Connections in Algebraic Geometry

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These are expanded notes of the lectures I delivered as a mini-course in IMPA in August 2022 and then again, in part, in Hanoi 2024. I structured the lecture assuming solely that students would be familiar with basic "Grothendieckean" algebraic geometry (e.g. schemes, fibre products, flatness, étale morphisms and some cohomology theory) and some commutative algebra here and there. At then end, in order to make visible a kind of example which is usually overlooked in the literature, I employed the theory of function fields, but provided substantial prerequisites.

The main point of the course was to provide four lectures on different and somewhat complementary points of view of the theory. Each one, with the exception of Lecture 1, can be greatly amplified.

Programme

Lecture 1. Introductory material: Smoothness, étale coordinates and differentials. Vector bundles, frames and co-cycles. Connections and their category. Systems of linear differential equations. Curvature and bracket.

Lecture 2. The Theorem of Atiyah-Weil on the existence of certain connections.

Lecture 3. Principal parts and stratifications (or connections in Grothendieck's language). Connections and $\mathcal{D}$-modules.

Lecture 4. The Gauss-Manin connection in the "birational setting". (I did not cover this in the lectures in Hanoi.)

Lecture 1 and 2

(30/10/24).

A word about vector bundles

$(X, \mathcal{O})$ is a ringed space. A vector bundle is a locally free sheaf of finite rank.

Let $\mathcal{E}$ be a vector bundle of rank r. Given $U \subset X$ open, a set $e_1, \dots, e_r \in \mathcal{E}(U)$ is called a local frame if $(f_1, \dots, f_r) \mapsto f_1 e_1 + \cdots + f_r e_r$ gives an iso. $\mathcal{O}_U^r \simeq \mathcal{E}|_U$.

Let $\{U_\alpha\}$ be an open covering of X. Assume that for each α, we have a local frame $\boldsymbol{e}^\alpha = (e_1^\alpha, \dots, e_r^\alpha)$ defining the trivialization $\tau_\alpha : \mathcal{O}_{U_\alpha}^{\oplus r} \xrightarrow{\sim} \mathcal{E}|U_\alpha$ as before. The isomorphisms

$$g^{\alpha\beta} := \tau_\alpha^{-1} \circ \tau_\beta : \mathcal{O}_{U_{\alpha\beta}}^{\oplus r} \xrightarrow{\sim} \mathcal{O}_{U_{\alpha\beta}}^{\oplus r},$$

give $r \times r$ matrices $(g_{ij}^{\alpha\beta})$. Note that

$$e_j^\beta = \sum_{i=1}^r g_{ij}^{\alpha\beta} \cdot e_i^\alpha, \tag{1}$$

which means that the jth column

$$\begin{pmatrix} g_{1j}^{\alpha\beta} \\ \vdots \\ g_{rj}^{\alpha\beta} \end{pmatrix}$$

is simply the coordinate expression of the vector $e_j^\beta | U_{\alpha\beta}$. Also, we clearly have

$$\begin{aligned} g^{\alpha\beta} g^{\beta\gamma} &= \tau_\alpha^{-1} \tau_\beta \tau_\beta^{-1} \tau_\gamma \\ &= g^{\alpha\gamma}. \end{aligned}$$

This is just the analogue of what you may have seen in the case of line bundles (=invertible sheaves)

$$\operatorname{Pic}(X) \simeq \check{H}^1(X, \mathcal{O}_X^*).$$

A word about Differentials and tangent vector fields.

What we'll use all the time: Let X/k and Y/k be scheme of finite type over a noetherian ring k. If $f : Y \to X$ etale of k-schemes, then $f^* \Omega^1_{X/k} \xrightarrow{\sim} \Omega^1_{Y/k}$. "Differentials on X and Y are the same thing." See for example [Li02, p.223]. (Liu works with restrictive hypothesis that can be removed, but they are good enough!)

Let X/k scheme. Say that X is smooth if: for each $P \in X$, there exists an open U and an etale map $\boldsymbol{x} = (x_1, \dots, x_n) : U \to \mathbf{A}_k^n$. Say that $\boldsymbol{x}$ are "etale" coordinates. In this case, $\Omega_U^1 = \bigoplus_i \mathcal{O}_U \, \mathrm{d}x_i$. The dual basis of $\{\mathrm{d}x_i\}$ is $\{\partial/\partial x_i\}$, or $\{\partial_{x_i}\}$ or $\{\partial_i\}$. The sheaf $\mathcal{T}_X := \mathcal{H}om(\Omega_X, \mathcal{O})$ is the tangent sheaf. In what follows, *a coordinate neighbourhood* $(U, \boldsymbol{x})$ will always mean an affine open together with $\boldsymbol{x} : U \to \mathbf{A}_k^n$ which is etale.

Connections: definition and basic properties

Those who took a course in Riemannian geometry have already encountered the theory of connections and understood that "it is a way to differentiate vector fields". Let X/k be a smooth scheme, where k is some ring. Let $\mathcal{E}$ be a quasi-coherent module.

Definition 1. Connection is k-map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{X/k}$ such that:

$$\nabla(ae) = e \otimes \mathrm{d}a + a\nabla(e), \qquad \forall\, a \in \mathcal{O}_X,\, e \in \mathcal{E}.$$

Example 2. 1) $(\mathcal{O}_X, \mathrm{d})$ is the trivial connection.

2) Let $k = \mathbf{C}$ and $X = \mathbf{A}^1$ and $\mathcal{E} = \mathcal{O}_X \boldsymbol{e}$. Define a connection

$$\nabla(a\boldsymbol{e}) = \boldsymbol{e} \otimes \mathrm{d}a + a\boldsymbol{e} \otimes \mathrm{d}x.$$

Example 3. If ∇ and ∇' are two connections, then

$$\nabla' - \nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_X$$

is an $\mathcal{O}_X$-linear morphism. *Fancy way:* The set of connections is a principal homogeneous space over $H^0(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X))$.

Since $\mathcal{O}_X$–modules of the form $\mathcal{O}_X^r$ always carry connections, we can usually reduce the work of finding a connection to that of giving a matrix.

The covariant derivative

A connection can also be seen as a way of *differentiating sections of* $\mathcal{E}$. Indeed, let $v \in \mathcal{T}_X = \mathcal{H}om(\Omega_X, \mathcal{O}_X)$ be a vector-field. Let $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ connection. Let $\langle -, v\rangle : \mathcal{E} \otimes \Omega_X \to \mathcal{E}$ be $\mathrm{id}_{\mathcal{E}} \otimes_{\mathcal{O}} v$. Define

$$\nabla_v(e) := \langle \nabla e, v\rangle.$$

Then

$$\nabla_v(ae) = v(a)e + a\nabla_v(e). \tag{$*$}$$

This produces an $\mathcal{O}_X$*-linear* map $D : \mathcal{T}_X \to \mathcal{E}nd_k(\mathcal{E})$. Conversely, if

$$D : \mathcal{T}_X \longrightarrow \mathcal{E}nd_k(\mathcal{E})$$

is $\mathcal{O}$-linear arrow s.t. $(*)$ holds, then we can define a connection as follows. Let $(U, \boldsymbol{x})$ be coordinate open of X and let $\{dx_i\}$ and $\{\partial_i\}$ be the dual bases of Ω^1_X and $\mathcal{T}_X$. Let $D_i = D(\partial_i)$; this is an element of $\Gamma(U, \mathcal{E}nd_k(\mathcal{E})) = \mathrm{End}_{\mathcal{O}_U}(\mathcal{E}|_U)$. We define for $e \in \mathcal{E}(V)$:

$$\nabla^U e = \sum_i D_i(e) \otimes dx_i$$

which is a connection on $\mathcal{E}|_U$. It is a simple matter to see that these local definitions glue to a connection ∇.

Categorical considerations

Let X/k be smooth.

Definition 4. Let $(\mathcal{E}, \nabla)$ and $(\mathcal{E}', \nabla')$ be connections. An arrow of $\mathcal{O}_X$-modules $\varphi : \mathcal{E} \to \mathcal{E}'$ is *horizontal* if

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\nabla} & \mathcal{E} \otimes \Omega \\ \varphi \downarrow & & \downarrow \varphi\otimes\mathrm{id} \\ \mathcal{E}' & \xrightarrow[\nabla']{} & \mathcal{E}' \otimes \Omega \end{array}$$

commutes. The category of connections has as

objects the connections and as

arrows the horizontal arrows.

This category shall be denoted by $\mathbf{MC}(X/k)$. (Modules with connections.)

A section $s \in \mathcal{E}(U)$ is *horizontal* if $\nabla s = 0$.

It is not difficult to see that *the category of connections is an abelian subcategory of the category of quasi-coherent sheaves*. This is *best proved by thinking of connections as means of differentiate sections*, as I explained above, and noting that $\Omega^1_{X/k}$ is locally free. Here is an example.

Example 5. Let $k = \mathbb{F}_p$, $X = \mathbf{A}^1$. Let $\mathcal{E} = \mathcal{O}_X\mathbf{e}$ and $\mathcal{F} = \mathcal{O}_X\mathbf{f}$. Let us give both $\mathcal{E}$ and $\mathcal{F}$ the trivial connection: $\nabla\mathbf{e} = \mathbf{e} \otimes 0$ and $\Box\mathbf{f} = \mathbf{f} \otimes 0$. Let $\varphi : \mathcal{E} \to \mathcal{F}$ be $\mathbf{e} \mapsto x^p\mathbf{f}$. Then φ is horizontal since

$$\begin{aligned} \Box(\varphi(\mathbf{e})) &= \Box(x^p\mathbf{f}) \\ &= x^p \cdot (\mathbf{f} \otimes 0) + \mathbf{f} \otimes d(x^p) \\ &= 0 \\ &= (\varphi \otimes \mathrm{id})\nabla\mathbf{e}. \end{aligned}$$

It then follows that $\mathcal{C}oker(\varphi)$, which is the skyscraper sheaf supported at 0 carries a connection.

Exercise 6. Keep the context of the previous example, but replace $k = \mathbb{F}_p$ by $k = \mathbf{C}$. Show that the horizontal arrows are $\mathbf{e} \mapsto c\mathbf{f}$, with $c \in \mathbf{C}$.

Example 7 (Typical of char. p)**.** X/k smooth over a perfect field of characteristic $p > 0$; let $F : X \to X$ be the absolute Frobenius morphism and for $\mathcal{O}_X$-module $\mathcal{H}$ consider $\mathcal{E} := F^*\mathcal{H}$. It is an important fact that F is a finite and flat morphism of schemes. We now endow $\mathcal{E}$ with a canonical connection, ∇^{can}.

Let U be affine open such that $F : \mathcal{O}(U) \to \mathcal{O}(U)$ is free; said otherwise

$$\mathcal{O}(U) = \oplus_{j=1}^{s} \mathcal{O}(U)^p y_j \tag{F}$$

for functions $\{y_j\}$. (For regular schemes, the absolute Frobenius morphism is flat. This can be verified by the "magical" property assuring flatness [M89, 23.1] or by passing to algebraic closures and completions.) For each $v \in \mathcal{T}(U)$, define:

$$\nabla_v^{\mathrm{can}}(\sum a_i \otimes h_i) = \sum v(a_i) \otimes h_i.$$

To eliminate any ambiguity this expression may cause, we use (F) to compute. It is clear that this local definition extends to give $\mathcal{E}$ a connection ∇^{can}.

Exercise 8. Let $k = \mathbb{F}_p$ and $X = \mathbf{P}^1$. Use the above technique to give $\mathcal{O}_{\mathbf{P}^1}(p)$ a connection.

The category **MC** is also "tensorial". This means that for $(\mathcal{E}, \nabla)$ and $(\mathcal{E}', \nabla')$, the module $\mathcal{E}'' = \mathcal{E} \otimes \mathcal{E}'$ comes with a connection ∇'' *if we differentiate in a Lie-like manner:*

$$\boxed{\nabla''_v(e \otimes e') = \nabla_v(e) \otimes e' + e \otimes \nabla'_v(e').}$$

Hence, we have also connections on symmetric products, alternating products, duals, etc. Most details on the conventions concerning these come in [Ka70], which is the main reference basic material.

Example 9 (Etale covering)**.** Let $\pi : Y \to X$ be an etale finite covering. Then $\mathcal{A} := \pi_*\mathcal{O}_Y$ comes with a connection defined by $\pi_*\mathrm{d} : \mathcal{A} \to \pi_*\Omega_Y$. Recall that $\pi^* : \pi^*\Omega_X^1 \xrightarrow{\sim} \Omega_Y^1$ is iso. so that $\mathrm{d} : \mathcal{A} \to \pi_*\Omega_Y \simeq \mathcal{A} \otimes_{\mathcal{O}_X} \Omega_X^1$ is a connection. Let me be concrete and work over $\mathbf{Q}$. Let $Y = (y^n - x) \subset (\mathbf{A}^1 \setminus \{0\})^2$ and $\pi : Y \to X = (\mathbf{A}^1 \setminus \{0\})$ be the first projection. On

$$A = \mathbf{Q}[y^{\pm}] = \mathbf{Q}[x^{\pm}] \oplus \cdots \oplus \mathbf{Q}[x^{\pm}]y^{n-1}$$

we have a connection determined by $\nabla y^\ell = \mathrm{d}(y^\ell) = \dfrac{\ell}{n} y^\ell \otimes \dfrac{\mathrm{d}x}{x}$. Note: $\pi_*\mathcal{O}_Y$ is in this case a direct sum $\mathcal{L}^{\otimes 0} \oplus \mathcal{L}^{\otimes 1} \oplus \cdots \oplus \mathcal{L}^{\otimes n-1}$, where $\mathcal{L}$ is the connection determined by "y".

Connections $\times$ differential equations

Let X/k smooth n-dimensional. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$–module of rank r endowed with a connection ∇. Let $(U, \boldsymbol{x})$ be an étale chart and $e_1, \ldots, e_r$ be a local frame for $\mathcal{E}|_U$. If

$$\nabla_{\partial_{x_\nu}}(e_j) = \sum_{i=1}^r a_{ij}^{(\nu)} e_i,$$

then we define

$$\mathrm{Mat}_{\boldsymbol{e}}(\nabla_{\partial_{x_\nu}}) := (a_{ij}^{(\nu)})_{1 \le i,j \le r}.$$

Writing

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}$$

and letting $\boldsymbol{e} \cdot \boldsymbol{y} = y_1 e_1 + \cdots + y_r e_r$, we conclude that

$$\boxed{\nabla_{\partial_{x_\nu}}(\boldsymbol{e} \cdot \boldsymbol{y}) = \boldsymbol{e} \cdot \partial_{x_\nu}(\boldsymbol{y}) + \boldsymbol{e} \cdot (A^{(\nu)} \cdot \boldsymbol{y}).}$$

Definition 10. The (system of linear) differential equations associated to ∇, are the equations required to define *horizontal* sections:

$$\nabla_{\partial_{x_\nu}} \sum_j y_j \cdot e_j = 0 \qquad (\nu = 1, \ldots, n).$$

This translates into

$$\begin{aligned} 0 &= \sum_j \partial_{x_\nu}(y_j) \cdot e_j + \sum_j y_j \cdot \sum_i a_{ij}^{(\nu)} \cdot e_i \\ &= \sum_i \left(\partial_{x_\nu}(y_i) + \sum_j a_{ij}^{(\nu)} \cdot y_j \right) \cdot e_i. \end{aligned}$$

Hence

$$y_1 e_1 + \cdots + y_r e_r \text{ horizontal} \iff \partial_{x_\nu} y_i + \sum_{j=1}^{r} a_{ij}^{(\nu)} \cdot y_j = 0.$$

More synthetically,

$$\boxed{\text{Horizontal sections } \leftrightsquigarrow \partial_{x_\nu}\boldsymbol{y} = -\mathrm{Mat}_{\boldsymbol{e}}(\partial_{x_\nu}) \cdot \boldsymbol{y}.}$$

These are the *matrix differential equations* associated to the connection ∇, the local frame $\boldsymbol{e}$ and the coordinate system $\boldsymbol{x}$.

Example 11. $X = \mathbf{A}^1_{\mathbf{C}}$. Consider $(\mathcal{O}\boldsymbol{e}, \nabla)$ with $\nabla \boldsymbol{e} = -\boldsymbol{e} \otimes \mathrm{d}x$. Then, a horizontal section $y\boldsymbol{e}$ amounts to a solution of $y' = y$.

Reciprocally, given matrix differential equations

$$\partial_{x_\nu}\boldsymbol{y} + A^{(\nu)} \cdot \boldsymbol{y} = 0 \qquad (\nu = 1, \ldots, n)$$

with $A^{(\nu)} \in \mathrm{Mat}_r(\mathcal{O}(U))$, we obtain a connection on the trivial bundle $\mathcal{O}_U^r$ such that the associated matrix DEs are simply the ones we started with.

