# Introduction to Abelian Varieties

Joao Pedro dos Santos

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## Programme

1) Preliminary material on complex analytic manifolds.
2) Invariant vector fields (and the Lie algebra).
3) Examples: Complex tori.
4) Quotients by a properly discontinuous action.
5) Preliminaries on algebraic geometry.
6) Beginning of the theory of algebraic group varieties: Invariant vector fields and differentials.
7) Line bundles and cohomology groups.
8) Line bundles on a complex torus.
9) Riemann forms and the Appell-Humbert theorem.
10) Fundamental results in the theory of line bundles over a complex torus.
11) The dual of a complex torus.
12) Further topics.

### Objective of the notes

These are hastily typed notes to support the lectures I deliver in the *International school on algebraic geometry and algebraic groups*, which takes place in November 2022 in Hanoi, Vietnam. *I shall make modifications as I see fit*, hoping to adapt the content to the needs of students. So please, take note of the version which you're reading.

The course was structured as follows: Five lectures of one hour and 40 minutes. In order to help the student absorb the material in due time, I've divided the notes by lectures, representing roughly what was covered in .

Finally, I'd like to explain what my motivations are. The theory of abelian varieties was responsible for the development of much mathematics and I used this theme to introduce the student to what I believe is a next step in his education.

## Prerequisites

One of the main tools for our study is the theory of complex manifolds. But the reader having a solid understanding of the case of smooth manifolds, such as the one presented in [Ch, Ch. III], will have absolutely no difficulty: the novelties being the analytic continuation principle and the maximum principle which are usually well-understood from course in complex analysis of one variable.

I assume as that the reader has met the fundamental group and the principle of lifting of paths, homotopies, maps, etc. This is well explained in [Gre, Chapters 5 and 6], for example.

I shall also require a certain acquaintance with the theory of *sheaves*. Although a brief revision is made, I don't think that it is enough to learn these by solely the amount of knowledge I present. Some gymnastics necessary and the reader having not met sheaves before is advised to do a supplementary effort by studying, say, [GH, Chapter 0, Section 3].

Finally, comes algebraic geometry. Here, I shall also require a certain contact with the theory explained in [MRB, Ch. I]. I highly recommend [Sh, Ch. I] as well. On the other hand, [Sh] makes little explicit use of the theory of sheaves.

# Lecture 1

(07/11/2022).

# 1 First steps in the theory of Complex Lie groups

## 1.1 Preliminaries on complex manifolds

Let $\mathbb{K}$ stand for $\mathbb{R}$ or $\mathbb{C}$. Let $X$ be a $\mathbb{K}$-analytic manifold. For each $p \in X$, denote by $\mathcal{O}_p$ the ring of germs* of analytic functions to $\mathbb{K}$ near $p$. Following [Ch, Ch. III, §IV], we call a tangent vector at $p$, any $\mathbb{K}$-linear map $v : \mathcal{O}_p \to \mathbb{K}$ satisfying Leibniz's rule:

$$v(fg) = f(p)v(g) + g(p)v(f). \tag{1.1}$$

Clearly, the set of tangent vectors is a $\mathbb{K}$-vector space, which is denoted by $T_pX$.

Let $\mathbf{z} : U \to \mathbb{K}^n$ be a local chart near $p$. For any open neigh. $V \subset U$ of $p$ and any $f \in \mathcal{O}(V)$, let us agree to call $f_{\mathbf{z}}$ the analytic function $\mathbf{z}(V) \to \mathbb{C}$ defined by $f \circ \mathbf{z}^{-1}$. Then, defining

$$\partial_{z_j}(p) : f \longmapsto \frac{\partial f_{\mathbf{z}}}{\partial z_i}(\mathbf{z}(p))$$

we obtain tangent vectors which form a basis of $T_pX$.

A *tangent field* on $X$ is a function

$$v : X \longrightarrow \bigsqcup_p T_pX$$

such that $v(p) \in T_pX$ for all $p$. Hence, given a tangent field $v$ on an open subset $U$ and a function $f : U \to \mathbb{K}$, we obtain a new function $v(f)$, *the directional derivative of* $f$, defined by $p \mapsto v(p)(f)$.

A tangent field is analytic if for every open subset $U$ and every $f \in \mathcal{O}(U)$, the directional derivative $v(f)$ is analytic. The vector space of analytic vector fields on $U$ is denoted by $\mathcal{T}_X(U)$.

**Example 1.1.** Let $\partial_i$ be the vector field on $\mathbb{C}^n$ which to every $f \in \mathcal{O}_{\mathbb{C}^n,p}$ associates $\dfrac{\partial f}{\partial z_i}(p) \in \mathbb{C}$. Show that a vector field is analytic if and only if it can be written in the form $\displaystyle\sum_{i=1}^n a_i\,\partial_i$, with $a_i$ analytic functions.

Let $\varphi : X \to Y$ be an analytic map. Then, we define its derivative

$$\mathrm{D}_p\varphi : T_pX \longrightarrow T_{\varphi(p)}Y$$

*This is the ring $\varinjlim_U \mathcal{O}(U)$, where $U$ ranges over the set of open neighbourhoods of $p$. The direct limit in this formula is one of those concepts that usually never has the time to appear in a lecture course. On the other hand, such a process is quite common in geometry and algebra, and the reader is urged to go through [AM, Ch. 2, Exercise 14], or [DF, Section 7.6, Exercise 8], where the notion is treated.

at a point $p \in X$ as follows. For $v \in T_pX$, let

$$\mathrm{D}_p\varphi \cdot v : \mathcal{O}_{\varphi p} \longrightarrow \mathbb{C}$$

send $g \in \mathcal{O}_{\varphi(p)}$ to $v(g \circ \varphi)$.

## 1.2 Complex Lie groups and their vector fields

A complex (analytic) Lie group is a complex manifold $G$ which is also a group, and for which the operations of multiplication $G \times G \to G$ and inversion $(-)^{-1} : G \to G$ are analytic.

The theory of complex Lie groups is a vast edifice and at each stage, various very particular properties appear. In these lectures, at first, we shall concentrate on the case of *compact* complex Lie groups, *or complex tori.*

Let $G$ be a complex Lie group; for each $\sigma \in G$, let

$$\tau_\sigma : G \longrightarrow G$$

be the analytic diffeomorphism defined by $g \mapsto \sigma g$. We say that a vector field $v$ on $G$ is left-invariant if

$$\mathrm{D}_g\tau_\sigma(v(g)) = v(\sigma g) \qquad \forall \sigma \in G.$$

It is then clear that a left-invariant vector field $v$ is determined simply by its value at $e \in G$ since $v(g) = \mathrm{D}_e\tau_g(v(e))$. This being so, for any given $v \in T_eG$, we define

$$v^\natural(g) := \mathrm{D}_e\tau_g \cdot v.$$

The reader should verify that $v^\natural$ is left-invariant.

**Proposition 1.2.** *The vector field $v^\natural$ is analytic.*

*Proof.* We show that $v^\natural$ is analytic near $e$. Let $D \subset \mathbb{C}^n$ be an open polydisk about $\mathbf{0}$ and let $\mathbf{z} : U \to D$ be a system of coordinates near $e$ such that $\mathbf{z}(e) = \mathbf{0}$. To ease notation, let $\varphi := \mathbf{z}^{-1}$. Let $v = \sum_k \lambda_k \cdot \partial_{z_k}(e)$ with $\lambda_k \in \mathbb{C}$. Note $v(f) = \sum_k \lambda_k \cdot \partial_k(f_{\mathbf{z}})$, where $f_{\mathbf{z}}$ is the local expression of $f$.

Since multiplication is analytic, there exists $V \ni e$ such that $V \cdot V \subset U$ and we have

$$z_j(gh) = M_j(\mathbf{z}(g)\,,\, \mathbf{z}(h))$$

where $M_j$ is analytic on $\mathbf{z}(V) \times \mathbf{z}(V)$. Fix $a \in D$. By definition,

$$v^\natural(\varphi(a)) : z_j \longmapsto \underbrace{v(z_j \circ \tau_{\varphi(a)})}_{\in\mathbb{C}}.$$

The function $z_j \circ \tau_{\varphi(a)} : V \to \mathbb{C}$ is $g \mapsto M_j(a, \mathbf{z}(g))$, so that its local expression is $b \mapsto M_j(a, b)$. Hence

$$v(z_j \circ \tau_{\varphi(a)}) = \sum_k \lambda_k M_j^{(k)}(a, 0)$$

where $M_j^{(k)}$ is the derivative of $M_j$ with respect to the $(n+k)$th coordinate. This being an analytic function of $a$ shows that $v^\natural$ is analytic. $\square$

An important Corollary of this result is the following.

**Corollary 1.3.** *Let $G$ be of dimension $n$. Then, there exist invariant vector fields $v_1, \dots, v_n \in \mathcal{T}(G)$ such that*

$$T_pG = \mathrm{Span}_{\mathbb{C}}\{v_1(p), \dots, v_n(p)\}$$

*for each $p$.*

Once this result has been showed, we may extend it considerably to other objects associated to tangent vectors: cotangent vectors and exterior differential forms. This is just a matter of giving proper notations.

Let $\Omega_p^m G$ stand for the space of alternating linear forms $\underbrace{T_pG \times \cdots \times T_pG}_{m} \to \mathbb{C}$. We leave to the reader the task of giving a reasonable definition of an analytic $m$-form on an open $U \subset G$ (or consult [Ch, p.147]).

**Definition 1.4.** An analytic $m$-form $\omega$ on $G$ is invariant if $\tau_g^*\omega = \omega$ for all $g \in G$.

We then have

**Proposition 1.5.** *Let $b = \binom{n}{m}$. There exist invariant global $m$–forms $\omega_1, \dots, \omega_b$ on $G$ such that*

$$\Omega_p^m G = \mathrm{span}_{\mathbb{C}}\{\omega_1(p), \dots, \omega_b(p)\}.$$

Using the language of sheaves of $\mathcal{O}_G$–modules, this can be translated in a rather interesting way. Let

$$\mathcal{T}_G(U) = \{\mathcal{O}_G(U)\text{–module of analytic vector fields}\}.$$

**Corollary 1.6.** *Let $v_1, \dots, v_n \in \mathcal{T}(G)$ be as in Cor. 1.3. For each open subset $U$, let*

$$\varphi(U) : \mathcal{O}(U)^r \longrightarrow \mathcal{T}_G(U)$$

*be defined by*

$$(f_1, \dots, f_n) \longmapsto f_1 v_1|_U + \cdots + f_n v_n|_U.$$

*Then $\varphi$ defines an isomorphism of $\mathcal{O}_G$-modules.*

We move on on our theory of complex Lie groups. Examples such as $\mathbf{GL}_n(\mathbb{C})$, $\mathbf{SL}_n(\mathbb{C})$, etc are easily spotted and we inquire if there exist *compact* complex Lie groups. The most famous examples are complex (one dimensional) tori, or elliptic curves.

*Remark* 1.7. Complex manifolds possessing global vector fields giving a basis of the associated tangent spaces allover are called “parallelizable”. A classical Theorem of [Wa] says that any parallelizable compact complex manifold is of the form $G/\Gamma$, where $G$ is a complex Lie group.

*Remark* 1.8. The analogues of the above stated results hold in the real analytic case (the case of Lie groups) with exactly the same proofs. From Corollary 1.3 it then follows immediately from the famous “hairy ball theorem” [Mil] that no *even* dimensional sphere can be a Lie group. Using your knowledge of quaternions, try to say something about $S^3$!

## 1.3 Dimension one: Elliptic curves

**Definition 1.9.** A lattice in $\mathbb{C}$ is an abelian subgroup of $\mathbb{C}$ of the form $\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$, where $\gamma_1$ and $\gamma_2$ are linearly independent over $\mathbb{R}$.

Given a lattice $\Gamma = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ in $\mathbb{C}$, let $G = \mathbb{C}/\Gamma$ be the quotient group and $\rho : \mathbb{C} \to G$ the map which assigns to $z \in \mathbb{C}$ its orbit under $\Gamma$. We now endow $G$ with a topology, namely, the quotient topology: $V \subset G$ is open if and only if $\rho^{-1}(V)$ is open.

**Exercise 1.10.** Show that $G$ is homeomorphic to $S^1 \times S^1$.

Now, we shall give $G$ the structure of a complex manifold. Let

$$V = \{t_1\gamma_1 + t_2\gamma_2 \,:\, 0 < t_1, t_2 < 1\}.$$

Clearly, no two distinct points of $V$ belong to the same orbit of $\Gamma$, so that $\rho : V \to \rho(V)$ is bijective.

In the same vein, for each $a \in \mathbb{C}$, the set $V_a := a + V$ also has the property that no two distinct points belong to the same orbit so that $\rho|_{V_a} : V_a \to \rho(V_a)$ is bijective. Moreover,

$$\rho^{-1}(\rho(V_a)) = \bigsqcup_{\gamma\in\Gamma} \gamma + V_a$$

$\Rightarrow \rho(V_a)$ is open and $\rho|_{V_a} : V_a \to \rho(V_a)$ is a homeomorphism.

**Lemma 1.11.** *The data $\{\rho : V_a \to \rho(V_a) \,:\, a \in \mathbb{C}\}$ defines on $G$ a complex analytic atlas.*

*Proof.* Write $U_a := \rho(V_a)$ and $\sigma_a : U_a \to V_a$ for the inverse of $\rho|_{V_a}$. For each $z \in \sigma_a(U_a \cap U_b)$, the element $\sigma_b\sigma_a^{-1}(z)$ lies in $z + \Gamma$; let us write it as

$$\sigma_b\sigma_a^{-1}(z) = z + \gamma_{ba}(z).$$

Hence, $\sigma_b\sigma_a^{-1} - \mathrm{id}$ is a function which takes values in $\Gamma$ and is continuous. Hence, $\sigma_b\sigma_a^{-1} - \mathrm{id}$ is constant on each connected component of $\sigma_a(U_a \cap U_b)$ and in particular, holomorphic. (Note that $\sigma_a(U_a \cap U_b)$ is an open subset of $\mathbb{C}$ and hence connected components are also open.) □

Complex manifolds such as $G$ are called elliptic curves and it is not hard to see that $G$ is a complex Lie group. The reason for the name *elliptic curve* may be mysterious, but this follows from the theory of the Weierstrass $\wp$-function [Ah, eq. (15), p. 276].

## 1.4 Complex analytic tori and complex manifolds obtained from properly discontinuous actions

The example of an elliptic curve is easily generalized to higher dimensions.

**Definition 1.12.** A subgroup $\Gamma$ of $\mathbb{C}^n$ a lattice if, as a subgroup, it is generated by elements $\{\gamma_i\}_{i=1}^{2n}$ which are $\mathbb{R}$-linearly independent.

Working as before, we see that $G = \mathbb{C}^n/\Gamma$ is a complex Lie group, which is homeomorphic to $\underbrace{S^1 \times \cdots \times S^1}_{2n}$. A complex analytic group as such as $G$ is a *complex analytic torus*; their study occupies a large and central part of what I want to teach you.

Now, the method of proof of Lemma 1.11 carried behind a very useful technique, which I wish to discuss right away, which is the method of looking at orbit spaces. Let then $\Gamma$ be a group which acts (on the left of) a topological space $Y$ by homeomorphisms.

**Definition 1.13.** The action of $\Gamma$ on $Y$ is properly discontinuous if for each $y \in Y$, there exists an open neighbourhood $U \ni y$ such that $\gamma(U) \cap U \neq \varnothing$ implies $\gamma = e$. It is convenient to call the aforementioned open neighbourhood a *distinguished neighbourhood.*

Let $X$ be the quotient set, i.e. the set of all $\Gamma$-orbits in $Y$. Let

$$\rho : Y \longrightarrow X$$

be the natural projection associating to $y \in Y$ its orbit $Gy$; obviously $\rho$ is a surjection. We then give $X$ the *quotient topology*: $U \subset X$ is open if and only if $\rho^{-1}(U)$ is open. It is not difficult so see that $\rho$ is then an open map. We shall say that $U \subset X$ is distinguished if it is the image of a distinguished neighbourhood $V$. In this case,

$$\rho^{-1}(U) = \bigsqcup_{\gamma \in \Gamma} \gamma(V).$$

Now, we suppose that $Y$ is a complex manifold. Let $V_1$ and $V_2$ be distinguished open subsets of $Y$ and let $U_i = \rho(V_i)$. Put $\sigma_i = (\rho|_{U_i})^{-1} : U_i \to V_i$. Let me write $U_{12} = U_1 \cap U_2$ and adopt the notations introduced in the diagram

$$\begin{tikzcd} \sigma_1(U_{12}) \arrow[r, "\sigma_2 \circ (\rho|_{V_1}) = \xi"] \arrow[d, hook] & \sigma_2(U_{12}) \arrow[d, hook] \\ V_1 & V_2. \end{tikzcd}$$

Note that $\rho\xi(y) = \rho(y)$ and hence, for each $y \in \sigma_1(U_{12})$, there exists a certain $\gamma_y \in \Gamma$ such that $\xi(y) = \gamma_y \cdot y$.

