|^2_{L_2(e)} \\ &\qquad\qquad + h_D\|\partial I_{k,D}(\zeta-\bar{\zeta})/\partial s\|^2_{L_2(\partial D)} \\ &= h_D^{-2}\|\Pi^0_{k-2,D}(\zeta-\bar{\zeta})\|^2_{L_2(D)} \\ &\qquad + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e} I_{k,D}(\zeta-\bar{\zeta})\|^2_{L_2(e)} \\ &\qquad\qquad + h_D\|\partial I_{k,D}\zeta/\partial s\|^2_{L_2(\partial D)} \\ &\lesssim \|\zeta-\bar{\zeta}\|^2_{L_\infty(D)} + h_D\|\partial I_{k,D}\zeta/\partial s\|^2_{L_2(\partial D)} \end{aligned}$$

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

$$\|\zeta - \bar{\zeta}\|^2_{L_\infty(D)} \lesssim h_D^{-1}\|\zeta - \bar{\zeta}\|^2_{L_2(D)} + |\zeta - \bar{\zeta}|^2_{H^1(D)} + h_D^2|\zeta - \bar{\zeta}|^2_{H^2(D)}$$

Sobolev's Inequality

$$\|\zeta\|_{L_\infty(D)} \lesssim h_D^{-(1/2)}\|\zeta\|_{L_2(D)} + |\zeta|_{H^1(D)} + h_D|\zeta|_{H^2(D)} \qquad \forall\, \zeta \in H^2(D)$$

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

$$\begin{aligned}\|\zeta-\bar{\zeta}\|^2_{L_\infty(D)} &\lesssim h_D^{-1}\|\zeta-\bar{\zeta}\|^2_{L_2(D)} + |\zeta-\bar{\zeta}|^2_{H^1(D)} + h_D^2|\zeta-\bar{\zeta}|^2_{H^2(D)}\\ &\lesssim |\zeta|^2_{H^1(D)} + h_D^2|\zeta|^2_{H^2(D)}\end{aligned}$$

Poincaré-Friedrichs Inequality

$$h_D^{-(1/2)}\|\zeta\|_{L_2(D)} \lesssim h_D^{-2}\Big|\int_D \zeta\,dx\Big| + |\zeta|_{H^1(D)}$$

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

$$\begin{aligned}\|\zeta-\bar{\zeta}\|^2_{L_\infty(D)} &\lesssim h_D^{-1}\|\zeta-\bar{\zeta}\|^2_{L_2(D)} + |\zeta-\bar{\zeta}|^2_{H^1(D)} + h_D^2|\zeta-\bar{\zeta}|^2_{H^2(D)} \\ &\lesssim |\zeta|^2_{H^1(D)} + h_D^2|\zeta|^2_{H^2(D)}\end{aligned}$$

$$\begin{aligned}h_D\|\partial(I_{k,D}\zeta)/\partial s\|^2_{L_2(\partial D)} &= h_D\sum_{e\in\mathcal{E}_D}\|\partial(I_{k,D}\zeta)/\partial s\|^2_{L_2(e)} \\ &\lesssim h_D\sum_{e\in\mathcal{E}_D}\|\partial\zeta/\partial s\|^2_{L_2(e)}\end{aligned}$$

Mean Value Theorem

$$\|\partial(I_{k,D}\zeta)/\partial s\|_{L_2(e)} \lesssim \|\partial\zeta/\partial s\|^2_{L_2(e)}$$

(The hidden constant only depends on k.)

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

$$
\begin{aligned}
\|\zeta-\bar{\zeta}\|^2_{L_\infty(D)} &\lesssim h_D^{-1}\|\zeta-\bar{\zeta}\|^2_{L_2(D)} + |\zeta-\bar{\zeta}|^2_{H^1(D)} + h_D^2|\zeta-\bar{\zeta}|^2_{H^2(D)} \\
&\lesssim |\zeta|^2_{H^1(D)} + h_D^2|\zeta|^2_{H^2(D)}
\end{aligned}
$$

$$
\begin{aligned}
h_D\|\partial(I_{k,D}\zeta)/\partial s\|^2_{L_2(\partial D)} &= h_D\sum_{e\in\mathcal{E}_D}\|\partial(I_{k,D}\zeta)/\partial s\|^2_{L_2(e)} \\
&\lesssim h_D\sum_{e\in\mathcal{E}_D}\|\partial\zeta/\partial s\|^2_{L_2(e)} \\
&\lesssim |\zeta|^2_{H^1(D)} + h_D^2|\zeta|^2_{H^2(D)}
\end{aligned}
$$

Trace Inequality

$$
\|\zeta\|^2_{L_2(\partial D)} \lesssim h_D^{-1}\|\zeta\|^2_{L_2(D)} + h_D|\zeta|^2_{H^1(D)} \qquad \forall\,\zeta\in H^1(D)
$$

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

$$\begin{aligned}\|\zeta-\bar{\zeta}\|^2_{L_\infty(D)} &\lesssim h_D^{-1}\|\zeta-\bar{\zeta}\|^2_{L_2(D)} + |\zeta-\bar{\zeta}|^2_{H^1(D)} + h_D^2|\zeta-\bar{\zeta}|^2_{H^2(D)}\\ &\lesssim |\zeta|^2_{H^1(D)} + h_D^2|\zeta|^2_{H^2(D)}\end{aligned}$$

$$\begin{aligned}h_D\|\partial(I_{k,D}\zeta)/\partial s\|^2_{L_2(\partial D)} &= h_D\sum_{e\in\mathcal{E}_D}\|\partial(I_{k,D}\zeta)/\partial s\|^2_{L_2(e)}\\ &\lesssim h_D\sum_{e\in\mathcal{E}_D}\|\partial\zeta/\partial s\|^2_{L_2(e)}\\ &\lesssim |\zeta|^2_{H^1(D)} + h_D^2|\zeta|^2_{H^2(D)}\end{aligned}$$

$$|I_{k,D}\zeta|_{H^1(D)} \lesssim |\zeta|_{H^1(D)} + h_D|\zeta|_{H^2(D)}$$

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

The stability estimates

$$|I_{k,D}\zeta|_{H^1(D)} \lesssim |\zeta|_{H^1(D)} + h_D|\zeta|_{H^2(D)}$$

$$\|I_{k,D}\zeta\|_{L_2(D)} \lesssim \|\zeta\|_{L_2(D)} + h_D|\zeta|_{H^1(D)} + h_D^2|\zeta|_{H^2(D)}$$

imply (Bramble-Hilbert)

$$|\zeta - I_{k,D}\zeta|_{H^1(D)} \lesssim h_D^{\ell}|\zeta|_{H^{\ell+1}(D)} \qquad \forall\,\zeta \in H^{\ell+1}(D)$$

$$\|\zeta - I_{k,D}\zeta\|_{L_2(D)} \lesssim h_D^{\ell+1}|\zeta|_{H^{\ell+1}(D)} \qquad \forall\,\zeta \in H^{\ell+1}(D)$$

where $1 \leq \ell \leq k$.

$$I_{k,D}q = q \qquad \forall\, q \in \mathbb{P}_k(D)$$

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

Stability Estimate in $\|\cdot\|_{L_\infty(D)}$

$$\|I_{k,D}\zeta\|_{L_\infty(D)} \lesssim \|I_{k,D}\zeta\|_{L_\infty(\partial D)} + |I_{k,D}\zeta|_{H^1(D)}$$

Maximum Principle

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

Stability Estimate in $\|\cdot\|_{L_\infty(D)}$

$$
\begin{aligned}
\|I_{k,D}\zeta\|_{L_\infty(D)} &\lesssim \|I_{k,D}\zeta\|_{L_\infty(\partial D)} + |I_{k,D}\zeta|_{H^1(D)} \\
&\lesssim \|\zeta\|_{L_\infty(\partial D)} + |I_{k,D}\zeta|_{H^1(D)}
\end{aligned}
$$

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

Stability Estimate in $\|\cdot\|_{L_\infty(D)}$

$$
\begin{aligned}
\|I_{k,D}\zeta\|_{L_\infty(D)} &\lesssim \|I_{k,D}\zeta\|_{L_\infty(\partial D)} + |I_{k,D}\zeta|_{H^1(D)} \\
&\lesssim \|\zeta\|_{L_\infty(\partial D)} + |I_{k,D}\zeta|_{H^1(D)} \\
&\lesssim h_D^{-1}\|\zeta\|_{L_2(D)} + |\zeta|_{H^1(D)} + h_D|\zeta|_{H^2(D)}
\end{aligned}
$$

Sobolev inequality and estimate for $I_{k,D}\zeta$

Estimates for $I_{k,D} : H^2(D) \longrightarrow \mathcal{Q}^k(D)$

Stability Estimate in $\|\cdot\|_{L_\infty(D)}$

$$\begin{aligned}\|I_{k,D}\zeta\|_{L_\infty(D)} &\lesssim \|I_{k,D}\zeta\|_{L_\infty(\partial D)} + |I_{k,D}\zeta|_{H^1(D)}\\ &\lesssim \|\zeta\|_{L_\infty(\partial D)} + |I_{k,D}\zeta|_{H^1(D)}\\ &\lesssim h_D^{-1}\|\zeta\|_{L_2(D)} + |\zeta|_{H^1(D)} + h_D|\zeta|_{H^2(D)}\end{aligned}$$

It follows that (Bramble-Hilbert)

$$\|\zeta - I_{k,D}\zeta\|_{L_\infty(D)} \lesssim h_D^\ell|\zeta|_{H^{\ell+1}(D)} \qquad \forall\, \zeta \in H^{\ell+1}(D)$$

where $1 \le \ell \le k$.

