Algebraic groups acting on varieties and their applications

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These are transcriptions of the lectures I delivered – via Zoom – for the "International School on Algebraic Geometry and Algebraic Groups" organized by the Institute of Mathematics of the Vietnamese Academy of Sciences in Novembre 2021. I structured the lecture notes assuming solely that students would be familiar with basic "Grothendieckean" algebraic geometry (e.g. schemes, fibre products and flatness). But I must say that in order to grasp the contents of these lectures, the reader should have a certain experience with the aforementioned "basic" algebraic geometry.

Finally, I must emphasize that these are rough lecture notes; they probably contain many mistakes and imprecisions.

Programme

1. Introduction: what kind of problems lead us to study groups acting on varieties?
2. Functors and Yoneda's Lemma.
3. Group schemes and their representations: the affine case.
4. Affine quotients: General remarks on finite generation and the case of a finite and constant group scheme.

Lecture 1

(5 Novembre 2021).

Some conventions

1) k = algebraically closed field.

2) All schemes are k-schemes. A morphism of schemes is a morphism of k-schemes. The category of schemes is denoted by $\mathbf{Sch}_k$. (I shall make a brief recall of category theory.)

3) An algebraic k-scheme = k-scheme X which is covered by a finite number of affine open subsets U_i s.t. $\mathcal{O}(U_i)$ is of finite type. That is, a k-scheme of finite type.

4) A point on an algebraic scheme is always a closed point, unless otherwise mentioned. The set of points on an algebraic k-scheme X is denoted by $X(k)$. (See below as well.)

5) If S is an algebraic scheme and s is a point in it, then we know that the inclusion $k \to \mathbf{k}(t) = \mathcal{O}_{S,s}/\mathfrak{m}_s$ is bijective (because of the Nullstellensatz). For a morphism $f : X \to S$, we define the fibre of f above s as being the k-scheme

$$X \times_S \operatorname{Spec} \mathbf{k}(s).$$

6) More generally. If $s : S' \to S$ and $f : X \to S$ are morphisms of algebraic schemes, then the fibre of f above s is $X \times_S S'$.

Exercise 0.1. Let $f : \mathbf{A}^2 \to \mathbf{A}^2$ be defined by $(a, b) \to ab$. Describe the schematic fibre $f^{-1}(0)$. Is it integral? Is it irreducible?

Let $g : \mathbf{A}^2 \to \mathbf{A}^2$ be defined by $(a, b) \mapsto (a, ab)$. Describe the schematic fibre $g^{-1}(0)$ and compare it with the other fibres $g^{-1}(a, b)$.

1 Constructing moduli via an example

Want to study "spaces" of algebro-geometric objects up to "equivalence" or "isomorphism". These are traditionally called "moduli spaces" following Riemann's first usage of this name in describing how many parameters the "moduli" of Riemann surfaces should have.

The path to constructing such objects will be the one provided by invariant theory, which roughly means:

I. Finding a space $\mathscr{U}$ whose points correspond to all possible structures.

II. Taking equivalence classes to identify structures.

Sometimes it is not possible to attain neither **I**, nor **II**.

I shall explain these ideas through a simple example. : Sets of *two points in* $\mathbf{C}$. Once we obtain the theory of representable functors, we shall see how these ideas can be made more precise.

Take

$$U = \mathbf{C}^2 \smallsetminus \{(a, a) : a \in \mathbf{C}\}.$$

Let $\varepsilon : \mathbf{C}^2 \to \mathbf{C}^2$ be $(a, b) \mapsto (b, a)$. Then U/ε is the set of two points in $\mathbf{C}$.

In geometry:

$$\begin{aligned} \mathscr{U} &= \mathbf{A}^2 \smallsetminus \Delta \\ &= \operatorname{Spec}\left(\mathbf{C}[x, y]\left[\frac{1}{x-y}\right]\right), \end{aligned}$$

where Δ is the diagonal. Clearly, $\mathscr{U}(\mathbf{C})$ is U. Moreover, we have an automorphism $\varepsilon : \mathscr{U} \to \mathscr{U}$ defined by exchanging x and y. Two problems:

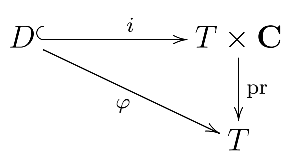
P1. What is $\mathscr{U}/\varepsilon$ in geometry?

P2. Construction is too set-theoretical and does not account for *families*.