I contend that $y \mapsto \gamma_y$ is a *locally constant function* no $\sigma_1(U_{12})$. Let $q \in \sigma_1(U_{12})$ and let $\mathcal{V} \ni q$ be such that $\gamma_q(\mathcal{V}) \subset \sigma_2(U_{12})$. For $y \in \mathcal{V}$, we have $\gamma_q(y) \in V_2$ (by construction of $\mathcal{V}$) and $\gamma_y(y) = \xi(y) \in V_2$. Therefore, $\gamma_y = \gamma_q$ because $V_2$ only contains one point in each orbit$^\dagger$. Hence, $\sigma_1(U_{12})$ is a union of open sets where $\xi$ is

$^\dagger$The argument here is much clearer if we introduce the notion of principal $\Gamma$-bundle.

simply the restriction of a certain element of $\Gamma$. Let us say this in other words: we obtain a locally constant map

$$\gamma_{21} : U_{12} \longrightarrow \Gamma.$$

Using the structure of complex manifold on $Y$, it becomes a simple matter to show that $X$ carries a structure of complex manifold such that $\rho$ is a local analytic diffeomorphism.

**Lemma 1.14.** *Suppose now that $U_1 \cap U_2$ is connected. Then $\gamma_{21} \in \Gamma$ is the* ***unique*** *element of $\Gamma$ such that*

$$\gamma_{21}(V_1) \cap V_2 \neq \varnothing.$$

*Proof.* Let $\tilde{\gamma} \neq \gamma$ and suppose that $\tilde{\gamma}(y_1) \in V_2$ with $y_1 \in V_1$. Then $\rho\tilde{\gamma}(y_1) = \rho(y_1) \in U_1$. Also, $\rho\tilde{\gamma}(y_1) \in \rho(V_2)$ and $x := \rho(y_1) \in U_1 \cap U_2$. But then $y_1 \in \sigma_1(U_1 \cap U_2)$ because $y_1 \in V_1$ and $\rho(y_1) \in U_1 \cap U_2$. Then, $\tilde{\gamma}(y_1) \in V_2$ and $\gamma_{21}(y_1) \in V_2$ by construction of $\gamma_{21}$. This is contradictory. $\square$

Let me end this section by recalling a central result in the theory of manifolds.

**Definition 1.15** ([BT, p.43])**.** A covering $\{U_i\}_{i \in I}$ of $X$ by open set is called *good* (too bad...) if for each $i_0, \ldots, i_p \in I$, the intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ is either empty or contractible.

**Theorem 1.16** (Whitehead)**.** *Let $\{U_i\}$ be a covering of a manifold $X$. Then $\{U_i\}$ has a refinement $\{V_j\}_{j \in J}$ which is good.*

The proof uses the notion of "convex" neighbourhoods in Riemannian geometry and can be found in [DC, 3.4].

# Lecture 2

(08 /11/2022).

## 2 First steps in the algebraic theory of group varieties

We are now presented with the task of rendering algebraic what was discussed previously. For that I shall rely on the theory of algebraic varieties as discussed by a master [MRB]. I am certainly convinced that *reading the following sections is not enough* to get started in this beautiful subject; I just hope that the student here will turn to [MRB, Ch. 1].

### 2.1 Irreducible algebraic sets over an algebraically closed field

Let $k$ be an algebraically closed field and $k[\boldsymbol{t}] = k[t_1, \ldots, t_n]$ be the ring of polynomials in $n$ variables. For each ideal $I \subset k[\boldsymbol{t}]$, we define its vanishing locus as

$$\mathcal{V}(I) = \{x \in k^n \,:\, f(x) = 0 \text{ for all } f \in I\}.$$

On the other direction, for a given subset $S \subset k^n$, we define its associated ideal as

$$\mathcal{I}(S) := \{f \in k[\boldsymbol{t}] \,:\, f(x) = 0 \text{ for all } x \in k^n\}.$$

Clearly, $\mathcal{V}(\mathcal{I}(S)) = S$. The relation between $I$ and $\mathcal{I}(\mathcal{V}(I))$ is more intricate.

**Theorem 2.1** (Hilbert's Nullstellensatz, [MRB, p. 5] ). *$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$.* □

We now move on to topology. The Zariski topology on $k^n$ is the topology whose closed sets are algebraic sets. In this manner, *each algebraic set $S$ inherits a topology from $k^n$*: closed sets of $S$ are intersections of closed sets from $k^n$. This topology is quite weak. So, the following definition becomes quickly important.

**Definition 2.2.** A topological space $X$ is irreducible if and only if whenever $X = C_1 \cup C_2$ with $C_1$ and $C_2$ closed, one necessarily must have either $C_1 = X$ or $C_2 = X$.

Using the Nullstellensatz, it is a simple matter to show that an algebraic set $S$ is irreducible if and only if $\mathcal{I}(S)$ is *prime*, cf [MRB, p. 7].

Let $S \subset k^n$ be an irreducible algebraic set. It then follows that

$$k[\boldsymbol{t}]/\mathcal{I}(S) \xrightarrow{\sim} \left\{ f : S \to k \,:\, \begin{array}{c} f \text{ is the restriction} \\ \text{of some } F \in k[\boldsymbol{t}] \end{array} \right\}.$$

The ring on the right-hand-side is called *the ring of $S$*; let us denote it by $\mathcal{O}(S)$. As $\mathcal{I}(S)$ is prime, we can form the quotient field $\mathcal{K}(S)$ of $\mathcal{O}(S)$. This is the field of *rational functions of $S$*.

**Definition 2.3.** For each $x \in S$, write

$$\mathcal{O}_{S,x} = \left\{ \frac{g}{h} \in \mathcal{K}(S) \,:\, h(x) \neq 0 \right\}.$$

The reader should have no difficulty in showing that $\mathcal{O}_x$ is a subring of $\mathcal{K}(X)$. In addition, it is a local ring.

**Definition 2.4** ([MRB, p.20])**.** If $U \subset S$ is open, define

$$\mathcal{O}(U) := \bigcap_{x \in U} \mathcal{O}_x.$$

This is the ring of regular functions on $U$ (cf Remark 2.5).

The attentive reader may here pause to note that the symbol $\mathcal{O}(S)$ has *two* definitions now! This is now a problem: see [MRB, Prp. 1, p. 20].

*Remark* 2.5. Elements of $\mathcal{O}(U)$ can be considered as genuine functions $U \to k$: if $f \in \mathcal{O}(U)$ and $x \in U$, then we can write $f = g/h$ with $h(x) \neq 0$ and hence $f(x) := g(x)/h(x)$ is a perfectly honest element of $k$. Also, note that the natural inclusions

$$\mathcal{O}(U) \longrightarrow \mathcal{O}_x, \quad x \in U$$

identify $\mathcal{O}_x$ with $\varinjlim_{U \ni x} \mathcal{O}(U)$.

## 2.2 Sheaves

As the theory advanced, algebraic geometers recognized the importance of the theory of *sheaves*, which is review briefly here.

Let $X$ be a topological space. For each open subset $U$, let $\mathcal{C}^0(U)$ stand for the ring of continuous functions $U \to \mathbb{C}$. It is clear that for each $V \subset U$, the restricting functions produces a morphism of rings

$$\mathrm{res}_{UV} : \mathcal{C}^0(U) \longrightarrow \mathcal{C}^0(V).$$

Obviously:

PS1. If $V = U$, then $\mathrm{res}_{UV} = \mathrm{id}$ and

PS2. If $W \subset V$, then $\mathrm{res}_{UW} = \mathrm{res}_{VW} \circ \mathrm{res}_{UV}$.

Now, if for some reason we wish to consider functions on $U$ which are $C^k$, real, holomorphic, etc, it is clear that PS1 and PS2 hold in the adapted context. Hence:

**Definition 2.6.** A pre-sheaf $\mathcal{P}$ on $X$ is a function which to every open subset $U \subset X$ associates an abelian group $\mathcal{P}(U)$, and to each couple of open subset $U \supset V$, a homomorphism $\mathrm{res}_{UV} : \mathcal{P}(U) \to \mathcal{P}(V)$ satisfying PS1 and PS2.

*Remark* 2.7. It is impossible to write consistently $\mathrm{res}_{UV}$ for the maps defining a pre-sheaf. One usually writes

$$\mathrm{res}_{UV}(s) = s|_V$$

instead, leaving the open subset $U$ implicit.

Here we have defined pre-sheaves as sheaves of abelian groups, but one can consider pre-sheaves of all kinds of algebraic objects: vector spaces, rings, fields, etc.

Carrying on our example of $\mathcal{C}^0$, we note that if $U = \cup_i U_i$ is an open covering, then a continuous function $U \to \mathbb{C}$ is the same thing as a family of continuous functions $f_i : U_i \to \mathbb{C}$ which coincide on overlaps. This is the property which distinguishes presheaves from sheaves.

**Definition 2.8.** A presheaf $\mathcal{F}$ is a sheaf if for each open $U$ and each covering $U = \cup U_i$, the map

$$\mathcal{F}(U) \longrightarrow \left\{ (f_i) \in \prod_i \mathcal{F}(U_i) \; : \; \mathrm{res}_{U_i, U_i \cap U_j}(f_i) = \mathrm{res}_{U_j, U_i \cap U_j}(f_j) \right\}$$

$$f \longmapsto (\mathrm{res}_{UU_i}(f))$$

is a bijection.

**Example 2.9.** Let $X = \mathbb{R}$. For each open $U \subset X$, define $\mathcal{C}^0(U)$ as the ring of continuous complex valued functions. This defines a sheaf. On the other hand, let $\mathcal{S}(U)$ be the ring of *constant* complex valued functions on $U$. Then $\mathcal{S}$ defines a pre-sheaf, but not a sheaf: for example, the element

$$(-1, +1) \in \mathcal{S}(\mathbb{R}_{<0}) \times \mathcal{S}(\mathbb{R}_{>0})$$

is not of the form $(f|_{\mathbb{R}_{>0}}, f|_{\mathbb{R}_{<0}})$ for a *constant* function $f \in \mathcal{S}(\mathbb{R}^*)$. Were we to define $\mathcal{S}(U)$ as the ring of *locally constant functions*, this problem would be removed.

**Example 2.10.** Let $S \subset k^n$ be an irreducible algebraic set. For each open $U$, let $\mathcal{O}(U)$ be its ring of regular functions as in Definition 2.4. For each open $V \subset U$, it is clear that $f \in \mathcal{K}(S)$ belongs to $\mathcal{O}(V)$ if it belongs to $\mathcal{O}(U)$ and hence $\mathcal{O}$ defines a pre-sheaf. In addition, it is not hard to see that this pre-sheaf is a sheaf.

The reader will have no difficulty in defining what a morphism of pre-sheaves should be [MRB, Def. 2, p.17]. Morphisms of sheaves are simply morphisms of pre-sheaves.

Here is one of the most important ways of constructing new sheaves from old: *pushing forward*. Given a continuous map $f : Y \to X$ between topological spaces and a sheaf $\mathcal{G}$ on $Y$, let us agree to write

$$f_*\mathcal{G}(U) = \mathcal{G}(f^{-1}(U)).$$

Then $f_*\mathcal{G}$ defines a sheaf on $X$, as a simple verification shows.

## 2.3 Varieties over an algebraically closed field

A ringed space is a couple of a topological space $X$ and a sheaf of rings $\mathcal{O}$. Varieties are a certain kind of ringed spaces.

**Example 2.11.** In the notations of example 2.9, The couple $(\mathbb{R}, \mathcal{C}^0)$ is a ringed space.

**Example 2.12.** Let $S \subset k^n$ be an irreducible algebraic set. The couple $(S, \mathcal{O})$ is a ringed space: $S$ is naturally a topological space and $\mathcal{O}$ is a sheaf.

In order to compare distinct ringed spaces, we need the notion of morphism.

**Definition 2.13.** Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be ringed spaces. A morphism $(Y, \mathcal{B}) \to (X, \mathcal{A})$ is a couple $(f, f^\#)$ consisting of a continuous ma $f : Y \to X$ and a morphism of sheaves of rings

$$f^\# : \mathcal{A} \longrightarrow f_* \mathcal{B}$$

usually called "the pull-back". An isomorphism of ringed spaces is a morphism which has a two sided inverse.

An *affine variety* $(X, \mathcal{O})$ is a topological space $X$ with a sheaf of $k$-algebras $\mathcal{O}$ which is isomorphic to some $(S, \mathcal{O})$ as in Example 2.12.

A pre-variety $(X, \mathcal{O})$ is a *connected* topological space $X$ with a sheaf of $k$-algebras $\mathcal{O}$ which is *locally, as a ringed space*, isomorphic to an affine variety [MRB, p.25].

Note that, in [MRB, Ch. 1], Mumford supposes that an algebraic variety over $k$ is immediately an *irreducible* topological space [MRB, p.25]. (It is possible to have a reasonable theory where this condition is removed.) Maps between varieties are simply maps of ringed spaces. Finally, a variety is a pre-variety which is, in addition, separated [MRB, Ch. 1]. This is not strictly necessary and the reader may compare such a requirement with the imposition that manifolds be *Hausdorff spaces.*

## 2.4 tangent vectors and differentials

I shall not assume knowledge of differential forms as in [MRB] — these are done using Kähler differentials —, instead I will do it as in [Sh].

If $x$ is a point on the algebraic variety $X$, let

$$\Omega_X(x) = \mathfrak{m}_x / \mathfrak{m}_x^2;$$

this is the *cotangent space of $X$ at $x$*. For each $f \in \mathcal{O}(X)$ and each $x \in X$, there exists one and only one $f(x) \in k$ s.t. $f - f(x) \in \mathfrak{m}_x$ and we define

$$\mathrm{d}_x f = \text{image of } f - f(x) \text{ in } \mathfrak{m}_x / \mathfrak{m}_x^2.$$

This is called the *differential of $f$ at $x$*.

For any given $x \in X$, define the *tangent*[‡] *space at $x$* as

$$\mathcal{T}_X(x) = (\Omega_X(x))^*.$$

(In [Sh], one finds the notation $\Theta_x$ for $\mathcal{T}_X(x)$. The reason for the somewhat pedantic notation $\mathcal{T}_X(x)$ is that we wish to distinguished between *fiber* and *stalk*!) Given a morphism of varieties

$$\varphi : Y \longrightarrow X$$

[‡]This is also called the *Zariski* tangent space.

we automatically obtain a morphism of tangent and cotangent spaces

$$\mathrm{D}_y\varphi : \mathcal{T}_Y(y) \longrightarrow \mathcal{T}_X(\varphi(y)) \quad \text{and} \quad \varphi_y^* : \Omega_X(\varphi(y)) \longrightarrow \Omega_Y(y),$$

whose specifications we leave to the reader.

**Exercise 2.14.** Let $\varphi : Y \to X$ be a morphism and let $x = \varphi(y)$. Prove that $\varphi_y^*(\mathrm{d}_x f) = \mathrm{d}_y(\varphi^\#(f))$ (here $\varphi^\#$ is the morphism of rings $\varphi^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$).

**Definition 2.15.** A differential form on an open subset $U$ is a function $\omega$ which to each $x \in U$ associates an element $\omega(x) \in \Omega_X(x)$. The set of differential forms is denoted by $\Phi(U)$.

Clearly, for each $U \subset X$ open, $\Phi(U)$ is a $\mathcal{O}(U)$-module.

**Definition 2.16.** We say that $\omega \in \Phi(U)$ is regular if each point has an open neighbourhood $V$ such that $\omega = f_1 \mathrm{d}g_1 + \cdots + \mathrm{d}g_m$, with $f_j, g_j \in \mathcal{O}(V)$. The $\mathcal{O}(U)$–submodule of $\Phi(U)$ consisting of regular forms is denoted by $\Omega_X(U)$.

**Exercise 2.17.** Let $\varphi : Y \to X$ be a morphism. Prove that $\varphi^*(\omega)$ is regular if $\omega$ is regular.

## 2.5 Algebraic groups

**Definition 2.18.** A group-variety, or an algebraic group[§], is an algebraic variety $X$ with a group structure such that multiplication and inversion are morphisms (of varieties).

An *abelian variety* is an algebraic group which is, as a variety, *complete* [MRB, ChI, §9].

**Example 2.19.** The most simple group varieties are $\mathbf{G}_a = k$ with the addition, and $\mathbf{G}_m = k^*$, with multiplication. Other celebrated examples are $\mathbf{GL}_n = \{A \in \mathrm{Mat}_n(k) \,:\, \det A \neq 0\}$, $\mathbf{SL}_n$, etc.[¶]

We define invariant vector fields as before and, in addition, go on to study *invariant differential forms*. A differential $\omega$ of $X$ is left invariant if for each $g \in G$, the pull-back

$$(\tau_g)_h^* : \Omega(gh) \longrightarrow \Omega(h)$$

maps $\omega(gh)$ to $\omega(h)$. The reader is invited to show that this amounts to

$$\tau_g^*\omega = \omega, \qquad \forall\, g \in G.$$

[§]Warning: Sometimes people can refer to algebraic groups which are not varieties, but schemes. Also, it is useful to extend the notion of variety to include non-irreducible ones, so that finite groups appear as group-varieties, cf [Br]. But I've opted out of this path to stay close to [MRB].

[¶]With this definition we leave finite and non-trivial groups out. Too bad!