Example 12. Consider a linear differential equation

$$\frac{dy}{dx} + Ay = 0, \quad A \in \mathrm{Mat}_r(\mathcal{O}(X)),$$

which we then transform into a connection on $\oplus_i^r \mathcal{O}_X e_i$ by

$$\nabla e_j = Ae_j \otimes \mathrm{d}x.$$

The determinant bundle $\wedge^r \mathcal{E}$ is $\mathcal{O}_X \underbrace{e_1 \wedge \cdots \wedge e_r}_{t}$. Now,

$$\begin{aligned} \nabla_{\partial_x} t &= Ae_1 \wedge e_2 \wedge \cdots \wedge e_r + e_1 \wedge Ae_2 \wedge \cdots \wedge e_r + \cdots e_1 \wedge \cdots \wedge Ae_r \\ &= a_{11}t + a_{22}t + \cdots + a_{rr}t \\ &= \mathrm{Tr}(A) \cdot t. \end{aligned}$$

This is, of course, in the theory of differential equations, known as "Liouville's formula", c.f. [Ha46, IV, Theorem 1.2].

Scalar differential equations

Let X be an open of $\mathbf{A}^1_k$. Let

$$L = y^{(r)} + a_{r-1} \cdot y^{(r-1)} \ldots + a_0 \cdot y$$

be a differential polynomial with coefficients in $\mathcal{O}(X)$. We then construct the differential system

$$\begin{aligned} y' &= y_1 \\ \vdots &= \vdots \\ y'_{r-1} &= -a_{r-1} \cdot y_{r-1} - \cdots - a_0 \cdot y. \end{aligned}$$

Hence, the linear differential system associated to L is defined by

$$\frac{d\boldsymbol{y}}{dx} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 0 & 1 \\ -a_0 & \cdots & \cdots & \cdots & -a_{n-1} \end{pmatrix} \cdot \boldsymbol{y},$$

and the connection attached to L on $\oplus_{i=1}^n \mathcal{O}_X e_i$, ∇^L, is defined by

$$\mathrm{Mat}_{\boldsymbol{e}}(\nabla^L) = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 0 & -1 \\ a_0 & \cdots & \cdots & \cdots & a_{n-1} \end{pmatrix}$$

(Here the conventions vary.)

Example 13. If $\mathcal{L}$ is as in Ex. 9, then we see that $\mathcal{L}$ is just associated to $L := y' + (1/nx)y$.

Connection matrices and change of local frame

Let X/k smooth. Let $\mathcal{E}$ be vb of rank r. Let $\boldsymbol{e} = \{e_1, \ldots, e_r\}$ and $\boldsymbol{e}' = \{e'_1, \ldots, e'_r\}$ be local frames on some open U of X. We write

$$\nabla e_j = \sum_i e_i \otimes \theta_{ij} \quad \text{and} \quad \nabla e'_j = \sum_i e'_i \otimes \theta'_{ij}.$$

The matrices θ and θ' are the *connection matrices*; they are 1-forms with values on $\mathfrak{gl}_n(k)$. Let g be the matrix of $\boldsymbol{e}'$ on the basis $\boldsymbol{e}$:

$$e'_j = \sum_i g_{ij} e_i.$$

Exercise 14. Show
$$\boxed{\theta' = g^{-1}\mathrm{d}g + \mathrm{Ad}_{g^{-1}}\theta,} \tag{2}$$

I only worked with locally free sheaves, but in many cases one has G-bundles, where G is some linear algebraic group. In these cases, one also has a *standard definition of connection*. In most treatises of Differential Geometry, this definition is quite long and has the disadvantage of talking about connections forms on the *total space* [KN63].

Here formula (2) can be useful in taking hold of things. The term $g^{-1}\mathrm{d}g$ is the pull-back under $g : U \to \mathbf{GL}_n$ of *the Maurer-Cartan form* [HN12, p.311]. Hence, we can say that a connection on a principal G-bundle boils down to a family of Lie G-valued 1-forms which behave according to (2). See [KN63, Prp. 1.4].

Curvature: definition

We now define a k–linear map
$$\nabla_\ell : \mathcal{E} \otimes \Omega^\ell \longrightarrow \mathcal{E} \otimes \Omega^{\ell+1}$$
by
$$\nabla_\ell(e \otimes \omega) = \nabla(e) \wedge \omega + e \otimes d\omega.$$

Remark 15. This is *not* the usual convention, but it is a less complicated version and follows [Be74, p.125].

Note that the above formula may depend on the way we represent elements of $\mathcal{E} \otimes \Omega^\ell$. To verify that this unambiguous, I need to show
$$\nabla(fe) \wedge \omega + fe \otimes \mathrm{d}\omega = \nabla(e) \wedge f\omega + e \otimes \mathrm{d}(f\omega). \tag{3}$$
and apply this to neighbourhoods where Ω^ℓ is free: But
$$\begin{aligned}\nabla(fe) \wedge \omega + fe \otimes \mathrm{d}\omega &= f\nabla(e) \wedge \omega + e \otimes \mathrm{d}f \wedge \omega + fe \otimes \mathrm{d}\omega \\ &= \nabla(e) \wedge \omega + e \otimes \mathrm{d}(f\omega),\end{aligned}$$
and we are done.

Proposition 16. *The composition $\nabla_{\ell+1} \circ \nabla_\ell$ is $\mathcal{O}_X$–linear.*

Proof. Let $f \in \mathcal{O}(U)$, $e \in \mathcal{E}(U)$ and $\omega \in \Omega^\ell_X(U)$. Then, if $\nabla(e) = \sum_i e_i \otimes \theta_i$, it follows that
$$\begin{aligned}\nabla_{\ell+1}\nabla_\ell(fe \otimes \omega) &= \nabla_{\ell+1}\left(\nabla(e) \wedge f\omega + e \otimes \mathrm{d}f \wedge \omega + e \otimes f\mathrm{d}\omega\right) \\ &= \nabla_{\ell+1}\left(\sum_i e_i \otimes f\theta_i \wedge \omega + e \otimes \mathrm{d}f \wedge \omega + e \otimes f\mathrm{d}\omega\right) \\ &= \sum_i \nabla(e_i) \wedge f\theta_i \wedge \omega + \sum_i e_i \otimes \mathrm{d}(f\theta_i \wedge \omega) + \\ &+ \sum_i e_i \otimes \theta_i \wedge \mathrm{d}f \wedge \omega + e \otimes \mathrm{d}(\mathrm{d}f \wedge \omega) + \\ &+ \sum_i e_i \otimes \theta_i f \wedge \mathrm{d}\omega + e \otimes \mathrm{d}(f\mathrm{d}\omega).\end{aligned}$$

Now $\mathrm{d}(f\mathrm{d}\omega) = \mathrm{d}f \wedge \mathrm{d}\omega$ while $\mathrm{d}(\mathrm{d}f \wedge \omega) = -\mathrm{d}f \wedge \omega$. Hence,

$$\nabla_{\ell+1}\nabla_\ell(fe \otimes \omega) = \sum_i \nabla(e_i) \wedge f\theta_i \wedge \omega + \sum_i e_i \otimes (\mathrm{d}(f\theta_i \wedge \omega) + \theta_i \wedge \mathrm{d}f \wedge \omega + f\theta_i \wedge \mathrm{d}\omega)\,.$$

Now

$$\begin{aligned} \mathrm{d}(f\theta_i \wedge \omega) &= \mathrm{d}(f\theta_i) \wedge \omega - f\theta_i \wedge \mathrm{d}\omega \\ &= \mathrm{d}f \wedge \theta_i \wedge \omega + f\mathrm{d}\theta_i \wedge \omega - f\theta_i \wedge \mathrm{d}\omega \\ &= -\theta_i \wedge \mathrm{d}f \wedge \omega + f\mathrm{d}\theta_i \wedge \omega - f\theta_i \wedge \mathrm{d}\omega. \end{aligned}$$

Therefore,

$$\begin{aligned} \nabla_{\ell+1}\nabla_\ell(fe \otimes \omega) &= \sum_i \nabla(e_i) \wedge f\theta_i \wedge \omega + \sum_i e_i \otimes f\mathrm{d}\theta_i \wedge \omega \\ &= f \cdot (\nabla_{\ell+1}\nabla_\ell(e \otimes \omega))\,. \end{aligned}$$

□

Definition 17. The $\mathcal{O}_X$–linear map $\nabla_1\nabla_0 : \mathcal{E} \to \mathcal{E} \otimes \Omega^2_X$ is called the curvature of ∇ and is denoted by R_∇. A connection is *integrable* when $R_\nabla = 0$.

Exercise 18 (Cartan's formula)**.** Let $\mathcal{E}$ be locally free, $e_1, \ldots, e_r$ be a local frame and $\theta = (\theta_{ij})$ the connection matrix. Let $R_\nabla e_j = \sum_i e_i \otimes \Theta_{ij}$ define the curvature matrix Θ. Show

$$\mathrm{d}\theta + \theta \wedge \theta.$$

Corollary 19. *Let $e_1, \ldots, e_r$ be a local frame of $\mathcal{E}$ over U, and assume that we have local etale coordinates $(\boldsymbol{x}, U)$. Let*

$$\theta = \sum_{\mu=1}^{n} A_\mu \,\mathrm{d}x_\mu$$

be the expression of the connection matrix on the basis $\{e_1, \ldots, e_r\}$. Then, the curvature matrix is

$$\sum_{\mu<\nu} \left\{\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right\} \,\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu.$$

Proof. Write $\theta_{ij} = \sum_\nu a^{(\nu)}_{ij} \,\mathrm{d}x_\nu$. Then $\mathrm{d}\theta_{ij} = \sum_\nu \mathrm{d}a^{(\nu)}_{ij} \wedge \mathrm{d}x_\nu$. This is $\sum_{\mu,\nu} \partial_\mu a^{(\nu)}_{ij} \,\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu$, which can be rewritten as $\sum_{\mu<\nu} \{\partial_\mu a^{(\nu)}_{ij} - \partial_\nu a^{(\mu)}_{ij}\} \,\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu$.

On the other hand, $\theta \wedge \theta = \sum_{\mu,\nu} A_\mu A_\nu \,\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu$, which is $\sum_{\mu<\nu}\{A_\mu A_\nu - A_\nu A_\mu\} \,\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu$. This finishes the proof. □

Vanishing curvature and a result about existence and uniqueness of differential systems

In order to understand the origin of "integrability", I shift to the complex analytic setting, and recall the following result. It is common to find a proof of this result which employs the Frobenius theorem for distributions, but it is at the end just a theorem on linear differential equations on the complex plane, that is, Cauchy's theorem [Ca85, VII].

Theorem 20. *Let $D \subset \mathbf{C}^n$ be an open polydisk about origin. Let $A^{(1)}, \ldots, A^{(n)} \in \mathrm{Mat}_r(\mathcal{O}(D))$. Consider the system of linear PDEs*

$$\partial_\mu \boldsymbol{y} + A^{(\mu)} \cdot \boldsymbol{y} = 0, \qquad \mu \in \{1, \ldots, n\}. \tag{I}$$

1) Consider the matrix $\theta = \sum_\mu A^{(\mu)} \, \mathrm{d}x_\mu$ and let ∇ be the connection on $\oplus_{i=1}^r \mathcal{O}_D e_i$ defined by θ. The integrability condition is

$$\left\| \begin{matrix} \partial_\mu & \partial_\nu \\ A^{(\mu)} & A^{(\nu)} \end{matrix} \right\| + [A^{(\mu)}, A^{(\nu)}] = 0.$$

2) Let $\boldsymbol{c} \in \mathbf{C}^r$. Then there exists a unique holomorphic

$$\phi : D \longrightarrow \mathbf{C}^r$$

satisfying all systems (I) and such that $\phi(\mathbf{0}) = \boldsymbol{c}$ □

Proof. Only (2) deserves attention. We work by induction on n. The case $n = 1$ is "well-known" and in addition, can be taken to depend upon parameters. By induction, let

$$\psi : D^{n-1} \longrightarrow \mathbf{C}^r$$

be such that for all $j = 2, \ldots, n$ we have

$$\partial_j \psi(\boldsymbol{x}) + A^{(j)}(0, \boldsymbol{x}) \psi(\boldsymbol{x}) = 0, \quad \text{and} \quad \psi(0) = \boldsymbol{c}.$$

Let now

$$\varphi : D \times D^{n-1} \longrightarrow \mathbf{C}^r$$

holomorphic s.t.

$$\partial_1 \varphi + A^{(1)} \cdot \varphi = 0 \quad \text{and} \quad \varphi(0, \boldsymbol{x}) = \psi(\boldsymbol{x}).$$

Note that for $j > 1$ we have $\partial_j \varphi(0, \boldsymbol{x}) + A^{(j)}(0, \boldsymbol{x}) \cdot \varphi(0, \boldsymbol{x}) = 0$. For fixed $j > 1$, consider the function $v = \partial_j \varphi + A^{(j)} \cdot \varphi$. Then, using the integrability condition, we have $\partial_1 v + A^{(1)} \cdot v = 0$ so that $v \equiv 0$. □

Curvature and the bracket

We now relate the above definition of curvature with its more established version in Differential Geometry, see [KN, III, Theorem 5.1] or [dC92, IV.2.1].

Let now $\nabla e = \sum_j e_j \otimes \theta_j$. We also write $a_{ij} = \langle \partial_i, \theta_j \rangle$, so that $\theta_j = \sum_i a_{ij}\, \mathrm{d}x_i$. Then

$$\begin{aligned}\nabla_{\partial_h} \nabla_{\partial_i}(e) &= \nabla_{\partial_h} \sum_j e_j \cdot a_{ij} \\ &= \sum_j \nabla_{\partial_h}(e_j) \cdot a_{ij} + e_j \cdot \partial_h(a_{ij}).\end{aligned}$$

And

$$\nabla_{\partial_i} \nabla_{\partial_h}(e) = \sum_j \nabla_{\partial_i}(e_j) \cdot a_{hj} + e_j \cdot \partial_i(a_{hj}).$$

On the other hand,

$$\begin{aligned}R_\nabla e &= \sum_j \nabla(e_j) \wedge \theta_j + \sum_j e_j \otimes \mathrm{d}\theta_j \\ &= \sum_j \left(\sum_i \nabla_{\partial_i}(e_j) \otimes \mathrm{d}x_i \right) \wedge \left(\sum_i a_{ij} \mathrm{d}x_i \right) + \sum_j e_j \otimes \left(\mathrm{d} \sum_i a_{ij} \mathrm{d}x_i \right) \\ &= \sum_j \left(\sum_{h<i} [\nabla_{\partial_h}(e_j) \cdot a_{ij} - \nabla_{\partial_i}(e_j) \cdot a_{hj}] \otimes \mathrm{d}x_h \wedge \mathrm{d}x_i \right) + \\ &\quad + \sum_j e_j \otimes \sum_{h<i} [\partial_h(a_{ij}) - \partial_i(a_{hj})]\, \mathrm{d}x_h \wedge \mathrm{d}x_i.\end{aligned}$$

Hence

$$\begin{aligned}\langle R_\nabla e \ ,\ \partial_h \wedge \partial_i \rangle &= \sum_j \nabla_{\partial_h}(e_j) \cdot a_{ij} - \nabla_{\partial_i}(e_j) \cdot a_{hj} + e_j \cdot \partial_h(a_{ij}) - e_j \cdot \partial_i(a_{hj}) \\ &= \nabla_{\partial_h} \nabla_{\partial_i}(e) - \nabla_{\partial_i} \nabla_{\partial_h}(e).\end{aligned}$$

This allows us to derive the usual expression relating the curvature with the Lie bracket. Recall that for derivations $v, w \in \mathcal{T}_X(U)$, their bracket $[v, w]$ is also a derivation.