(The hidden constant only depend on k and ρ_D.)

Stabilization Estimates

Stabilization Estimates

$$a_h(w,v) = \sum_{D\in\mathcal{T}_h} \Big[a^D(\Pi^{\nabla}_{k,D} w, \Pi^{\nabla}_{k,D} v) + S^D(w - \Pi^{\nabla}_{k,D} w, v - \Pi^{\nabla}_{k,D} v) \Big]$$

$$a^D(w,v) = \int_D \nabla w \cdot \nabla v \, dx$$

Stabilization Estimates

$$a_h(w,v) = \sum_{D\in\mathcal{T}_h} \Big[a^D(\Pi^\nabla_{k,D} w, \Pi^\nabla_{k,D} v) \\ + S^D(w - \Pi^\nabla_{k,D} w, v - \Pi^\nabla_{k,D} v)\Big]$$

$$a^D(w,v) = \int_D \nabla w \cdot \nabla v \, dx$$

The Null Space of $\Pi^\nabla_{k,D}$

$$\mathcal{N}(\Pi^\nabla_{k,D}) = \{v \in \mathcal{Q}^k(D) : \ \Pi^\nabla_{k,D} v = 0\} \\ = \{v - \Pi^\nabla_{k,D} v : \ v \in \mathcal{Q}^k(D)\}$$

$$\mathcal{Q}^k(D) = \mathbb{P}_k(D) \oplus \mathcal{N}(\Pi^\nabla_{k,D})$$

Stabilization Estimates

$$a_h(w,v) = \sum_{D\in\mathcal{T}_h} \Big[a^D(\Pi^{\nabla}_{k,D} w, \Pi^{\nabla}_{k,D} v) + S^D(w - \Pi^{\nabla}_{k,D} w, v - \Pi^{\nabla}_{k,D} v) \Big]$$

$$a^D(w,v) = \int_D \nabla w \cdot \nabla v \, dx$$

The Null Space of $\Pi^{\nabla}_{k,D}$

$$\begin{aligned} \mathcal{N}(\Pi^{\nabla}_{k,D}) &= \{v \in \mathcal{Q}^k(D) : \ \Pi^{\nabla}_{k,D} v = 0\} \\ &= \{v - \Pi^{\nabla}_{k,D} v : \ v \in \mathcal{Q}^k(D)\} \end{aligned}$$

$$\mathcal{Q}^k(D) = \mathbb{P}_k(D) \oplus \mathcal{N}(\Pi^{\nabla}_{k,D})$$

Key The relation between $S^D(\cdot,\cdot)$ and $a^D(\cdot,\cdot)$ on $\mathcal{N}(\Pi^{\nabla}_{k,D})$.

Stabilization Estimates

$$\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} \lesssim h_D \sum_{e\in\mathcal{E}_D} \|\Pi^0_{k-1,e}v\|^2_{L_2(e)} \qquad \forall\, v \in \mathcal{N}(\Pi^\nabla_{k,D})$$

(The hidden constant only depends on k and ρ_D.)

Stabilization Estimates

$$\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} \lesssim h_{\scriptscriptstyle D} \sum_{e\in\mathcal{E}_D} \|\Pi^0_{k-1,e}v\|^2_{L_2(e)} \qquad \forall\, v \in \mathcal{N}(\Pi^\nabla_{k,D})$$

(The hidden constant only depends on k and $\rho_{\scriptscriptstyle D}$.)

Computation of $\Pi^\nabla_{k,D}$

$$\int_D \nabla\Pi^\nabla_{k,D}v \cdot \nabla q\,dx = \int_{\partial D} v\frac{\partial q}{\partial n}ds - \int_D v(\Delta q)dx \quad \forall\, q \in \mathbb{P}_k(D)$$

implies, for $v \in \mathcal{N}(\Pi^\nabla_{k,D})$,

$$\int_D v(\Delta q)\,dx = \sum_{e\in\mathcal{E}_D} \int_e v\frac{\partial q}{\partial n}\,ds \qquad \forall\, q \in \mathbb{P}_k(D)$$

Stabilization Estimates

$$\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} \lesssim h_D \sum_{e\in\mathcal{E}_D} \|\Pi^0_{k-1,e}v\|^2_{L_2(e)} \qquad \forall\, v \in \mathcal{N}(\Pi^\nabla_{k,D})$$

(The hidden constant only depends on k and ρ_D.)

Computation of $\Pi^\nabla_{k,D}$

$$\int_D \nabla\Pi^\nabla_{k,D}v\cdot\nabla q\,dx = \int_{\partial D} v\frac{\partial q}{\partial n}ds - \int_D v(\Delta q)dx \quad \forall\, q \in \mathbb{P}_k(D)$$

implies, for $v \in \mathcal{N}(\Pi^\nabla_{k,D})$,

$$\int_D v(\Delta q)\,dx = \sum_{e\in\mathcal{E}_D}\int_e v\frac{\partial q}{\partial n}\,ds \qquad \forall\, q \in \mathbb{P}_k(D)$$

$$\Delta q \in \mathbb{P}_{k-2}(D) \quad \text{and} \quad \frac{\partial q}{\partial n} \in \mathbb{P}_{k-1}(e)$$

Stabilization Estimates

$$\|\Pi^0_{k-2,D} v\|^2_{L_2(D)} \lesssim h_D \sum_{e\in\mathcal{E}_D} \|\Pi^0_{k-1,e} v\|^2_{L_2(e)} \qquad \forall\, v \in \mathcal{N}(\Pi^\nabla_{k,D})$$

(The hidden constant only depends on k and ρ_D.)

Computation of $\Pi^\nabla_{k,D}$

$$\int_D \nabla \Pi^\nabla_{k,D} v \cdot \nabla q\, dx = \int_{\partial D} v \frac{\partial q}{\partial n} ds - \int_D v(\Delta q) dx \quad \forall\, q \in \mathbb{P}_k(D)$$

$$\int_{\partial D} \Pi^\nabla_{k,D} v\, ds = \int_{\partial D} v\, ds$$

Stabilization Estimates

$$\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} \lesssim h_D \sum_{e\in\mathcal{E}_D} \|\Pi^0_{k-1,e}v\|^2_{L_2(e)} \qquad \forall\, v \in \mathcal{N}(\Pi^\nabla_{k,D})$$

(The hidden constant only depends on k and ρ_D.)

Computation of $\Pi^\nabla_{k,D}$

$$\int_D \nabla\Pi^\nabla_{k,D}v \cdot \nabla q\,dx = \int_{\partial D} v\frac{\partial q}{\partial n}ds - \int_D v(\Delta q)dx \quad \forall\, q \in \mathbb{P}_k(D)$$

$$\int_{\partial D} \Pi^\nabla_{k,D}v\,ds = \int_{\partial D} v\,ds$$

implies

$$\int_{\partial D} v\,ds = 0 \qquad \forall\, v \in \mathcal{N}(\Pi^\nabla_{k,D})$$

Stabilization Estimates

$$S^D(w, v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$$

Stabilization Estimates

$$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$$

$$\begin{aligned} a^D(v,v) &= |v|^2_{H^1(D)} \\ &\lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} \\ &\qquad + h_D\|\partial v/\partial s\|^2_{L_2(D)} \end{aligned}$$

Inverse Estimate

$$\begin{aligned} |v|^2_{H^1(D)} &\lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} \\ &\qquad + h_D\|\partial v/\partial s\|^2_{L_2(\partial D)} \qquad \forall\, v \in \mathcal{Q}^k(D) \end{aligned}$$