What are families? Suppose that T is a set and that $\Phi : T \to U/\varepsilon$ is a map. Then $\Phi(t)$ gives me a couple of two points in $\mathbf{C}$ and we construct a *family parametrised by* T :

$$D_\Phi = \{(t, c) \,:\, c \in \Phi(t)\} \subset T \times \mathbf{C}.$$

Alternatively, consider the diagram:



(⋆)

where i is inclusion and $\#\varphi^{-1}(t) = 2$. This gives a map $\Phi_D : T \to U/\varepsilon$.

A particular case of interest is when Φ is the identity and we obtain the *universal family*:

$$D_{\mathrm{id}} = \{(m, a) \,:\, a \in m\} \subset U/\varepsilon \times \mathbf{C}.$$

Now: if everything in $(\star)$ is algebraic/analytic/C^∞, etc, is it the case that Φ_D also has these properties? Analytic and algebraic geometry are very well suited to handle these problems since singularities are part of the theory.

To tackle (P1), note : *If* $f : \mathscr{U}/\varepsilon \to \mathbf{A}^1$ *is a function* $\Rightarrow f \circ \varepsilon = f$. It is then reasonable to look at the ring

$$A = \{f \in \mathcal{O}(\mathscr{U}) \,:\, \varepsilon^{\#}(f) = f\}$$

and

$$\mathscr{M} = \operatorname{Spec} A.$$

Exercise 1.1. Let $\xi = x + y$, $\eta = xy$ and $\delta = x - y$. Show that $A = \mathbf{C}[\xi, \eta][1/\delta^2]$ and that $\delta^2 = \xi^2 - 4\eta$.

The universal family is a bit subtler (and I'll hide the reasoning). Take

$$\mathscr{D} = \operatorname{Spec} A[X]/(X^2 - \xi X + \eta).$$

We now have a diagram

$$\begin{array}{ccc} \mathscr{D} & \longrightarrow & \mathscr{M} \times \mathbf{A}^1 \\ & {\scriptstyle\chi}\searrow & \downarrow {\scriptstyle \mathrm{pr}} \\ & & \mathscr{M}. \end{array}$$

Exercise 1.2. Show that for each closed point m of $\mathscr{M}$, the fibre $\chi^{-1}(m)$ is $\operatorname{Spec} \mathbf{C} \sqcup \operatorname{Spec} \mathbf{C}$. Show that $\mathscr{U} \simeq \mathscr{D}$.

An important fact is that the ring $\mathcal{O}(\mathscr{D})$ is a free $\mathcal{O}(\mathscr{M})$–module of rank two.

Exercise 1.3. (1) Let T be affine and algebraic and consider

$$\begin{array}{ccc} D & \xhookrightarrow{\ i\ } & T \times \mathbf{C} \\ & {\scriptstyle\varphi}\searrow & \downarrow {\scriptstyle \mathrm{pr}} \\ & & T \end{array}$$

where we suppose that

- $\mathcal{O}(D)$ is, as an $\mathcal{O}(T)$–module, free of rank two.
- For each $t \in T$, the fibre $\varphi^{-1}(t)$ is $\operatorname{Spec} \mathbf{C} \sqcup \operatorname{Spec} \mathbf{C}$.

Then, there exists a unique morphism $\Phi_D : T \to \mathscr{M}$ such that

$$\mathscr{D} \underset{\chi, \mathscr{M}, \Phi}{\times} T = D.$$

Hint: Since $\mathcal{O}(D) = \mathcal{O}(T)v \oplus \mathcal{O}(T)w$, we can write $\mathcal{O}(D) = \mathcal{O}(T)[X]/(X^2 - \alpha X + \beta)$. This means that D "depends on two parameters". The fact that $\varphi^{-1}(t)$ has two points puts a relation between α and β.

Thus we obtain a complete answer to our problem. We can say that the space of two points in $\mathbf{C}$ is, in algebraic geometry, the scheme $\mathscr{M}$ and, in addition, that

$$\operatorname{Mor}_k(T, \mathscr{M}) = \{\text{certain families of two points over } T\}.$$

This point of view shall lead to category theory, which is, as taught by Grothendieck, a very important tool for doing mathematics.

Lecture 2

(5 Novembre 2021).

2 Brief overview of category theory

A fundamental fact of pure mathematics unveiled in the XX century was the use of category theory. This started to flourish on the hands of the algebraic topologists, but took a enormous impetus in the hands of A. Grothendieck. It is now a fundamental way of communicating. The best reference on the subject is [ML98], but it may be a bit impressive in a first look (at least that is the impression I had when I was a student). Students will also appreciate [Le14].