In particular, an invariant differential form is determined by its value on $e$. Let us check the analogue of Proposition 1.2. Hence, given $\eta \in \Omega_X(e)$ and writing $\overline{x}$ instead of $x^{-1}$, we define

$$\begin{aligned}\eta^\natural(x) &:= (\tau_{\overline{x}})^*_x(\eta) \\ &= \text{Image of } \eta \text{ via } \Omega(e) \xrightarrow{(\tau_{\overline{x}})^*_x} \Omega(x)\end{aligned}$$

It is for the reader to verify that

$$\tau_x^*(\eta^\natural) = \eta^\natural.$$

**Lemma 2.20.** *The differential form $\eta^\natural$ is regular.*

*Proof.* Let $U \subset X$ be an affine open, let $V \subset U$ be affine open such that $VV \subset U$. Multiplication $V \times V \to U$ induces

$$\mathcal{O}(U) \longrightarrow \mathcal{O}(V \times V) = \mathcal{O}(V) \otimes \mathcal{O}(V).$$

‖ This has the following translation in terms of functions: If $f \in \mathcal{O}(V)$, then

$$f(g'g'') = \sum_j f'_j(g') \cdot f''_j(g'')$$

for all $g', g'' \in V$, where $f'_j$ and $f''_j$ are regular on $V$. Hence, if $\overline{x} \in V$ we have

$$f(\overline{x}g'') = \sum_j f'_j(\overline{x}) \cdot f''_j(g''),$$

which means

$$\tau_{\overline{x}}^{\#}(f) = \sum_j f'_j(\overline{x}) \cdot f''_j. \tag{2.1}$$

Consider

$$(\mathrm{d}_e f)^\natural : x \longmapsto \tau_{\overline{x}}^*(\mathrm{d}_e f).$$

By Exercise 2.14,

$$\tau_{\overline{x}}^* : \Omega_X(e) \longrightarrow \Omega_X(x)$$

sends $\mathrm{d}_e f$ to $\mathrm{d}_x(\tau_{\overline{x}}^{\#}(f))$. Equation (2.1) implies: $\Rightarrow$

$$\begin{aligned}(\mathrm{d}_e f)^\natural(x) =& \tau_{\overline{x}}^*(\mathrm{d}_e f) \\ &= \mathrm{d}_x(\tau_{\overline{x}}^{\#}(f)) \\ &= \sum_j f'(\overline{x}) \cdot \mathrm{d}_x f''_j.\end{aligned}$$

‖Here, I've gone fast: The fact that the product of affine varieties is an affine variety *and* that its ring of functions is the tensor product is a beautiful fact from algebraic geometry, which is explained in [MRB, Proposition 1, p. 34].

This means

$$(\mathrm{d}_e f)^\natural = \sum_j f'_j \circ (-)^{-1} \cdot \mathrm{d} f''_j$$

on $V \cap V^{-1}$. We conclude that if $f$ is regular on some open neighbourhood of $e$, then $(\mathrm{d}_e f)^\natural$ is regular on some open neighbourhood of $e$.

Now, write $\eta = \sum_i c_i \mathrm{d}_e f_i$ with $f_i$ regular in some neighbourhood of $e$. Then $\eta^\natural$ is regular on some open neighbourhood of $e$. This implies that $\eta^\natural$ is regular allover. $\square$

## 2.6 Smoothness

Let $X$ be a variety. Recall that a point $x \in X$ is regular if $\mathcal{O}_{X,x}$ is a regular local ring. Now, varieties are always regular on some non-empty subset. Hence, if $X$ is a group-variety, the fact that $\mathcal{O}_x \simeq \mathcal{O}_e$ (via $\tau_x^\# : \mathcal{O}_x \to \mathcal{O}_e$) then $X$ is regular allover.

So regularity is something common to all group-varieties.

## 2.7 Basic structure of Abelian varieties

Imposing that a group-variety be complete puts a rather unexpected structure on it.

**Theorem 2.21** (Rigidity Lemma). *Let $X$ complete variety. Let $f : X \times T \to Y$ be morphism. Suppose $f|_{X \times \{t_0\}}$ is constant for some $t_0 \in T$. Then there exists $\tilde{f} : T \to Y$ such that $\tilde{f} \circ \mathrm{pr}_T = f$; in particular, $f|_{X \times \{t\}}$ is constant for any $t \in T$.*

*Proof.* Let $x_0 \in X$. Define $\tilde{f}(t) = f(x_0, t)$. We shall show that $t_0$ has an open and non-empty neighbourhood $T_0$ such that

$$f(x,t) = f(x_0,t) \quad \forall (x,t) \in X \times T_0,$$

or that $f = \tilde{f} \circ \mathrm{pr}_T$ on $X \times T_0$. Using Exercise 10.7, we shall be done. Another way to proceed is to note that the closed subset $\{f = \tilde{f} \circ \mathrm{pr}_T\} \subset X \times T$ will also be open.

Let $Y_0$ affine neighbourhood of $y_0 := f(x_0, t_0)$. Let $C = Y \setminus Y_0$. Since $X$ is complete, $\mathrm{pr}_T(f^{-1}(C))$ is closed in $T$. In addition, if $t \notin \mathrm{pr}_T(f^{-1}(C)) \Rightarrow$

$$f(x,t) \notin C, \qquad \forall x \in X,$$

which means that

$$f(x,t) \in Y_0, \qquad \forall x \in X.$$

Obviously, $t_0 \notin \mathrm{pr}_T(f^{-1}(C))$, so

$$T_0 := T \setminus \mathrm{pr}_T(f^{-1}(C))$$

is open and non-empty. Then, if $t \in T_0$, we have

$$f(X \times \{t\}) \subset Y_0.$$

As $X$ is complete and $Y_0$ affine, $f(X \times \{t\}) = f(x_0, t)$. $\square$

*Remark* 2.22. Show that the above proof goes through if we remplace "variety", respectively complete, by "complex manifold", respectively "compact."

**Corollary 2.23.** *Abelian varieties are abelian groups.*

*Proof.* Consider the commutator

$$c : X \times X \to X, \quad (x, y) \longmapsto xyx^{-1}y^{-1}.$$

Obviously, $c(X \times \{e\}) = \{e\}$ and hence $c(X \times \{y\})$ is always a point by the Rigidity Lemma. It is easy to see that this point is $e$. □

Using the rigidity Lemma, we can also show:

**Corollary 2.24.** *Let $X$ and $Y$ be abelian varieties and $f : X \to Y$ a morphism of* varieties*. Then, there exists $y_0 \in Y$ and a homomorphism $h : X \to Y$ such that $f(x) = h(x) + y_0$ for all $x$.*

*Proof.* It's enough to suppose $f(e) = e$ and prove that $f$ is a homomorphism. Consider $\varphi(x, y) := f(x + y) - f(x) - f(y)$ and note that $\varphi(X \times 0) = 0$ and hence $\varphi(x, y) = \varphi(x', y)$ for all $x, x', y \in X$. Since $\varphi(0 \times X) = 0$, we also conclude that $\varphi(x, y) = \varphi(x, y')$ for all $x, y, y' \in X$ and hence $\varphi(x', y') = \varphi(x, y') = \varphi(x, y)$. □

The next important result in the theory concerns *divisors*, or line bundles. Recall that for an elliptic curve $E$, Abel's theorem says that for any $x \in E$, the map

$$\phi : E \longrightarrow \frac{\text{divisors of degree } 0}{\text{principal divisors}} =: \mathrm{Pic}^0(E)$$

$$x \longmapsto [x] - [O]$$

is a morphism of groups allowing us to identify $E$ with $\mathrm{Pic}^0(E)$. See [Si, Proposition 3.4, p.66].

Now, in greater dimension, it is not so simple to produce divisors on $X$ starting from *points* of $X$. The very important result which we are then looking for is the construction of a group morphism

$$\phi : X \longrightarrow \mathrm{Div}(X).$$

The algebraic way requires delicate analysis of cohomology; although it is masterfully explained in [MAV], it certainly deviates us from the theory of abelian varieties by increasing considerably the algebraic geometric technology *and* the maturity necessary to structure this material. Our strategy will then be to work with *complex compact Lie groups* following [MAV] and [BL]. The first step is to recognise that these are complex tori.

# Lecture 3

(09/11/22).

# 3 Structure of complex compact groups

Let $G$ be a complex analytic group. A *1-parameter subgroup*, of $G$ is a morphism of analytic groups $\theta : \mathbb{C} \to G$. Then, $\theta'(0) \in T_eG$.

**Theorem 3.1.** *For each $v \in T_eG$, there exits a unique 1-parameter subgroup $\theta : \mathbb{C} \to G$ such that $\theta'(0) = v$.*

*Proof.* Let $v^\natural$ be the left-invariant analytic vector field obtained from $v$. For each $r > 0$ and $x \in G$, consider the problem on the disk $D_r(0)$:

$$\begin{cases} f : D_r(0) \to G \text{ is holomorphic} \\ f'(t) = v^\natural(f(t)) \\ f(0) = x. \end{cases} \tag{$*_{r,x}$}$$

The theorem of existence of differential equations shows that there exists an $\varepsilon > 0$ s.t. $*_{\varepsilon,e}$ has a solution $\theta$.

On the other hand, *uniqueness* assures the following. If $\varphi$ and $\psi$ are solutions to $(*_{r,x})$, then $\varphi = \psi$. See [A, p.8] for details.

Now, let us show that $\theta$ extends to $\mathbb{C}$. Fix $|s| < \varepsilon/2$ and let $y = \theta(s)$. Consider $\varphi(t) = y\theta(t)$ and $\psi(t) = \theta(s + t)$. Then

$$\varphi'(t) = v^\natural(\varphi(t))$$

because $v^\natural$ is left-invariant. Also,

$$\psi'(t) = v^\natural(\psi(t)).$$

Hence, $\varphi$ and $\psi$ are solutions to $(*_{\varepsilon/2,y})$. Then $\psi = \varphi$ on $D_{\varepsilon/2}(0) \Rightarrow$

$$\theta(s)\theta(t) = \theta(s + t), \quad \text{if } \max(|s|, |t|) < \varepsilon/2. \tag{†}$$

We now extend $\theta$ to $\mathbb{C}$. Given $t \in \mathbb{C}$, there let $k \in \mathbb{N}$ be s.t. $t/k \in D_{\varepsilon/2}(0)$ and define

$$\widetilde{\theta}(t) = \theta(t/k)^k.$$

This is independent of $k$ because of (†). It is then simple to show that $\widetilde{\theta}$ is a homomorphism and that it is holomorphic. □

Let $v \in T_eG$ and let $\theta_v$ the 1-parameter-subgroup passing by $v$. Define

$$\exp(v) := \theta_v(1).$$

A basic fact concerning the exponential is:

**Theorem 3.2.** *The map $\exp : T_eG \to G$ is analytic.*

Let us now shift attention to a compact complex Lie group $X$. Let

$$p : V \longrightarrow X$$

be the universal covering of $X$ and let $0 \in p^{-1}(e)$. Then, $V$ inherits from $X$, via $p$, the structure of a complex manifold so that $p$ is locally a biholomorphism. Because $\pi_1(V \times V) = 0$,

$$p \times p : V \times V \longrightarrow X \times X$$

is the universal covering of $X \times X$ so that multiplication $X \times X \to X$ lifts:

$$\begin{tikzcd} V \times V \arrow[r, dashed, "\tilde{m}"] \arrow[d, "p\times p"'] & V \arrow[d, "p"] \\ X \times X \arrow[r] & X. \end{tikzcd}$$

See [Ch, Proposition 1, Ch.II, §VIII] or [Gre, Theorem 6.1]. Using an element of $\pi_1(X, e)$, which acts on $p^{-1}(e)$ transitively, we can in addition assume that $\tilde{m}(0, 0) = 0$. Moreover, the uniqueness part concerning liftings allows us to say:

**Proposition 3.3.** *The complex manifold $V$ inherits the structure of a complex group with multiplication $\tilde{m}$. The analytic map $p$ is a morphism of complex groups. This group structure is abelian.* □

*Remark* 3.4. Let $G$ be a Lie group and $p : U \to G$ its universal covering. It is then the case that $U$ has a structure of Lie group, that $p$ is a morphism of Lie groups and $\mathrm{Ker}(p)$ is contained in the centre of $U$. See [Ch, Ch. I, Sect. VII, Prp. 2].

Since $V$ is abelian, it is a simple matter to show that exp is a homomorphism [A, 2.19]. It turns out to be a covering, and hence an analytic diffeomorphism. Hence:

**Theorem 3.5.** *The complex analytic group $V$ is isomorphic (as a complex group) to the vector space $T_eG$.* □

Let now $\Gamma = \mathrm{Ker}(p)$: This is a subgroup of $V$. In addition, since $p$ is a covering, $\Gamma$ is discrete.

**Exercise 3.6.** Let $G \subset \mathbb{R}^m$ be a discrete subgroup. Show that $G = \mathbb{Z}g_1 + \cdots + \mathbb{Z}g_r$ with $g_1, \ldots, g_r$ linearly independent over $\mathbb{R}$. Show, in addition, that if $\mathbb{R}^m/G$ is compact, then $r = m$.

*Solution.* We proceed by induction on $\dim V$. Hence, it is enough to suppose that $G$ is not contained in a hyperplane. Let then $g_1, \ldots, g_m \in G$ be a basis. If $W = \mathrm{span}_{\mathbb{R}}(g_1, \ldots, g_{n-1})$, then $W \cap G = \mathbb{Z}h_1 + \ldots + \mathbb{Z}h_r$, with $h_1, \ldots, h_r$ linearly independent over $\mathbb{R}$.Clearly, $r \le m - 1$ and since $\{g_1, \ldots, g_{m-1}\} \subset \mathrm{span}_{\mathbb{R}}(h_1, \ldots, h_r)$, we have $m - 1 = r$. Replacing, can suppose $\mathbb{Z}g_1 + \cdots + \mathbb{Z}g_{m-1} = G \cap W$.

Consider the finite set

$$T = \left\{ \sum_{j=1}^{m} t_j g_j \in G \,:\, 0 \le t_1, \ldots, t_{m-1} < 1,\, 0 \le t_m \le 1 \right\}.$$

If $\{\varphi_j\}$ is a dual basis to $\{g_j\}$, let $x = \sum_j x_j g_j$ b e an element of $T$ such that $\varphi_m(x) = \min\{\varphi(y) : y \in T \setminus W\}$. (Note that $g_n \in T$.) Then $\{g_1, \ldots, g_{n-1}, x\}$ is a linearly independent set. We claim $G = \mathbb{Z}g_1 + \cdots + \mathbb{Z}g_{m-1} + \mathbb{Z}x$. Indeed, write $G' = \mathbb{Z}g_1 + \cdots + \mathbb{Z}g_{m-1} + \mathbb{Z}x$ and let $y \in G$. Then

$$y \equiv \sum_{j=1}^{m-1} y_j g_j + cx \mod G'$$

with $0 \leq y_j < 1$ and $0 \leq c < 1$. Hence,

$$y' = \sum_{j=1}^{m-1} y_j g_j + cx \in T$$

But $0 \leq \varphi_m(y') = cx_n < x_m$. Hence, $c = 0$. We conclude that $y' \in W \cap G$ and hence that $y' \in \mathbb{Z}g_1 + \cdots + \mathbb{Z}g_{m-1} \subset G'$. $\square$

# 4 Generalities on line bundles over complex manifolds

## 4.1 Cech cohomology: Fixing notations.

Let $X$ be a topological space and $\mathcal{F}$ a *pre-sheaf* of abelian groups on $X$. Let $\underline{U} = \{U_i : i \in I\}$ be an open covering of $X$, indexed by a totally ordered set $I$, and abbreviate $U_{i_0} \cap \cdots \cap U_{i_n}$ to $U_{i_0 \cdots i_n}$. Define

$$C^n(\underline{U}, \mathcal{F}) = \prod_{i_0 < \ldots < i_n} \mathcal{F}(U_{i_0 \cdots i_n}).$$

Define also

$$d^n : C^n(\underline{U}, \mathcal{F}) \longrightarrow C^{n+1}(\underline{U}, \mathcal{F})$$

by

$$(d^n f)_{i_0 \cdots i_{n+1}} \longmapsto \underbrace{f_{i_1 \cdots i_{n+1}}}_{\text{restrict}} - \underbrace{f_{i_0 i_2 \cdots i_{n+1}}}_{\text{restrict}} + \cdots + (-1)^{n+1} \underbrace{f_{i_0 \cdots i_n}}_{\text{restrict}}.$$

Then

$$\{C^n(\underline{U}, \mathcal{F}), d^n\}_{n \geq 0}$$

is a complex and

$$H^n(\underline{U}, \mathcal{F})$$

is its $n$th cohomology group. If $\underline{V} = \{V_j\}$ is a refinement [GH, p.39] of $\underline{U}$, we then obtain a morphism of groups $H^n(\underline{U}, \mathcal{F}) \to H^n(\underline{V}, \mathcal{F})$ which allows us to pass to the limit:

$$H^n(X, \mathcal{F}) = \varinjlim_{\underline{U}} H^n(\underline{U}, \mathcal{F}).$$

The process of taking limits renders computation almost impossible, hence the theory is redeemed by

**Theorem 4.1** (Leray's Theorem)**.** *If $\underline{U}$ is a covering such that $H^{>0}(U_{i_0\cdots i_n}, \mathcal{F}) = 0$, then $H^n(X, \mathcal{F}) = H^n(\underline{U}, \mathcal{F})$.* □

Another fundamental feature of Cech cohomology is the notion of *long exact sequence* associated to a short exact sequence. Indeed, Cech cohomology comes as a way to understand when a surjective morphism of sheaves fails to be a surjection on global section.