Lemma 21. *Let $e \in \mathcal{E}(U)$, $v, w \in \mathcal{T}_X(U)$. Then*

$$\langle R_\nabla(e), v \wedge w \rangle = [\nabla_v, \nabla_w](e) - \nabla_{[v,w]}(e).$$

Proof. If $v = \partial_h$ and $w = \partial_i$ this formula is proved above. For the general case, we only need to note that the expressions

$$K(e, v, w) := \langle R_\nabla(e), v \wedge w \rangle$$

and

$$L(e, v, w) := \nabla_v \nabla_w(e) - \nabla_w \nabla_v(e) - \nabla_{[v,w]}(e)$$

are tri-linear. That K is tri-linear is obvious. As for L we begin by recalling

$$\begin{aligned} [v, gw] &= g \cdot [v, w] + v(g) \cdot w \\ [fv, w] &= -[w, fv] \\ &= -f[w, v] - w(f)v \\ &= f[v, w] - w(f)v. \end{aligned}$$

Then,

$$\begin{aligned} \nabla_{fv}\nabla_w - \nabla_w\nabla_{fv} - \nabla_{[fv,w]} &= f\nabla_v\nabla_w - f\nabla_w\nabla_v - w(f)\nabla_v - f\nabla_{[v,w]} + w(f)\nabla_v \\ &= f\{\nabla_v\nabla_w - \nabla_w\nabla_v - \nabla_{[v,w]}\}. \end{aligned}$$

Since $L(e, v, w) = -L(e, w, v)$, we conclude that L is linear on the second variable. Finally, we compute

$$\begin{aligned} L(re, v, w) &= \nabla_v \left(w(r)e + r\nabla_w(e)\right) - \nabla_w \left(v(r)e + r\nabla_v(e)\right) \\ &\quad - r\nabla_{[v,w]}(e) - [v, w](r)e \\ &= vw(r)e + w(r)\nabla_v(e) + v(r)\nabla_w(e) + r\nabla_v\nabla_w(e) - \\ &\quad - wv(r)e - v(r)\nabla_w(e) - w(r)\nabla_v(e) - r\nabla_w\nabla_v(e) - \\ &\quad - r\nabla_{[v,w]}(e) - [v, w](r)e \\ &= [v, w](r)e + r\nabla_v\nabla_w(e) - r\nabla_w\nabla_v(e) - r\nabla_{[v,w]}(e) - [v, w](r)e \\ &= rL(e, v, w). \end{aligned}$$

□

Corollary 22. *The curvature R_∇ vanishes if and only if the covariant derivative*

$$T_X \xrightarrow{v \mapsto \nabla_v} \mathcal{E}nd_k(\mathcal{E})$$

is compatible with the bracket.

Lecture 3 and 4

(31/10/2024).

The Atiyah-Weil theorem

Let X/k smooth over algebraically closed field of characteristic $p \geq 0$. Let $\mathcal{E}$ a vector bundle of rank r on X.

Recall that a vector bundle $\mathcal{E}$ on X is *decomposable* if it can be written as a direct sum $\mathcal{E}' \oplus \mathcal{E}''$, where $\operatorname{rk} \mathcal{E}'$ and $\operatorname{rk} \mathcal{E}''$ are both $< \operatorname{rk} \mathcal{E}$. A vb. is *indecomposable* if it is not decomposable. It is clear that any vector bundle can be written as a direct sum of indecomposable vector bundles. A simple result tells that this decomposition is essentially unique:

Exercise 23 (see [At56])**.** Suppose that X is proper. Let $\mathcal{E}_1, \dots, \mathcal{E}_s$, $\mathcal{F}_1, \dots, \mathcal{F}_t$ be indecomposable vector bundles. If

$$\bigoplus_{i=1}^{s} \mathcal{E}_i \simeq \bigoplus_{j=1}^{t} \mathcal{F}_j$$

then $s = t$ and there exists $\sigma \in S_t$ such that $\mathcal{E}_{\sigma(i)} \simeq \mathcal{F}_i$.

Another ingredient is necessary to state the Atiyah-Weil theorem: the *degree of a vector bundle*. Suppose that X *is a proper curve*. Recall that $\det \mathcal{E} = \wedge^r \mathcal{E}$; this is an invertible sheaf. Let $\deg(\mathcal{E})$ be the degree of a divisor D such that $\det(\mathcal{E}) \simeq \mathcal{O}(D)$.

Theorem 24 (Atiyah-Weil)**.** *Let X/k be a smooth and proper curve. Suppose that $\deg(\mathcal{F}) \equiv 0 \mod p$ for each indecomposable factor $\mathcal{F}$ of $\mathcal{E}$. Then $\mathcal{E}$ admits a connection. Conversely, if $\mathcal{E}$ carries a connection, then each indecomposable component of $\mathcal{E}$ has a degree which is divisible by p.*

The proof of this result will occupy the following pages; it hinges on the *Atiyah class*. Curiously enough, the most difficult part of the explanation is unravelling Serre duality on X in terms which are sufficiently explicit. Therefore, I've made some choices as to what the student listening to this was supposed to know, and these are largely based on the sentence from [H77, p.243]: "*A weakness of the duality theorem as we have proved it is that even for a nonsingular projective variety X, we don't have much information about the trace map $t : H^n(X, \omega) \to k$.*"

Theorem 24 comes in a long lineage of results (Weil's paper mentions Abel…). One of its heirs, the Narasimhan–Seshadri comes from the fact that there are ways to *canonically* give certain vector bundles preferred connections. At the end I shall comment on them.

The Atiyah class

At this point, it is not necessary to assume that X is a curve, only *a smooth k-scheme*. Any locally free $\mathcal{O}_X$-module of finite rank admits locally a connection:

the trivial one. Therefore, we introduce an obstruction to the existence of a global connection: the Atiyah class.

Let $\{U_\alpha\}$ be an open covering. Let ∇_α be a connection on $\mathcal{E}|_{U_\alpha}$. Then, for each couple of indices α, β, the element

$$A_{\alpha\beta} := \nabla_\alpha - \nabla_\beta : \mathcal{E}|_{U_{\alpha\beta}} \longrightarrow \mathcal{E}|_{U_{\alpha\beta}} \otimes \Omega^1|_{U_{\alpha\beta}}$$

is $\mathcal{O}_{U_{\alpha\beta}}$–linear. A simple calculation shows that

$$A_{\beta\gamma} - A_{\alpha\gamma} + A_{\alpha\beta} = 0$$

on triple intersections and hence defines a 1–cocycle with values on $\mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes \Omega^1)$. If there exists a 0–cochain $\lambda = (\lambda_\alpha)$ with values in $\mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes \Omega^1)$ such that

$$\lambda_\beta - \lambda_\alpha = \delta(\lambda) = A_{\alpha\beta},$$

then

$$\nabla_\alpha + \lambda_\alpha$$

is a connection on $\mathcal{E}|_{U_\alpha}$ which glues to a global connection on $\mathcal{E}$.

If we start with a different family of connections on the open sets U_α, say ∇'_α, then there exists an $\mathcal{O}$–linear map $\theta_\alpha : \mathcal{E}|_{U_\alpha} \to \mathcal{E}|_{U_\alpha} \otimes \Omega^1$ such that $\nabla_\alpha + \theta_\alpha = \nabla'_\alpha$. Therefore,

$$\nabla'_\alpha - \nabla'_\beta = \nabla_\alpha - \nabla_\beta + (\theta_\beta - \theta_\alpha),$$

so that the 1–cocyle $\{\nabla'_\alpha - \nabla'_\beta\}$ differs from the 1–cocyle $\{\nabla_\alpha - \nabla_\beta\}$ by the a coboundary $\delta(\theta_\alpha) = (\theta_\beta - \theta_\alpha)$.

This allows us to put forward the following:

Definition 25. The class of $\{\nabla_\alpha - \nabla_\beta\}$ in $H^1(X, \mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes \Omega^1))$ is called the Atiyah class of $\mathcal{E}$. It will be denoted by $A(\mathcal{E})$.

We have proved above:

Lemma 26. *The Atiyah class is $A(\mathcal{E}) \in \check{H}^1(X, \mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes \Omega^1))$ is independent of the choice of the local connections ∇_α.*

Theorem 27. *The vector bundle $\mathcal{E}$ admits a connection if and only if $A(\mathcal{E}) = 0$.*

Remark 28. It is usually interesting to call $\mathcal{E} \otimes \mathcal{E}^*$ the *adjoint bundle of $\mathcal{E}$* and denote it by $\mathrm{Ad}(\mathcal{E})$. Since $\mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes \Omega^1) \simeq \mathrm{Ad}(\mathcal{E}) \otimes \Omega^1$, the Atiyah class is then an element of $\check{H}^1(\mathrm{Ad}(\mathcal{E}) \otimes \Omega^1)$.

Remark 29. In [At57], Atiyah introduces two Cech cohomology classes, "*a*" and "*b*" (see pages 188 and 194). In [At57, p. 195] these are proved to be, up to normalisations, the same. What we introduce here is just a simpler variant of "*b*".

In view of this proposition, we can chose the simplest possible connections in order to compute $A(\mathcal{E})$. If a frame $\{e_i^\alpha\}$ of $\mathcal{E}$ over $\{U_\alpha\}$ is given and if $g_{\alpha\beta}$ is the associated 1–cocycle, that is

$$e_j^\beta = \sum_i g_{\alpha\beta,ij} \cdot e_i^\alpha,$$

then we can take for ∇_α the direct sum connection on $\mathcal{E}|_{U_\alpha} \simeq \mathcal{O}_{U_\alpha}^n$. We obtain

$$\begin{aligned}
\left(\nabla_\alpha - \nabla_\beta\right)\left(e_\ell^\alpha\right) &= -\nabla_\beta \left(\sum_{j=1}^n g_{\beta\alpha,j\ell} \cdot e_j^\beta \right) \\
&= -\sum_{j=1}^n e_j^\beta \otimes \mathrm{d}g_{\beta\alpha,j\ell} \\
&= -\sum_{j=1}^n \sum_{i=1}^n g_{\alpha\beta,ij} \cdot e_i^\alpha \otimes \mathrm{d}g_{\beta\alpha,j\ell} \\
&= -\sum_{i=1}^n e_i^\alpha \otimes \left(\sum_j g_{\alpha\beta,ij} \cdot \mathrm{d}g_{\beta\alpha,j\ell} \right) \\
&= \sum_{i=1}^n e_i^\alpha \otimes \left(\sum_j \mathrm{d}g_{\alpha\beta,ij} \cdot g_{\beta\alpha,i\ell} \right),
\end{aligned}$$

so that the Atiyah class is given by

$$\boxed{A(\mathcal{E}) = \left(\mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1} \right).}$$

This gives immediately:

Theorem 30. *Suppose that $\mathcal{E}$ is the direct sum of vector bundles $\mathcal{E}' \oplus \mathcal{E}''$. Then $A(\mathcal{E}) = A(\mathcal{E}') \oplus A(\mathcal{E}'')$ in*

$$H^1(\mathrm{Ad}(\mathcal{E}) \otimes \Omega^1) = \begin{array}{ll} & H^1(\mathrm{Ad}(\mathcal{E}') \otimes \Omega^1) \\ \oplus & H^1(\mathrm{Ad}(\mathcal{E}') \otimes \Omega^1) \\ \oplus & H^1(\mathcal{H}om(\mathcal{E}', \mathcal{E}'')) \otimes \Omega^1) \\ \oplus & H^1(\mathcal{H}om(\mathcal{E}'', \mathcal{E}') \otimes \Omega^1). \end{array}$$

In particular, if $A(\mathcal{E}) = 0$, then $A(\mathcal{E}') = A(\mathcal{E}'') = 0$.

We wish to find conditions for $A(\mathcal{E}) = 0$ and to achieve this goal we shall study this cohomology class by means of Serre's duality theorem. Indeed, $A(\mathcal{E})$ gives rise to an element of $H^0(\mathrm{Ad}(\mathcal{E}))^*$, by Serre duality.

Exercise 31. Show that the Atiyah class is functorial.

Digression on Serre duality

In your Algebraic Geometry course, you probably learned about the famous Serre duality: $H^i(\mathcal{F})^* \simeq H^{m-i}(\mathcal{F}^* \otimes \omega_X)$. There are, of course, many ways to get to this fundamental result, but I'd like to explain one which is less evident nowadays. (Grothendieck throws the weight behind a formal definition of ω_X, the dualizing sheaf.)

Let $(M, \mathcal{A})$ be a ringed space. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{A}$-modules. Let $\underline{U} = (U_\alpha)$ be an open covering of M and let $C^\bullet(\underline{U}, -)$ stand for the Cech complexes. Given cocyles $f \in C^p(\mathcal{F})$ and $g \in C^q(\mathcal{G})$, we define

$$(f \cup g)_{\alpha_0,\dots,\alpha_{p+q}} := f_{\alpha_0,\dots,\alpha_p} \otimes g_{\alpha_p,\cdots,\alpha_{p+q}} \in C^{p+q}(\mathcal{F} \otimes \mathcal{G}).$$

This gives us a pairing

$$H^p(\mathcal{F}) \times H^q(\mathcal{G}) \longrightarrow H^{p+q}(\mathcal{F} \otimes \mathcal{G}).$$

Let us now suppose that M is an irreducible and regular m-dimensional, projective scheme over k. Let $\mathcal{V}$ be a vector bundle over M. Then, the cup-product produces

$$\begin{tikzcd} H^i(\mathcal{V}) \times H^{m-i}(\mathcal{V}^* \otimes \Omega^m) \arrow[r, "\cup"] \arrow[rr, bend left, "\Sigma"] & H^m(\mathcal{V} \otimes \mathcal{V}^* \otimes \Omega^m) \arrow[r, "\mathrm{Contr.}\otimes\mathrm{id}"] & H^m(\Omega^m). \end{tikzcd}$$

Serre duality now reads:

Theorem 32 (Serre duality). *(a)* $\dim H^m(\Omega^m) = 1$.

(b) The pairing Σ is perfect and defines $H^i(\mathcal{V}^* \otimes \Omega^m) \simeq H^{m-i}(\mathcal{V})^*$.

These are quickly explained by Serre in [Se54, §2, Theorem 4] and proceed by induction on m; the case $m = 1$ being a consequence of the proof employing residues and the proper identifications. See also [Se88, II.10].

The cup product of the Atiyah class and an endomorphism

Suppose X is again a curve. Fix a vector bundle $\mathcal{E}$. We note that $\mathrm{Ad}(\mathcal{E})^* \simeq \mathrm{Ad}(\mathcal{E})$. Explicitly, we have an arrow of $\mathcal{O}_X$-modules

$$\mathrm{Ad}(\mathcal{E}) \otimes \mathrm{Ad}(\mathcal{E}) \longrightarrow \mathcal{O}_X, \quad u \otimes v \longmapsto \mathrm{Tr}(uv), \tag{§}$$

which gives

$$\tau : \mathrm{Ad}(\mathcal{E}) \xrightarrow{\sim} \mathrm{Ad}(\mathcal{E})^*, \quad u \longmapsto (\mathrm{Tr}_u : v \mapsto \mathrm{Tr}(uv)).$$

Obviously, τ is an isomorphism. This being so,

$$\mathrm{Contr.} : \mathrm{Ad}(\mathcal{E}) \otimes \mathrm{Ad}(\mathcal{E}) \longrightarrow \mathcal{O}_X$$

is just (§)

In the above setting, we get

$$H^1(\mathrm{Ad}(\mathcal{E}) \otimes \Omega^1) \times H^0(\mathrm{Ad}(\mathcal{E})) \xrightarrow{\quad\Sigma\quad} H^1(\Omega^1),$$

which we know, given that $H^1(\Omega^1) \simeq k$, gives us a perfect pairing. *To simplify notation, in what follows, we shall write $A \cup T$ instead of $\Sigma(A, T)$.*

We now wish to compute, for each $T \in H^0(X, \mathrm{Ad}(\mathcal{E}))$ the cup product

$$A(\mathcal{E}) \cup T.$$

For that purpose, we observe that the endomorphisms of $\mathcal{E}$ have a particular decomposition.

Lemma 33. *Any endomorphism T can be decomposed as $c \cdot \mathrm{Id}_{\mathcal{E}} + N$, where $c \in k$ and N is a nilpotent endomorphism.*

Remark 34. It is here that we use that $\mathcal{E}$ is indecomposable. It is also used here that k is algebraically closed. The Atiyah-Weil theorem is true if k is perfect, but false otherwise. See [Bi05].