A category $\mathcal{C}$ is the data of a set of objects, denoted usually by $\operatorname{Ob}\mathcal{C}$, a set* of arrows $\operatorname{Arr}\mathcal{C}$, two maps

$$s, t : \operatorname{Arr}\mathcal{C} \longrightarrow \operatorname{Ob}\mathcal{C}$$

called the source and the target. In addition, we also have composition rules and an identity. That is, letting

$$\begin{aligned} \operatorname{CArr}(\mathcal{C}) &= \operatorname{Arr}(\mathcal{C}) \times_{s, \operatorname{Ob} C, t} \operatorname{Arr}(\mathcal{C}) \\ &= \{(g, f) \in \operatorname{Arr}(\mathcal{C}) \times \operatorname{Arr}(\mathcal{C}) \, : \, t(f) = s(g)\} \end{aligned}$$

be the set of all “ composable couples”, we have maps

$$\begin{array}{ccc} \operatorname{Ob}\mathcal{C} \xrightarrow{\text{id}} \operatorname{Arr}\mathcal{C} & \text{and} & \circ : \operatorname{CArr}\mathcal{C} \longrightarrow \operatorname{Arr}\mathcal{C}, \\ c \longmapsto \operatorname{id}_c & & (g, f) \longmapsto g \circ f, \end{array}$$

which are subjected to the axioms of *associativity* and *unity*. These axioms are

$$h \circ (g \circ f) = h \circ (g \circ f) \quad \text{and} \quad f \circ \operatorname{id} = \operatorname{id} \circ f.$$

An arrow f having source a and target b is represented by $f : a \to b$. The set of all arrows from a to b, which is $s^{-1}(a) \cap t^{-1}(b)$, is denoted by $\operatorname{Hom}_{\mathcal{C}}(a, b)$.

One can say a lot about categories in the abstract [ML98], but here we shall simply use this idea in order to communicate and to prove the Yoneda lemma. Hence, it is fait to say that the reader will be well prepared to handle what comes in meditating on the following examples.

Example 2.1. The category of groups has for objects all the possible groups and for arrows the group morphisms.

Example 2.2. The category **Top** of topological spaces and continuous maps between them.

Example 2.3. The category of k-schemes, $\mathbf{Sch}_k$, which has for objects all k-schemes and whose arrows are morphisms of k-schemes.

*I shall be sloppy in dealing with set theoretical issues here. Details are in [ML98]

Now, another very important concept is that of a functor.

Definition 2.4. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. A functor is the data of two maps $F : \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{C}'$ and $F : \operatorname{Arr}\mathcal{C} \to \operatorname{Arr}\mathcal{C}'$ (no notational distinction is usually made!) such that

$$F(\mathrm{id}_c) = \mathrm{id}_{F(c)} \quad \text{and} \quad F(g) \circ F(f) = F(g \circ f).$$

(On the latter equation, one has to assume that g and f are composable.)

There are numerous examples of functors.

Example 2.5. Let **Rng** be the category of associative rings with identity. Then define a functor $U : \mathbf{Rng} \to \mathbf{Ab}$ by associating to a ring A the underlying abelian group and for a ring-morphism $f : A \to A'$ the morphism of abelian groups $f : A \to A'$. This is usually called *a forgetful functor*. (Because we forget that there was an extra structure.)

Example 2.6. Let $U : \mathbf{Sch}_k \to \mathbf{Top}$ be the functor associating to the scheme $(X, \mathcal{O}_X)$ the topological space X. This is a forgetful functor.

Exercise 2.7. Define **Top** to be the category of topological spaces and **Set** the category of sets. Construct two distinct functors $D : \mathbf{Set} \to \mathbf{Top}$.

Many interesting functors invert the direction of arrows. For this reason, one introduces:

Definition 2.8. If $\mathcal{C}$ is a category, we define $\mathcal{C}^{\mathrm{op}}$ as the category with the same set of objects, but such that $\operatorname{Hom}_{\mathcal{C}^{\mathrm{op}}}(a, b) = \operatorname{Hom}_{\mathcal{C}}(b, a)$. It is called the opposed category. It is usually never really used other than to give a name to *functors which invert arrows*. Such functors are called *contra-variant* functors.

Finally, the last pillar of category theory is the notion of natural transformation.