**Definition 4.2** (Surjective maps of sheaves)**.** Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We say that $\alpha$ is *injective* if $\alpha(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all $U$. On the other hand, we say that $\alpha$ is *surjective* if for each $g \in \mathcal{G}(U)$, there exists a covering $U = \cup U_i$ and sections $f_i \in \mathcal{F}(U_i)$ such that

$$g|_{U_i} = f_i.$$

The reader encountering this definition for the first time may be puzzled by the *a priori* strange definition of *surjectivity.* Why not require the arrows $\alpha(U)$ to be surjective? But this is precisely the interest: we encounter very many situations where above definition occurs. Let me give an example.

**Example 4.3.** Let $X = \mathbb{C} \setminus \{0\}$. As one learns in complex analysis, if $D \subset X$ is a disk, then it is possible to define a branch of the logarithm on $D$ [Ah, Ch. 4, §4.4, Cor. 2], that is, there exists $\ell : D \to \mathbb{C}$ holomorphic such that $e^{\ell(z)} = z$ for all $z \in D$. Let $\mathcal{O}$ be the sheaf of holomorphic functions on $X$ and $\mathcal{O}^*$ the sheaf of invertible holomorphic functions. We then have a morphism of sheaves $\exp : \mathcal{O} \to \mathcal{O}^*$ which, by the considerations above *is surjective.* Now, one also learns in a course of complex analysis that there is no holomorphic function $L : X \to \mathbb{C}$ such that $e^{L(z)} = z$ for all $z \in X$.

Now, what is the obstruction to finding a globally defined logarithm? Let us cover $X$ by disks, $X = \cup_i D_i$, and let $\ell_i$ be a branch of the logarithm defined on $D_i$. Let now $\mathbb{Z}_X$ be the sheaf of locally constant integer valued functions on $X$. Then, $\delta_{ij} := \ell_j - \ell_i \in 2\pi\mathbf{i}\mathbb{Z}_X(D_i \cap D_j)$ and $(\delta_{ij})$ becomes an element of $Z^1(X, 2\pi\mathbf{i}\mathbb{Z}_X)$.

Now, with this notion of surjective morphism, we can define short exact sequences of sheaves in the usual sense. Furthermore, if

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0$$

is a short exact sequence of sheaves on $X$, then we obtain a long exact sequence

$$0 \longrightarrow \mathcal{E}(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{G}(X) \xrightarrow{\delta}$$

$$\xrightarrow[\delta]{} H^1(\mathcal{E}) \longrightarrow H^1(\mathcal{F}) \longrightarrow H^1(\mathcal{G}) \xrightarrow{\delta}$$

$$\cdots$$

The maps

$$\delta_n : H^n(\mathcal{G}) \longrightarrow H^{n+1}(\mathcal{E})$$

are defined in parallel to the situation in Example 4.3. On starts with $g \in Z^n(\underline{U}, \mathcal{G})$; when restricted to a finer unspecified covering, we can suppose that $g = \beta(f)$ with $f \in C^n(\mathcal{F})$. Now $d^n f \in C^{n+1}(\mathcal{F})$ is such that $\beta(d^n f) = d^n \beta(f) = 0$ and hence, passing to a finer unspecified covering, we have $d^n f = \alpha(e)$, and the element $e \in C^{n+1}(\mathcal{E})$ "is" $\delta f$. (Details are often omitted in the literature and for complete proofs one needs to go to [Se, Ch. I, §3].)

Finally, to end this brief review of Cech cohomology, we state one of the two main foundational results of this theory. These are labelled according to the orignal problems giving rise to the techniques of Cech cohomology.

**Theorem 4.4** ("Mittag-Leffler's Problem")**.** *$H^m(\mathbb{C}^n, \mathcal{O}) = 0$ for $m \geq 1$.*

*Proof.* See [GH, pages 46-7]. □

**Theorem 4.5** ("Cousin's Problem")**.** *$H^m(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $m \geq 1$.*

*Proof.* See [GH, pages 46-7]. □

## 4.2 Line bundles in the complex case

We begin with generalities. See [GH, 0.5, pp. 66-69] for further references. Let $X$ be a complex manifold. Recall:

**Definition 4.6.** A line bundle is a complex manifold $L$ and an analytic morphism $\pi : L \to X$ such that the following hold. (For expedience, we write $L|_S := \pi^{-1}(S)$.)

(1) There exists a covering $\{U_i\}$ of $X$ and isomorphisms $\sigma_i : U_i \times \mathbb{C} \to L|_{U_i}$ such that $\pi \circ \sigma_i(u, c) = u$.

(2) The arrow $\sigma_i^{-1}\sigma_j : U_{ij} \times \mathbb{C} \to U_{ij} \times \mathbb{C}$ is of the form $(u, c) \mapsto (u, g_{ij}(u) \cdot c)$ with $g_{ij} \in \mathcal{O}_X(U_{ij})^*$.

For the sake of discussion, we shall call the data of the covering $\{U_i\}$ and the isomorphisms $\sigma_i : U_i \times \mathbb{C} \to L|_{U_i}$ a *trivializing atlas for* $L$ and $(g_{ij})$ the cocycle associated to $L$ (and the atals). Because of Definition 4.6-(2), each $L|_x := \pi^{-1}(x)$ is a complex vector space of dimension one: just express $\ell \in L|_x$ as $\sigma_j(x, c)$ and define, for each $\lambda \in \mathbb{C}$, the element $\lambda\ell := \sigma_j(x, \lambda c)$.

*Remark* 4.7 (Mnemonic for $g_{ij}$'s). In the notations of Definition 4.6, let $s_i(x) := \sigma_i(x, 1)$; clearly $\pi s_i(x) = x$. Then

$$s_j(x) = g_{ij}(x) s_i(x),$$

which means that $g_{ij}(x)$ is the "coordinate" of $s_j(x)$ with respect to the "vector" $s_i(x)$.

Obviously, the functions $\{g_{ij}\}$ satisfy $1 = g_{jk}g_{ik}^{-1}g_{ij}$ and $g_{ij}g_{ji} = 1$, so that $(g_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O}_X^*)$. This is called the cocycle associated to $L$.

A very important invariant associated to a line bundle is its *Chern class*, whose definition relies on Exercise 10.11. Consider now the exponential exact sequence of sheaves on $X$:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{e^{2\pi \mathbf{i}(-)}} \mathcal{O}_X^* \longrightarrow 1.$$

(Here $\mathbb{Z}$ is the sheaf of locally constant integer valued functions.) The long exact cohomology sequence then gives us

$$\delta : H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}),$$

and if $L \to X$ is a line bundle with isomorphism class $[L] \in H^1(X, \mathcal{O}_X^*)$, we define its Chern class $c_1(L)$ as $\delta[L]$.

Let us make this explicit. If $\{g_{ij}\}$ is a cocycle for the covering $\mathfrak{U} = \{U_i\}$, where each $U_{ij}$ is simply connected, let $\lambda_{ij} : U_{ij} \to \mathbb{C}$ be a "logarithm", that is,

$$g_{ij} = e^{2\pi \mathbf{i} \lambda_{ij}}.$$

Then

$$(\lambda_{jk} - \lambda_{ik} + \lambda_{ij}) \in Z^2(\mathfrak{U}, \mathbb{Z})$$

represents $c_1(L)$.

Finally, I ended with one technical construction : pull-backs of line bundles. (More details are in [FG, IV, p. 180]) Now, given a map of complex manifolds $f : Y \to X$ and a line bundle $\pi : L \to X$, we obtain a new line bundle $f^*(\pi) : f^{-1}L \to Y$ as follows. As a space, $f^{-1}L$ is the fibre product

$$Y \times_X L = \{(y, \ell) \in Y \times L \, : \, \pi(\ell) = f(y)\}, \tag{4.1}$$

which has the topology given by the inclusion $L \times Y$. Define $f^*(\pi)$ as being the restriction of $\mathrm{pr}_Y : Y \times L \to Y$.

## 4.3 $\mathrm{Pic}^0(X)$ and the Neron-Severi group.

To study $\mathrm{Pic}(X)$, we shall break it up using the Chern classes. (In what follows, we shall denote the sheaves of locally constant integer valued, real valued, etc, functions by $\mathbb{Z}$, $\mathbb{R}$, etc.)

**Definition 4.8.** We define $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ as being $\mathrm{Ker}\, c_1$ and $\mathrm{NS}(X) \subset H^2(X, \mathbb{Z})$, the Neron-Severi group, as being $\mathrm{Im}\, c_1$.

One usually thinks as $\mathrm{Pic}^0(X)$ as being the "continuous" part of $\mathrm{Pic}(X)$, and $\mathrm{NS}(X)$ as being its "connected components". A very important piece I wish to explain to you is the structure of $\mathrm{Pic}^0(X)$, which will be identified with $\mathrm{Hom}(\Gamma, S^1)$.

# 5 Line bundles on a complex torus

We give ourselves a compact analytic and connected group $X$. It then follows that its universal covering space $V$ is a just a vector space and hence that $X \simeq V/\Gamma$ for a certain lattice $\Gamma$.

Theorem 4.5 will enable us to express line bundles on $X$ in terms of group-theoretical information because

$$V \times_X L \xrightarrow{\sim} V \times \mathbb{C}.$$

To go on, we need:

## 5.1 Interlude on group cohomology

I shall give some explanations concerning the cohomology of groups. I follow [AW] and [GS]; the latter is rather concrete.

Let $G$ be an arbitrary group and $M$ an abelian group. A map

$$G \times M \longrightarrow M, \quad (g, m) \longmapsto g \cdot m$$

gives $M$ the structure of $G$-module if $\mu$ is an action by automorphisms of abelian groups. That is, $g \cdot (m + m') = g \cdot m + g \cdot m'$, and $g \cdot (g' \cdot m) = (gg') \cdot m$.

Clearly, $G$-modules are simply left $\mathbb{Z}G$-modules, where $\mathbb{Z}G$ is the *group ring*. As such, we have cohomological theories associated to these, as we have to any category of modules over a ring.

**Definition 5.1.** In the category of $\mathbb{Z}G$-modules, the $\mathbb{Z}G$-module $\mathbb{Z}$ has the following resolution by free $\mathbb{Z}G$–modules, called the *standard resolution*, or *unnormalized bar resolutions*. For each $n \in \mathbb{N}$, let $B_n$ be the $\mathbb{Z}G$-module $\mathbb{Z}G^{n+1}$; said differently, $g \cdot (g_0, \ldots, g_n) = (gg_0, \ldots, gg_n)$. Let

$$\partial_n : B_n \longrightarrow B_{n-1}$$

be defined by

$$\partial_n(g_0, \ldots, g_n) = (g_1, \ldots, g_n) - (g_0, g_2, \ldots, g_n) + \cdots + (-1)^n (g_0, \ldots, g_{n-1}).$$

**Exercise 5.2.** Show that

$$B_\bullet : \qquad \cdots \longrightarrow B_n \longrightarrow B_{n-1} \longrightarrow \cdots \longrightarrow B_0$$

is exact and $H_0(B_\bullet) \simeq \mathbb{Z}$. (Details are on p. 72 of [GS].)

Let $M$ be a $G$-module. Write

$$\partial^n := \mathrm{Hom}_G(\partial_n, M)$$

and consider the *complex*

$$\cdots \longrightarrow \mathrm{Hom}_G(B_{n+1}, M) \xrightarrow{\partial^n} \mathrm{Hom}_G(B_n, M) \longrightarrow \mathrm{Hom}_G(B_{n-1}, M) \longrightarrow \cdots$$

By definition, the quotient groups

$$\frac{\operatorname{Ker}(\partial^n)}{\operatorname{Im}(\partial^{n-1})}$$

are the cohomology groups of $M$ and are denoted by

$$H^n(G,M).$$

Note that $\operatorname{Hom}_G(B_n, M)$ is the group of all maps $\psi : G^{n+1} \to M$ satisfying $\psi(gg_0,\ldots,g_n) = g\psi(g_0,\ldots,g_n)$ and we then obtain an isomorphism of $\mathbb{Z}$-modules

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbb{Z}G}(B_n, M) & \xrightarrow{\zeta_n} & \operatorname{Map}(G^n, M) \\ & & \\ \psi & \longmapsto & (g_1,\ldots,g_n) \mapsto \psi(e,g_1,\ldots,g_n) \end{array} .$$

For the sake of tradition, let

$$C^n(G,M) = \operatorname{Map}(G^n, M);$$

under the bijection $\zeta_n$ we see that

$$\partial^n : C^n(G,M) \longrightarrow C^{n+1}(G,M)$$

is determined by

$$\begin{aligned} \partial^n\psi(g_1,\ldots,g_{n+1}) = {} & g_1\cdot\psi(g_2,\ldots,g_{n+1}) - f(g_1g_2,g_3,\ldots,g_{n+1}) + \cdots \\ & + (-1)^n\psi(g_1,g_2\ldots,g_ng_{n+1}) \\ & + (-1)^{n+1}\psi(g_1,\ldots,g_{n+1}). \end{aligned}$$

As usual, it is worth writing these formulas down in case $n$ is small:

$$\begin{aligned} \partial_0\psi(g_1) &= g_1\cdot\psi(e) - f(e) \\ \partial_1\psi(g_1,g_2) &= g_1\cdot\psi(g_2) - \psi(g_1g_2) + \psi(g_2) \\ \partial_2\psi(g_1,g_2,g_3) &= g_1\cdot\psi(g_2,g_3) - \psi(g_1g_2,g_3) + \psi(g_1,g_2g_3) - \psi(g_1,g_2). \end{aligned}$$

Due to the explicitness of these formulae, we usually introduce "the" group of 1–cocyles as being

$$Z^1(G,M) = \{g : G \to M\ : \psi(g_1g_2) =\ g_1\cdot\psi(g_2) - \psi(g_1g_2) + \psi(g_2)\},$$

"the" 1–coboundaries

$$B^1(G,M) = \{\psi : G \to M\ :\ \psi(g) = g\cdot\psi(e) - \psi(e)\}$$

"the" 2-cocyles

$$Z^2(G,M) = \{\psi : G^2 \to M\ :\ \psi(g_1g_2)\},$$

and so on. Due to their prominence, elements of $Z^1(G,M)$ are usually called *crossed homomorphisms*.

Hence, we can re-define:

**Definition 5.3.** For each $G$-module $M$, we define

$$\boxed{H^n(G,M) := \frac{Z^n(G,M)}{B^n(G,M)}.}$$

A fundamental fact of homological algebra (see [AW, §1] or for a longer discussion [GS, 3.1]) is that if $\{P_n\}_{n\geq 0}$ are free $\mathbb{Z}G$-modules and

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} 0$$

is a diagram such that $\mathrm{Ker}(d_n) = \mathrm{Im}(d_{n+1})$ for $n \geq 0$ and

$$\frac{P_0}{\mathrm{Im}(d_1)} \simeq \mathbb{Z},$$

then, letting $d^n := \mathrm{Hom}_G(d^n, M)$, there exists a canonical isomorphism

$$H^n(G,M) \xrightarrow{\sim} \frac{\mathrm{Ker}(d^n)}{\mathrm{Im}(d^{n-1})}.$$

Finally, we need some results concerning the connecting homomorphisms.

**Lemma 5.4** (Explicit descriptions)**.** *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

*be exact sequence of $G$-modules. Let*

$$\delta^0 : H^0(G,M'') \longrightarrow H^1(G,M')$$

*and*

$$\delta^1 : H^1(G,M'') \longrightarrow H^2(G,M')$$

*be the connecting homomorohsns.*

1. *Let $m'' \in (M'')^G = H^0(G,M'')$ be given and let $m \in M$ be above it. Define*

   $$\psi : g \longmapsto gm - m.$$

   *Then $\psi(g) \in M'$ and $\psi : G \to M'$ is an element of $Z^1(G,M')$ such that*

   $$[\psi] = [\delta^0 m''].$$

2. *Let $m'' : G \to M'' \in Z^1(G,M'')$ be given and let $m : G \to M$ be above it. Define*

   $$\psi(g,h) \longmapsto g \cdot (m(h)) - m(gh) + m(h).$$

   *Then $\psi \in Z^2(G,M')$ and*

   $$[\psi] = [\delta^1 m''].$$

*Proof.* Can be directly seen from Example 3.2.3, on p. 73 of of [GS]. □

## 5.2 $\mathrm{Pic}(X)$ as cohomology of $\Gamma$

We return to the geometric situation in the beginning of Section 5.

Give $\mathcal{O}(V)$ its natural structure of $\Gamma$-module by $sf : z \mapsto f(z+s)$. This induces on the abelian group $\mathcal{O}(V)^*$ a natural action of $\Gamma$ by group morphisms: $\mathcal{O}(V)^*$ is a $\Gamma$-module.