Proof. We let $\{e_1^\alpha, \ldots, e_n^\alpha\}$ be trivialising sections of $\mathcal{E}$ over some open subset U_α. Moreover, we write

$$e_j^\beta = \sum_{i=1}^n g_{\alpha\beta,ij} \cdot e_i^\alpha,$$

where $g_{\alpha\beta}$ are elements of $\mathrm{GL}_n(\mathcal{O}(U_{\alpha\beta}))$. Let $T_\alpha = (t_{\alpha,ij})$ be a $n \times n$ matrix with entries on $\mathcal{O}(U_\alpha)$ defining the restriction of T to U_α. That is,

$$T(e_j^\alpha) = \sum_{i=1}^n t_{\alpha,ij} \cdot e_i^\alpha.$$

Therefore, $T_\alpha \cdot g_{\alpha\beta} = g_{\alpha\beta} \cdot T_\beta \Rightarrow$ if χ_α stands for the characteristic polynomial of T_α – which is a polynomial in $\mathcal{O}(U_\alpha)[\lambda]$ – it follows that $\chi_\alpha|_{U_{\alpha\beta}} = \chi_\beta|_{U_{\alpha\beta}}$, so that $\chi_\alpha = \chi$ for some $\chi \in k[\lambda]$. Let

$$\chi(\lambda) = \prod_{i=1}^m (\lambda - c_i)^{\mu_i} = 0.$$

Since $\mathcal{E}$ is indecomposable, we conclude that $m = 1$ and $\chi = (\lambda - c)^N$. Hence, $T - c \cdot \mathrm{Id}$ is a nilpotent endomorphism. □

Proposition 35. *Let $N : \mathcal{E} \to \mathcal{E}$ be a nilpotent endomorphism. Then $A(\mathcal{E}) \cup N = 0$.*

Proof. This is taken directly from [At57, Prp. 18, p. 202].
Step 1. Assume that $\mathcal{E}$ sits in an exact sequence

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

of *locally free sheaves*. This means that we can find a covering of X together with cocycles for $\mathcal{E}$ which have the form

$$g_{\alpha\beta} = \begin{bmatrix} g'_{\alpha\beta} & * \\ 0 & g''_{\alpha\beta} \end{bmatrix},$$

so that

$$\mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1} = \begin{bmatrix} \mathrm{d}g'_{\alpha\beta} & * \\ 0 & \mathrm{d}g''_{\alpha\beta} \end{bmatrix} \cdot \begin{bmatrix} g'^{-1}_{\alpha\beta} & * \\ 0 & g''^{-1}_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} \mathrm{d}g'_{\alpha\beta} \cdot g'^{-1}_{\alpha\beta} & * \\ 0 & \mathrm{d}g''_{\alpha\beta} \cdot g''^{-1}_{\alpha\beta} \end{bmatrix}.$$

Assume that $N(\mathcal{E}) \subset \mathcal{E}'$, in particular $N(\mathcal{E}') \subset \mathcal{E}'$. Then, the matrix of N in $U_{\alpha\beta}$ will have the following shape

$$\begin{bmatrix} N' & N'' \\ 0 & 0 \end{bmatrix}.$$

Consequently,

$$\mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1} \cdot N = \begin{bmatrix} \mathrm{d}g'_{\alpha\beta} \cdot g'^{-1}_{\alpha\beta} \cdot N' & * \\ 0 & 0 \end{bmatrix},$$

which implies that

$$\mathrm{Tr}\left(\mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1} \cdot N\right) = \mathrm{Tr}\left(\mathrm{d}g'_{\alpha\beta} \cdot g'^{-1}_{\alpha\beta} \cdot N'\right).$$

In other words, we arrive at

$$A(\mathcal{E}) \cup N = A(\mathcal{E}') \cup (N|_{\mathcal{E}'}).$$

Step 2. Let

$$0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

be a filtration of $\mathcal{E}$ by subsheaves such that

1. $\mathcal{E}_i/\mathcal{E}_{i-1}$ is locally free as an $\mathcal{O}_X$–module, and
2. $N(\mathcal{E}_{i+1}) \subset \mathcal{E}_i$.

Then

$$\begin{aligned} A(\mathcal{E}) \cup N &= A(\mathcal{E}_{l-1}) \cup N|_{\mathcal{E}_{l-1}} \\ &\cdots \\ &= A(\mathcal{E}_0) \cup N|_{\mathcal{E}_0} \\ &= 0. \end{aligned}$$

Step 3. Define $\mathcal{E}_i$ be the saturation of $\mathrm{Ker}(N^i)$ in $\mathcal{E}$. That is,

$$\mathcal{E}_i(U) = \left\{ \begin{array}{c} e \in \mathcal{E}(U) \text{ such that for each } x \in U, \text{ there} \\ \text{exists } f_x \in \mathcal{O}_x \setminus \{0\} \text{ with } N^i(f_x e_x) = 0 \end{array} \right\}.$$

Then $0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$ is a filtration as in Step 2. □

We now concentrate on the calculation of $A(\mathcal{E}) \cup \mathrm{Id}$.

Lemma 36. $A(\mathcal{E}) \cup \mathrm{id} = \deg(\mathcal{E})$.

Proof. Some hints. First, $H^1(\Omega^1)$ is a one dimensional vector space with a canonical generator [Se88, Proposition 3, p.12]. The composition

$$\underbrace{H^1(\mathcal{O}_X^*)}_{\mathrm{Pic}} \xrightarrow{\quad \mathrm{dlog} \quad} H^1(\Omega^1) \xrightarrow[\sim]{\mathrm{can}} k\,.$$

sends, for a closed point P, the class $\mathcal{O}(P)$ to 1. Hence, it sends $\mathcal{L}$ to $\deg \mathcal{L}$.

The equality $A(\mathcal{E}) \cup \mathrm{id} = A(\det \mathcal{E})$ is true. Indeed, for any etale chart (U, x) and any function $g \in \mathbf{GL}_r(\mathcal{O}(U))$, we have, due to Lemma 37

$$\begin{aligned} \mathrm{Trace}\left(\mathrm{d}g \cdot g^{-1}\right) &= \frac{\mathrm{d}(\det(g))}{\det(g)} \\ &= \mathrm{dlog}(\det(g)). \end{aligned}$$

Composing with the canonical arrow we obtain $A(\mathcal{E}) \cup \mathrm{id} = \deg \mathcal{E}$. $\square$

Lemma 37. *Let $G = (g_{ij}) \in \mathbf{GL}_n(\mathcal{O}(U))$; let $\Delta = \det(g_{ij})$. Then*

$$\mathrm{Trace}\left(G^{-1} \cdot \frac{\mathrm{d}G}{\mathrm{d}x}\right) = \Delta^{-1} \frac{\mathrm{d}\Delta}{\mathrm{d}x}.$$

Proof. If A is a matrix, then we write $A_{[ij]}$ for the matrix obtained from A by deleting the ith row and the jth column. Then, it is known that

$$\widetilde{A} := ((-1)^{i+j} \det(A_{[ji]}))$$

satisfies

$$\widetilde{A} \cdot A = A \cdot \widetilde{A} = \det(A) \cdot \mathrm{Id}.$$

(See Lang's "Algebra".) Hence, $G^{-1} = \widetilde{G}\Delta^{-1}$ and we are required to prove that

$$\mathrm{Trace}\left(\widetilde{G} \cdot G'\right) = \Delta',$$

where we write $(-)'$ instead of $\dfrac{\mathrm{d}}{\mathrm{d}x}$. Now:

$$\left(\widetilde{G} \cdot G'\right)_{ij} = \sum_{\ell} (-1)^{i+\ell} G_{[\ell i]} \cdot g'_{\ell j}$$

so that

$$\mathrm{Trace}\,(\widetilde{G} \cdot G') = \sum_{i,k} (-1)^{i+\ell} G_{[\ell i]} \cdot g'_{\ell i}.$$

Now, we consider det as a regular function on $\mathbf{A}^{n^2}$. Expanding according to some line, for fixed i, we have:

$$\det = (-1)^{i+1} z_{i1} \det(A_{[i1]}) + \cdots + (-1)^{i+n} z_{in} \det(A_{[in]}).$$

Then

$$\frac{\partial \det}{\partial z_{ij}} = (-1)^{i+j} \det(A_{[ij]}).$$

The chain rule gives

$$\Delta' = \sum_{i,j} (-1)^{i+j} G_{[ij]} \cdot g'_{ij}.$$

which gives what we want. * $\square$

Exercise 38. Let $\operatorname{char} k = 0$. On $\mathbf{P}^1$, show that the only algebraic connections are the trivial ones using Grothendieck's classification theorem. It is possible to prove that if V is a smooth, projective and rationally-connected variety, then there are also "no" algebraic (not necessarily integrable) connections [BdS13].

The Narasimhan-Seshadri theorem

Let $k = \mathbf{C}$. We now view X as a projective complex manifold (Riemann-Surface) and identify, by GAGA, complex analytic coherent sheaves with their algebraic counterparts. Let π be the fundamental group of X at some point and $\psi : \widetilde{X} \to X$ the universal covering, which I regard as a principal π-bundle with π acting on the right. For each representation $\varrho : \pi \to \mathbf{GL}_n(\mathbf{C})$, we get a locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{V}(\rho)$ by

$$\mathcal{V}(\varrho)(U) = \left\{ f : \psi^{-1}(U) \to \mathbf{C}^n \ : \ \begin{array}{c} f(\tilde{u} \cdot \gamma) = \varrho(\gamma)^{-1} \cdot f(\tilde{u}), \\ \forall \tilde{u} \in \psi^{-1}U, \, \forall \gamma \in \pi \end{array} \right\}.$$

These sheaves come with a natural connection ∇^{can}: For each distinguished neighbourhood U, each $v \in \mathcal{T}_X(U)$ and each $f \in \Gamma(U, \mathcal{V}(\varrho))$, we define, by letting $\tilde{v} \in \mathcal{T}_{\tilde{X}}(\widetilde{X}|_U)$ lift v,

$$\nabla^{\mathrm{can}}_v (f_1, \ldots, f_n) = (\tilde{v}(f_1), \ldots, \tilde{v}(f_n)).$$

This connection can also be interpreted by considering the "local system" (a sheaf of $\mathbf{C}$-spaces which is, locally, isomorphic to the constant sheaf $\underline{\mathbf{C}}^r$)

$$\mathbf{V}(\varrho)(U) = \left\{ f : \psi^{-1}(U) \to \mathbf{C}^r \ : \ \begin{array}{c} f \text{ is locally constant and} \\ f(\tilde{u} \cdot \gamma) = \varrho(\gamma)^{-1} \cdot f(\tilde{u}), \ \forall \tilde{u} \in \psi^{-1}U \end{array} \right\}.$$

In addition, each vector bundle $\mathcal{E}$ having a connection ∇ must be of the form $\mathcal{V}(\varrho)$ for some ϱ by the monodromy representation.

* Recall that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Remark 39. The construction $\mathcal{V}(\varrho)$ given above is called the "twisting construction" and is basic in the theory of principal and vector bundles. Another way to understand it is as follows. Interpret $\widetilde{X} \to X$ as a principal π-bundle or π-torsor: There is an action of π on the right of $\widetilde{X}$ (letting π act on the right is traditional). Let now $\{U_\alpha\}$ be a cover of X by "distinguished" neighbourhoods, that is, there exists $V_\alpha \subset \psi^{-1}U_\alpha$ such that $\psi : V_\alpha \xrightarrow{\sim} U_\alpha$. In addition, let us suppose that $U_{\alpha\beta}$ is connected.

Let $\tau_\alpha = (\psi|_{V_\alpha})^{-1} : U_\alpha \to \widetilde{X}$. Now, for $u \in U_{\alpha\beta}$ we have $\tau_\beta(u) = \tau_\alpha(u) \cdot \gamma_{\alpha\beta}(u)$, with $\gamma_{\alpha\beta}(u) \in \pi$. It is not difficult to see that $\gamma_{\alpha\beta}$ is constant on the connected space $U_{\alpha\beta}$. The cocycle associated to $\mathcal{V}(\varrho)$ is $\varrho(\gamma_{\alpha\beta})$.

This construction $\varrho \mapsto \mathcal{V}(\varrho)$ is then upgraded to a functor

$$\mathcal{V}(-) : \mathrm{Rep}_{\mathbf{C}}(\pi) \longrightarrow \mathbf{MIC}.$$

A wonderful fact behind this functor is that it preserves all constructions of linear algebra: It is an exact, **C***-linear tensor functor.*

Now, if we compose $\mathcal{V}$ with the forgetful $\mathbf{MIC} \to \mathbf{VB}$, we obtain another functor $\mathcal{W} : \mathrm{Rep}_{\mathbf{C}}(\pi) \to \mathbf{VB}$ and "Atiyah-Weil" $\Rightarrow$ all vector bundles with indecomposable factors of degree zero must lie in $\mathrm{Im}\,\mathcal{W}$. But it does not allow us to see $\mathrm{Rep}_{\mathbf{C}}(\pi)$ "inside **VB**" because $\mathcal{W}$ is not *full*. For example, it may be the case that in

$$\begin{tikzcd} \underbrace{(\mathbf{C})^{\pi}}_{\pi\text{-invariant}} \arrow[r] \arrow[d, equal] & \mathcal{V}(\varrho)^{\nabla} \arrow[d, equal] \\ \mathrm{Hom}_{\pi}(\mathbf{C}, \mathbf{C}^n) \arrow[r] & \mathrm{Hom}_{\mathbf{MC}}(\mathcal{O}_X, \mathcal{V}(\varrho)) \end{tikzcd} \qquad (*)$$

the horizontal arrows are *not* surjective. The Narasimhan-Seshadri theorem comes to throw light on this situation.

Definition 40 (Mumford)**.** A vector bundle $\mathcal{E}$ is stable (of degree zero) if for each proper subbundle[†] $\mathcal{F} \subset \mathcal{E}$ we have $\deg \mathcal{F} < 0$.

Theorem 41 (Narasimhan-Seshadri)**.** *The functor $\mathcal{W}$ gives an equivalence between the* irreducible *representations $\pi \to \mathbf{U}_n$ of π and the stable bundles of degree zero.*

The original paper [NS65] is difficult. A masterful summary can be found in [S82]. A work which I found rather useful in comprehending the Theorem is [NS64], where the authors study locally the "moduli space" of unitary representations showing that it resembles the "moduli space" of stable bundles.

This theorem was generalised to higher dimensional varieties by means of fundamental theorems of Donaldson and Kobayashi, Mehta-Ramanathan [MR84,

[†]It is usual to employ the term "sub-bundle" for $\mathcal{O}_X$-submodules which have a locally free quotient.

Thm 5.1] and a very complete, but delicate[‡] picture was reached, after the efforts of many mathematician, in the Simpson correspondence [Si92].

[‡]Not only in the sense that the result itself is difficult to obtain, but also that it reveals a complex picture. The Simpson correspondence says that the category of integrable connections is equivalent to the category of semi-stable Higgs bundles (of "some Chern classes"). But it is *not* true that the underlying vector bundles remain unaltered in the correspondence.

Lecture 5

(12 August 2022).

Stratifications (Grothendieck's take on connections)

I shall work on the affine case. The general one is standard and requires maturity with sheaves and the like, so might be off putting for novices. The main references for this part are [Be74] (well organized but demanding) and then [BO78] (a bit more cursory). Grothendieck's way of looking at connections is through the notion of "stratification".

Here are is Grothendieck's reinterpretation:

a) To associate to integrable connections a natural structure of $\mathcal{D}$-module. This can be easily done in complex analysis by simply understanding differential operators as "compositions of derivations", but it much less suitable to other contexts.

b) To associate to integrable connections the idea of an "action of a groupoid" and hence extend it to broader contexts.

Let $k \to A$ be arrow of rings. (Soon we shall introduce more hypothesis on this couple.) Let $\mathcal{P}_A = A \otimes_k A$ and let

$$I_\Delta = \mathrm{Ker\,mult.} : \mathcal{P}_A \longrightarrow A$$

be the ideal of the diagonal in $\mathcal{P}_A$.

Define

$$\mathcal{P}_A^\nu = \mathcal{P}_A / I^{\nu+1}.$$

The rings $\mathcal{P}_A$ and $\mathcal{P}_A^\nu$ have two structures of A-algebras: $\mathrm{d}_0 : a \mapsto a \otimes 1$ and $\mathrm{d}_1 : a \mapsto 1 \otimes a$ [EGA IV_4, 17]. Let us agree to call

$$\mathrm{d}_0 \quad \begin{array}{l} \bullet \text{ the canonical one} \\ \bullet \text{ the one on } \textit{the left}, \end{array}$$

and

$$\mathrm{d}_1 =: \tau \quad \begin{array}{l} \bullet \text{ the Taylor structure} \\ \bullet \text{ the one on } \textit{the right}, \end{array}$$

(Grothendieck calls it "d" in [EGA IV_4, §17].) In what follows, the elements

$$\mathrm{d}a := \tau a - a$$

shall also play an important role. As the reader probably noticed, I abuse notation, so that $\mathrm{d}a$ may belong to $\mathcal{P}_A^\nu$ for some ν or to $\mathcal{P}_A$, or even to $\Omega^1_{A/k}$. Moreover, the reader has to be *careful* : $\mathrm{d}x$ may not mean "the differential of x" in some contexts!