Stabilization Estimates

$$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$$

$$\begin{aligned} a^D(v,v) &= |v|^2_{H^1(D)} \\ &\lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} \\ &\qquad + h_D\|\partial v/\partial s\|^2_{L_2(D)} \\ &\lesssim h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + h_D\|\partial v/\partial s\|^2_{L_2(\partial D)} \end{aligned}$$

$$\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} \lesssim h_D\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} \qquad \forall\, v\in\mathcal{N}(\Pi^\nabla_{k,D})$$

Stabilization Estimates

$$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$$

$$\begin{aligned}
a^D(v,v) &= |v|^2_{H^1(D)} \\
&\lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} \\
&\qquad + h_D\|\partial v/\partial s\|^2_{L_2(D)} \\
&\lesssim h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + h_D\|\partial v/\partial s\|^2_{L_2(\partial D)} \\
&\le h_D^{-1}\|v\|^2_{L_2(\partial D)} + h_D\|\partial v/\partial s\|^2_{L_2(\partial D)}
\end{aligned}$$

Stabilization Estimates

$$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$$

$$\begin{aligned}
a^D(v,v) &= |v|^2_{H^1(D)} \\
&\lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} \\
&\qquad + h_D\|\partial v/\partial s\|^2_{L_2(D)} \\
&\lesssim h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + h_D\|\partial v/\partial s\|^2_{L_2(\partial D)} \\
&\leq h_D^{-1}\|v\|^2_{L_2(\partial D)} + h_D\|\partial v/\partial s\|^2_{L_2(\partial D)} \\
&\lesssim h_D\|\partial v/\partial s\|^2_{L_2(\partial D)}
\end{aligned}$$

Poincaré Inequality in 1D

$$\|v\|_{L_2(\partial D)} \lesssim h_D\|\partial v/\partial s\|_{L_2(\partial D)} \qquad \text{if } \int_{\partial D} v\,ds = 0$$

Stabilization Estimates

$$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$$

$$\begin{aligned}
a^D(v,v) &= |v|^2_{H^1(D)} \\
&\lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} \\
&\qquad + h_D\|\partial v/\partial s\|^2_{L_2(D)} \\
&\lesssim h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)} + h_D\|\partial v/\partial s\|^2_{L_2(\partial D)} \\
&\leq h_D^{-1}\|v\|^2_{L_2(\partial D)} + h_D\|\partial v/\partial s\|^2_{L_2(\partial D)} \\
&\lesssim h_D\|\partial v/\partial s\|^2_{L_2(\partial D)}
\end{aligned}$$

$$a^D(v,v) \lesssim S^D(v,v) \qquad \forall\, v \in \mathcal{N}(\Pi^\nabla_{k,D})$$

(The hidden constant only depends on k and ρ_D.)

Stabilization Estimates

$$S^D(w,v) = \sum_{p\in\mathcal{N}_{\partial D}} w(p)v(p)$$

$$a^D(v,v) \lesssim \ln(1+\tau_D)S^D(v,v) \qquad \forall\, v \in \mathcal{N}(\Pi^{\nabla}_{k,D})$$

where

$$\tau_D = \frac{\max_{e\in\mathcal{E}_D} h_e}{\min_{e\in\mathcal{E}_D} h_e}$$

and the hidden constant depends only on k, ρ_D and $|\mathcal{E}_D|$.

Inverse Estimate

$$|v|^2_{H^1(D)} \lesssim h_D^{-2}\|\Pi^0_{k-2,D}v\|^2_{L_2(D)} + h_D^{-1}\sum_{e\in\mathcal{E}_D}\|\Pi^0_{k-1,e}v\|^2_{L_2(e)}$$
$$+ \ln(1+\tau_D)\|v\|^2_{L_\infty(\partial D)}$$

Stabilization Estimates

$$a^D(v,v) \lesssim \alpha_D S^D(v,v) \qquad \forall\, v \in \mathcal{N}(\Pi^{\nabla}_{k,D})$$

where

$$\alpha_D = \begin{cases} 1 & \text{if } S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)} \\ \ln(1+\tau_D) & \text{if } S^D(w,v) = \sum_{p\in\mathcal{N}_{\partial D}} w(p)v(p) \end{cases}$$

$$\tau_D = \frac{\max_{e\in\mathcal{E}_D} h_e}{\min_{e\in\mathcal{E}_D} h_e}$$

The hidden constant depends only on k and ρ_D if

$$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$$

and also $|\mathcal{E}_D|$ if

$$S^D(w,v) = \sum_{p\in\mathcal{N}_{\partial D}} w(p)v(p)$$

Stabilization Estimates

$$a^D(v,v) \lesssim \alpha_D S^D(v,v) \qquad \forall\, v \in \mathcal{N}(\Pi^{\nabla}_{k,D})$$

where

$$\alpha_D = \begin{cases} 1 & \text{if } S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)} \\ \ln(1+\tau_D) & \text{if } S^D(w,v) = \sum_{p\in\mathcal{N}_{\partial D}} w(p)v(p) \end{cases}$$

$$\tau_D = \frac{\max_{e\in\mathcal{E}_D} h_e}{\min_{e\in\mathcal{E}_D} h_e}$$

For both choices (Poincaré-Friedrichs)

$$S^D(v,v) \lesssim h_D^{-1}\|v\|^2_{L_2(\partial D)} \lesssim |v|^2_{H^1(D)} = a^D(v,v)$$

for all $v \in \mathcal{N}(\Pi^{\nabla}_{k,D})$, since $\int_{\partial D} v\, ds = 0$

Error Estimates in
H^1 and L_2

Global Spaces and Operators

$\mathcal{T}_h =$ partition of Ω into polygonal subdomains

$$\mathcal{Q}_h^k = \{v \in H_0^1(\Omega) : v\big|_D \in \mathcal{Q}^k(D) \quad \forall\, D \in \mathcal{T}_h\}$$

$$\mathcal{P}_h^k = \{v \in L_2(\Omega) : v\big|_D \in \mathbb{P}_k(D) \quad \forall\, D \in \mathcal{T}_h\}$$

Global Spaces and Operators

$\mathcal{T}_h =$ partition of Ω into polygonal subdomains

$$\mathcal{Q}_h^k = \{v \in H_0^1(\Omega) : \, v\big|_D \in \mathcal{Q}^k(D) \quad \forall\, D \in \mathcal{T}_h\}$$

$$\mathcal{P}_h^k = \{v \in L_2(\Omega) : \, v\big|_D \in \mathbb{P}_k(D) \quad \forall\, D \in \mathcal{T}_h\}$$

The operators

$$\Pi_{k,h}^{\nabla} : H^1(\Omega) \longrightarrow \mathcal{P}_h^k$$

$$\Pi_{k,h}^{0} : H^1(\Omega) \longrightarrow \mathcal{P}_h^k$$

$$I_{k,h} : H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow \mathcal{Q}_h^k$$

are defined in terms of their local counterparts.

Global Shape Regularity Assumptions

All the local estimates can be extended to global estimates under the following global shape regularity assumptions.

Assumption 1 There exists a positive number $\rho \in (0,1)$, independent of h, such that

$$\rho_D \geq \rho \qquad \forall\, D \in \mathcal{T}_h$$

This is the only assumption we need for

$$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$$

Global Shape Regularity Assumptions

All the local estimates can be extended to global estimates under the following global shape regularity assumptions.

Assumption 1 There exists a positive number $\rho \in (0,1)$, independent of h, such that

$$\rho_{D} \geq \rho \qquad \forall\, D \in \mathcal{T}_h$$

Assumption 2 There exists a positive integer N, independent of h, such that

$$|\mathcal{E}_{D}| \leq N \qquad \forall\, D \in \mathcal{T}_h$$

We need both assumptions for

$$S^D(w,v) = \sum_{p \in \mathcal{N}_{\partial D}} w(p)v(p)$$

Global Shape Regularity Assumptions

All the local estimates can be extended to global estimates under the following global shape regularity assumptions.

Assumption 1 There exists a positive number $\rho \in (0,1)$, independent of h, such that

$$\rho_{D} \geq \rho \qquad \forall\, D \in \mathcal{T}_h$$

Assumption 2 There exists a positive integer N, independent of h, such that

$$|\mathcal{E}_{D}| \leq N \qquad \forall\, D \in \mathcal{T}_h$$

From here on all the hidden constants only depend on k and ρ for the first stabilization and also N for the second stabilization.