**Theorem 5.5.** *There is an isomorphism of groups*

$$\mathrm{Pic}(X) \xrightarrow{\sim} H^1(\Gamma, \mathcal{O}(V)^*).$$

This result is a particular case of "Cousin's Theorem", Theorem 4.5, applied to Theorem 9.3 below. But for the sake of concreteness, we shall explain it in more detail.

Let then

$$\chi : V \times \mathbb{C} \longrightarrow L \times_X V$$

be an isomorphism of line bundles. In particular, $\mathrm{pr}_V \chi = \mathrm{pr}_V$. Let $\tau_s : V \to V$ be the translation by $s \in \Gamma$. We have an automorphism of

$$\mathrm{id}_L \times \tau_s : L \times_X V \longrightarrow L \times_X V.$$

Note that $\mathrm{pr}_V \circ (\mathrm{id}_L \times \tau_s) = \tau_s \circ \mathrm{pr}_V$, so that $\mathrm{id}_L \times \tau_s$ is *not* an automorphism of lines bundles over $V$! Define $\varphi_s$ by rendering

$$\begin{tikzcd} V \times \mathbb{C} \arrow[r, "\varphi_s"] \arrow[d, "\chi"'] & V \times \mathbb{C} \arrow[d, "\chi"] \\ L \times_X V \arrow[r, "(\mathrm{id}_L, \tau_s)"'] & L \times_X V \end{tikzcd}$$

commutative. Then

$$\varphi_s(z, c) = (z + s, \psi_s(z, c)).$$

It is not hard to see that $\psi_s(z, c) = c\psi_s(z, 1)$ and so, abbreviating, we write

$$\varphi_s(z, c) = (z + s, \psi_s(z) \cdot c).$$

In addition, observe that $\psi_s \in \mathcal{O}(V)^*$. Since

$$\varphi_t(\varphi_s(z, c)) = \varphi_{s+t}(z, c),$$

we see that

$$\psi_{s+t} = (s\psi_t) \cdot \psi_s. \tag{5.1}$$

Said differently in the terminology of p. 24:

**Corollary 5.6.** *$\psi : \Gamma \to \mathcal{O}(V)^*$ is a crossed homomorphism, or an element of $Z^1(\Gamma, \mathcal{O}(V)^*)$.* □

**Definition 5.7.** The 1–crossed homomorphism $(\psi_s)_{s \in \Gamma}$ is said to be *a multiplier* or *automorphy factor* for $L$.

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It should be noted by that the construction of $(\psi_s) \in Z^1(\Gamma, \mathcal{O}(V)^*)$ depends on the trivialization $\chi : V \times \mathbb{C} \to L \times_X \mathbb{C}$. Now, if $\chi'$ is another such trivialization, then $\chi^{-1}\chi'(z,c) = (z, u(z)c)$ with $u \in \mathcal{O}(V)^*$. Hence, if $\varphi'_s$ and $\psi'_s$ are defined as $\varphi_s$ and $\psi_s$ were, but by using $\chi'$, we conclude that

$$\psi'_s = \psi_s \cdot \frac{s \cdot u}{u}.$$

Therefore, $(\psi'_s)_{s\in\Gamma}$ and $(\psi_s)_{s\in\Gamma}$ differ by an element of $B^1(\Gamma, \mathcal{O}(V)^*)$.

**Definition 5.8.** The cohomology class $[\psi] \in H^1(\Gamma, \mathcal{O}(V)^*)$ is independent of the trivialization $\chi : Z \times \mathbb{C} \to Z \times_X L$.

Let us now show how to construct from a given $\psi \in Z^1(\Gamma, \mathcal{O}(V)^*)$ a line bundle $L_\psi \to X$. Let $\Gamma$ act on $V \times \mathbb{C}$ via

$$(z,c) \bullet s = (z+s, \psi_s(z)c).$$

The fact that $\psi_{s+t}(z) = \psi_t(z+s) \cdot \psi_s(z)$ for all $z \in V$ and $s,t \in \Gamma$ assures that this is an action:

$$\begin{aligned}
[(z,c) \bullet s] \bullet t &= (z+s, \psi_s(z) \cdot c) \bullet t \\
&= (z+s+t, \psi_s(z+s) \cdot \psi_t(z)c) \\
&= (z,c) \bullet (s+t).
\end{aligned}$$

Note, in addition, that if $D \subset V$ is a distinguished open subset for the action of $\Gamma$, then $D \times \mathbb{C}$ is also a distinguished open for the action of $\Gamma$ in $V \times \mathbb{C}$. Consequently,

$$L_\psi := (V \times \mathbb{C})/\Gamma$$

is a complex manifold and the first projection induces an analytic map

$$L_\psi \longrightarrow X.$$

**Proposition 5.9.** *$L_\psi \to X$ is a line bundle.*

These constructions show Thm. 5.5.

## 5.3 The first Chern class of a line bundle

We now know that line bundles on $X$ are defined by elements in $H^1(\Gamma, \mathcal{O}(V)^*)$. Let

$$\psi \in Z^1(\Gamma, \mathcal{O}(V)^*)$$

correspond to $L \in \mathrm{Pic}(X)$.

Since the sequence $0 \to \underline{\mathbb{Z}} \to \mathcal{O}(V) \to \mathcal{O}(V)^* \to 1$ is exact, we obtain from Theorem 9.3 that

$$\begin{array}{ccc} H^1(\Gamma, \mathcal{O}(V)^*) & \xrightarrow{\delta} & H^2(\Gamma, \mathbb{Z}) \\ \downarrow\wr & & \downarrow\wr \\ H^1(X, \mathcal{O}_X^*) & \xrightarrow[c_1]{} & H^2(X, \mathbb{Z}) \end{array}$$

is commutative. Hence, $\delta(\psi)$ corresponds to $c_1(L)$. In order to make the link between Chern classes in geometry and the Chern class $\delta\psi$, let us agree to write

$$\boxed{c_1(\psi) := \delta\psi.}$$

We give a more explicit description of $c_1(\psi)$.

Write

$$\psi_s = e^{2\mathbf{i}\lambda_s}, \qquad \lambda_s \in \mathcal{O}(V).$$

Hence, from eq. (5.1)

$$\lambda_{s+t} \equiv s\lambda_t + \lambda_s \mod \mathbb{Z}. \tag{5.2}$$

The above equation holds for all and we can define

$$c_\psi : \Gamma \times \Gamma \longrightarrow \mathbb{Z},$$

$$(s,t) \longmapsto s\lambda_t + \lambda_s - \lambda_{s+t}.$$

Due to Lemma 5.4, we have $c_\psi \in Z^2(\Gamma, \mathbb{Z})$ and in fact

$$[\delta\psi] = [c_\psi] \text{ in } H^2(\Gamma, \mathbb{Z}).$$

We now need to have a good hold of $H^2(\Gamma, \mathbb{Z})$. Since our working definition of this group is by means of *inhomogeneous* cochains, we shall need:

**Lemma 5.10.** *Let $f \in Z^2(\Gamma, \mathbb{Z})$ and define*

$$Af(s,t) = f(s,t) - f(t,s).$$

*Then:*

*i) $Af$ belongs to* $\mathrm{Alt}^2(\Gamma, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\wedge^2\Gamma, \mathbb{Z})$.

*ii) The function $A : Z^2(\Gamma, \mathbb{Z}) \to \mathrm{Alt}^2(\Gamma, \mathbb{Z})$ is surjective and*

*iii) defines an isomorphism of groups*

$$H^2(\Gamma, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Alt}^2(\Gamma, \mathbb{Z}).$$

*Proof.* (i) Clearly $Af(s,t) = -Af(t,s)$. We now need to show

$$Af(s+t,u) = Af(s,u) + Af(t,u).$$

Unravelling, we need to prove that

$$f(s+t,u)-f(u,s+t)+f(u,s)-f(s,u)+f(u,t)-f(t,u) \tag{5.3}$$

is zero. From $f \in Z^2(\Gamma,\mathbb{Z})$, we know that

$$\begin{aligned} d^2f(s,t,u)=f(t,u)-f(s+t,u)+f(s,t+u)-f(s,t) &= 0 && (5.4)\\ d^2f(u,s,t)=f(s,t)-f(u+s,t)+f(u,s+t)-f(u,s) &= 0 && (5.5)\\ d^2f(s,u,t)=f(u,t)-f(s+u,t)+f(s,u+t)-f(s,u) &= 0. && (5.6)\end{aligned}$$

Then (5.4)+(5.5)−(5.6) equals the negative of (5.3) and vanishes.

(ii) Let $\ell,\ell' \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z})$. Define an element $\ell\ell' \in Z^1(\Gamma,\mathbb{Z})$ by $\ell\ell' : (\gamma,\gamma') \mapsto \ell(\gamma)\ell'(\gamma')$. (Recall that $\Gamma$ acts trivially on $\mathbb{Z}$.)

Under the canonical identification

$$\wedge^2\mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}\left(\wedge^2\Gamma,\mathbb{Z}\right) = \mathrm{Alt}^2_{\mathbb{Z}}(\Gamma,\mathbb{Z}),$$

$A(\ell\ell') \in \mathrm{Alt}^2(\Gamma,\mathbb{Z})$ corresponds to $\ell\wedge\ell'$. Now $\{\ell\wedge\ell' : \ell,\ell' \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z})\}$ generated $\mathrm{Alt}^2(\Gamma,\mathbb{Z})$, and we are done.

(iii) We show that $B^1(\Gamma,\mathbb{Z}) \subset \mathrm{Ker}\,A$. By definition, for $f \in C^1(\Gamma,\mathbb{Z})$, we have $df(s,t)=f(t)-f(s+t)+f(s)$ and hence

$$A(df)(s,t)=f(t)-f(s+t)+f(s)-f(t)+f(t+s)-f(t)=0.$$

So $A$ factors through $H^2(\Gamma,\mathbb{Z})$ and we obtain a surjective morphism $A : H^2(\Gamma,\mathbb{Z}) \to \mathrm{Alt}^2(\Gamma,\mathbb{Z})$.

We know that $H^2(\Gamma,\mathbb{Z})$ is isomorphic to $\mathrm{Alt}^2(\Gamma,\mathbb{Z})$ and a surjective endomorphism of free $\mathbb{Z}$-modules of equal rank is always an isomorphism. □

Let us revise what has been done so far. Starting from a line bundle $L \to X$, we obtained its multipliers: a 1–cocyle $\psi \in Z^1(\Gamma,\mathcal{O}(V)^*)$. From it, we derived an element

$$c_\psi \in Z^2(\Gamma,\mathbb{Z}).$$

We now pass to the *alternating form*

$$C_\psi = A(c_\psi),$$

which, is explicitly defined by

$$\boxed{C_\psi(s,t)=\lambda_s-\lambda_t+s\lambda_t-t\lambda_s \in \mathrm{Alt}^2_{\mathbb{Z}}(\Gamma,\mathbb{Z}).} \tag{5.7}$$

Up until now, we have made no use of $L$, just of $\psi$. A key feature of $C_\psi$, Theorem 5.11 below, requires geometry. Since $C_\psi : \Gamma\times\Gamma \to \mathbb{Z}$ is alternating, it can be extended to

$$C_\psi : V\times V \longrightarrow \mathbb{R}.$$

**Theorem 5.11.** *For each $z,w \in V$, we have*

$$C_\psi(\mathbf{i}z,\mathbf{i}w)=C_\psi(z,w).$$

In order to prove this result, we embark on a deeper study cohomology. In any case, we already identified a fundamental object, the space of Riemann forms:

$$\mathcal{R}(\Gamma) := \left\{ E \in \mathrm{Alt}^2_{\mathbb{R}}(V, \mathbb{R}) \;:\; \begin{array}{c} E \text{ is integral on } \Gamma \text{ and} \\ E(\mathbf{i}v, \mathbf{i}, w) = E(v, w) \\ \text{for all } v, w \in V \end{array} \right\}, \tag{5.8}$$

where $C_\psi$ is to live. Note that $\mathcal{R}(\Gamma)$ can also be defined as

$$\mathcal{R}(\Gamma) = \left\{ E \in \mathrm{Alt}^2_{\mathbb{Z}}(\Gamma, \mathbb{Z}) \;:\; \begin{array}{c} \text{The extension of } E \text{ to } V \text{ admits} \\ \mathbf{i} : V \to V \text{ as automorphism} \end{array} \right\}.$$

# 6 Deeper study of the cohomology of a complex analytic torus

To begin, I'd like to explain a bit about de Rham cohomology of compact Lie groups. Formally it shall not give us what we need to prove Theorem 5.11, but it gives us the idea behind the theory and not simply a proof.

## 6.1 Cohomology of compact Lie groups

Let us consider a *compact Lie group* $G$ of dimension $n$. Define $\mathcal{A}^m_{\mathrm{inv}}(G)$ as the $\mathbb{C}$-space of complex-valued *invariant* differentiable $m$-forms on $G$. Write

$$\mathcal{A}^\bullet_{\mathrm{inv}}(G) = \bigoplus_{0 \le m \le n} \mathcal{A}^m_{\mathrm{inv}}(G).$$

Now, it is not difficult to deduce from Corollary 1.3, or at least its $C^\infty$ version, that

$$\mathcal{A}^m_{\mathrm{inv}}(G) \xrightarrow{\text{evaluation at } e} A^m_e(G)$$

is an isomorphism.

**Theorem 6.1.** *Define on $A^\bullet_e G$ the differential*

$$D_m : A^m_e G \longrightarrow A^{m+1}_e G$$

*given by***

$$D_m\omega(v_1, \ldots, v_{m+1}) = \sum_{i<j} (-1)^{i+j} \omega([v_i, v_j], v_1, \ldots, \underbrace{v_i}_{omit}, \ldots, \underbrace{v_i}_{omit}, \ldots, v_{m+1})$$

*Then:*

*a) For each $\omega \in \mathcal{A}^m_{\mathrm{inv}}(G)$, we have*

$$d\omega(e) = D_m(\omega(e)).$$

**Here I suppose that the reader is at ease with the notion of Lie bracket on $T_eG$. This is obtained by using the Lie bracket of vector fields on $G$ and then bringing it back via invariance.

*b) The inclusion*

$$\mathcal{A}^\bullet_{\mathrm{inv}}(G) \longrightarrow \mathcal{A}^\bullet(G)$$

*together with the isomorphism* $\mathcal{A}^\bullet_{\mathrm{inv}}(G) \simeq A^\bullet_e G$ *define an isomorphism*

$$\frac{\operatorname{Ker} D_m}{\operatorname{Im} D_{m-1}} \simeq H^m_{\mathrm{dR}}(G;\mathbb{C}).$$

*c) If $G$ is abelian, then*

$$H^\bullet(G;\mathbb{C}) \simeq A^\bullet_e G.$$

*Sketch of proof.* Details are well explained in [Bre, V.12]. The main actors are the Haar integral and the process of taking average.

*Invariant integral.* Let $\psi_1, \dots, \psi_n \in \mathcal{A}^1_{\mathrm{inv}}(G)$. Let

$$\Psi = \psi_1 \wedge \cdots \wedge \psi_n \in \mathcal{A}^n_{\mathrm{inv}}(G).$$

Dividing $\Psi$ by a constant, we assume that $\int_G \Psi = 1$. We then get a linear functional

$$C^\infty(G,\mathbb{C}) \xrightarrow{I} \mathbb{C}$$

$$f \longmapsto \int_G f\Psi.$$

Let $L_g : C^\infty(G,\mathbb{C}) \to C^\infty(G,\mathbb{C})$ and $R_g : C^\infty(G,\mathbb{C}) \to C^\infty(G,\mathbb{C})$ be "left" and "right" translations. Then $I(f) = I(L_g f) = I(R_g f)$. Invariance "on the left" is simple because $\Psi$ is left invariant; right invariance comes with compactness.

*Taking the mean.* Let $\omega \in \mathcal{A}^m(G)$. Given $v_1, \dots, v_m \in T_pG$, construct a function

$$F_{\omega,v_1,\dots,v_m} : g \longmapsto [\tau_g^*\omega](p)(v_1, \dots, v_n)$$

We define $\mathrm{M}\omega(p) \in A^m_p(G)$ by

$$(v_1, \dots, v_m) \longmapsto I(F_{\omega,v_1,\dots,v_m})$$

A simple calculation shows that $\mathrm{M}\omega \in \mathcal{A}^m_{\mathrm{inv}}(G)$. Working a bit further, we have

$$\underbrace{\mathrm{M}(\omega) - \omega}_{\text{exact}}$$

which shows that

$$H^m_{\mathrm{dR}}(G/\mathbb{C}) \simeq \frac{\text{closed forms in } \mathcal{A}^m_{\mathrm{inv}}(G)}{d\mathcal{A}^{m-1}_{\mathrm{inv}}(G)}.$$

*Conclusion.* We recall that

$$A^m_e G \xrightarrow{\sim} \mathcal{A}^m_{\mathrm{inv}}(G),$$

and take for granted that the differential $d : \mathcal{A}^m(G) \to \mathcal{A}^{m+1}(G)$ satisfies $d\mathcal{A}^m_{\mathrm{inv}}(G) \subset \mathcal{A}^{m+1}_{\mathrm{inv}}(G)$, and is given by the desired expression once transported back to $A^m_e G$. $\square$

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## 6.2 Linear algebra and forms

Let $E$ be a *real* vector space of dimension $2n$. Let

$$A^m = A^m(E) = \left\{\text{Alternating forms } \omega : E^{\times m} \to \mathbb{C}\right\}.$$

Note: $A^m(E)$ is a complex subspace of $\mathrm{Hom}_{\mathbb{R}}\left(E^{\otimes n}, \mathbb{C}\right)$ of dimension $\binom{2n}{m}$.