Recall from your algebraic geometry course that $\Omega^1_{A/k} = I_\Delta / I_\Delta^2$ and that the *universal differential* $\mathrm{d} : A \to \Omega^1_{A/k}$ is $\mathrm{d}_1 - \mathrm{d}_0$. On each $\mathcal{P}_A^\nu$ we have the image of the "augmentation ideal" $I_\Delta \cdot \mathcal{P}_A^\nu$ which will be denoted simply by I_Δ. In case $\nu = 1$, $I_\Delta^2 = 0$, if we consider it as an A-module via the d_0 or d_1, the result is the same, and is none other than $\Omega^1_{A/k}$.

Example 42. Let $A = k[x,y]$, k a field $\Rightarrow \mathcal{P}$ polynomial ring on four variables. Using the notation introduced above: $\mathcal{P} = k[x,y,\mathrm{d}x,\mathrm{d}y]$. Note that d_0 is just the inclusion and d_1 is $x \mapsto x + \mathrm{d}x$, etc. Then $I = (\mathrm{d}x, \mathrm{d}y)$ and

$$\mathcal{P}^\nu = k[x,y][\mathrm{d}x,\mathrm{d}y]/(\mathrm{d}x,\mathrm{d}y)^{\nu+1}.$$

Hence

$$\tau f = f(x+dx, y+dy) = \text{Taylor series.}$$

Exercise 43. Show that I_Δ is generated by $\{\mathrm{d}a \,:\, a \in A\}$.

If E is an A-module, then we obtain now two $\mathcal{P}^\nu_A$-modules via the distinct algebra structures:

$$\mathcal{P}^\nu \underset{\tau,A}{\otimes} E \quad \text{and} \quad \mathcal{P}^\nu \underset{A}{\otimes} E.$$

To take advantage of the notation, $E \otimes_A \mathcal{P}^\nu_A$ is the tensor product using the morphism "on the left" $\mathrm{d}_0 : A \to \mathcal{P}^\nu$, and $\mathcal{P}^\nu \otimes_A E$ is the tensor product using the structure "on the right" $\tau = \mathrm{d}_1$.

From now on we assume that k and A are Noetherian. Example 42 can be generalized:

Theorem 44. *Suppose that $x_1, \ldots, x_n \in A$ define etale coordinates $\boldsymbol{x} : \operatorname{Spec} A \to \mathbf{A}^n$. Then*

$$\mathcal{P}^\nu_A = \bigoplus_{q_1+\cdots+q_n \le \nu} A\ (\mathrm{d}x_1)^{q_1} \cdots (\mathrm{d}x_n)^{q_n}.$$

Proof. The idea behind the proof relies entirely on the relation between regular immersions and smooth morphisms.

Recall from your Commutative Algebra course the notions of regular sequence and quasi-regular sequence [M89, Section 16]. (Where you should have realised that "quasi-regular" is the correct concept since it "localizes".) If B is a ring and $J = (b_1, \ldots, b_n)$ is an ideal, by definition, the sequence $b_1, \ldots, b_n$ is quasi-regular if the obvious morphism

$$B/J[Y_1, \ldots, Y_n] \longrightarrow B/J \oplus J/J^2 \oplus J^2/J^3 \oplus \cdots$$

$$Y \longmapsto b \mod J^2.$$

is an isomorphism.

If $X := \operatorname{Spec} A$ and $P = \operatorname{Spec} \mathcal{P}_A$, then I_Δ corresponds to the diagonal immersion $\Delta : X \to P$. Both schemes are k-smooth. Then Δ is regular immersion, as is well-explained in Chapter 6, Proposition 3.31, p. 230, of [Li02]. We do not know a priori that $I_\Delta = (\mathrm{d}x_1, \ldots, \mathrm{d}x_n)$, just that I_Δ is "locally" generated by a quasi-regular sequence. But on $\mathcal{P}^\nu_A$, the ideal I_Δ is nilpotent and Nakayama's Lemma tells us that I_Δ is generated by $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ since $I/I^2 = \Omega^1_A$ is generated by them. Hence, I^j/I^{j+1} is free on the basis $\{(\mathrm{d}x_1)^{q_1} \cdots (\mathrm{d}x_n)^{q_n} \,:\, q_1 + \cdots + q_n = j\}$. Counting ranks we arrive at the desired formula. $\square$

It is clear that $\mathcal{P}^1_A = A \oplus \Omega^1_A$, and that the *ring structure* is given by letting Ω^1_A be an ideal of square zero. Now comes the *fundamental observation*.

Proposition 45. *A connection on E is equivalent to an isomorphism of $\mathcal{P}^1_A$–modules*

$$\varepsilon : \mathcal{P}^1 \underset{\tau,A}{\otimes} E \xrightarrow{\sim} E \underset{A}{\otimes} \mathcal{P}^1$$

which gives the identity modulo $I \subset \mathcal{P}^1_A$.

Proof. Let $\nabla : E \to E \otimes \Omega^1$ connection. Define $\theta : E \to E \otimes \mathcal{P}^1$ by $\theta e = e \otimes \mathbf{1} + \underbrace{\nabla e}_{\in E \otimes \Omega}$. Then

$$\begin{aligned} \theta(ae) &= ae \otimes \mathbf{1} + \nabla(ae) \\ &= e \otimes a + a \cdot (\nabla(e)) + e \otimes \mathrm{d}a \\ &= e \otimes \tau a + a\nabla(e). \end{aligned}$$

What is $a\nabla(e)$? This is just $\tau(a) \cdot \nabla(e)$ because the elements of $\Omega^1_A \subset \mathcal{P}^1_A$ are all of square zero. Hence, $\theta(ae) = \theta(e) \cdot \tau(a)$. By the universal property of the tensor product, we can therefore construct a linear map:

$$\varepsilon : \mathcal{P}^1 \underset{\tau,A}{\otimes} E \longrightarrow E \underset{A}{\otimes} \mathcal{P}^1$$

$$\eta \otimes e \longmapsto \eta \cdot \theta(e).$$

It is not hard to see that ε induces the identity via the augmentation $\mathcal{P}^1 \to A$.

On the other direction, if $\varepsilon : \mathcal{P}^1_A \otimes_A E \to E \otimes_A \mathcal{P}^1_A$ is $\mathcal{P}^1_A$–linear and gives the identity modulo I_Δ. Define $\nabla : e \mapsto \varepsilon(\mathbf{1} \otimes e) - e \otimes \mathbf{1}$; this belongs to $E \otimes \Omega^1_A \subset E \otimes \mathcal{P}^1_A$. A simple computation shows that it is a connection. □

Corollary 46 (André)**.** *Suppose that Ω^1 is free and. Assume that the trivial connection (A, d) has "no" differential ideals. If $M \in$ **MC** is of finite presentation $\Rightarrow$ it is projective.*

Proof. Let Φ_j be the rth Fitting ideal of M [Ei95, Definition 20.4, p.493].[§] By a fundamental property of these ideals [Ei95, Corollary 20.5], the isomorphism in

$$M \otimes_A \mathcal{P}^1_A \xrightarrow{\sim} \mathcal{P}^1_A \otimes_A M$$

says that $\tau(\Phi_j)\mathcal{P}^1_A = \Phi_j \mathcal{P}^1_A$. We conclude that for $\varphi \in \Phi_j$, the element $(\varphi, d\varphi)$ belongs to $\Phi_j \mathcal{P}^1_A$. Hence, for $\varphi \in \Phi_r$, $d\varphi = \sum_i \varphi_i \omega_i$ with $\omega_i \in \Omega^1_A$ and $\varphi_i \in \Phi_r \Rightarrow d(\Phi_r) \subset \Phi_j \otimes \Omega^1_A$. So $\Phi_j = (0)$ or (1). Another fundamental result of Fitting is that in this case M is projective [Ei95, Prp. 20.8]. □

With this idea, the proof of the following is not very difficult.

[§]Let $M = \mathrm{Coker}(\varphi)$, where $\varphi \in \mathrm{Mat}_{r\times s}$. The ideal generated by the $(r-j)$-minors of φ depends only on M and j and is the jth Fitting ideal Φ_j of M. (By convention $\Phi_{<0} = 0$ and $\Phi_\infty = A$.)

Corollary 47 (André)**.** *Assume k is a field of char. zero and* Spec A *smooth. Then any $M \in \mathbf{MC}$ is locally free. (No need for integrability here!)*[¶] $\square$

This prompts the definition:

Definition 48. An n-connection is a family of isomorphisms

$$\varepsilon_\nu : \mathcal{P}^\nu_A \underset{\tau,A}{\otimes} E \xrightarrow{\sim} E \underset{A}{\otimes} \mathcal{P}^\nu_A, \qquad \nu = 1, \ldots, n$$

which are compatible via $\mathcal{P}^\nu_A \to \mathcal{P}^\mu_A$ and induce the identity when reduced modulo $I \subset \mathcal{P}^\nu_A$. An ∞*-connection* or a *pseudo-stratification* is defined in the obvious manner and becomes an isomorphism

$$\widehat{\mathcal{P}}_A \otimes_A E \longrightarrow E \otimes_A \widehat{\mathcal{P}}_A.$$

Remark 49. Conceptual advantage: can define "pseudo-stratifications" for objects other than modules over A. This was understood from the start [Be74, II.1.2]. Let me give some interesting examples. In them, I shall in fact talk about stratifications, which will be defined below.

C. Simpson explored this point of view to define a "non-Abelian" Gauss-Manin. Suppose $k = \mathbf{C}$. Let Z/A be smooth and projective. Let $\mathbf{M}^\natural_{dR}$ be the functor which to each A-algebra B associates the isomorphism classes of vector bundles of rank r on $Z \otimes_A B$ having an integrable B-linear connection. Then this functor is co-represented by a quasi-projective A–scheme $\mathbf{M}_{dR}$. This scheme comes with a stratification.

Remark 50. [‖]Let $X = \operatorname{Spec} A$ be a smooth $\mathbf{C}$-scheme, and $\Omega^1_{\mathcal{F}}$ a projective quotient of Ω^1_X; we think of $\Omega^1_{\mathcal{F}}$ as being the sheaf of 1–forms along a smooth distribution $\mathcal{F}$ on X. For a given finite A–module E, we define a partial connection as being a $\mathbf{C}$-linear arrow $\nabla : E \to E \otimes \Omega^1_{\mathcal{F}}$ satisfying Leibniz's rule: Letting $\mathrm{d}_{\mathcal{F}} : A \to \Omega^1_{\mathcal{F}}$ stand for the composition of $\mathrm{d} : A \to \Omega^1_X$ and $\Omega^1_X \to \Omega^1_{\mathcal{F}}$, then, for each $e \in E$ and $a \in A$ we have $\nabla(ae) = e \otimes \mathrm{d}_{\mathcal{F}}(a) + a\nabla(e)$. Let now $\mathcal{P}^1_{\mathcal{F}}$ be the A-module $A \oplus \Omega^1_{\mathcal{F}}$. Interpreting $\Omega^1_{\mathcal{F}}$ as an ideal of square zero, we obtain on $\mathcal{P}^1_{\mathcal{F}}$ two distinct structures of A–algebra: one via $a \mapsto (a, 0)$ and the other via $a \mapsto (a, \mathrm{d}_{\mathcal{F}}(a))$.

As in Proposition 45, we can then say that if E carries a partial connection, then $\mathcal{P}^1_{\mathcal{F}} \otimes E \simeq E \otimes \mathcal{P}^1_{\mathcal{F}}$, as $\mathcal{P}^1_{\mathcal{F}}$–modules.

Now, suppose that $\Omega^1_{\mathcal{F}}$ is the co-tangent sheaf of a smooth foliation $\mathcal{F}$, so that we can talk about leaves of $\mathcal{F}$. If the foliation $\mathcal{F}$ (on the analytic space X^{an} associated to X) "has no $\mathcal{F}$-saturated algebraic subsets", then the existence of a partial connection on E implies that E is projective. Indeed, in this case, if $I \subset A$ is an ideal such that $\mathrm{d}_{\mathcal{F}}(I) \subset I \cdot \Omega^1_{\mathcal{F}}$, then the subscheme Y cut-out by I is saturated.

[¶]I. Biswas reminds me that this result is already explained, with unnecessary hypothesis, in [B+87, p.211]. On the other hand, André's point of view is useful, as seen in the case of partial connections.

[‖]Thanks are due to J. V. Pereira for raising this point.

Stratifications and differential operators

We now define

$$\mathcal{D}^{\leq \nu}_{A/k} := \mathrm{Hom}_A\left(\mathcal{P}^{\nu}_A, A\right),$$

which are *the differential operators* of order $\leq \nu$ and

$$\mathcal{D}_{A/k} = \varinjlim \left(\cdots \longrightarrow \mathrm{Hom}_A\left(\mathcal{P}^{\mu}_A, A\right) \xrightarrow{\mathrm{Hom}_A(\mathrm{pr}, A)} \mathrm{Hom}_A\left(\mathcal{P}^{\nu}, A\right) \longrightarrow \cdots \right).$$

Each $\mathrm{Hom}_A\left(\mathcal{P}^{\mu}_A, A\right)$ has an obvious A-module structure, so at the moment that is all we have. But there is a way to multiply (=compose).

Let me note right from the start that if $(\varepsilon_\mu)_{\mu \in \mathbf{N}}$ is a pseudo-stratification of E, then we can let each $\partial \in \mathcal{D}^{\leq \mu}$ "act" on E by

$$E \xrightarrow{\mathbf{1} \otimes \mathrm{id}} \mathcal{P}_A \otimes E \xrightarrow{\varepsilon_\mu} E \otimes \mathcal{P}_A \xrightarrow{\mathrm{id} \otimes \partial} E\,.$$

In order to have the proper notion of action, it is necessary to be able to compose in $\mathcal{D}^{\leq \mu}$.

Let

$$\delta : \mathcal{P}_A \longrightarrow \mathcal{P}_A \otimes_A \mathcal{P}_A$$
$$a_0 \otimes a_1 \longmapsto (a_0 \otimes 1) \otimes (1 \otimes a_2).$$

This is hard to understand algebraically. BUT, geometrically, it makes a lot of sense. Let $X = \mathrm{Spec}\, A$ and $P = \mathrm{Spec}\, \mathcal{P}_A$. Then δ corresponds to the morphism

$$P \underset{\mathrm{pr}_1, X, \mathrm{pr}_0}{\times} P \longrightarrow P, \qquad (x_0, x_1, x_1, x_2) \longmapsto (x_0, x_2).$$

Then, using that

$$\boxed{\delta(\mathrm{d}x_i) = dx_i \otimes \mathbf{1} + \mathbf{1} \otimes dx_i}$$

it is a simple matter to see that $\delta(I^{\mu+\nu+1}) \subset I^{\mu+1} \otimes \mathcal{P}_A + \mathcal{P}_A \otimes I^{\nu+1}$ and we obtain

$$\delta^{\mu,\nu} : \mathcal{P}^{\mu+\nu}_A \longrightarrow \mathcal{P}^{\mu}_A \underset{\tau, A}{\otimes} \mathcal{P}^{\nu}_A.$$

Definition 51. Let $\varphi \in \mathcal{D}^{\leq \mu}$ and $\varphi \in \mathcal{D}^{\leq \nu}$. Define $\varphi\psi$ as the element of $\mathcal{D}^{\leq \mu+\nu}$ s.t.

$$\mathcal{P}^{\mu+\nu}_A \xrightarrow{\delta} \mathcal{P}^{\mu} \otimes_A \mathcal{P}^{\nu} \xrightarrow{\mathrm{id} \otimes \psi} \mathcal{P}^{\mu} \xrightarrow{\varphi} A.$$

With this property,

$$\mathcal{D}_A := \cup_\mu \mathcal{D}^{\leq \mu}$$

is an associative A-algebra.

Example 52. Take $A = k[x]$. We know that $\mathcal{P}^{\mu}_A = k[x][dx]/(dx)^{\mu+1}$. Let ∂_q be the dual basis of $\{(dx)^q\}$. Cause

$$\delta((dx)^q) = \sum_{i+j=q} \binom{q}{i} (\mathrm{d}x)^j \otimes (\mathrm{d}x)^i$$

we get

$$\partial_r \partial_s = \binom{r+s}{s} \partial_{r+s}.$$

Suppose $\mathbb{F}_p \subset k$. Then,

$$q! \cdot \partial_q = (\partial_1)^q.$$

Then,

$$(\partial_1)^p = 0.$$

In this case, $\mathcal{D}_A$ is not f.g.

This example can be generalized with the introduction of even more notation and the help of Thm. 44.