The Global Parameter α_h

We define

$$\alpha_h = \max_{D \in \mathcal{T}_h} \alpha_D$$

For $S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$

$$\alpha_h = 1$$

For $S^D(\cdot,\cdot) = \sum_{p \in \mathcal{N}_{\partial D}} w(p)v(p)$

$$\begin{aligned}\alpha_h &= \ln(1 + \max_{D \in \mathcal{T}_h} \tau_D) \\ &= \ln\left(1 + \max_{D \in \mathcal{T}_h} \frac{\max_{e \in \mathcal{E}_D} h_e}{\min_{e \in \mathcal{E}_D} h_e}\right)\end{aligned}$$

The Bilinear Form $a_h(\cdot,\cdot)$

$$a_h(w,v) = \sum_{D\in\mathcal{T}_h} \Big[a^D(\Pi^\nabla_{k,D}\, w, \Pi^\nabla_{k,D}\, v) + S^D(w - \Pi^\nabla_{k,D}\, w, v - \Pi^\nabla_{k,D}\, v)\Big]$$

$$a^D(w,v) = \int_D \nabla w \cdot \nabla v \, dx$$

The Bilinear Form $a_h(\cdot,\cdot)$

$$a_h(w,v) = \sum_{D\in\mathcal{T}_h} \Big[a^D(\Pi^{\nabla}_{k,D} w, \Pi^{\nabla}_{k,D} v) + S^D(w - \Pi^{\nabla}_{k,D} w, v - \Pi^{\nabla}_{k,D} v) \Big]$$

$$\begin{aligned} a(v,v) &= |v|^2_{H^1(\Omega)} \\ &= \sum_{D\in\mathcal{T}_h} \Big[a^D(\Pi^{\nabla}_{k,D} v, \Pi^{\nabla}_{k,D} v) + a^D(v - \Pi^{\nabla}_{k,D} v, v - \Pi^{\nabla}_{k,D} v) \Big] \end{aligned}$$

The Bilinear Form $a_h(\cdot,\cdot)$

$$a_h(w,v) = \sum_{D\in\mathcal{T}_h} \Big[a^D(\Pi^\nabla_{k,D} w, \Pi^\nabla_{k,D} v) + S^D(w - \Pi^\nabla_{k,D} w, v - \Pi^\nabla_{k,D} v) \Big]$$

$$\begin{aligned}
a(v,v) &= |v|^2_{H^1(\Omega)} \\
&= \sum_{D\in\mathcal{T}_h} \Big[a^D(\Pi^\nabla_{k,D} v, \Pi^\nabla_{k,D} v) + a^D(v - \Pi^\nabla_{k,D} v, v - \Pi^\nabla_{k,D} v) \Big] \\
&\lesssim \sum_{D\in\mathcal{T}_h} \Big[a^D(\Pi^\nabla_{k,D} v, \Pi^\nabla_{k,D} v) + \sum_{D\in\mathcal{T}_h} \alpha_D S^D(v - \Pi^\nabla_{k,D} v, v - \Pi^\nabla_{k,D} v) \Big] \\
&\lesssim \alpha_h a_h(v,v) \qquad \forall\, v \in \mathcal{Q}^k_h
\end{aligned}$$

The Bilinear Form $a_h(\cdot,\cdot)$

$$a_h(w,v)=\sum_{D\in\mathcal{T}_h}\Big[a^D(\Pi^\nabla_{k,D}w,\Pi^\nabla_{k,D}v)\\
+S^D(w-\Pi^\nabla_{k,D}w,v-\Pi^\nabla_{k,D}v)\Big]$$

The coercivity

$$a_h(v,v)\gtrsim\alpha_h^{-1}|v|^2_{H^1(\Omega)}\qquad\forall\,v\in\mathcal{Q}^k(D)$$

implies the existence of a unique $u_h\in\mathcal{Q}_h^k$ that satsifies the discrete problem

$$a_h(u_h,v)=(f,\Xi_hv)\qquad\forall\,v\in\mathcal{Q}_h^k$$

The Mesh-Dependent Energy Norm $\|\cdot\|_h$

We will derive an error estimate in the mesh-dependent norm

$$\begin{aligned}\|v\|_h^2 &= a_h(v,v)\\ &= \sum_{D\in\mathcal{T}_h}\Big[a^D(\Pi^\nabla_{k,D}v,\Pi^\nabla_{k,D}v)\\ &\qquad + S^D(v-\Pi^\nabla_{k,D}v, v-\Pi^\nabla_{k,D}v)\Big]\end{aligned}$$

that is well-defined on $\big[H^2(\Omega)\cap H^1_0(\Omega)\big]+\mathcal{Q}_h^k+\mathcal{P}_h^k$.

The Mesh-Dependent Energy Norm $\|\cdot\|_h$

We will derive an error estimate in the mesh-dependent norm

$$\begin{aligned}\|v\|_h^2 &= a_h(v,v)\\ &= \sum_{D\in\mathcal{T}_h}\Big[a^D(\Pi^\nabla_{k,D}v,\Pi^\nabla_{k,D}v)\\ &\qquad\qquad + S^D(v-\Pi^\nabla_{k,D}v, v-\Pi^\nabla_{k,D}v)\Big]\end{aligned}$$

that is well-defined on $\big[H^2(\Omega)\cap H^1_0(\Omega)\big]+\mathcal{Q}_h^k+\mathcal{P}_h^k$.

Note that

$$|v|_{H^1(\Omega)} \lesssim \sqrt{\alpha_h}\|v\|_h \qquad \forall\, v\in\mathcal{Q}_h^k$$

The Mesh-Dependent Energy Norm $\|\cdot\|_h$

We will derive an error estimate in the mesh-dependent norm

$$\begin{aligned}\|v\|_h^2 &= a_h(v,v)\\ &= \sum_{D\in\mathcal{T}_h}\Big[a^D(\Pi^\nabla_{k,D}v,\Pi^\nabla_{k,D}v)\\ &\qquad\qquad + S^D(v-\Pi^\nabla_{k,D}v, v-\Pi^\nabla_{k,D}v)\Big]\end{aligned}$$

that is well-defined on $\big[H^2(\Omega)\cap H^1_0(\Omega)\big]+\mathcal{Q}_h^k+\mathcal{P}_h^k$.

Note that

$$|v|_{H^1(\Omega)}\lesssim\sqrt{\alpha_h}\|v\|_h \qquad \forall\, v\in\mathcal{Q}_h^k$$

We will also use the norm

$$|v|_{h,1}=\Big(\sum_{D\in\mathcal{T}_h}|v|^2_{H^1(D)}\Big)^{\frac{1}{2}}$$

An Abstract Error Estimate

We have a standard error estimate

$$\|u - u_h\|_h \leq \inf_{v \in \mathcal{Q}_h^k} \|u - v\|_h + \sup_{v \in \mathcal{Q}_h^k} \frac{a_h(u, v) - (f, \Xi_h v)}{\|v\|_h}$$

since the virtual finite element method is defined in terms of a non-inherited SPD bilinear form.

Berger-Scott-Strang

An Abstract Error Estimate

We have a standard error estimate

$$\|u - u_h\|_h \leq \inf_{v \in \mathcal{Q}_h^k} \|u - v\|_h + \sup_{v \in \mathcal{Q}_h^k} \frac{a_h(u, v) - (f, \Xi_h v)}{\|v\|_h}$$

The first term on the right-hand side can be bounded by

$$\inf_{v \in \mathcal{Q}_h^k} \|u - v\|_h \leq \|u - I_{k,h} u\|_h$$

The key is to control the numerator in the second term on the right-hand side.

An Abstract Error Estimate

We have a standard error estimate

$$\|u - u_h\|_h \leq \inf_{v \in \mathcal{Q}_h^k} \|u - v\|_h + \sup_{v \in \mathcal{Q}_h^k} \frac{a_h(u,v) - (f, \Xi_h v)}{\|v\|_h}$$

$$a_h(u,v) - (f, \Xi_h v) = \sum_{D \in \mathcal{T}_h} \Big[a^D(\Pi_{k,D}^{\nabla} u - u, v) + S^D(u - \Pi_{k,D}^{\nabla} u, v - \Pi_{k,D}^{\nabla} v) \Big] + (f, v - \Xi_h v)$$

Beirão da Veiga-Brezzi-Cangiani-Manzini-Marini-Russo (2013)

An Abstract Error Estimate

We have a standard error estimate

$$\|u - u_h\|_h \leq \inf_{v \in \mathcal{Q}_h^k} \|u - v\|_h + \sup_{v \in \mathcal{Q}_h^k} \frac{a_h(u, v) - (f, \Xi_h v)}{\|v\|_h}$$

$$\begin{aligned} a_h(u, v) - (f, \Xi_h v) = \sum_{D \in \mathcal{T}_h} \Big[& a^D(\Pi^{\nabla}_{k,D} u - u, v) \\ & + S^D(u - \Pi^{\nabla}_{k,D} u, v - \Pi^{\nabla}_{k,D} v) \Big] \\ & + (f, v - \Xi_h v) \end{aligned}$$