A complex structure on $E$ is an endomorphism $J \in \mathrm{End}_{\mathbb{R}}(E)$ s.t. $J^2 = -\mathrm{id}$; multiplication by $\mathbf{i}$ is then defined by $J$.

**Lemma 6.2.** *Let $J$ be a complex structure on $E$. Then*

$$A^1(E) = A^{1,0}(E) \oplus A^{0,1}(E),$$

*where $A^{1,0} = \{\varphi \in A^1(E) : \varphi J = \mathbf{i}\varphi\}$ and $A^{0,1} = \{\varphi \in A^1(E) : \varphi J = -\mathbf{i}\varphi\}$.* □

Still supposing $E$ to have a complex structure $J$, let

$$A^{r,0}(E) = \begin{array}{c}\mathbb{C}\text{-space spanned}\\ \text{by the image of}\end{array} \quad A^{1,0} \times \cdots \times A^{1,0} \xrightarrow{\wedge} A^{r,0},$$

$$A^{0,s}(E) = \begin{array}{c}\mathbb{C}\text{-space spanned}\\ \text{by the image of}\end{array} \quad A^{0,1} \times \cdots \times A^{0,1} \xrightarrow{\wedge} A^{0,s},$$

and finally

$$A^{r,s}(E) = \begin{array}{c}\mathbb{C}\text{-space spanned}\\ \text{by the image of}\end{array} \quad A^{r,0} \times A^{0,s} \xrightarrow{\wedge} A^{r+s}(E).$$

This is the space of alternating forms of type $(r, s)$. A simple characterization of these is as follows.

Let $\varphi_1, \ldots, \varphi_n : E \to \mathbb{C}$ form a basis (over $\mathbb{C}$) of the space $A^{1,0}(E)$. Then $\overline{\varphi}_1, \ldots, \overline{\varphi}_n$ is a basis of $A^{0,1}E$. Then, a basis for $A^{r,s}$ is

$$\varphi_{j_1} \wedge \cdots \wedge \varphi_{j_r} \wedge \overline{\varphi}_{k_1} \wedge \cdots \wedge \overline{\varphi}_{k_s}, \qquad \begin{array}{l} j_1 < \cdots < j_r \\ k_1 < \cdots < k_s \end{array}$$

If should be noted that on $A^m(E)$ have the $\mathbb{R}$-linear involution defined by $\alpha \mapsto \overline{\alpha}$ and that $\overline{A^{r,s}} = A^{s,r}$.

Further ahead, we shall require the following Lemma.

**Lemma 6.3.** *A real-valued 2-form $\omega \in A^2(E)$ belongs to $A^{1,1}(E)$ if and only if $\omega(Je, Je') = \omega(e, e')$ for all $e, e' \in E$.* □

Once these observations have been made, we move to geometry.

## 6.3 Forms of mixed type on complex manifolds

Let $M$ be a real analytic manifold of dimension $2n$. Write, for a point $p \in M$,

$$A_p^k M := A^k(T_p M).$$

If $\mathbf{x} : U \to \mathbb{R}^{2n}$ is a local chart, then

$$\{\mathrm{d}_p x_{i_1} \wedge \cdots \wedge \mathrm{d}_p x_{i_k} \; : \; 1 \leq i_1 < \cdots < i_k \leq 2n\}$$

is a $\mathbb{C}$-basis of $A_p^k M$. A function which associates to each $p \in U$ an element $\omega(p) \in A_p^m M$ is said to be differentiable if it can be written as

$$\sum_{1 \leq i_1 < \cdots < i_k \leq 2n} \underbrace{f_{i_1 \cdots i_k}}_{C^\infty} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}.$$

These are called *complex-valued differentiable $m$-forms on $U$*. With little effort, the reader can give the correct definition of differentiable $m$-forms on arbitrary open subsets.

If $M$ happens to be a *complex manifold*, the *real* vector space $T_p M$ inherits a structure of *complex* vector space. Indeed, if $\mathbf{z} : U \to \mathbb{C}^n$ is a local chart about $p$, let us write $z_j = x_j + \mathbf{i} y_j$ we define

$$\mathbf{i}\partial_{x_j} := \partial_{y_j} \quad \text{and} \quad \mathbf{i}\partial_{y_j} = -\partial_{x_j}.$$

We decompose

$$A_p^1 M = A_p^{1,0} M \oplus A_p^{0,1} M$$

where $A_p^{1,0} M$, resp. $A_p^{0,1} M$, is the subspace of 1–forms which are complex linear, resp. anti-linear. Similarly,

$$A_p^k M = \bigoplus_{r+s=k} A_p^{r,s} M.$$

Clearly,

$$\{\mathrm{d}_p z_j := \mathrm{d}_p x_j + \mathbf{i} \, \mathrm{d}_p y_j\}_{j=1}^n$$

is a basis for $A_p^{1,0} M$ and

$$\{\mathrm{d}_p \overline{z}_j = \mathrm{d}_p x_j - \mathbf{i} \, \mathrm{d}_p y_j\}_{j=1}^n$$

Finally, we define $\mathcal{A}_M^{r,s} \subset \mathcal{A}_M^{r+s}$ as the sheaf of all complex and smooth differential forms of type $(r, s)$.

## 6.4 Dolbeault cohomology

In the case of complex analytic manifolds, along with de Rham cohomology, we have Dolbeault cohomology, which are cohomology groups obtained from forms of mixed type.

We can decompose exterior differentiations as follows:

$$\begin{tikzcd}
 & \mathcal{A}_M^{r+1,s} \arrow[dr] & \\
\mathcal{A}_M^{r,s} \arrow[ur, "\partial"] \arrow[rr, "\mathrm{d}"'] \arrow[dr, "\overline{\partial}"'] & & \mathcal{A}_M^{r+1,s} \oplus \mathcal{A}_M^{r,s+1} \\
 & \mathcal{A}_M^{r,s+1}. \arrow[ur] &
\end{tikzcd}$$

There is a $\overline{\partial}$–Poincaré Lemma saying that the map

$$\overline{\partial} : \mathcal{A}^{r,s}(M) \longrightarrow \mathcal{A}^{r,s+1}(M)$$

is surjective if $M$ is a ball [GH, p25]. Hence, on the level of *sheaves* we have a long exact sequence

$$0 \longrightarrow \Omega_M^r \xrightarrow{\overline{\partial}} \mathcal{A}_M^{r,1} \xrightarrow{\overline{\partial}} \mathcal{A}_M^{r,2} \xrightarrow{\overline{\partial}} \cdots \tag{6.1}$$

**Definition 6.4.** Define the Dolbeault cohomology (space) as

$$H_{\mathrm{Dol}}^{r,s}(M) = \frac{\operatorname{Ker} \mathcal{A}^{r,s}(M) \xrightarrow{\overline{\partial}} \mathcal{A}^{r,s+1}(M)}{\overline{\partial} \mathcal{A}^{r,s-1}(M)},$$

Once these groups have been defined, comes the main observation. The proof is not difficult once all the ingredients are in place. Indeed, all one needs is to break up the Dolbeault exact sequence (6.1) and use:

**Proposition 6.5** ([GH, p.42])**.** $H^m(\mathcal{A}_M^{r,s}) = H^m(M, \mathcal{A}_M^k) = 0$ *for* $m \geq 1$.

Then

**Theorem 6.6** ([GH, p.45])**.** *There are isomorphisms*

$$H^s(\Omega^r) \simeq H_{\mathrm{Dol}}^{r,s}(M).$$

## 6.5 Invariance in cohomology Dolbeault cohomology

In the same direction as with Theorem 6.1, we have

**Theorem 6.7.** *Suppose that $X$ is a complex torus. The inclusion*

$$\mathcal{A}_{\mathrm{inv}}^{0,s} X \longrightarrow \mathcal{A}^{0,s}(X)$$

*induces an isomorphism*

$$A_e^{0,s}(X) \xrightarrow{\sim} H_{\mathrm{Dol}}^{0,s}(X)$$

*Consequently, we have isomorhisms*

$$A_e^{0,s}(X) \simeq H^s(\mathcal{O}).$$

A complete proof of this can be found in [MAV, pp 4-7].

# 7 Return to the theory of line bundles on a torus

*Proof of Theorem 5.11.* From the long exact sequence $c_1(L)$ lies in the kernel of the natural map

$$\alpha : H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X).$$

Use the factorization of $\alpha$:

$$H^2(X,\mathbb{Z}) \xrightarrow{u} H^2(X,\mathbb{C}) \xrightarrow{p} H^2(X,\mathcal{O}_X).$$

Then, we move on to the identifications which are expressed in the commutative diagram

$$\begin{tikzcd}
H^2(X,\mathbb{Z}) \arrow[r, "u"] \arrow[d, "\sim"] & H^2(X,\mathbb{C}) \arrow[r, "p"] \arrow[d, "\sim"] & H^2(X,\mathcal{O}_X) \arrow[d, "\sim"] \\
\mathrm{Alt}^2_{\mathbb{Z}}(\Gamma,\mathbb{Z}) \arrow[r, "u"'] & A^2(V) \arrow[r, "\mathrm{proj}"'] & A^{0,2}(V).
\end{tikzcd}$$

Using the decomposition

$$A^2(V) = A^{2,0} \oplus A^{1,1} \oplus A^{0,2}$$

let us write $u(c_1(L)) = c^{0,2} + c^{2,0} + c^{1,1}$. It then follows that $c^{0,2} = 0$. Since $\overline{c^{2,0}} = c^{0,2}$, we conclude that $u(c_1(L)) \in A^{1,1}$. From Lemma 6.3, $u(c_1(L)) \in A^{1,1}$. $\square$

In order to take full advantage of the Riemann form $C_\psi$, we require some gymnastics in hermitian forms. We present the necessary facts as Exercise 10.13.

We can then rephrase the definition of $\mathcal{R}(\Gamma)$ in eq. (5.8):

$$\mathcal{R}(\Gamma) := \left\{ \begin{array}{l} \text{Hermitian forms } H \text{ on } V \\ \text{s.t. } \Im H \text{ integral on } \Gamma \end{array} \right\}. \tag{7.1}$$

**Definition 7.1.** Let $H \in \mathcal{R}(\Gamma)$ have imaginary part $E$.

1) A map $\chi : \Gamma \to S^1$ is a *semi-character for $H$* if

$$\frac{\chi(s+t)}{\chi(s)\chi(t)} = e^{\pi \mathbf{i} E(s,t)}.$$

2) The set of couples $(H, \chi)$, where $H \in \mathcal{R}(\Gamma)$ and $\chi$ is a semi-character for $H$ is the abstract Picard set $\mathcal{P}(\Gamma)$. Endowing $\mathcal{P}(\Gamma)$ with multiplication

$$(H,\chi) \cdot (H',\chi') = (H + H', \chi\chi')$$

we obtain on $\mathcal{P}(\Gamma)$ a group structure; $\mathcal{P}(\Gamma)$ is the *abstract Picard group*.

Define the "Chern class".

$$C : \mathcal{P}(\Gamma) \longrightarrow \mathcal{R}(\Gamma), \quad (H,\chi) \longmapsto H.$$

**Definition 7.2.** For $(H, \chi) \in \mathcal{P}(\Gamma)$, define the *canonical factor of automorphy* as

$$\boxed{k(H, \chi)_s(z) = \chi(s) \exp\left\{\pi H(z, s) + \frac{\pi}{2} H(s, s)\right\}.}$$

**Proposition 7.3.** *The following assertions are true.*

*1)* $k(H, \chi) \in Z^1(\Gamma, \mathcal{O}(V)^*)$.

*2) The map* $k : \mathcal{P}(\Gamma) \to H^1(\Gamma, \mathcal{O}(V)^*)$ *is a morphism of groups.*

*3) The abstract Chern class* $C_{k(H,\chi)}$ *is* $H$*. In particular, any* $H \in \mathcal{R}(\Gamma)$ *is an abstract Chern class.*

*Proof.* (1) and (2) are left to the reader.
(3)We abbreviate $k(H, \chi)$ to $k$ and $\Im H$ to $E$. Write $\chi(s) = e^{2\pi \mathbf{i} \ell(s)}$. Then

$$k_s(z) = \exp 2\pi\mathbf{i} \left[\underbrace{\ell(s) - \frac{\mathbf{i}}{2} H(z, s) - \frac{\mathbf{i}}{4} H(s, s)}_{\lambda_s(z)}\right],$$

and hence, according to eq. (5.7), we have

$$\begin{aligned} E(s, t) &= \lambda_s(z) - \lambda_t(z) + s\lambda_t(z) - t\lambda_s(z) \\ &= -\frac{\mathbf{i}}{2} H(z + s, t) - \frac{\mathbf{i}}{4} H(t, t) - \frac{\mathbf{i}}{2} H(z, s) - \frac{\mathbf{i}}{4} H(s, s) \\ &+ \frac{\mathbf{i}}{2} H(z + t, s) + \frac{\mathbf{i}}{4} H(s, s) + \frac{\mathbf{i}}{2} H(z, t) + \frac{\mathbf{i}}{4} H(t, t) \\ &= \frac{1}{2\mathbf{i}} (H(s, t) - H(t, s)) \\ &= E(s, t). \end{aligned}$$

□

Put

$$\mathcal{P}^0(\Gamma) = \mathrm{Ker}\ \mathcal{P}(\Gamma) \xrightarrow{C} \mathcal{R}(\Gamma).$$

**Theorem 7.4.** *The restriction of*

$$k : \mathcal{P}(\Gamma) \longrightarrow \mathrm{Pic}(X)$$

*to* $\mathcal{P}^0(\Gamma)$ *defines an isomorphism*

$$\mathcal{P}^0(\Gamma) \xrightarrow{\sim} \mathrm{Pic}^0(X).$$

*Proof.* We proceed in two steps working on the side of Galois cohomology.

*Injectivity.* If $\chi, \chi' \in Z^1(\Gamma, S^1) = \mathrm{Hom}(\Gamma, S^1)$ induce isomorphic line bundles, then

$$\chi'_s(z) = \chi_s(z)\Lambda(z + s)\Lambda(z)^{-1}$$

where $\Lambda \in \mathcal{O}(V)^* \Rightarrow |\Lambda(z)| = |\Lambda(z+s)|$ for all $s \in \Gamma \Rightarrow \Lambda$ is bounded $\Rightarrow$ it is constant by the maximum principle.

*Surjectivity.* We work on the side of Galois cohomology. Let

$$\psi \in \mathrm{Ker}\ \ H^1(\Gamma, \mathcal{O}(V)^*) \longrightarrow H^2(\Gamma, \mathbb{Z}).$$

From the exponential exact sequences we have the following commutative diagram

$$\begin{tikzcd}
H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathcal{O}) \arrow[r, two heads, "e^{2\pi\mathbf{i}(-)}"] & \mathrm{Pic}^0(X) \\
H^1(\Gamma,\mathbb{Z}) \arrow[r] \arrow[u, "\sim"] & H^1(\Gamma,\mathcal{O}(V)) \arrow[r, two heads, "e^{2\pi\mathbf{i}(-)}"'] \arrow[u, "\sim"] & \mathrm{Ker}\left(H^1(\Gamma,\mathcal{O}(V)^*) \to H^2(\Gamma,\mathbb{Z})\right). \arrow[u, "\sim"]
\end{tikzcd}$$

It then follows that $[\psi] = [e^{2\pi\mathbf{i}\lambda}]$, with $\lambda \in Z^1(\Gamma, \mathcal{O}(V))$.

Now, we use the diagram (which depends on Theorem 9.3)

$$\begin{tikzcd}
H^1(X,\mathbb{C}) \arrow[r, two heads] & H^1(X,\mathcal{O}) \\
H^1(\Gamma,\mathbb{C}) \arrow[r] \arrow[u, "\sim"] & H^1(\Gamma,\mathcal{O}(V)) \arrow[u, "\sim"']
\end{tikzcd}$$

to conclude that the the inclusion $\mathbb{C} \subset \mathcal{O}(V)$ gives a *surjection*

$$H^1(\Gamma, \mathbb{C}) \longrightarrow H^1(\Gamma, \mathcal{O}(V)).$$

Hence, we may suppose that

$$\psi = e^{2\pi\mathbf{i}\lambda}$$

for a certain $\lambda \in Z^1(\Gamma, \mathbb{C}) = \mathrm{Hom}(\Gamma, \mathbb{C})$.

We now show how to find $\Lambda \in \mathcal{O}(V)$ such that modifying $\lambda \in Z^1(\Gamma, \mathcal{O}(V))$ by the 1-coboundary

$$s \longmapsto \Lambda(z+s) - \Lambda(z)$$

gives us a new 1–cocyle $\mu$ *with values on* $\mathbb{R} \Rightarrow \psi$ can be taken to belong to $\mathrm{Hom}(\Gamma, S^1)$.