Theorem 53 ([BO78, 2.6])**.** *Let $\boldsymbol{x} : \operatorname{Spec} A \to \mathbf{A}^n_k$ be étale coordinates. Let*

$$\{\partial_{\boldsymbol{q}} \,:\, q_1 + \cdots + q_n \leq \nu\}$$

be the dual basis of $(\mathrm{d}x_1)^{q_1} \cdots (\mathrm{d}x_n)^{q_n}$. Then

$$\mathcal{D}^{\leq \nu}_{A/k} = \bigoplus_{q_1+\cdots+q_n \leq \nu} A\, \partial_{\boldsymbol{q}}.$$

Exercise 54. For each $\boldsymbol{q}, \boldsymbol{r} \in \mathbf{N}^n$, let $\binom{\boldsymbol{q}}{\boldsymbol{r}} = \prod_i \binom{q_i}{r_i}$ and $\boldsymbol{q}! = q_1! \cdots q_n!$. Prove the following formula.

1) $\partial_{\boldsymbol{q}} \partial_{\boldsymbol{r}} = \binom{\boldsymbol{q}+\boldsymbol{r}}{\boldsymbol{r}} \partial_{\boldsymbol{q}+\boldsymbol{r}}$.

2) Suppose now that $\mathbf{Q} \subset k$. Write

$$\partial_i := \partial_{\underbrace{(0, \ldots, 1, \ldots, 0)}_{i \text{ th place}}}$$

Show that

$$\partial_{\boldsymbol{q}} = \frac{1}{\boldsymbol{q}!} \partial^{\boldsymbol{q}},$$

where I have adopted the usual notation in PDEs: $\partial^{\boldsymbol{q}} = \partial_1^{q_1} \cdots \partial_n^{q_n}$. *In particular, $\mathcal{D}_A$ is generated by the differential operators of order ≤ 1.*

Once $\mathcal{D} := \cup_{\mu} \mathcal{D}^{\leq \mu}$ gains structure of ring, we can now introduce Grothendieck's version of what a connection is.

Definition 55. A pseudo-stratification $\{\varepsilon_\nu\}$ on M is a *stratification* when it satisfies *the cocycle condition*, which is the following. The arrow of $\mathcal{P}_A^{\mu+\nu}$-modules

$$\varepsilon_{\mu+\nu} : \mathcal{P}^{\mu+\nu} \underset{A}{\otimes} M \xrightarrow{\sim} M \underset{A}{\otimes} \mathcal{P}^{\mu+\nu},$$

when pulled-back via $\delta^{\mu,\nu} : \mathcal{P}^{\mu+\nu} \to \mathcal{P}^\mu \otimes \mathcal{P}^\nu$:

$$(\mathcal{P}^\mu \otimes \mathcal{P}^\nu) \bigotimes_{\mathcal{P}^{\mu+\nu}} \left(\mathcal{P}^{\mu+\nu} \otimes M\right) \quad \longrightarrow \quad \left(M \otimes \mathcal{P}^{\mu+\nu}\right) \bigotimes_{\mathcal{P}^{\mu+\nu}} (\mathcal{P}^\mu \otimes \mathcal{P}^\nu)$$

is "what you wanted to be", namely

$$\mathcal{P}^\mu \otimes \mathcal{P}^\nu \otimes M \quad \xrightarrow{\mathrm{id}\otimes\varepsilon_\nu} \quad \mathcal{P}^\mu \otimes (M \otimes \mathcal{P}^\nu) \quad \xrightarrow{\varepsilon_\mu\otimes\mathrm{id}} \quad M \otimes \mathcal{P}^\mu \otimes \mathcal{P}^\nu.$$

Theorem 56 (Grothendieck)**.** *If A/k is smooth: Stratifications and actions of $\mathcal{D}$ are one and the same objects.*

Idea behind the proof. Let E be an A-module. If E has a pseudo-stratification, on p. 27 I already explained how to make $\mathcal{D}$ act on E. The cocycle condition assures that this is indeed an action.

More details are in [BO78, 2.11]. □

Example 57. Let $k = \mathbb{F}_3$. $A = k[x]$. Let $\mathcal{E} = \mathcal{O}\boldsymbol{e}$. Let $\nabla \boldsymbol{e} = x\boldsymbol{e} \otimes \mathrm{d}x$ define a connection. This gives a 1-stratification. Suppose that this connection extends to a stratification. Then $(\partial_1)^3(\boldsymbol{e}) = (\partial_x)^3(e) = 0$, which it false. Even if $k = \mathbf{Z}$, then $3 \mid \partial_1^3\boldsymbol{e}$, which is again false.

On the positive side, let me take $k = \mathbf{Z}[1/2]$, $A = k[x^\pm]$; let $B = k[y^\pm]$ be the A-algebra defined by $x \mapsto y^2$. That is, $B = A \oplus yA$. Using that $\binom{1/2}{\nu} \in k$, we can give B an action of $\mathcal{D}$ which is compatible with the action of $\dfrac{\mathrm{d}}{\mathrm{d}x}$ introduced in Example 9.

Lecture 4

(16 August 2022).

"Gauss-Manin"

Many interesting linear differential equations come from "differentiating under the integral sign with respect to a parameter". This is an idea which has a very long history; in algebraic and analytic geometry by introducing further understanding and technology, we arrive at the notion of "Gauss-Manin" connections.

In most of modern algebraic literature this is presented following an idea of Mumford [KO68] and needs the notion of "algebraic de Rham cohomology", spectral sequences and filtrations. These are heavy objects. Another way to talk about these is by means of Analysis. See [MVL21, Ch. 9] for more. I shall present another way, due to Chevalley [Ch51] and employed by Manin in his original treatment [Ma]. The reason for this choice is two-fold: I wanted to throw light on a construction which should be better known and I wanted to explain Messing's computation in [Me72]. All relies on *function fields* and is very thoroughly explained in [Ch51]. This will also give me the possibility of talking about *differential modules*, which is an important and much practiced method of studying connections.

Differential modules

Let K be a field and $\delta : K \to K$ be a derivation: $\delta(fg) = \delta(f)g + f\delta(g)$.

Definition 58. A differential module over K is a a couple (E, ∇) consisting of a finite dimensional vector space E and an additive arrow $\nabla : E \to E$ which satisfies Leibniz's rule $\nabla(fe) = \delta(f)e + f\nabla(e)$ for all $f \in K$ and $e \in E$.

Following the pattern of Definition 63, the reader will have no difficulty in defining the category of modules with connections $\mathbf{MC}(K)$.

It is possible to work in more generality and instead of fixing one derivation δ, one deals with a finite set of commuting ones.

Examples of differential modules can easily be obtained from connections on open subsets of $\mathbf{A}^1$.

Although less geometric in embryo, differential modules are and have always been actively studied due to their simplicity. I strongly recommend [SvdP].

Basics of function fields

An extension of fields F/K is a *function field* (in one variable) over K if there exists $x \in F$ which is transcendent over K and such that $F/K(x)$ is finite and separable. (It is possible to work out a theory where separability is eschewed as [Ch51, p.1] hints, but this is rather pathologic.) An element functioning as x above is a *separating variable*. We shall assume that K is algebraically closed in F (the algebraic closure of K in F is called in [Ch51, p.1], the field of constants of F).

Of course, the reader is required to think about fields of rational functions on complete curves.

Here are some ingredients which we shall require.

Definition 59. (a) The Zariski-Riemann space $\mathbf{S}_{F|K}$ is the set of discrete valuations, or places, $v : F^* \to \mathbf{Z}$ which are trivial on K.

(b) For $v \in \mathbf{S}_{F|K}$, we let $\mathcal{O}_v = \{f \in F : v(f) \geq 0\}$ stand for its valuation ring and $\mathfrak{m}_v$ for its maximal ideal, the residue field is denoted by K_v.

(c) For $v \in \mathbf{S}_F$, we let $\widehat{F}_v$ be the v-adic completion of F and let $\widehat{\mathcal{O}}_v$ be the closure of the DVR $\mathcal{O}_v$.

(d) A divisor of F is a finite sum $\sum_v n_v\,[v]$, where $n_v \in \mathbf{Z}$.

(e) The ring of adèles (resp. complete adèles) of F/K is the restricted product

$$A = \left\{ \alpha \in \prod_v F \;:\; \alpha_v \in \mathcal{O}_v \text{ except for finitely many } v \right\}.$$

resp.

$$\widehat{A} = \left\{ \alpha \in \prod_v \widehat{F}_v \;:\; \alpha_v \in \widehat{\mathcal{O}}_v \text{ except for finitely many } v \right\}.$$

In [Ch51], A is called the space of repartitions as is $\widehat{A}$. See pages 25 and 46 in [Ch51]. This is naturally an F-vector space, and also a K vector space.

(f) For $D = \sum n_v\,[v]$, let $A(D)$ be the space of adèles α s.t. $v(\alpha) \geq -n_v$. Note: $D \leq D' \Rightarrow A(D) \subset A(D')$.

W–differentials

Define the space of W–differentials (Weil differentials) as

$$W = \{\theta \in \operatorname{Hom}_K(A, K) \;:\; \theta \text{ vanishes on } F + A(D) \text{ for some } D\}.$$

See [Ch51, p. 30]. It is standard to write $W(D) := \{\theta \in W \;:\; \theta|_{F+A(D)} = 0\}$. If $D \leq D' \Rightarrow$ the natural arrow $W(D') \to W(D)$ is injection.

Since $D \leq D'$ implies $W(D') \subset W(D)$, the following definition makes sense.

Definition 60. For $\theta \in W$, there exists a divisor $\operatorname{div}(\theta)$ such that

$$\theta \in W(\operatorname{div}(\theta)) \text{ but } \theta \notin W(D) \text{ if } D > \operatorname{div}(\theta).$$

The divisor $\operatorname{div}(\theta)$ is the *divisor of the W–differential* θ. Given $v \in S_F$, we let $v(\theta) = v(\operatorname{div}(\theta))$ stand for the *order of* θ *at* v. The place v is a *pole* of θ if $v(\theta) < 0$ and a *zero* if $v(\theta) > 0$. A W–differential is *regular* if it has no poles.

The definition of Weil-differential is by far the worst for those who have never met this approach before. It is one of the thorns of the theory. Simple examples of W–differentials are difficult to write down. The link between these and "Kähler" differentials is given by the *theory of residues*, which is a delicate matter if the ground fields are not algebraically closed.

Definition 61 (Local components). Let $\theta \in W$ and $v \in \mathbf{S}_F$. If $\iota_v : F \to A$ stands for the inclusion of the vth coordinate, we define θ_v, *the local component of* θ, as being $\theta \circ \iota_v$.

Exercise 62. Let $\theta \in W$ and $v \in \mathbf{S}_F$. Show that $\theta_v : F \to K$ is continuous, where F is given the v-adic topology and K the discrete one. Deduce that θ_v can be extended to $\widehat{F}_v$.

Given $f \in F$ and $\theta \in W$: define $f\theta$ by $\alpha \mapsto \theta(f\alpha)$. This endows W with the structure of an F-vector space.

Theorem 63 ([Ch51, p.31]). *The F-space W is one dimensional.* □

Corollary 64. *The divisors of any two W–differentials are equivalent. The class of any divisor* $\operatorname{div}(\theta)$ *is called the canonical class.*

We now produce W–differentials.

Example 65 (Key example, [Ch51, p.102]). Let $F = K(x)$. Define $\mathrm{d}x \in W$: If ∞ is the place at infinity, then

$$[\mathrm{d}x]_\infty : x^i \longmapsto \begin{cases} 0, & i \neq -1 \\ -1, & i = -1. \end{cases}$$

Let $\pi \in K[x]$ irreducible, monic, of degree m, and $v \in \mathbf{S}_F$ the associated zero. Decompose $f \in F$ in partial fractions:

$$f = \frac{f_r}{\pi^r} + \cdots + \frac{f_1}{\pi} + g,$$

where $f_i \in K[x]$, $\deg f_i < \deg f$ and $v(g) \geq 0$. Then

$$[\mathrm{d}x]_v(f) = \text{coefficient of } x^{\deg f - 1} \text{ in } f_1.$$

Let F/K be a function field and $x \in F$ a separating variable. Write $R := K(x)$. Following [Ch51, p. 67], let us define

$$\mathrm{Tr} : \widehat{A}_F \longrightarrow \widehat{A}_R.$$

by

$$(\mathrm{Tr}\,\alpha)_v = \sum_{w|v} \underbrace{\mathrm{Tr}_{\widehat{F}_w/\widehat{R}_v}(\alpha_w)}_{\in \widehat{R}_v}.$$

This is called the trace of α.

Now, if $\eta \in W_{R/K}$, we then define its *co-trace*,

$$\mathrm{Cotr}(\eta) : \widehat{A} \longrightarrow K, \quad \alpha \longmapsto \eta(\mathrm{Tr}(\alpha)).$$

As [Ch51, pp 105-6] argue, $\mathrm{Cotr}(\eta)$ is a W-differential. For future use, we shall observe that if we identify $\widehat{F}_w$ with the adèles supported at w, then

$$\mathrm{Cotr}(\eta)_w : \widehat{F}_w \longrightarrow K \quad f \longmapsto \eta\left[\mathrm{Tr}_{\widehat{F}_w/\widehat{R}_v}(f)\right]$$

In this way, for each separating variable $x \in F$, we can define

$$[\mathrm{d}x]_F := \mathrm{Cotr}(\mathrm{d}x) \in W_F,$$

which together with Theorem 63 allows us to write

$$W = F \cdot [\mathrm{d}x]_F.$$

If $x \in K$, we shall put $[\mathrm{d}x]_F = 0$, and if $x \in F \smallsetminus K$ is not separating, we also decree that $[\mathrm{d}x]_F = 0$.

The relation between W and the usual $\Omega^1_{F/K}$ is given by Theorem 9 on p. 116 of [Ch51]. To understand this result, recall that the usual derivation $\dfrac{\mathrm{d}}{\mathrm{d}x}$ on $K(x)$ can be uniquely extended to a derivation, denote likewise, of F. This being so, for any element $y \in F$, we have

$$\mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x. \tag{4}$$

Hence,

$$[\mathrm{d}-]_F : F \longrightarrow W$$

identifies W with $\Omega^1_{F/K}$.

Differentiating W-differentials

Let F/K be a function field with separating variable x. Let $\delta : K \to K$ be a derivation. We extend δ to

$$\delta^x : F \longrightarrow F$$

by the rule $\delta^x(x) = 0$. This is called a "horizontal" extension.

For each $f \in F$, define

$$\delta^x(f\mathrm{d}x) = \delta^x(f)\,\mathrm{d}x.$$

In this way, we obtain an additive arrow

$$\delta^x : W \longrightarrow W.$$

This is rather silly and arbitrary for the moment and depends on a the many choices.

Let $y \in F$ be another separating element. By eq. (4)

$$\begin{aligned}\delta^y(\mathrm{d}x) &= \delta^y\left(\frac{\mathrm{d}x}{\mathrm{d}y}\,\mathrm{d}y\right)\\ &= \delta^y\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)\,\mathrm{d}y.\end{aligned}$$

Lemma 66. *Let* $\frac{d}{dx} : F \to F$ *be the K–linear derivation mapping x to 1. Then*

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}, \delta^x\right] = 0.$$

Proof. $\left[\frac{\mathrm{d}}{\mathrm{d}x}, \delta^x\right]$ vanishes on K and on x, hence of F. □

Using the Lemma, we continue our calculation of $\delta^y(\mathrm{d}x)$:

$$\begin{aligned} \delta^y(\mathrm{d}x) &= \delta^y\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)\mathrm{d}y \\ &= \frac{\mathrm{d}}{\mathrm{d}y}(\delta^y(x))\,\mathrm{d}y \\ &= \mathrm{d}(\delta^y(x)). \end{aligned} \tag{5}$$

Theorem 67 (Chevalley)**.** *Let* $y \in F$ *separating. Then*

$$\delta^x - \delta^y : W \longrightarrow W$$

*has its image in the **exact** W-differentials; precisely:*

$$[\delta^x - \delta^y](f\,\mathrm{d}x) = -\mathrm{d}(f\delta^y(x))$$

Proof. Consider the derivation

$$\delta^x - \delta^y + \delta^y(x)\frac{\mathrm{d}}{\mathrm{d}x}.$$

It vanishes on K and on x. Hence vanishes on $F \Rightarrow$

$$\delta^x - \delta^y = -\delta^y(x)\frac{\mathrm{d}}{\mathrm{d}x}. \tag{6}$$

Then

$$\begin{aligned} (\delta^x - \delta^y)(f\,\mathrm{d}x) &= (\delta^x(f) - \delta^y(f))\,\mathrm{d}x - f\delta^y(\mathrm{d}x) && (\delta^y \text{ is derivation}) \\ &= (\delta^x(f) - \delta^y(f))\,\mathrm{d}x - f\mathrm{d}(\delta^y(x)) && (\text{see eq. (5)}) \\ &= -(\delta^y(x)\frac{\mathrm{d}}{\mathrm{d}x}f)\mathrm{d}x - f\,\mathrm{d}(\delta^y(x)) && \text{because of eq. (6)} \\ &= -\delta^y(x)\,\mathrm{d}f - f\,\mathrm{d}(\delta^y(x)) \\ &= -\mathrm{d}(f\delta^y(x)). \end{aligned}$$

□

For geometric purposes, the "de Rham" space $W/\mathrm{d}W$ is much too big. To get something reasonable, we require *differentials of second kind* and the associated de Rham space.