It follows that

$$\begin{aligned} \|u - u_h\|_h \lesssim & \|u - I_{k,h} u\|_h + \|u - \Pi^{\nabla}_{k,h} u\|_h \\ & + \sqrt{\alpha_h} \Big[|u - \Pi^{\nabla}_{k,h} u|_{h,1} + \sup_{w \in \mathcal{Q}_h^k} \frac{(f, w - \Xi_h w)}{|w|_{H^1(\Omega)}} \Big] \end{aligned}$$

Concrete Error Estimates

Theorem Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\|u - u_h\|_h \lesssim \sqrt{\alpha_h} h^{\ell} |u|_{H^{\ell+1}(\Omega)}$$

Concrete Error Estimates

Theorem   Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\|u - u_h\|_h \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$

Corollary   Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$|u - \Pi^\nabla_{k,h} u_h|_{h,1} \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$
$$|u - \Pi^0_{k,h} u_h|_{h,1} \lesssim \alpha_h h^\ell |u|_{H^{\ell+1}(\Omega)}$$

$$|v|_{h,1} = \Big(\sum_{D \in \mathcal{T}_h} |v|^2_{H^1(D)} \Big)^{\frac{1}{2}}$$

Concrete Error Estimates

Theorem Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\|u - u_h\|_h \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$

Corollary Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$|u - \Pi_{k,h}^\nabla u_h|_{h,1} \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$
$$|u - \Pi_{k,h}^0 u_h|_{h,1} \lesssim \alpha_h h^\ell |u|_{H^{\ell+1}(\Omega)}$$

$$\begin{aligned} |u - \Pi_{k,h}^\nabla u_h|_{h,1} &\le |u - \Pi_{k,h}^\nabla u|_{h,1} + |\Pi_{k,h}^\nabla (u - u_h)|_{h,1} \\ &\le |u - \Pi_{k,h}^\nabla u|_{h,1} + \|u - u_h\|_h \end{aligned}$$

apply the Theorem and estimates for $\Pi_{k,h}^\nabla$

Concrete Error Estimates

Theorem Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\|u - u_h\|_h \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$

Corollary Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$|u - \Pi^\nabla_{k,h} u_h|_{h,1} \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$
$$|u - \Pi^0_{k,h} u_h|_{h,1} \lesssim \alpha_h h^\ell |u|_{H^{\ell+1}(\Omega)}$$

$$\begin{aligned} |u - \Pi^0_{k,h} u_h|_{h,1} &\le |u - \Pi^0_{k,h} u|_{h,1} + |\Pi^0_{k,h}(u - u_h)|_{h,1} \\ &\lesssim |u - \Pi^0_{k,h} u|_{h,1} + |u - u_h|_{h,1} \end{aligned}$$

$$|\Pi^0_{k,D} \zeta|_{H^1(D)} \lesssim |\zeta|_{H^1(D)} \qquad \forall\, \zeta \in H^1(D)$$

Concrete Error Estimates

Theorem Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\|u - u_h\|_h \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$

Corollary Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$|u - \Pi^\nabla_{k,h} u_h|_{h,1} \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$

$$|u - \Pi^0_{k,h} u_h|_{h,1} \lesssim \alpha_h h^\ell |u|_{H^{\ell+1}(\Omega)}$$

$$\begin{aligned} |u - \Pi^0_{k,h} u_h|_{h,1} &\leq |u - \Pi^0_{k,h} u|_{h,1} + |\Pi^0_{k,h}(u - u_h)|_{h,1} \\ &\lesssim |u - \Pi^0_{k,h} u|_{h,1} + |u - u_h|_{h,1} \\ &\lesssim |u - \Pi^0_{k,h} u|_{h,1} + \sqrt{\alpha_h} \|u - u_h\|_h \end{aligned}$$

$$|v|_{H^1(\Omega)} \lesssim \sqrt{\alpha_h} \|v\|_h$$

Concrete Error Estimates

Theorem Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\|u - u_h\|_h \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$

Corollary Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$|u - \Pi^{\nabla}_{k,h} u_h|_{h,1} \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$
$$|u - \Pi^{0}_{k,h} u_h|_{h,1} \lesssim \alpha_h h^\ell |u|_{H^{\ell+1}(\Omega)}$$

$$\begin{aligned} |u - \Pi^0_{k,h} u_h|_{h,1} &\leq |u - \Pi^0_{k,h} u|_{h,1} + |\Pi^0_{k,h}(u - u_h)|_{h,1} \\ &\lesssim |u - \Pi^0_{k,h} u|_{h,1} + |u - u_h|_{h,1} \\ &\lesssim |u - \Pi^0_{k,h} u|_{h,1} + \sqrt{\alpha_h}\|u - u_h\|_h \end{aligned}$$

apply the Theorem and estimates for $\Pi^0_{k,h}$

Concrete Error Estimates

Theorem Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\|u-u_h\|_{L_2(\Omega)} \lesssim \alpha_h h^{\ell+1}|u|_{H^{\ell+1}(\Omega)}$$

Concrete Error Estimates

Theorem Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\|u - u_h\|_{L_2(\Omega)} \lesssim \alpha_h h^{\ell+1} |u|_{H^{\ell+1}(\Omega)}$$

Corollary Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\|u - \Pi^0_{k,h} u_h\|_{L_2(\Omega)} + \|u - \Pi^\nabla_{k,h} u_h\|_{L_2(\Omega)} \lesssim \alpha_h h^{\ell+1} |u|_{H^{\ell+1}(\Omega)}$$

Error Estimates in L_∞

L_∞ Estimate for u_h

Observe that

$$\|u - u_h\|_h \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$

implies

$$\sum_{D \in \mathcal{T}_h} S^D\big((u-u_h) - \Pi^\nabla_{k,D}(u-u_h), (u-u_h) - \Pi^\nabla_{k,D}(u-u_h)\big)$$
$$\lesssim \alpha_h h^{2\ell} |u|^2_{H^{\ell+1}(\Omega)}$$

$$\begin{aligned}\|v\|_h^2 &= a_h(v,v) \\ &= \sum_{D \in \mathcal{T}_h} \Big[a^D(\Pi^\nabla_{k,D} v, \Pi^\nabla_{k,D} v) \\ &\qquad + S^D(v - \Pi^\nabla_{k,D} v, v - \Pi^\nabla_{k,D} v)\Big]\end{aligned}$$

L_∞ Estimate for u_h

Observe that

$$\|u - u_h\|_h \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$

implies

$$\sum_{D \in \mathcal{T}_h} S^D\big((u - u_h) - \Pi^\nabla_{k,D}(u - u_h), (u - u_h) - \Pi^\nabla_{k,D}(u - u_h)\big)$$
$$\lesssim \alpha_h h^{2\ell} |u|^2_{H^{\ell+1}(\Omega)}$$

We can use this information to obtain an estimate for

$$\max_{e \in \mathcal{E}_h} \|u - u_h\|_{L_\infty(e)}$$

where $\mathcal{E}_h$ is the set of the edges in the partition $\mathcal{T}_h$.

L_∞ Estimate for u_h

$S^D(w,v) = h_D(\partial w/\partial s, \partial v/ps)_{L_2(\partial D)}$

We begin with the estimate

$$\sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_D\|\partial[(u-u_h)-\Pi^{\nabla}_{k,D}(u-u_h)]/\partial s\|^2_{L_2(e)} \lesssim h^{2\ell}|u|^2_{H^{\ell+1}(\Omega)}$$

that is a part of the estimate for $\|u-u_h\|_h$.

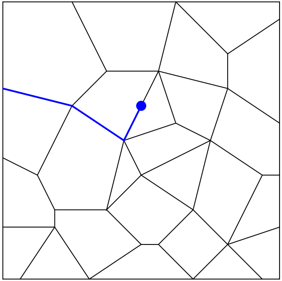
L_∞ Estimate for u_h

$S^D(w,v) = h_D(\partial w/\partial s, \partial v/ps)_{L_2(\partial D)}$

We begin with the estimate

$$\sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_D\|\partial[(u-u_h)-\Pi^\nabla_{k,D}(u-u_h)]/\partial s\|^2_{L_2(e)} \lesssim h^{2\ell}|u|^2_{H^{\ell+1}(\Omega)}$$

We can connect any point in an edge $e\in\mathcal{E}_h$ to $\partial\Omega$, where $u-u_h=0$, by a path along the edges in $\mathcal{E}_h$.