Definition 2.9. Given $F, G : \mathcal{C} \to \mathcal{A}$ two functors. A natural transformation φ from F to G, denoted by $\varphi : F \Rightarrow G$, is a family of arrows

$$\varphi_c : F(c) \longrightarrow G(c)$$

such that for all arrows $f : c \to d$ in $\operatorname{Arr}(\mathcal{C})$, the diagram

$$\begin{array}{ccc} Fc & \xrightarrow{\varphi_c} & Gc \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ Fd & \xrightarrow[\varphi_d]{} & Gd. \end{array}$$

commutes.

Let me show the utility of these concepts with an example.

Example 2.10. Let **vect** be the category of vector spaces. We then have the functor $F : \mathbf{vect} \to \mathbf{vect}$ given by $F(V) = \operatorname{Hom}_k(k, V)$. We all know that a linear map $k \to V$ is "just the choice of a vector". In categorical terms, this comes with more precision. We have a natural transformation $\varepsilon : F \Rightarrow \operatorname{id}$ given by

$$\varepsilon_V : F(V) \longrightarrow V$$

$$\alpha \mapsto \alpha(1).$$

Obviously, for each $f : V \to W$, the diagram

$$\begin{array}{ccc} \operatorname{Hom}_k(k, V) & \xrightarrow{F(f)} & \operatorname{Hom}_k(k, W) \\ \downarrow{\scriptstyle \varepsilon_V} & & \downarrow{\scriptstyle \varepsilon_W} \\ V & \xrightarrow[f]{} & W \end{array}$$

commutes since, the element $\alpha \in \operatorname{Hom}_k(k, V)$ behaves as

$$\begin{array}{ccc} \alpha & \longrightarrow & f \circ \alpha \\ \downarrow & & \downarrow \\ \alpha(1) & \xrightarrow[f]{} & f\alpha(1). \end{array}$$

3 Representable functors

We saw that to construct "spaces of structures" in geometry, we needed the notion of *quotient* and of *families*. In addition, we noted that if $\mathscr{M}$ is a certain "space of structures", then it is reasonable to interpret $\operatorname{Mor}_k(T, \mathscr{M})$ as a certain set of families of that structure. For this study, we need more category theory.

Let $\mathcal{C}$ be a category. For each $M \in \mathcal{C}$, let

$$\boxed{h_M : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}}$$

stand for the functor defined by

$$T \mapsto \operatorname{Hom}_{\mathcal{C}}(T, M).$$

It is called the functor of points of M. Let me explain why this functor has such a geometric name. (At this point you should also consult [Mu66,].)

Example 3.1. Let

$$M = \operatorname{Spec} k[T_1, \ldots, T_m]/(f_1, \ldots, f_n).$$

For $X = \operatorname{Spec} A$, an element of $\operatorname{Mor}_k(X, M)$ is determined bya morphis of k-algebras

$$k[T_1, \ldots, T_m]/(f_1, \ldots, f_n) \longrightarrow A,$$

which amounts to $(a_1, \ldots, a_m) \in A^m$ such that $f_i(a_1, \ldots, a_m) = 0$ for all i. That is, a point of M with values on A.

Definition 3.2. Functors $F : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ naturally isomorphic to some h_M are called representable. If we have $F \simeq h_M$, then we say that M is represents F.

Exercise 3.3. Let $\mathcal{C}$ = algebraic k-schemes. Define $\mathbf{G}_a : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ by $T \mapsto \mathcal{O}(T)$. Then $\mathbf{G}_a$ is represented by $\mathbf{A}^1$.

Example 3.4. Let $\mathcal{C} = \mathbf{ASch}^{\text{op}}_{\mathbf{C}}$, the category of algebraic $\mathbf{C}$-schemes. Let

$$[2](T) = \left\{ \begin{array}{c} \text{closed subscheme } D \subset T \times \mathbf{A}^1 \\ \text{such that the } \mathcal{O}_T\text{-module} \\ \text{pr}_*(\mathcal{O}_D) \text{ is locally free of rank two} \\ \text{and } D \cap \{t\}\mathbf{A}^1 \text{ has two points.} \end{array} \right\}$$

This defines a contra-variant functor from $\mathbf{ASch}^{\text{op}}_k$ to $\mathbf{Set}$: If $u : T' \to T$ is an arrow of algebraic $\mathbf{C}$-schemes, then

$$[2](u) : [2](T) \longrightarrow [2](T')$$

takes the closed subscheme $D \subset T \times \mathbf{A}^1$ to its base-change:

$$\begin{aligned} T' \times_T D &\subset T' \times_T (T \times \mathbf{A}^1) \\ &= T' \times \mathbf{A}^1. \end{aligned}$$

Exercise 3.5. This is a good exercise on fibre products: Show that for each point t' of T', the fibre of $T' \times_T D$ has only two points.