Differentials of the second kind and residues

It is here where working with W–differentials comes cheaper, since it allows directly the use of residues. For the sake of economy, we write δ in place of δ^x.

Let $v \in \mathbf{S}_{F|K}$. Let $K_{v,\mathrm{s}} \subset K_v$ the separable closure of K inside K_v. Hensel's Lemma assures the existence of a copy of $K_{v,\mathrm{s}} \subset \widehat{\mathcal{O}}_v$ containing K [Ch51, Theorem 1, p.44]. Consequently,

$$\theta_v(f) = \mathrm{Tr}_{K_{v,\mathrm{s}}/K}\left(\mathrm{res}_v(\theta)\cdot f\right), \qquad \forall\, f \in K_{v,\mathrm{s}}$$

for a unique $\mathrm{res}_v(\theta) \in K_{v,\mathrm{s}}$. This is called the *residue of θ at v*. See [Ch51, p.48]. It goes without saying that the actual determination of $\mathrm{res}_v(\theta)$ is not a trivial matter. On the other hand, in special circumstances, we have a satisfactory picture:

Theorem 68 ([Ch51, Corollary, p.110])**.** *Let K be of characteristic zero and $v \in \mathbf{S}_{F|K}$ be of degree one. Let $t \in F$ be a uniformizer at v.*

Then:

1) *We have an isomorphism $\widehat{F}_v \simeq K((t))$.*

2) *The local component $[\mathrm{d}t]_v$ is*

$$\sum_{i \gg -\infty} c_i t^i \longmapsto c_{-1}.$$

3) *For $y \in F$ such that $y = \sum_{i\gg-\infty} c_i t^i$, we have*

$$\mathrm{res}_v(y\,\mathrm{d}t) = c_{-1}.$$

Definition 69. A W–differential θ is of the second kind if $\mathrm{res}_v(\theta) = 0$ for all v. Notation: W^{sk}.

Theorem 70 (Chevalley)**.** *$\delta(W^{\mathrm{reg}}) \subset W^{\mathrm{reg}}$ and $\delta(W^{\mathrm{sk}}) \subset W^{\mathrm{sk}}$.*

The proof is a consequence of:

Theorem 71. *i) Let $v \in \mathbf{S}_{F|K}$ and $\theta \in W$. Then,*

$$[\delta\theta]_v = \delta \circ \theta_v - \theta_v \circ \delta. \tag{7}$$

*ii) For each $v \in \mathbf{S}_F$ and each $\theta \in W$, we have $\mathrm{res}_v(\delta(\theta)) = \delta(\mathrm{res}_v(\theta))$. (*Residues and δ commute.*)*

Proof. (i) Let $\theta = y\,\mathrm{d}x$. Let $f \in \widehat{F}_v$. We evaluate the LHS of eq. (7) on f:

$$\begin{aligned}
[\delta\theta]_v(f) &= [\delta(y\,\mathrm{d}x)]_v\,(f) \\
&= [\delta(y)\,\mathrm{d}x]_v\,(f) && \text{(definition of } \delta \text{ acting on } W) \\
&= [\mathrm{d}x]_v(\delta(y)\cdot f) && \text{(definition of } F \text{ acting on } W).
\end{aligned}$$

Now:

$$\delta\left(\theta_v(f)\right) = \delta\left([\mathrm{d}x]_v(fy)\right)$$

and

$$\theta_v(\delta(f)) = [\mathrm{d}x]_v(y \cdot \delta(f)),$$

so that we want

$$[\mathrm{d}x]_v(\delta(y) \cdot f) = \delta\left([\mathrm{d}x]_v(f \cdot y)\right) - [\mathrm{d}x]_v(y \cdot \delta(f)).$$

We then only need to prove

$$\delta\left([\mathrm{d}x]_v(g)\right) = [\mathrm{d}x]_v\left(\delta(g)\right) \tag{8}$$

in case $g = f \cdot y \in \widehat{F}_v$ and use that δ is derivation.

Let $R := K(x)$ and let $w \in \mathbf{S}_{R/K}$ be the image of v. We know that derivations commute with traces (Exercise 76), in the present case

$$\mathrm{Tr}_{\widehat{F}_v/\widehat{R}_w} \circ \delta = \delta \circ \mathrm{Tr}_{\widehat{F}_v/\widehat{R}_w}.$$

Now, we know that $\mathrm{d}x \in W_{F/K}$ is the co-trace of $\mathrm{d}x \in W_{R/K}$. This means:

$$[\mathrm{d}x]_v = [\mathrm{d}x]_w \circ \mathrm{Tr}_{\widehat{F}_v/\widehat{R}_w}.$$

Then, eq. (8) follows from

$$\delta\left([\mathrm{d}x]_w(h)\right) = [\mathrm{d}x]_w\left(\delta\left(h\right)\right) \tag{9}$$

applied to $h = \mathrm{Tr}_{\widehat{F}_v/\widehat{R}_w}(g) \in \widehat{R}_w$.

The case "$w = \infty$." – Write $h = \sum_{i \gg -\infty}^{\infty} h_i x^{-i}$ with $h_i \in K$. In this case, $[\mathrm{d}x]_w(x^i)$ is -1 if $i = -1$ and zero otherwise. Because $\delta(x^i) = 0 \Rightarrow \delta(h) = \sum_{i \gg -\infty}^{\infty} \delta(h_i) x^{-i}$ and we are done.

The case "$w \neq \infty$" – We shall rely on the previous case. Pick $h_1 \in R$ such that $w(h - h_1) \geq 0$ and use the fact that $\mathrm{div}(\mathrm{d}x) = -2[\infty]$ to obtain

$$[\mathrm{d}x]_w(h) = [\mathrm{d}x]_w(h_1).$$

By strong approximation applied to the set $\{\infty, w\}$ and the element h_1, there exists $h_2 \in R$ such that $w(h_1 - h_2) = 0$ and $\overline{w}(h_2) \geq 0$ for all $\overline{w} \notin \{w, \infty\}$. In particular

$$\begin{aligned}[\mathrm{d}x]_w(h) &= [\mathrm{d}x]_w(h_1)\\ &= [\mathrm{d}x]_w(h_2).\end{aligned}$$

Exercise 72. For each $\overline{w} \in \mathbf{S}_R$, we have $\delta(\mathcal{O}_{\overline{w}}) \subset \mathcal{O}_{\overline{w}}$.

We know that h_2 and $\delta(h_2)$ have poles at most on $\{\infty, w\}$ and that $[\mathrm{d}x]_{\overline{w}}(\mathcal{O}_{\overline{w}}) = 0$ if $\overline{w} \neq \infty$. Then

$$\begin{aligned}
[\mathrm{d}x]_w(\delta(h)) &= [\mathrm{d}x]_w(\delta(h_1)) && \text{(because } w(\delta(h)-\delta(h_1)) \geq 0\text{)}\\
&= [\mathrm{d}x]_w(\delta(h_2)) && \text{(because } w(\delta(h_1)-\delta(h_2)) \geq 0\text{)}\\
&= -[\mathrm{d}x]_\infty(\delta h_2) && \text{(because of Residue Thm.),}\\
&= -\delta([\mathrm{d}x]_\infty(h_2)) && \text{(previous case),}\\
&= \delta([\mathrm{d}x]_w(h_2)) && \text{(Residue Thm.),}\\
&= \delta([\mathrm{d}x]_w(h)) && \text{(because } w(h-h_2) \geq 0\text{).}
\end{aligned}$$

(ii) Let $v \in \mathbf{S}_F$ and put $\rho = \mathrm{res}_v(\theta)$. Write $\mathrm{Tr} := \mathrm{Tr}_{K_{v,s}/K}$.

Let $f \in K_{v,s}$ be arbitrary. We shall show that

$$\mathrm{Tr}(\delta(\rho)\cdot f) = [\delta\theta]_v(f),$$

which proves $\mathrm{res}_v(\delta(\theta)) = \delta(\rho)$ by the very definition of res_v. Given (i), we need to prove

$$\mathrm{Tr}(\delta(\rho)\cdot f) = \delta(\theta_v(f)) - \theta_v(\delta(f)).$$

But

$$\begin{aligned}
\delta(\theta_v(f)) &= \delta(\mathrm{Tr}(\rho f)) && \text{(because } \delta \text{ commutes with traces)}\\
&= \mathrm{Tr}(\delta(\rho)\cdot f + \rho\cdot\delta(f))\\
&= \mathrm{Tr}(\delta(\rho)\cdot f) + \theta_v(\delta(f))
\end{aligned}$$

which gives the equality we want.

□

Theorem 73 ([Ch51, Cor. 1, p.130])**.** *If* char. $K = 0$*, then*

$$\dim_K W^{\mathrm{sk}}/\mathrm{d}W = 2 \times \textit{genus}.$$

This is false in char. p.

The above space is called H^1_{dR}. (It is the first algebraic de Rham cohomology of the proper smooth modem of F.)

Definition 74. The differential module (H^1_{dR}, δ^x) is called the Gauss-Manin connection [Ma].

Remark 75. It is in general possible to develop a nice theory of differentials of the second kind in complex geometry; see [GH78, 454 ff].

Exercise 76. Let E/F be a finite extension of fields and $d : E \to E$ a derivation which preserves F.

1) Assume that E/F is purely inseparable. Show that $\mathrm{Tr}_{E/F} \circ d = d \circ \mathrm{Tr}_{E/F}$.

2) Assume that E/F is separable. Passing to the Galois closure, show that $\mathrm{Tr}_{E/F} \circ d = d \circ \mathrm{Tr}_{E/F}$.

A noble series

For each $s \in \mathbf{C} \smallsetminus \mathbf{Z}_{\leq 0}$ and $n \in \mathbf{Z}_{>0}$, define the *Pochammer symbol*:

$$(s)_n = s(s+1)\cdots(s+n-1).$$

By convention $(s)_0 = 1$. Following Wallis, Euler and Gauss we set out to study the *hypergeometric* * * series

$$F(a,b,c \mid \lambda) = 1 + \sum_{n \geq 1} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n(1)_n}\lambda^n.$$

This series, Euler had remarked before, was the solution to the hypergeometric differential equation

$$\lambda(1-\lambda)\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} + \{\gamma - (\alpha+\beta+1)\lambda\}\frac{\mathrm{d}}{\mathrm{d}\lambda} - \alpha\beta = 0. \tag{HGDE}$$

The associated differential module over $\mathbf{Q}(\lambda)$ shall be denoted by $\mathrm{HG}(\alpha,\beta,\gamma)$.

The Gauss-Manin connection of certain hyperelliptic curves and the hypergeometric differential equation

Let now $a,b,c,n \in \mathbf{Z}_{>0}$. Suppose:

$$\gcd(a,n) = \gcd(b,n) = \gcd(c,n) = \gcd(a+b+c,n) = 1.$$

Let $\boldsymbol{\mu}_n$ be the group of nth roots of unity and set $K = \mathbf{Q}(\boldsymbol{\mu}_n, \lambda)$. Define

$$\boxed{F = K(x,y) \quad \text{with} \quad y^n = x^a(x-1)^b(x-\lambda)^c.}$$

Said differently, if

$$R = K(x),$$

then $F|R$ is a *cyclic extension* and $\mathrm{Gal}(F|R) \simeq \boldsymbol{\mu}_n$ by letting $\varepsilon \in \boldsymbol{\mu}_n$ act as the identity on R and as usual, $\varepsilon(y) = \varepsilon y$, on y. This action gives rise to an action on $\mathbf{S}_{F|K}$ and on $\widehat{A}_{F|K}$ as follows. For each $v \in \mathbf{S}_{F|K}$ we set $\varepsilon(v) = v \circ \varepsilon^{-1}$. Note: we obtain isomorphisms $\varepsilon : \mathcal{O}_v \xrightarrow{\sim} \mathcal{O}_{\varepsilon v}$ and $\varepsilon : \widehat{\mathcal{O}}_v \xrightarrow{\sim} \widehat{\mathcal{O}}_{\varepsilon v}$. For $\alpha \in \widehat{A}_{F|K}$, we put

$$[\varepsilon(\alpha)]_v = \varepsilon(\alpha_{\varepsilon^{-1}(v)}) = \varepsilon(\alpha_{v\circ\varepsilon}).$$

* * As explained in [Gou36], the term hyperomgetric has the following meaning. Take a series $\sum a_n\lambda^n$ with non-vanishing coefficients and such that the d'Alembert limit $\lim_n \frac{a_{n+1}}{a_n}$ is 1. Suppose one can write $\frac{a_{n+1}}{a_n} = \frac{P_m(n)}{(n+1)Q_{m-1}(n)}$, where P_m is a polynomial of degree m and Q_{m-1} of degree $m-1$. If $m = 1$, we recover the usual series of the form $(1-\lambda)^\alpha$. The case $m = 2$ gives us hypergeometric series. See also §14.1 in [WW15].

In particular, $f \in \widehat{F}_v$ is considered as an adele, then $\varepsilon(f)$ is the adele which is zero allover, except at εv, where it is $\varepsilon(f)$.

The action on the adeles, in turn, induces an action on $W_{F|K}$ defined by

$$\varepsilon(\theta) : \widehat{A}_{F|K} \longrightarrow K, \quad \alpha \longmapsto \theta \circ \varepsilon^{-1}.$$

In particular, on the level of local components, we have

$$[\varepsilon(\theta)]_v = \theta_{\varepsilon^{-1}(v)} \circ \varepsilon^{-1}.$$

In addition, it is not difficult to see that each cotrace Weil-differential $\mathrm{Cotr}(\eta)$ is invariant under the action of $\boldsymbol{\mu}_n$. In particular, $\varepsilon(f\mathrm{d}x) = \varepsilon(f)\mathrm{d}x$. Hence, using that $F = \bigoplus_{\ell=0}^{n-1} Ry^\ell$ and that $W_{F|K} = F\,\mathrm{d}x$, we conclude that

$$W = \bigoplus_{\ell=0}^{n-1} W_\ell,$$

where $\boldsymbol{\mu}_n$ acts on W_ℓ via the character $\varepsilon \mapsto \varepsilon^\ell$.

Lemma 77. *The following statements are true.*

1) For each $\varepsilon \in \boldsymbol{\mu}_n$ and $f \in F$, we have $\varepsilon(\mathrm{d}f) = \mathrm{d}\varepsilon(f)$.

2) For each $\varepsilon \in \boldsymbol{\mu}_n$, $\theta \in W$ and $v \in \mathbf{S}_F$, the formula $\mathrm{res}_{\varepsilon v}(\varepsilon\theta) = \varepsilon(\mathrm{res}_v(\theta))$ holds.

3) The group $\boldsymbol{\mu}_n$ acts by K-linear automorphisms on H^1_{dR}.

Proof. (1) We have

$$\begin{aligned} \varepsilon(\mathrm{d}f) &= \varepsilon\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\mathrm{d}x \\ &= \frac{\mathrm{d}(\varepsilon f)}{\mathrm{d}x}\,\mathrm{d}x, \end{aligned}$$

where for the last equality we used that $\varepsilon\dfrac{\mathrm{d}}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\varepsilon$ are derivations of F which agree on $R = K(x)$.

(2) By definition, $\varepsilon : F \to F$ gives $\mathcal{O}_v \xrightarrow{\sim} \mathcal{O}_{\varepsilon v}$, which induces $\varepsilon : \widehat{\mathcal{O}}_v \xrightarrow{\sim} \widehat{\mathcal{O}}_{\varepsilon v}$. In particular, the Cohen field $\widetilde{K}_v \subset \widehat{\mathcal{O}}_v$ is sent isomorphically to the Cohen field $\widetilde{K}_{\varepsilon v}$.