L_∞ Estimate for u_h

$S^D(w,v) = h_D(\partial w/\partial s, \partial v/ps)_{L_2(\partial D)}$

We begin with the estimate

$$\sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_D\|\partial[(u-u_h)-\Pi^{\nabla}_{k,D}(u-u_h)]/\partial s\|^2_{L_2(e)} \lesssim h^{2\ell}|u|^2_{H^{\ell+1}(\Omega)}$$

We can connect any point in an edge $e \in \mathcal{E}_h$ to $\partial\Omega$, where $u-u_h=0$, by a path along the edges in $\mathcal{E}_h$.

$$\|u-u_h\|^2_{L_\infty(e)} \lesssim \sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_e\|\partial(u-u_h)/\partial s\|^2_{L_2(e)}$$

Sobolev's Inequality in 1D

L_∞ Estimate for u_h

$S^D(w,v) = h_D(\partial w/\partial s, \partial v/ps)_{L_2(\partial D)}$

We begin with the estimate

$$\sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_D\|\partial[(u-u_h)-\Pi^\nabla_{k,D}(u-u_h)]/\partial s\|^2_{L_2(e)} \lesssim h^{2\ell}|u|^2_{H^{\ell+1}(\Omega)}$$

We can connect any point in an edge $e\in\mathcal{E}_h$ to $\partial\Omega$, where $u-u_h=0$, by a path along the edges in $\mathcal{E}_h$.

$$\begin{aligned}\|u-u_h\|^2_{L_\infty(e)} &\lesssim \sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_e\|\partial(u-u_h)/\partial s\|^2_{L_2(e)}\\ &\lesssim \sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_D\|\partial[(u-u_h)-\Pi^\nabla_{k,D}(u-u_h)]/\partial s\|^2_{L_2(e)}\\ &\qquad + \sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_D\|\partial[\Pi^\nabla_{k,D}(u-u_h)]/\partial s\|^2_{L_2(e)}\end{aligned}$$

L_∞ Estimate for u_h

$S^D(w,v) = h_D(\partial w/\partial s, \partial v/ps)_{L_2(\partial D)}$

We begin with the estimate

$$\sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_D\|\partial[(u-u_h)-\Pi^\nabla_{k,D}(u-u_h)]/\partial s\|^2_{L_2(e)} \lesssim h^{2\ell}|u|^2_{H^{\ell+1}(\Omega)}$$

We can connect any point in an edge $e \in \mathcal{E}_h$ to $\partial\Omega$, where $u - u_h = 0$, by a path along the edges in $\mathcal{E}_h$.

$$\begin{aligned}\|u-u_h\|^2_{L_\infty(e)} &\lesssim \sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_e\|\partial(u-u_h)/\partial s\|^2_{L_2(e)}\\ &\lesssim \sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_D\|\partial[(u-u_h)-\Pi^\nabla_{k,D}(u-u_h)]/\partial s\|^2_{L_2(e)}\\ &\qquad + \sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_D\|\partial[\Pi^\nabla_{k,D}(u-u_h)]/\partial s\|^2_{L_2(e)}\end{aligned}$$

$\boldsymbol{L_\infty}$ Estimate for u_h

$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$

$$\sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_D\|\partial[\Pi^\nabla_{k,D}(u-u_h)]/\partial s\|^2_{L_2(e)}$$

$$\lesssim \sum_{D\in\mathcal{T}_h}\big(|\Pi^\nabla_{k,D}(u-u_h)|^2_{H^1(D)} + h_D^2|\Pi^\nabla_{k,D}(u-u_h)|^2_{H^2(D)}\big)$$

trace inequality

L_∞ Estimate for u_h

$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$

$$\sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_D\|\partial[\Pi^\nabla_{k,D}(u-u_h)]/\partial s\|^2_{L_2(e)}$$

$$\lesssim \sum_{D\in\mathcal{T}_h}\big(|\Pi^\nabla_{k,D}(u-u_h)|^2_{H^1(D)} + h_D^2|\Pi^\nabla_{k,D}(u-u_h)|^2_{H^2(D)}\big)$$

$$\lesssim \sum_{D\in\mathcal{T}_h}|\Pi^\nabla_{k,D}(u-u_h)|^2_{H^1(D)}$$

scaling argument for polynomials

L_∞ Estimate for u_h

$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$

$$\sum_{D\in\mathcal{T}_h}\sum_{e\in\mathcal{E}_D} h_D\|\partial[\Pi^\nabla_{k,D}(u-u_h)]/\partial s\|^2_{L_2(e)}$$
$$\lesssim \sum_{D\in\mathcal{T}_h}\big(|\Pi^\nabla_{k,D}(u-u_h)|^2_{H^1(D)} + h_D^2|\Pi^\nabla_{k,D}(u-u_h)|^2_{H^2(D)}\big)$$
$$\lesssim \sum_{D\in\mathcal{T}_h}|\Pi^\nabla_{k,D}(u-u_h)|^2_{H^1(D)}$$
$$\le \|u-u_h\|_h^2$$
$$\lesssim h^{2\ell}|u|^2_{H^{\ell+1}(\Omega)}$$

L_∞ Estimate for u_h

$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$

Theorem Assuming that u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\max_{e\in\mathcal{E}_h} \|u-u_h\|_{L_\infty(e)} \le Ch^\ell |u|_{H^{\ell+1}(\Omega)}$$

where the positive constant C only depends on k and ρ.

L_∞ Estimate for u_h $\quad S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$

Theorem Assuming that u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\max_{e\in\mathcal{E}_h} \|u - u_h\|_{L_\infty(e)} \le Ch^\ell |u|_{H^{\ell+1}(\Omega)}$$

where the positive constant C only depends on k and ρ.

Corollary Assuming that u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\|u - \Pi^\nabla_{k,h} u_h\|_{L_\infty(\Omega)} + \|u - \Pi^0_{k,h} u_h\|_{L_\infty(\Omega)} \le Ch^\ell |u|_{H^{\ell+1}(\Omega)}$$

where the positive constant C only depends on k and ρ.

L_∞ Estimate for u_h $\quad S^D(w,v) = \sum_{p \in \mathcal{N}_{\partial D}} w(p)v(p)$

Definition The partition $\mathcal{T}_h$ is quasi-uniform if there exists a positive constant γ independent of h such that

$$h_D \geq \gamma h \qquad \forall\, D \in \mathcal{T}_h.$$

L_∞ Estimate for u_h $S^D(w,v) = \sum_{p\in\mathcal{N}_{\partial D}} w(p)v(p)$

Definition The partition $\mathcal{T}_h$ is quasi-uniform if there exists a positive constant γ independent of h such that

$$h_D \geq \gamma h \qquad \forall\, D \in \mathcal{T}_h.$$

Theorem Assuming $\mathcal{T}_h$ is quasi-uniform and u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\max_{e\in\mathcal{E}_h} \|u-u_h\|_{L_\infty(e)} \leq C\ln(1+\max_{D\in\mathcal{T}_h} \tau_D)h^\ell |u|_{H^{\ell+1}(\Omega)},$$

where

$$\tau_D = \frac{\max_{e\in\mathcal{E}_D} h_e}{\min_{e\in\mathcal{E}_D} h_e}$$

and the positive constant C only depends on k, ρ, N, and γ.

$|\mathcal{E}_D| \leq N \qquad \forall\, D \in \mathcal{T}_h$

L_∞ Estimate for u_h

$S^D(w,v) = \sum_{p\in\mathcal{N}_{\partial D}} w(p)v(p)$

Definition The partition $\mathcal{T}_h$ is quasi-uniform if there exists a positive constant γ independent of h such that

$$h_D \geq \gamma h \qquad \forall\, D \in \mathcal{T}_h.$$

Theorem Assuming $\mathcal{T}_h$ is quasi-uniform and u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\max_{e\in\mathcal{E}_h} \|u-u_h\|_{L_\infty(e)} \leq C \ln(1+\max_{D\in\mathcal{T}_h} \tau_D) h^\ell |u|_{H^{\ell+1}(\Omega)},$$

Corollary

$$\|u-\Pi^\nabla_{k,h}u_h\|_{L_\infty(\Omega)} + \|u-\Pi^0_{k,h}u_h\|_{L_\infty(\Omega)}$$

$$\leq C \ln(1+\max_{D\in\mathcal{T}_h} \tau_D) h^\ell |u|_{H^{\ell+1}(\Omega)}$$

Extensions to 3D

Local Virtual Element Spaces

D is a bounded polyhedron.

$\mathcal{F}_D$ is the set of the faces of D.

$\mathcal{E}_F$ is the set of the edges of the face F.