As $\Im\lambda \in \mathrm{Hom}(\Gamma, \mathbb{R}) \Rightarrow$ extend to $\Im\lambda \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. There is a *unique* $\Lambda \in \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ with

$$\Im\Lambda = -\Im\lambda.$$

(If $f : V \to \mathbb{R}$ is $\mathbb{R}$-linear and $h = g + \mathbf{i}f$ is $\mathbb{C}$ linear, then $-f(v) + \mathbf{i}g(v) = g(\mathbf{i}v) + \mathbf{i}f(\mathbf{i}v) \Rightarrow g(v) = f(\mathbf{i}v)$.) Now,

$$\mu_s(z) := \overbrace{\underbrace{\lambda_s}_{\text{const.}} + \underbrace{\Lambda(z+s) - \Lambda(z)}_{\Lambda(s)}}^{\text{real}}.$$

In conclusion,

$$[\psi] = [e^{2\pi\mathbf{i}\mu}]$$

with $\mu$ *real-valued* so that $k : H^1(\Gamma, S^1) \to \mathrm{Pic}^0(X)$ is surjective.

$\square$

**Theorem 7.5** (The Appell-Humbert Theorem)**.** *We have a commutative diagram of isomorphisms*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{P}^0(\Gamma) \arrow[r] \arrow[d, "k"', "\sim"] & \mathcal{P}(\Gamma) \arrow[r, "C"] \arrow[d, "k"', "\sim"] & \mathcal{R}(\Gamma) \arrow[r] \arrow[d, "\sim"] & 0 \\ 0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r, "c_1"'] & \mathrm{NS}(X) \arrow[r] & 0. \end{tikzcd}$$

*Proof.* We already know that $\mathcal{P}^0(\Gamma) \to \mathrm{Pic}^0 X$ is an isomorphism. From the commutative diagram

$$\begin{tikzcd} \mathcal{P}(\Gamma) \arrow[rr, "C"] \arrow[d, "k"] & & \mathcal{R}(\Gamma) \arrow[d, hook] \\ H^1(\Gamma, \mathcal{O}(V)^*) \arrow[rr, "\delta"'] \arrow[urr, dashed] & & H^2(\Gamma, \mathbb{Z}) \arrow[r, equal] & \mathrm{Alt}^2_{\mathbb{Z}}(\Gamma, \mathbb{Z}) \end{tikzcd}$$

we conclude that

$$\mathrm{Image}\ H^1(\Gamma, \mathcal{O}(V)^*) \xrightarrow{\delta} H^2(\Gamma, \mathbb{Z})$$

is $\mathcal{R}(\Gamma)$. The snake Lemma finishes the proof. $\square$

The following is true and can be proved with Theorem 9.3.

**Lemma 7.6.** *For each every $x \in X$ and each $(H, \chi)$, define*

$$\chi_x : s \longmapsto \chi(s) \cdot e^{2\pi \mathbf{i} E(x,s)},$$

*where $E = \Im H$. Then $\tau_x^*(L(H, \chi)) \simeq L(H, \chi_x)$.*

**Corollary 7.7** (The theorem of the square)**.** *For each $L \in \mathrm{Pic}(X)$, the map*

$$\phi_L : x \longmapsto \tau_x^*(L) \otimes L^{-1}$$

*is a homomorphism of groups.*

*Proof.* Let $L \in \mathrm{Pic}(X)$ have canonical factor $(H, \chi)$. Write as usual $E = \Im H$. Then $\phi_L(x)$ has canonical factor $(0, e^{2\pi \mathbf{i} E(x,-)})$ and the result follows from the additivity $E(x + y, -) = E(x, -) + E(y, -)$. $\square$

# 8 Duality

†† We have encountered $\mathrm{Pic}^0(X) = \mathrm{Hom}(\Gamma, S^1)$ as an abstract group. We shall now give it a natural structure of complex torus. At the start, the fact that this structure is natural will not be evident.

Let $A(V) = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ and decompose it as $A^{1,0} \oplus A^{0,1}$. Note that $A^{0,1}$ and $A^{1,0}$ are both $n$ dimensional complex vector spaces.

†† I was unable to cover this in the lectures

In addition, given $f \in A^{0,1}(V)$, it is not difficult to see that

$$\Im f(v) = \Re f(\mathbf{i}v) \quad \text{and} \quad \Re f(v) = -\Im f(\mathbf{i}v).$$

Hence,

$$\Im : A^{0,1} \longrightarrow V^*, \quad f \longmapsto \Im f$$

is an *isomorphism* of real vector spaces. Consequently, the pairing

$$\langle \cdot, \cdot \rangle : A^{01}(V) \times V \longrightarrow \mathbb{R}$$

$$(\varphi, v) \longmapsto \Im\varphi(v)$$

is non-degenerate. We now write

$$\boxed{\check{V} := A^{0,1} \quad \text{and} \quad \check{\Gamma} = \{\varphi \in \check{V} \,:\, \langle \varphi, \Gamma \rangle \subset \mathbb{Z}\}}.$$

Since $\langle \cdot, \cdot \rangle$ is non-degenerate, $\check{\Gamma}$ is a lattice.

**Lemma 8.1.** *The canonical map*

$$\check{V} \longrightarrow \operatorname{Hom}\left(\Gamma, S^1\right)$$

*defined by sending* $\varphi$ *to* $e^{2\pi \mathbf{i}\langle \varphi, - \rangle} = e^{2\pi \mathbf{i} \Im \varphi}$ *induces an isomorphism* $\check{V}/\check{\Gamma} \simeq \operatorname{Hom}(\Gamma, S^1)$.

*Proof.* Any $\chi \in \operatorname{Hom}(\Gamma, S^1)$ is of the form $e^{2\pi \mathbf{i} \lambda}$ with $\lambda \in \operatorname{Hom}(\Gamma, \mathbb{R})$. Now, any element of $\operatorname{Hom}(\Gamma, \mathbb{R})$ gives rise to a certain $\lambda \in V^* = \operatorname{Hom}(V, \mathbb{R})$ and in turn to a certain $\varphi \in A^{01}$ s.t. $\Im\varphi = \lambda$. $\square$

**Definition 8.2.** The dual complex torus associated to $X$ is $\check{X} = \check{V}/\check{\Gamma}$.

In order for the above isomorphism of groups to be natural, we require something more. This is the notion of Poincaré bundle.

**Definition 8.3.** A line bundle $P \to X \times \check{X}$ is a Poincaré bundle if for each $\check{x} \in \check{X}$, we have $P|_{X \times \check{x}} \in \check{x}$ and $P|_{e \times \check{X}} \simeq X \times \mathbb{C}$.

To define a Poincaré bundle on $X \times \check{X}$, it is required to find a canonical factor on $V \times \check{V} \times V \times \check{V}$. It is not hard to see that on this complex vector space there is a canonical Hermitian form defined by

$$\boldsymbol{H}(z, \check{z}, w, \check{w}) = \check{z}(w) + \overline{\check{w}(z)}.$$

**Theorem 8.4.** *Define on* $V \times \check{V} \times V \times \check{V}$ *the Hermitian form*

$$\boldsymbol{H}(z, \check{z}, w, \check{w}) = \check{z}(w) + \overline{\check{w}(z)}.$$

*Then*

*1)* $\boldsymbol{\chi}(s, \check{s}) = e^{\pi \mathbf{i} \langle s, \check{s} \rangle}$ *is a semi-character for* $\Gamma \times \check{\Gamma}$.

*2) The line bundle associated to* $(\boldsymbol{H}, \boldsymbol{\chi})$*, call it* $P$*, is a Poincaré bundle.*

*Proof.* 1) Simple.

2) The canonical factor of $(\boldsymbol{H}, \boldsymbol{\chi})$ is

$$k_{s,\check{s}}(z, \check{z}) = e^{\pi \mathbf{i} \langle \check{s}, s\rangle} \cdot \exp \pi \boldsymbol{H}(z, \check{z}, s, \check{s}) + \frac{\pi}{2} \boldsymbol{H}(s, \check{s}, s, \check{s}).$$

Consider the commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & V \times \check{V} \\ \downarrow & & \downarrow \\ X \times \{\check{z}_0\} & \longrightarrow & X \times \check{X} \end{array}$$

and apply Theorem 9.3: The 1-cocycle associated to $P|_{X \times \{\check{z}_0\}}$ is

$$\begin{aligned} k_{s,0}(z) &= \exp \pi \boldsymbol{H}(z, \check{z}_0, s, 0) + \frac{\pi}{2} \boldsymbol{H}(s, 0, s, 0) \\ &= e^{\pi \check{z}_0(s)}. \end{aligned}$$

Now, according to Lemma 8.1, the point $\check{z}_0 \in \operatorname{Pic}^0(X) = \check{V}/\check{\Gamma}$ corresponds to the 1-cocyle

$$\begin{aligned} \psi_s &= e^{2\pi \mathbf{i}\, \Im \check{z}_0(s)} \\ &= e^{\pi \check{z}_0(s)} \cdot e^{-\pi \overline{\check{z}_0(s)}} \\ &= e^{\pi \check{z}_0(s)} \cdot \underbrace{e^{-\pi \overline{\check{z}_0(z+s)}} e^{\pi \overline{\check{z}_0(z)}}}_{\in B^1} \end{aligned}$$

hence showing that the automorphic factor $k_s$ gives produces the line bundle $\check{z}_0$. $\square$

# 9 The cohomology of a quotient

‡‡

Let $Y$ be a topological space on which a group $\Gamma$ acts by homeomorphisms.

In addition, let us suppose that the action is *properly discontinuous*: For each $y \in Y$, there exists an open neighbourhood $V \ni y$ such that $\gamma(V) \cap V \neq \varnothing$ implies $\gamma = e$. It is convenient to call the aforementioned open neighbourhood a *distinguished neighbourhood*.

Let $X$ be the quotient set, the set of all $\Gamma$-orbits in $Y$, for the action. Let $\rho : Y \to X$ be the natural projection associating to $y \in Y$ its orbit $\Gamma y$; obviously $\rho$ is a surjection. We then give $Y$ the *quotient topology*: $U \subset X$ is open if and only if $\rho^{-1}(U)$ is open.

Finally, note that $\rho : Y \to X$ is a topological covering: each $x \in X$ has an open neighbourhood $U$ such that

$$\rho^{-1}(U) = \bigsqcup V_i$$

‡‡I was unable to cover this in the lectures

where $\rho|_{V_i} : V_i \to U$ is a homeomorphism. Any such neighbourhood shall be called *evenly covered.*

We are now interested in studying the cohomology of $X$ by means of the cohomology of $Y$ *and* group cohomology.

Let $\mathcal{F}$ be a sheaf of abelian groups on $X$. Then, if $V \subset Y$ is an open subset, we have an abelian group $\mathcal{F}(\rho(V))$, which gives us a pre-sheaf $\rho^0\mathcal{F}$ on $Y$.

**Definition 9.1.** We define $\rho^*\mathcal{F}$ as the sheaf associated to the pre-sheaf $\rho^0\mathcal{F}$.

*Remark* 9.2. We *warn the reader* that $\rho^*\mathcal{F}$ is, among algebraic geometers, usually denoted by $\rho^{-1}\mathcal{F}$.

It should be noted that $\rho^0\mathcal{F}$ is not a "bad" presheaf since it is at least "separated". For any open covering $V = \cup_i V_i$, the restriction obvious homomorphism of groups $\mathcal{F}(V) \to \prod_i \mathcal{F}(V_i)$ is injective.

Hence, any section $s \in \rho^*\mathcal{F}(Y)$ must be given by $\{V_i, s_i\}$, where $\{V_i\}$ is an open covring of $Y$, $s_i \in \mathcal{F}(\rho(V_i))$ and the restrictions coincide. Hence, on $\rho^*\mathcal{F}$ we have an obvious action of $\Gamma$: to $\{V_i, s_i\}$, we associate $\{\gamma V_i, s_i\}$.

**Theorem 9.3.** *For every sheaf $\mathcal{F}$ on $X$, there exists a homomorphism of abelian groups*

$$u_n^{\mathcal{F}} : H^n(\Gamma, H^0(Y, \rho^*\mathcal{F})) \longrightarrow H^n(X, \mathcal{F})$$

*enjoying the following properties.*

*(Funct) If $f : \mathcal{F} \to \mathcal{G}$ is a map of sheaves, then*

$$\begin{tikzcd} H^n(X,\mathcal{F}) \arrow[r, "H^n(f)"] & H^n(X,\mathcal{G}) \\ H^n(\Gamma\,,\, \rho^*\mathcal{F}(Y)) \arrow[r] \arrow[u, "u"] & H^n(\Gamma\,,\, \rho^*\mathcal{G}(Y)) \arrow[u, "u"'] \end{tikzcd}$$

*commutes.*

*(Iso) If $\rho^*\mathcal{F}$ is acyclic, then each $u$ is an isomorphism.*

*(Long) Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be an exact sequence of sheaves on $X$ for which*

$$0 \longrightarrow \rho^*\mathcal{E}(Y) \longrightarrow \rho^*\mathcal{F}(Y) \longrightarrow \rho^*\mathcal{G}(Y) \longrightarrow 0$$

*is exact. Then the diagram*

$$\begin{tikzcd} \cdots \arrow[r] & H^n(\Gamma\,,\, \rho^*\mathcal{G}(Y)) \arrow[r, "\delta"] \arrow[d, "u_n"] & H^{n+1}(\Gamma\,,\, \rho^*\mathcal{E}(Y)) \arrow[r] \arrow[d, "u_n"] & H^{n+1}(\Gamma\,,\, \rho^*\mathcal{F}(Y)) \arrow[d, "u_{n+1}^{\mathcal{F}}"] \\ \cdots \arrow[r] & H^n(X\,,\, \mathcal{G}) \arrow[r, "\delta"] & H^{n+1}(X\,,\, \mathcal{E}) \arrow[r] & H^{n+1}(X\,,\, \mathcal{F}) \end{tikzcd}$$

*is commutative.*

(Funct.2) *Let $\tilde{\Gamma}$ be a group acting properly on $\tilde{Y}$ with quotient $\tilde{\rho} : \tilde{Y} \to \tilde{X}$. Let $\varphi : \tilde{\Gamma} \to \Gamma$ be a homomorphism, $\beta : \tilde{Y} \to Y$ be a continuous map such that $g(\tilde{\gamma}\tilde{y}) = \varphi(\tilde{\gamma})g(\tilde{y})$. Let $\alpha : \tilde{X} \to X$ be the induced map on quotient spaces.*

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\beta"] \arrow[d, "\tilde{\rho}"'] & Y \arrow[d, "\rho"] \\ \tilde{X} \arrow[r, "\alpha"'] & X \end{tikzcd}$$

*Let $\mathcal{F}$ be a sheaf on $X$. Let $\tilde{\mathcal{F}} = f^*\mathcal{F}$. Then*

$$\begin{tikzcd} H^n(\Gamma, \rho^*\mathcal{F}(Y)) \arrow[r] \arrow[d, "u"'] & H^n(\tilde{\Gamma}, \tilde{\rho}^*\tilde{\mathcal{F}}) \arrow[d, "u"] \\ H^n(X, \mathcal{F}) \arrow[r, "\text{pull-back}"'] & H^n(\tilde{X}, \tilde{\mathcal{F}}) \end{tikzcd}$$

*commutes.*

*Sketch.* The details are in the Appendix to §2 in [MAV, Ch.1]. Those willing to see the bigger picture behind this result, should consult [GT, Ch. V].

Let $X = \cup U_i$ be an open covering of $X$ by distinguished open subsets. Refine this covering to a good covering (Theorem 1.16), which is denoted likewise. For each $i$, pick $V_i \subset \rho^{-1}(U_i)$ s.t.

$$\rho : V_i \longrightarrow U_i$$

is homeomorphism. Denote its inverse by

$$\sigma_i : U_i \longrightarrow V_i.$$

If $U_i \cap U_j \neq \varnothing$, let $\gamma_{ij} \in \Gamma$ be the unique element such that

$$\gamma_{ij}(V_j) \cap V_i \neq \varnothing.$$

See section 1.4 for the existence of $\gamma_{ij}$.