For each $f \in \widetilde{K}_v \subset \widehat{\mathcal{O}}_v$, we have

$$\begin{aligned} (\varepsilon\theta)_v(f) &= \theta_{\varepsilon^{-1}(v)}(\varepsilon^{-1}(f)) \\ &= \mathrm{Tr}_{\varepsilon^{-1}\widetilde{K}_v|K}(\varepsilon^{-1}(f) \cdot \mathrm{res}_{\varepsilon^{-1}(v)}(\theta)) \\ &= \mathrm{Tr}_{\widetilde{K}_v|K}(f \cdot \varepsilon(\mathrm{res}_{\varepsilon^{-1}(v)}(\theta))) \\ &\Rightarrow \\ \mathrm{res}_v(\varepsilon\theta) &= \varepsilon(\mathrm{res}_{\varepsilon^{-1}(v)}(\theta)) \\ &\Rightarrow \\ \mathrm{res}_{\varepsilon(v)}(\varepsilon\theta) &= \varepsilon(\mathrm{res}_v(\theta)). \end{aligned}$$

(3) This is clear. □

In order to go further, we let v_0, v_1, v_λ and v_∞ be the obvious rational places of $R = K(x)$.

Lemma 78. *Only v_0, v_1, v_λ and v_∞ ramify in F and each one has ramification index n (that is, it is totally ramified). The genus of $F|K$ is $n-1$.*

Proof. From [St09, 6.3.1, p.227] we know that v_0, v_1, v_λ are all totally ramified in F, and that if w_∞ is above v_∞, the ramification index is

$$\frac{n}{\gcd(n, a+b+c)}.$$

The hypothesis then shows that v_∞ is totally ramified. From this and the formula of [St09, 6.3.1(b), p.227], the genus of F/K is $n-1$ □

We then have the following situation:

$$\begin{array}{cccccc}
\mathbf{S}_{F|K} & \ni & w_0 & w_1 & w_\lambda & w_\infty \\
 & & | & | & | & | \\
\mathbf{S}_{R|K} & \ni & v_0 & v_1 & v_\lambda & v_\infty,
\end{array}$$

all places in question are *rational*.

We then have

	w_0	w_1	w_λ	w_∞
x	n	0	0	$-n$
y	a	b	c	$-(a+b+c)$

Now, recall that *in characteristic zero*, $w(\mathrm{d}x) = v(x) - 1$ if $w(x) \neq 0$ and $w(\mathrm{d}x) \geq 0$ if $w(x) = 0$. From this we deduce

	w_0	w_1	w_λ	w_∞
$\mathrm{d}x$	$n-1$	$n-1$	$n-1$	$-n-1$
$y^{-1}\mathrm{d}x$	$n-a-1$	$n-b-1$	$n-c-1$	$a+b+c-n-1$
$xy^{-1}\mathrm{d}x$	$2n-a-1$	$2n-b-1$	$2n-c-1$	$a+b+c-2n-1$

If

$$n - 1 \geq \max(a, b, c),$$

then differentials

$$x^i \frac{\mathrm{d}x}{y} \quad i = 0, 1, \ldots$$

only have poles at w_∞. In particular, each one of these is of the second kind. Also, we see that a canonical divisor is given by

$$(n-1)\{[w_0] + [w_1] + [w_\lambda]\} - (n+1)[w_\infty].$$

Lemma 79. *Assume that $n \nmid \ell$. Then*

$$\begin{aligned}\mathrm{res}_{w_0}\ x^m \frac{\mathrm{d}x}{y^\ell} &= \mathrm{res}_{w_1}\ (x-1)^m \frac{\mathrm{d}x}{y^\ell} \\ &= \mathrm{res}_{w_\lambda}\ (x-\lambda)^m \frac{\mathrm{d}x}{y^\ell} \\ &= 0\end{aligned}$$

for each $m \geq 0$.

Before entering the proof, let us make some preliminary comments on differentials.[††] Let $\tilde{K}|K$ be a finite extension. Let $\tilde{F} := F\tilde{K}$ and let $\tilde{w} \in \mathbf{S}_{\tilde{F}|\tilde{K}}$ be above $w \in \mathbf{S}_{F|K}$. According to [Ch51, Thm 11, p.119], for any differential $\theta \in W_{F|K}$, we have

$$\mathrm{res}_{\tilde{w}} \left(\mathrm{Cotr}_{\tilde{F}|F}(\theta)\right) = \mathrm{res}_w \theta. \tag{10}$$

(The reader who actually consults loc.cit. will see that this is an intricate proof. At the end, it is possible to simplify the matter adding assumptions on the extension $\tilde{K}|K$, but we shall leave details as they are.) This shall allow us to verify easily that certain differentials are of second type.

Proof. Let π be a local parameter at w_0. Let $\xi = x/\pi^n + \mathfrak{m}_{w_0}$ and $\eta = (-1)^{b+c}\lambda^c$, and consider $\tilde{K} = K(\sqrt[n]{\xi}, \sqrt[n]{\eta})$.

Let $\tilde{F} = F\tilde{K}$; there is a unique place $\tilde{w}_0 \in \mathbf{S}_{\tilde{F}|\tilde{K}}$ above w_0. (This follows from the fact that $K_{w_0} = K$.) We shall show that

$$\mathrm{Cotr}\left(x^m \frac{\mathrm{d}x}{y^\ell}\right) \in W_{\tilde{F}|\tilde{K}}$$

have vanishing residues on $\tilde{w}_0$. Since $x/\pi^n \in \mathcal{O}_{\tilde{w}_0}$ is invertible and its class class is an nth power in $\mathcal{O}_{\tilde{w}_0}/\mathfrak{m}_{\tilde{w}_0}$, Hensel's Lemma tells us that there exists $t \in \widehat{\mathcal{O}}_{\tilde{w}_0}$ such that $x = t^n$.

We then have that

$$y^n = t^{na}(x-1)^b(x-\lambda)^c.$$

Hence,

$$\left(\frac{t^a}{y}\right)^n = \frac{1}{(x-1)^b(x-\lambda)^c}.$$

Now,

$$(x-1)^b(x-\lambda)^c = (-1)^{b+c}\lambda^c + \left\{(-1)^{b+c-1}c + (-1)^{b+c-1}\lambda^c b\right\} \cdot x + \cdots$$

and we conclude, again by Hensel's Lemma, that $(x-1)(x-\lambda)^c$ is an nth power. In fact, there exist

$$r = r_0 + r_1 x^1 + \cdots \in \tilde{K}((t))$$

[††]Thanks are due to K.-O. Stöhr for discussing this point with me.

whose nth power equals $(x-1)^b(x-\lambda)^c$. Therefore,

$$\frac{1}{y} = t^{-a}\left(s_0 + s_1 x + \ldots\right).$$

Hence,

$$x^m \frac{\mathrm{d}x}{ny^\ell} = t^{n-1-\ell a}\left(s_0' x^m + s_1' x^{m+1} + \ldots\right)\mathrm{d}t$$

Since $x = t^n$, we know that the exponents of t in the above power series are always of the form

$$n - 1 - \ell a + \mu n, \qquad \mu \geq m.$$

Such an expression cannot equal -1. Indeed if this is the case, then $n \mid \ell a$, which is impossible since $\gcd(n, a) = 1$ and $n \nmid \ell$.

The proof of the fact that $(x-1)^m \dfrac{\mathrm{d}x}{y^\ell}$, resp. $(x-\lambda)^m \dfrac{\mathrm{d}x}{y^\ell}$, has no residue on w_1, resp. w_λ, is similar, since this differential is really $(x-1)^m \dfrac{\mathrm{d}(x-1)}{y^\ell}$, resp. $(x-1)^m \dfrac{\mathrm{d}(x-\lambda)}{y^\ell}$. □

Let

$$H^1_{dR} = \bigoplus_{\ell=0}^{n-1} H_{-\ell},$$

be the isotypic decomposition, where $\boldsymbol{\mu}_n$ acts on H_ℓ by $\varepsilon \mapsto \varepsilon^\ell$.

Corollary 80. *Let ℓ be a positive integer not divisible by n. Then*

$$\varphi_\ell := \frac{\mathrm{d}x}{y^\ell}, \quad \psi_\ell =: x\frac{\mathrm{d}x}{y^\ell}$$

are of the second kind and induce elements of $H_{-\ell}$.

Proof. The only poles of φ_ℓ and ψ_ℓ are among w_0, w_1, w_λ or w_∞. We show that their residues on w_0, w_1 or w_λ vanish, so that these differentials are of the second kind. We already know that φ_ℓ and ψ_ℓ have no residues at w_0. By Lemma 79 applied with $m = 1$, we conclude

$$\mathrm{res}_{w_1} x\frac{\mathrm{d}x}{y^\ell} = \mathrm{res}_{w_1} \frac{\mathrm{d}x}{y^\ell} = 0.$$

Analogously,

$$\mathrm{res}_{w_\lambda} x\frac{\mathrm{d}x}{y^\ell} = \lambda \cdot \mathrm{res}_{w_\lambda} \frac{\mathrm{d}x}{y^\ell} = 0.$$

□

Proposition 81. *Assume that $n \nmid \ell$. The classes of φ_ℓ and ψ_ℓ in H^1_{dR} are linearly independent. In particular,*

$$H^1_{dR} = \bigoplus_{\ell=1}^{n-1} H_{-\ell},$$

and $H_{-\ell} = K\varphi_\ell + K\psi_\ell$.

Proof. We decompose $W = \oplus_{\ell=0}^{n-1} W_\ell$ into isotypic components for the $\boldsymbol{\mu}_n$-action and note that $\mathrm{d}(K(x)y^\ell) \subset W_\ell$. Hence, if our first statement is false, we should have that there exists $f \in K(x)$, $\alpha, \beta \in K$, such that

$$\mathrm{d}(fy^{-\ell}) = (\alpha x + \beta) \cdot \frac{\mathrm{d}x}{y^\ell}.$$

This gives

$$\frac{\mathrm{d}f}{y^\ell} - \ell \frac{f}{y^\ell}\frac{\mathrm{d}y}{y} = (\alpha x + \beta) \cdot \frac{\mathrm{d}x}{y^\ell},$$

so that we have the following equality in $W_{K(x)|K}$:

$$\frac{\mathrm{d}f}{f} - \ell \left[\frac{a/n}{x} + \frac{b/n}{x-1} + \frac{c/n}{x-\lambda} \right] \mathrm{d}x = (\alpha x + \beta)\frac{\mathrm{d}x}{f}. \tag{\&}$$

We now make several claims.

The poles of f are in $\{v_0, v_1, v_\lambda, v_\infty\}$. Suppose this is false. Let $v \neq v_0, v_1, v_\lambda, v_\infty$ be a place of $K(x)$ which is a pole of f. Then $f^{-1}\mathrm{d}f$ has a pole at v, which contradicts (&).

The only pole of f is v_∞. Assume that v_0 is a pole of f. Then, using Theorem 68, the LHS of (&) has residue

$$v_0(f) - \frac{\ell a}{n}.$$

But the RHS of (&) is regular at v_0, so that we should conclude that $n \mid \ell a$, which is false. The same reasoning applies for v_1, v_λ.

Each v_0, v_1, v_λ is a zero of f. Assume that f is invertible in $\mathcal{O}_{v_0}$. Then equation (&) reads

regular at v_0 + pole of order 1 at v_0 = regular at v_0.

This is absurd. The same reasoning applies to v_1, v_λ and the proof is finished.

From the last claim, we see that $\deg(f) \geq 3$. It then follows that the RHS of (&) is regular at v_∞. We now show that the residue of the LHS of (&) at v_∞ is $\neq 0$ to arrive at a contradiction and therefore show that f does not exist.

If $\xi = 1/x$ and $f = \xi^\mu f_1$, with $f_1 \in \mathcal{O}^*_{v_\infty}$, then the LHS of (&) reads

$$\frac{\mathrm{d}f_1}{f_1} + \mu \cdot \frac{d\xi}{\xi} + \ell \left[\frac{a}{n}\xi + \frac{b}{n}\frac{\xi}{1-\xi} + \frac{c}{n}\frac{\xi}{1-\lambda\xi} \right] \frac{\mathrm{d}\xi}{\xi^2}.$$

It follows that the LHS of (&) has residue

$$\mu + \frac{\ell(a+b+c)}{n}$$

at v_∞. This cannot equal zero since $n \nmid \ell$. Hence, the LHS of (&) is not regular at v_∞ while the RHS is, as wanted.

We have now concluded that the subspace of H^1_{dR} spanned by the classes φ_ℓ and ψ_ℓ is of dimension two. Since $\dim_K H^1_{dR} = 2g$, and F has genus $n-1$ (Lemma 78), the final claim follows without effort. $\square$

We shall now determine the differential module structure of $H_{-\ell}$ given by the Gauss-Manin connection. Let $\delta = \dfrac{\mathrm{d}}{\mathrm{d}\lambda}$; this is a derivation of K which is extended to a derivation of F, called likewise, such that $\delta x = 0$. We shall now require some results on the calculus which once obtained are easily verified.

Proposition 82 (see p. 6 of [Gou81])**.** *Let α, β and γ be real numbers and let*

$$V_{\alpha,\beta,\gamma}(u,\lambda) := u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-\lambda u)^{-\alpha}.$$

Then

$$\left\{\lambda(1-\lambda)\frac{\partial^2}{\partial\lambda^2} + (\gamma-(\alpha+\beta+1)\lambda)\frac{\partial}{\partial\lambda} - \alpha\beta\right\} V_{\alpha,\beta,\gamma} = -\alpha\frac{\partial}{\partial u}\left[\frac{u(1-u)}{1-\lambda u}V_{\alpha,\beta,\gamma}\right].$$

$\square$

Now, let $u = 1/x$; in this case

$$V_{\alpha,\beta,\gamma} = x^{2+\alpha-\gamma}(x-1)^{\gamma-\beta-1}(x-\lambda)^{-\alpha}.$$

since $\dfrac{\partial}{\partial u} = -x^2\dfrac{\partial}{\partial x}$, we have

$$\left\{\lambda(1-\lambda)\frac{\partial^2}{\partial\lambda^2} + (\gamma-(\alpha+\beta+1)\lambda)\frac{\partial}{\partial\lambda} - \alpha\beta\right\} (\widetilde{V}_{\alpha,\beta,\gamma}) = \alpha\frac{\partial}{\partial x}(\cdots),$$

where $\widetilde{V}_{\alpha,\beta,\gamma} = V_{\alpha,\beta+2,\gamma+2}$. Now, letting

$$\alpha_\ell = \frac{\ell c}{n}, \quad \beta_\ell = \frac{\ell(a+b+c)}{n} - 1 \quad \text{and} \quad \gamma_\ell = \frac{\ell(a+c)}{n},$$

we conclude that

$$\begin{aligned} y^{-\ell} &= x^{-\frac{\ell a}{n}}(x-1)^{-\frac{\ell b}{n}}(x-\lambda)^{\frac{-\ell c}{n}} \\ &= \widetilde{V}_{\alpha_\ell,\beta_\ell,\gamma_\ell}, \end{aligned}$$

Hence,

$$\left\{\lambda(1-\lambda)\delta^2 + (\gamma-(\alpha+\beta+1)\lambda)\delta - \alpha\beta\right\} \varphi_\ell = 0 \tag{11}$$

in H^1_{dR}.

Theorem 83. *Let n, a, b, c be positive integers such that each one of a, b, c, and $a+b+c$ is prime to n. Then H^1_{dR}, with its Gauss-Manin connection, is isomorphic to*

$$\bigoplus_{\ell=1}^{n-1} \mathrm{HG}\,(\alpha_\ell, \beta_\ell, \gamma_\ell)$$

where

$$\begin{aligned}\alpha_\ell &= \frac{\ell c}{n}\\ \beta_\ell &= \frac{\ell a + \ell b + \ell c}{n} - 1\\ \gamma_\ell &= \frac{\ell a + \ell c}{n}.\end{aligned}$$

In particular, the differential equation associated to the Legendre family

$$y^2 = x(x-1)(x-\lambda)$$

is $\mathrm{HG}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Proof. From Corollary 80, Proposition 81 and eq. (11), all we need is to show that ψ_ℓ belongs to $K \cdot \varphi_\ell + K \cdot \delta\varphi_\ell$ in H^1_{dR}. This is a consequence of

$$\mathrm{d}\left(\frac{x(x-1)}{y^\ell}\right) = \left(\frac{\ell(a+c)}{n} - 1 - \frac{\ell c \lambda}{n}\right)\varphi_\ell + \left(2 - \frac{\ell(a+b+c)}{n}\right)\psi_\ell + \lambda(\lambda-1)\cdot\delta\varphi_\ell,$$

found in [Ka72, 6.8.3.1], and the fact that $\gcd(a+b+c, n) = 1$. □

Remark 84. Note that the computations in [Ka72, 6.8.0] are carried out for the affine curve

$$X = \{y^n = x^a(x-1)^b(x-\lambda)^c\} \smallsetminus \{(0,0)\}$$

over $\mathbf{C}(\lambda)$ and $H^1_{\mathrm{dR}}(X/\mathbf{C}(\lambda))$ is not completely described. In [Me72], one finds a description of the direct summand H_{-1}.

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