$\mathcal{Q}^k(F)$ is the virtual element space on F.

$$\mathcal{Q}^k(\partial D) = \{v \in C(\partial D) : v\big|_F \in \mathcal{Q}^k(F) \quad \forall\, F \in \mathcal{F}_D\}$$

Local Virtual Element Spaces

$\Pi^{\nabla}_{k,D}$ is the projection from $H^1(D)$ onto $\mathbb{P}_k(D)$ with respect to the inner product

$$(\!(\zeta,\eta)\!) = \int_D \nabla\zeta\cdot\nabla\eta\,dx + \Big(\int_{\partial D}\zeta\,dS\Big)\Big(\int_{\partial D}\eta\,dS\Big)$$

or equivalently,

$$\int_D \nabla(\Pi^{\nabla}_{k,D}\zeta)\cdot\nabla q\,dx = \int_D \nabla\zeta\cdot\nabla q\,dx \qquad \forall\, q\in\mathbb{P}_k(D)$$

$$\int_{\partial D}\Pi^{\nabla}_{k,D}\zeta\,dS = \int_{\partial D}\zeta\,dS$$

Local Virtual Element Spaces

$\Pi^{\nabla}_{k,D}$ is the projection from $H^1(D)$ onto $\mathbb{P}_k(D)$ with respect to the inner product

$$(\!(\zeta,\eta)\!) = \int_D \nabla\zeta\cdot\nabla\eta\,dx + \Big(\int_{\partial D}\zeta\,dS\Big)\Big(\int_{\partial D}\eta\,dS\Big)$$

$\Pi^{0}_{k,D}$ is the projection from $L_2(D)$ onto $\mathbb{P}_k(D)$.

Local Virtual Element Spaces

$\Pi^{\nabla}_{k,D}$ is the projection from $H^1(D)$ onto $\mathbb{P}_k(D)$ with respect to the inner product

$$(\!(\zeta,\eta)\!) = \int_D \nabla\zeta\cdot\nabla\eta\,dx + \Big(\int_{\partial D}\zeta\,dS\Big)\Big(\int_{\partial D}\eta\,dS\Big)$$

$\Pi^{0}_{k,D}$ is the projection from $L_2(D)$ onto $\mathbb{P}_k(D)$.

Virtual Element Space $\mathcal{Q}^k(D)$ $\quad (k \geq 1)$

$v \in H^1(D)$ belongs to $\mathcal{Q}^k(D)$ if and only if

- The trace of v on ∂D belongs to $\mathcal{Q}^k(\partial D)$.
- The distribution Δv belongs to $\mathbb{P}_k(D)$.
- $\Pi^0_{k,D}v - \Pi^{\nabla}_{k,D}v \in \mathbb{P}_{k-2}(D)$

Local Virtual Element Spaces

Properties of $v \in \mathcal{Q}^k(D)$

- v is uniquely determined by $v\big|_{\partial D}$ and $\Pi^0_{k-2,D} v$.

Local Virtual Element Spaces

Properties of $v \in \mathcal{Q}^k(D)$

- v is uniquely determined by $v\big|_{\partial D}$ and $\Pi^0_{k-2,D} v$.
- The dofs of $\mathcal{Q}^k(D)$ consist of
  - The values of v at the vertices of D and nodes on the interior of each edge of D that determine a polynomial of degree k on the edge.
  - The moments of $\Pi^0_{k-2,F} v$ on each face F of D.
  - The moments of $\Pi^0_{k-2,D} v$ on D.

Local Virtual Element Spaces

Properties of $v \in \mathcal{Q}^k(D)$

- v is uniquely determined by $v\big|_{\partial D}$ and $\Pi^0_{k-2,D}v$.
- The dofs of $\mathcal{Q}^k(D)$ consist of
  - The values of v at the vertices of D and nodes on the interior of each edge of D that determine a polynomial of degree k on the edge.
  - The moments of $\Pi^0_{k-2,F}v$ on each face F of D.
  - The moments of $\Pi^0_{k-2,D}v$ on D.
- $\Pi^{\nabla}_{k,D}v$ and $\Pi^0_{k,D}v$ are computable.

because $\Pi^0_{k,F}v$ is available

Local Virtual Element Spaces

Properties of $v \in \mathcal{Q}^k(D)$

- v is uniquely determined by $v\big|_{\partial D}$ and $\Pi^0_{k-2,D}v$.
- The dofs of $\mathcal{Q}^k(D)$ consist of
  - The values of v at the vertices of D and nodes on the interior of each edge of D that determine a polynomial of degree k on the edge.
  - The moments of $\Pi^0_{k-2,F}v$ on each face F of D.
  - The moments of $\Pi^0_{k-2,D}v$ on D.
- $\Pi^\nabla_{k,D}v$ and $\Pi^0_{k,D}v$ are computable.
- v is continuous on $\bar{D}$.

Virtual Element Methods

$\mathcal{T}_h$ is a partition of Ω into polyhedral subdomains.

$\mathcal{F}_h$ is the set of all the faces of the subdomains of $\mathcal{T}_h$.

$$\mathcal{Q}_h^k = \{v \in H_0^1(\Omega) : \, v\big|_D \in \mathcal{Q}^k(D) \quad \forall\, D \in \mathcal{T}_h\}$$

Virtual Element Methods

Find $u_h \in \mathcal{Q}_h^k$ such that

$$a_h(u_h, v) = (f, \Xi_h v) \qquad \forall\, v \in \mathcal{Q}_h^k$$

where

$$a_h(w, v) = \sum_{D \in \mathcal{T}_h} \Big[a^D(\Pi_{k,D}^{\nabla} w, \Pi_{k,D}^{\nabla} v) + S^D(w - \Pi_{k,D}^{\nabla} w, v - \Pi_{k,D}^{\nabla} v) \Big]$$

$$a^D(w, v) = \int_D \nabla w \cdot \nabla v \, dx$$

and

$$\Xi_h = \begin{cases} \Pi_{1,h}^0 & \text{if } k = 1, 2, \\ \Pi_{k-2,h}^0 & \text{if } k \geq 3. \end{cases}$$

Virtual Element Methods

Stabilization bilinear form

$$S^D(w,v) = h_D \sum_{F\in\mathcal{F}_D} \Big[h_F^{-2} (\Pi^0_{k-2,F} w, \Pi^0_{k-2,F} v)_{L_2(F)} + \sum_{p\in\mathcal{N}_{\partial F}} w(p)v(p) \Big]$$

where $\mathcal{N}_{\partial F}$ is the set of the nodes along ∂F associated with the degrees of freedom of a virtual element function.

Virtual Element Methods

Stabilization bilinear form

$$S^D(w,v) = h_D \sum_{F\in\mathcal{F}_D} \Big[h_F^{-2}(\Pi^0_{k-2,F}w, \Pi^0_{k-2,F}v)_{L_2(F)} + \sum_{p\in\mathcal{N}_{\partial F}} w(p)v(p)\Big]$$

where $\mathcal{N}_{\partial F}$ is the set of the nodes along ∂F associated with the degrees of freedom of a virtual element function.

This stabilization is equivalent to the one in the 2013 paper by Ahmad-Alsaedi-Brezzi-Marini-Russo.

Local Shape Regularity Assumptions

The (open) polyhedron D is star-shaped with respect to a ball $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

Local Shape Regularity Assumptions

The (open) polyhedron D is star-shaped with respect to a ball $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

All the consequences of the star-shaped assumption in 2D (Sobolev inequalities, Bramble-Hilbert estimates, Poincaré-Friedrichs inequalities, etc.) also hold in 3D after adjusting for the difference in dimensions.

Local Shape Regularity Assumptions

The (open) polyhedron D is star-shaped with respect to a ball $\mathfrak{B}_D \subset D$ with radius $\rho_D h_D$, where h_D is the diameter of D.

All the consequences of the star-shaped assumption in 2D (Sobolev inequalities, Bramble-Hilbert estimates, Poincaré-Friedrichs inequalities, etc.) also hold in 3D after adjusting for the difference in dimensions.

Each face F of D is star-shaped with respect to a disc with radius $\rho_F h_F$, where h_F is the diameter of F.