**Definition 9.4.** For each $n$-cochain

$$f : \Gamma^n \longrightarrow \rho^*\mathcal{F}(Y),$$

we define

$$u_n^{\mathcal{F}}(f)_{i_0,\dots,i_n} := \begin{array}{l} \text{Element of } \mathcal{F}(U_{i_0\cdots i_n}) \text{ corresponding to} \\ f(\gamma_{i_0i_1}, \dots, \gamma_{i_{n-1}i_n})\big|_{\sigma_{i_0}(U_{i_0\cdots i_n})} \text{ under the isomorphism} \\ H^0(\sigma_{i_0}(U_{i_0\cdots i_n}), \rho^*(\mathcal{F})) \simeq \mathcal{F}(U_{i_0\cdots i_n}) \end{array}$$

With all these indices, the above definition can be quite complicated. But one can simplify matters by looking at the case of $n = 1$. Hence, we have a 1-cocyle $f : \Gamma \to \rho^*\mathcal{F}(Y)$. For every $i \neq j$, we consider the open $\sigma_i(U_{ij}) = \gamma_{ij}(V_j) \cap V_i$. Then

$$u(f)_{ij} = f(\gamma_{ij})\big|_{\sigma_i(U_{ij})}.$$

□

# 10 Some exercises

**Exercise 10.1.** Let $\mathbf{GL}_n(\mathbb{C})$ have the obvious coordinate functions $\mathbf{z} : \mathbf{GL}_n(\mathbb{C}) \to \mathbb{C}^{n^2}$. It then follows that all analytic vector fields on $\mathbf{GL}_n(\mathbb{C})$ are of the form $\sum_{(i,j)} A_{ij}\partial_{ij}$ with $\partial_{ij} = \dfrac{\partial}{\partial z_{ij}}$ and $A_{ij}$ analytic. Writing $\partial_{ij}(e)^\natural = \sum_{k,\ell} A_{ij,k\ell}\partial_{k\ell}$, find an expression for $A_{ij,k\ell}$.

**Exercise 10.2.** Let $X$ be a compact Riemann surface of genus $g \geq 2$. Show that $X$ does not have a structure of analytic group.

**Exercise 10.3.** Let $G$ be a complex analytic group. Let $v_1, \dots, v_n \in \mathcal{T}(G)$ be invariant vector fields as in Corollary 1.3. For each $g \in G$, define $\{\omega_1(g), \dots, \omega_n(g)\} \subset T_g^*G$ as being the dual basis of $\{v_1(g), \dots, v_n(g)\}$. Show that each $\omega_i$ is invariant.

**Exercise 10.4.** Let $G$ be a complex analytic group. Let $f : \mathbb{P}^1 \to G$ be a map of complex manifolds.

1) Recall that $\Omega_G$ is the sheaf of analytic 1–forms on $G$. Show that for each $\omega \in \Omega(G)$, the form $f^*\omega$ vanishes.

2) Using Proposition 1.5, show that $\mathrm{D}_x f = 0$ for each $x \in \mathbb{P}^1$.

3) Conclude that $f$ is constant.

4) Let $F : \mathbb{P}^n \to G$ be an analytic map. Show that $F$ is constant.

**Exercise 10.5.** Let $X$ be an algebraic variety. Let $v \in \mathcal{T}_X(x)$ be given. For each $f \in \mathcal{O}_x$, define

$$v(f) := v(\mathrm{d}_x f).$$

Show that $v$ satisfies the analogue of equation (1.1). Show that any $k$-linear map satisfying the analogue of eq. (1.1) is of the above form.

**Exercise 10.6.** Let $X$ be a compact and connected analytic group. Let

$$c : X \times X \longrightarrow X, \quad (x, y) \longmapsto [x, y]$$

be the commutator.

i) Let $U \subset e$ be a coordinate neighbourhood. Show that for each $x \in X$, there exist an open neighbourhood $V_x$ of $x$ and an open neighbourhood $W_x$ of $e$ such that $c(V_x \times W_x) \subset U$.

ii) Using the maximum principle [FG, I.4.11], show that there exists a neighbourhood $W$ of $e$ such that $c(X \times W) = \{e\}$.

   *Solution.* Let $V_{x_1} \cup \cdots \cup V_{x_m}$ cover $X$. Consider $W = W_{x_1} \cap \cdots \cap W_{x_m}$. Then, if $w \in W$, know that $c(V_{x_i} \times \{w\}) \subset U \Rightarrow c(X \times \{w\}) \subset U \Rightarrow c(X \times \{w\}) = c(e, w) = e \Rightarrow c|_{X\times W}$ agrees with $e$ on $X \times W$. $\square$

iii) Deduce that $c = e$ allover. (You can apply here the identity principle [FG, 4.10].) Deduce that $X$ is abelian.

iv) Give an example of a compact, connected and *real* analytic group which is not abelian? Where does the above proof break down?

**Exercise 10.7.** Let $X$ and $Y$ be varieties, and $f, g : X \to Y$ morphisms. Show that if $f = g$ on an open and non-empty subset $U \subset X$, then $f = g$.

**Exercise 10.8.** Using Corollary 2.24, give another proof of Corollary 2.23.

**Exercise 10.9.** Let $\mathbb{P}^n$ be the set of all lines passing by the origin in $\mathbb{C}^{n+1}$ and endow it with the quotient topology coming from the surjection

$$\begin{aligned} \rho : \mathbb{C}^{n+1} \setminus \{0\} &\longrightarrow \mathbb{P}^n \\ z &\longmapsto \mathbb{C}z. \end{aligned}$$

1) Let $H_j = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \ : \ z_j = 0\}$ be the "coordinate hyperplanes". For each $j \in \{0, \ldots, n\}$, define

$$A_j := \{\ell \in \mathbb{P}^n \ : \ \ell \text{ not contained in } H_j\}.$$

Show that $A_j$ is open and using the intersections $A_j \cap A_k$, give $\mathbb{P}^n$ the structure of a complex manifold.

*Solution.* Note that $\rho^{-1}(A_j) = \mathbb{C}^{n+1} \setminus H_j$, which is open and hence $A_j$ is open. Let $T_j$ be the hyperplane $z_j = 1$. We define

$$\varphi_j : T_j \longrightarrow A_j, \qquad \varphi_j(t) = \mathbb{C}t.$$

This is a bijection $T_j \to A_j$, with inverse

$$\varphi_j^{-1} : \ell \longmapsto \ell \cap T_j.$$

This inverse can be described analytically as

$$\varphi_j^{-1} : \ell \longmapsto \left(\frac{z_0}{z_j}, \ldots, \frac{z_n}{z_i}\right),$$

where $(z_0, \ldots, z_n) \in \ell \setminus \{0\}$. Now, let $i \neq j$ and let $\ell \in A_i \cap A_j$. Let $z = (z_0, \ldots, z_n) \in T_j$, so that $z_j = 1$. Suppose $\varphi_j(z) \in A_i$ also, which means that $z_i \neq 0$. Then

$$\begin{aligned} \mathbb{C}z &= \mathbb{C}\left(\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}\right) \\ &= \varphi_i\left(\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}\right). \end{aligned}$$

Hence, $\varphi_i^{-1}\varphi_j : T_j \setminus H_i \to T_i \setminus H_j$ is

$$\varphi_i^{-1}\varphi_j\left(z_0, \ldots, z_n\right) = \left(\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}\right).$$

This is analytic. $\square$

2) Define $O(-1)$ as the set of couples $(\ell, v)$, where $\ell \in \mathbb{P}^n$ and $v \in \ell$. Let $\pi : O(-1) \to \mathbb{P}^n$ be the map $\pi(\ell, v) = \ell$. Using the sets $L|_{A_j}$, give $O(-1)$ the structure of analytic line bundle over $\mathbb{P}^n$. *This is known as the "tautological line bundle"*. Describe the class $[O(-1)] \in H^1(\mathbb{P}^n, \mathcal{O}^*)$.

*Solution.* We'll use previous notations. Let $i \neq j$. On $A_i$, we have the analytic "coordinate" functions $\zeta_{ij} : \ell \mapsto z_j/z_i$, where $(z_0, \ldots, z_n)$ is an arbitrary element of $\ell \setminus \{0\}$.

Construct

$$\sigma_i : A_i \times \mathbb{C} \longrightarrow O(-1)$$

by

$$(\ell, c) \longmapsto (\ell, c \cdot \varphi_i^{-1}(\ell)).$$

Then

$$\sigma_i^{-1}\sigma_j : A_{ij} \times \mathbb{C} \longrightarrow A_{ij} \times \mathbb{C}$$

is

$$(\ell, c) \longmapsto (\ell, c\zeta_{ij}(\ell)^{-1}).$$

Indeed, if $z \in \ell \setminus \{0\}$, then $\varphi_i^{-1}(\ell) = (z_0/z_i, \ldots, z_n/z_i)$ and $\zeta_{ij}(\ell) = z_j/z_i$, which gives

$$\begin{aligned}
\sigma_i(\ell, c\zeta_{ij}(\ell)^{-1}) &= (\ell, c\zeta_{ij}(\ell)^{-1} \cdot \varphi_i^{-1}(\ell)) \\
&= (\ell, (z_i/z_j)c \cdot (z_0/z_i, \ldots, z_n/z_i)) \\
&= (\ell, c\varphi_j^{-1}(\ell)).
\end{aligned}$$

$\square$

3) Is it possible to find an analytic section $s : \mathbb{P}^n \to O(-1)$ to $\pi : O(-1) \to \mathbb{P}^n$?

*Solution.* No! This is because we have a morphism of complex manifolds

$$f : O(-1) \longrightarrow \mathbb{P}^n \times \mathbb{C}^{n+1}$$

defined by the fact that $O(-1)$ is a subset of $\mathbb{P}^n \times \mathbb{C}^{n+1}$. Hence, a section $f$ would produce a holomorphic function $f : \mathbb{P}^n \to \mathbb{C}^{n+1}$. As such, it would be constant. Now, the only $v \in \mathbb{C}^n$ belonging to each $\ell \in \mathbb{P}^n$ is 0. Hence, $[0] : \mathbb{P}^n \to O(-1)$ defined by $\ell \mapsto (\ell, 0)$ is the only section. $\square$

**Exercise 10.10.** Let $\mathrm{Pic}(X)$ be the group of isomorphism classes of line bundles over $X$. If $\mathcal{U} = \{U_i, \tau_i\}$ is a trivializing atlas with cocycle $(g_{ij})$, let $\gamma(L)$ be the image of $(g_{ij})$ in $H^1(X, \mathcal{O}_X^*)$.

1. Show that if $\mathcal{V} = \{V_i, \sigma_i\}$ is another trivializing atlas, the $\gamma_{\mathcal{U}}(L) = \gamma_{\mathcal{V}}(L)$.

2. Show that $\gamma : \mathrm{Pic}(X) \to H^1(X, \mathcal{O}_X^*)$ is bijective. Hint: Given $\{g_{ij}\} \in Z^1(\{U_i\}, \mathcal{O}_X^*)$, we consider on the set

$$\tilde{L} := \coprod_i U_i \times \mathbb{C}$$

the following equivalence relation: $(u, \lambda) \sim (v, \mu) \Leftrightarrow u$ belongs to a certain $U_i$, $v$ belongs to a certain $U_j$, and $\mu = g_{ij}(u)\lambda$. Let $L$ be the quotient space endowed with the quotient topology (open in $L$ if and only if its inverse image is open in $\tilde{L}$). Show that $L$ is a vector bundle, and that $\{g_{ij}\}$ is a cocyle for $L$.

3. For a line bundle $L \to X$, denote by $[L]$ its isomorphism classes in Pic. Endowing Pic with the multiplication $[L] \cdot [L']$, describe the multiplication induced on $H^1(X, \mathcal{O}_X^*)$ via the bijection $\gamma$.

**Exercise 10.11** (Test your knowledge $\pi_1$)**.** Let $X$ be a simply connected complex manifold and $f : X \to \mathbb{C}$ a nowhere vanishing holomorphic function. Use your knowledge of covering spaces an homotopies to show that $f$ has a logarithm, that is, there exists $\lambda : X \to \mathbb{C}$ holomorphic such that $e^{2\pi\sqrt{-1}\lambda} = f$. Does this still hold if we omit the simply connectedness assumption? Use cohomology theory to show that the hypothesis "simply connected" can be replaced by $H^1(X, \mathbb{Z}) = 0$.

**Exercise 10.12.** Let $X$ be compact. Show from the exponential sequence that $H^1(X, \mathbb{Z})$ has no torsion.

**Exercise 10.13.** Let $E$ be a real vector space of dimension $2n$. The reader should recall that the structure of a complex vector space on $E$ (compatible with its structure of real space) amounts to an isomorphism $J : E \to E$ s.t. $J^2 = -\mathrm{id}$.

1) Let $\omega \in \mathrm{Alt}^2_{\mathbb{R}}(E, \mathbb{R})$. Let $J$ be a complex structure such that $J$ is "orthogonal" with respect to $\omega$, that is, $\omega(Je, Je') = \omega(e, e')$. Show that $S(e, e') = \omega(Je, e')$ is real and symmetric, and that $J$ is orthogonal with respect to $S$. In addition, prove that $H = S + \mathbf{i}\omega$ is Hermitian with respect to the complex structure $(E, J)$.

2) Let us again endow $E$ with a complex structure $J$. Let $H$. be an hermitian form on $E$. Show that $\omega = \Im H$ is real, alternating and $J$ is "orthogonal" with respect to $\omega$. Show that $S = \Re H$ is real, symmetric and $J$ is "orthogonal" with respect to $S$.

**Exercise 10.14.** Let $X$ be a complex manifold and $\pi : L \to X$ be a holomorphic line bundle over it. Let $\{U_i\}_{i \in I}$ be a trivializing atlas for $L$ (see Definition 4.6). Denote the associated 1-cocycle is $\{g_{ij}\}$.

Let $f : Y \to X$ be an analytic map from another complex manifold $Y$. For each $y \in Y$, let $i(y) \in I$ be such that $f(y) \in U_{i(y)}$. Let $V_{i(y)} \ni y$ be an open neighbourhood of $y$ such that $f(V_{i(y)}) \subset U_i$.

Using the covering of $Y$ hence obtained, show that $L \times_X Y \to Y$ (see eq. (4.1)) is a line bundle and prove that if a 1-cocyle associated to it is $(g_{ij} \circ f)$.

**Exercise 10.15.** 1) Let $G$ be a cyclic group of order $n$ generated by $g$. In the ring $\mathbb{Z}G$, consider the element $I = 1 + g + \cdots + g^{n-1}$.

a) Show that $\mathbb{Z}I = (\mathbb{Z}G)^G$.

b) Let $q = g - 1 \in \mathbb{Z}G$. Using that $qI = 0$, prove that

$$\mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}G \xleftarrow{q} \mathbb{Z}G \xleftarrow{N} \mathbb{Z}G \xleftarrow{q} \cdots$$

is a free resolution of $\mathbb{Z}$. Deduce:

$$H^q(G, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0 \\ 0 & q \text{ is odd.} \\ \mathbb{Z}/n, & \text{otherwise.} \end{cases}$$

2) Let $G$ be a free abelian group with generators $x_1, \ldots, x_n$ so that

$$\mathbb{Z}G = \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}].$$

As this is a commutative ring, the computation of $H^\bullet(G, \mathbb{Z})$ relies on homological theory of commutative algebra.

Define $q_j = x_j - 1$ in $\mathbb{Z}G$. Note that giving $\mathbb{Z}$ the trivial structure of $\mathbb{Z}G$ module amounts to saying that for each $k \in \mathbb{Z}$, we have $q_j \cdot k = 0$, so $\mathbb{Z} = \mathbb{Z}G/(q_1, \ldots, q_n)$, as $\mathbb{Z}G$-modules.

For each $1 \leq p \leq n$, let

$$K_p = \bigoplus_{i_1 < \cdots < i_p} \mathbb{Z}G\, \mathbf{e}_{i_1 \ldots i_p};$$

this is a free $\mathbb{Z}G$-module of rank $\binom{n}{p}$. Define $K_0 = \mathbb{Z}G$. Let, for $p > 1$,

$$d_p : K_p \longrightarrow K_{p-1}$$

be defined by

$$\mathbf{e}_{i_1 \cdots i_p} \longmapsto q_{i_1} \mathbf{e}_{i_2 \cdots i_p} - q_{i_2} \mathbf{e}_{i_1 i_3 \cdots i_p} + \cdots$$

and $d_1$ by $\mathbf{e}_i \mapsto q_i$. The complex

$$K_\bullet : \qquad \underbrace{K_0}_{\mathbb{Z}G} \longleftarrow \underbrace{K_1}_{(\mathbb{Z}G)^n} \longleftarrow \cdots$$

is called the Koszul complex and is a *free resolution* of $\mathbb{Z}G/(q_1, \ldots, q_n)$. (The proof is not difficult, but needs a clever point of view, see [Mat, §16].) Using $K_\bullet$, show that $H^\nu(G, \mathbb{Z})$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(\wedge^\nu G, \mathbb{Z})$.

**Exercise 10.16.** 1) Show that if $V \subset Y$ is open, then $\rho(V)$ is equally open.

2) Let $U \subset X$ be evenly covered and let $V \subset \rho^{-1}(U)$ be such that $\rho|_V : V \to U$ is a homeomorphism. Show that $\rho^{-1}(V) = \sqcup_{\gamma \in \Gamma} \gamma(V)$.

3) Show that any open subset $U \subset X$ is of the form $\rho(V)$, where $V \subset Y$ is open and stable under the action of $\Gamma$.

4) Show that the action is free: given $y \in Y$, the stabilizer subgroup $\mathrm{St}(y) = \{\gamma \in \Gamma : \gamma y = y\}$ is $\{e\}$.

**Exercise 10.17.** Is $\rho^0 \mathcal{F}$ a sheaf?

**Exercise 10.18.** Let $\rho : \mathbb{C} \to \mathbb{C}^\times$ be the covering $z \mapsto e^{2\pi\sqrt{-1}z}$. Describe $\rho^*(\mathcal{O}_\mathbb{C})$.

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