Global Shape Regularity Assumptions

Assumption 1 There exists a positive number $\rho \in (0,1)$, independent of h, such that

$$\rho_{_D} \geq \rho \qquad \forall\, D \in \mathcal{T}_h$$

$$\rho_{_F} \geq \rho \qquad \forall\, F \in \mathcal{F}_h$$

Global Shape Regularity Assumptions

Assumption 1 There exists a positive number $\rho \in (0,1)$, independent of h, such that

$$\rho_{_D} \geq \rho \qquad \forall\, D \in \mathcal{T}_h$$

$$\rho_{_F} \geq \rho \qquad \forall\, F \in \mathcal{F}_h$$

Assumption 2 There exists a positive integer N, independent of h, such that

$$|\mathcal{F}_{_D}| \leq N \qquad \forall\, D \in \mathcal{T}_h$$

$$|\mathcal{E}_{_F}| \leq N \qquad \forall\, F \in \mathcal{F}_h$$

Error Estimates

Theorem Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\|u-u_h\|_h \leq C\sqrt{\beta_h}h^{\ell}|u|_{H^{\ell+1}(\Omega)}$$

where the positive constant C depends only on k, ρ and N, and

$$\beta_h = \ln(1+\max_{F\in\mathcal{F}_h}\tau_F)$$

$$\tau_F = \frac{\max_{e\in\mathcal{E}_F} h_e}{\min_{e\in\mathcal{E}_F} h_e}$$

$$\|v\|_h^2 = a_h(v,v)$$

Error Estimates

Theorem Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\|u - u_h\|_h \leq C\sqrt{\beta_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$

where the positive constant C depends only on k, ρ and N, and

$$\beta_h = \ln(1 + \max_{F \in \mathcal{F}_h} \tau_F)$$

$$\tau_F = \frac{\max_{e \in \mathcal{E}_F} h_e}{\min_{e \in \mathcal{E}_F} h_e}$$

Remark The existence of small faces does not necessarily affect the performance of the virtual element method. It is the relative sizes of the edges on each face that matter.

Error Estimates

Theorem Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\|u - u_h\|_h \le C\sqrt{\beta_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$

Corollary

$$|u - \Pi^{\nabla}_{k,h} u_h|_{h,1} \le C\sqrt{\beta_h} h^\ell |u|_{H^{\ell+1}(\Omega)}$$

$$|u - \Pi^{0}_{k,h} u_h|_{h,1} \le C\beta_h h^\ell |u|_{H^{\ell+1}(\Omega)}$$

Error Estimates

Theorem Assuming the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, there exists a positive constant C, depending only on k, ρ and N, such that

$$\|u-u_h\|_{L_2(\Omega)}+\|u-\Pi^0_{k,h}u_h\|_{L_2(\Omega)}+\|u-\Pi^\nabla_{k,h}u_h\|_{L_2(\Omega)} \leq C\beta_h h^{\ell+1}|u|_{H^{\ell+1}(\Omega)}$$

Error Estimates

Definition The partition $\mathcal{T}_h$ is quasi-uniform if there exists a positive constant γ independent of h such that

$$h_D \geq \gamma h \qquad \forall\, D \in \mathcal{T}_h.$$

Theorem Assuming $\mathcal{T}_h$ is quasi-uniform and the solution u belongs to $H^{\ell+1}(\Omega)$ for some ℓ between 1 and k, we have

$$\max_{e\in\mathcal{E}_h} \|u - u_h\|_{L_\infty(e)} + \|u - \Pi^{\nabla}_{k,h} u_h\|_{L_\infty(\Omega)} + \|u - \Pi^{0}_{k,h} u_h\|_{L_\infty(\Omega)} \\ \leq C\beta_h h^{\ell-(1/2)} |u|_{H^{\ell+1}(\Omega)}$$

where $\mathcal{E}_h$ is the set of the edges of the faces of $\mathcal{T}_h$, and the positive constant C only depends on k, ρ, N and γ.

Error Estimates

The proofs are similar to the 2D case. But we also need additional 2D estimates such as

$$\inf_{q\in\mathbb{P}_\ell} |\zeta - q|_{H^m(D)} \lesssim h_D^{\ell+\frac{1}{2}-m} |\zeta|_{H^{\ell+\frac{1}{2}}(D)} \qquad \forall\,\zeta \in H^{\ell+\frac{1}{2}}(D)$$

$$h_D |\zeta|^2_{H^1(e)} \lesssim |\zeta|^2_{H^1(D)} + h_D |\zeta|^2_{H^{3/2}(D)} \qquad \forall\,\zeta \in H^{3/2}(D)$$

$$|\zeta - I_{k,D}\zeta|_{H^1(D)} \lesssim h_D^{\ell-\frac{1}{2}} |\zeta|_{H^{\ell+\frac{1}{2}}(D)} \qquad \forall\,\zeta \in H^{\ell+\frac{1}{2}}(D)$$

$$\|\zeta - I_{k,D}\zeta\|_{L_2(D)} \lesssim h_D^{\ell+\frac{1}{2}} |\zeta|_{H^{\ell+\frac{1}{2}}(D)} \qquad \forall\,\zeta \in H^{\ell+\frac{1}{2}}(D)$$

$$\|\zeta - I_{k,D}\zeta\|_{L_\infty(D)} \lesssim C\,h_D^{\ell-\frac{1}{2}} |\zeta|_{H^{\ell+\frac{1}{2}}(D)} \qquad \forall\,\zeta \in H^{\ell+\frac{1}{2}}(D)$$

Concluding Remarks

We have demonstrated that it is possible to analyze virtual element methods on polytopal meshes with small edges/faces by using well-known estimates (Sobolev's inequality, Poincaré-Friedrichs inequality, Bramble-Hilbert estimates, trace inequalities, etc.) that can be controlled by the star-shaped assumption.

We have demonstrated that it is possible to analyze virtual element methods on polytopal meshes with small edges/faces by using well-known estimates (Sobolev's inequality, Poincaré-Friedrichs inequality, Bramble-Hilbert estimates, trace inequalities, etc.) that can be controlled by the star-shaped assumption.

The projection $\Pi^{\nabla}_{k,D} : H^1(D) \longrightarrow \mathbb{P}_k(D) \ (k \geq 2)$ can also be defined with respect to the inner product

$$((\zeta, \eta)) = \int_D \nabla\zeta \cdot \nabla\eta \, dx + \Big(\int_D \zeta \, dx \Big) \Big(\int_D \eta \, dx \Big)$$

and our analysis can be extended to this choice of $\Pi^{\nabla}_{k,D}$.

We have demonstrated that it is possible to analyze virtual element methods on polytopal meshes with small edges/faces by using well-known estimates (Sobolev's inequality, Poincaré-Friedrichs inequality, Bramble-Hilbert estimates, trace inequalities, etc.) that can be controlled by the star-shaped assumption.

The projection $\Pi^{\nabla}_{k,D} : H^1(D) \longrightarrow \mathbb{P}_k(D) \ (k \geq 2)$ can also be defined with respect to the inner product

$$((\zeta, \eta)) = \int_D \nabla\zeta \cdot \nabla\eta \, dx + \Big(\int_D \zeta \, dx\Big)\Big(\int_D \eta \, dx\Big)$$

and our analysis can be extended to this choice of $\Pi^{\nabla}_{k,D}$.

The extension of our analysis to stabilizations that involve degrees of freedom inside the domain does not pose additional difficulties.

Virtual element methods in 2D (allowing small edges) satisfy optimal error estimates under the stabilization

$$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$$

and nearly optimal (up to a log factor) error estimates under the stabilization

$$S^D(w,v) = \sum_{p \in \mathcal{N}_{\partial D}} w(p)v(p)$$

Virtual element methods in 2D (allowing small edges) satisfy optimal error estimates under the stabilization

$$S^D(w,v) = h_D(\partial w/\partial s, \partial v/\partial s)_{L_2(\partial D)}$$

and nearly optimal (up to a log factor) error estimates under the stabilization

$$S^D(w,v) = \sum_{p\in\mathcal{N}_{\partial D}} w(p)v(p)$$

Virtual element methods in 3D (allowing small edges and faces) satisfy nearly optimal (up to a log factor) error estimates under the stabilization

$$S^D(w,v) = h_D \sum_{F\in\mathcal{F}_D} \Big[h_F^{-2}(\Pi^0_{k-2,F}w, \Pi^0_{k-2,F}v)_{L_2(F)} + \sum_{p\in\mathcal{N}_{\partial F}} w(p)v(p)\Big]$$

A Conjecture

Virtual element methods in 3D (allowing small edges and faces) satisfy optimal error estimates under the stablization

$$S^D(v,w) = \sum_{F\in\mathcal{F}_D} h_F(\nabla_F \Pi^{\nabla}_{k,F} v, \nabla_F \Pi^{\nabla}_{k,F} w)_{L_2(F)}$$

$$+ \sum_{F\in\mathcal{F}_D} h_F \sum_{e\in\mathcal{E}_F} h_e\big(\partial(v-\Pi^{\nabla}_{k,F} v)/\partial s, \partial(w-\Pi^{\nabla}_{k,F} w)/\partial s\big)_{L_2(e)}$$

We have obtained L_∞ error estimates under the assumption that u belongs to $H^{\ell+1}(\Omega)$.

Error estimates under the assumption that u belongs to $W^\ell_\infty(\Omega)$ (*á la* Scott, Natterer, Rannacher, Schatz, Walhbin, ...) remain an open problem.

Acknowledgement

NSF