We saw that $[2] \simeq h_{\mathscr{M}}$. More precisely, we saw that there exists

$$\begin{array}{ccc} \mathscr{D} & \longrightarrow & \mathscr{M} \times \mathbf{A}^1 \\ & {\scriptstyle \chi} \searrow & \downarrow {\scriptstyle \text{pr}} \\ & & \mathscr{M} \end{array} \qquad \in [2](\mathscr{M})$$

such that the natural transformation

$$\text{Mor}_k(T, \mathscr{M}) \longrightarrow [2](T)$$

$$(T \xrightarrow{u} \mathscr{M}) \longmapsto T \times_{\mathscr{M}} \mathscr{D}$$

is a bijection.

Lecture 3

(9 Novembre 2021).

Last time we put under the light functors of the form $T \mapsto \mathrm{Hom}_{\mathcal{C}}(T, M)$; the representable functors. With this, we can give a more definite version of what a *moduli problem* is. Let $\mathbf{ASch}_k$ be the category of algebraic schemes and

$$F : \mathbf{ASch}_k^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

a functor. This can be seen as a "moduli problem" and the "moduli space" is a scheme representing F. Now there are many such interesting and relevant functors which are not representable, so this is not a definite goal and many times needs to be weakened.

In some sense, the "theory of moduli" can now be thought of a "membership" problem: Structures give rise to functors and we want to know which of these are what we already know (the representables). It then becomes important to know that passing from schemes to functors $\mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Set}$ will not cause any loss on the geometric side. This is solved by Yoneda's Lemma, a fundamental fact of category theory, deeply explored by Grothendieck.

Let $\mathcal{C}$ be a category. Let $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ be the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$: Objects are functors and arrows between objects are natural transformations.

Consider now the functor

$$h_\bullet : \mathcal{C} \longrightarrow \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}),$$

which sends $X \in \mathcal{C}$ to h_X, and sends the arrow $u : X \to Y$ to the natural transformation

$$h_u \;=\; \{\; u \circ (-) : h_X(T) \longrightarrow h_Y(T) \;\}_{T \in \mathcal{C}}.$$

Theorem 3.6 (Yoneda's Lemma)**.** *The arrow*

$$h_\bullet : \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Nat}(h_X, h_Y)$$

is bijective

Proof. This is a triviality. The inverse to $h_\bullet$ is

$$\varepsilon : \mathrm{Nat}(h_X, h_Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Y),$$

$$\Phi \longmapsto \Phi_X(\mathrm{id}_X).$$

That $\varepsilon h_u = u$ for all $u : X \to Y$ is obvious. We show that

$$\mathrm{Nat}(h_X, h_Y) \xrightarrow{\varepsilon} \mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{h_\bullet} \mathrm{Nat}(h_X, h_Y)$$

is the identity, Let $\Phi : h_X \Rightarrow h_Y$ be given. By definition, for each $f : T \to U$, the diag.

$$\begin{array}{ccc} h_X(U) & \xrightarrow{\Phi_U} & h_Y(U) \\ {\scriptstyle (-)\circ f}\downarrow & & \downarrow{\scriptstyle (-)\circ f} \\ h_X(T) & \xrightarrow[\Phi_T]{} & h_Y(T) \end{array}$$

commutes. That is: for $\alpha \in h_X(U)$, we have

$$\Phi_U(\alpha) \circ f = \Phi_T(\alpha \circ f).$$

Apply to $U = X$ and $\alpha = \mathrm{id}_X$ to get

$$\underbrace{\underbrace{\Phi_X(\mathrm{id}_X)}_{\varepsilon(\Phi)} \circ f}_{h_{\varepsilon(\Phi)}(f)} = \Phi_T(f).$$

□

Hence, the schemes can be known by their functor of points. This makes certain constructions and definitions very natural. My favourite is:

Exercise 3.7. Let $f : X \to S$ and $g : Y \to S$ be arrow of **sets**. Define the fibre product, denoted $X \times_S Y$, as the subset of $X \times Y$ given by the couples (x, y) such that $f(x) = g(y)$. Recall that Prof. Hai explained that in the exercise session on November 5th.

Let us now suppose that f and g are morphisms of *schemes*. Define the functor

$$T \xmapsto{h_X \times_{h_S} h_Y} h_X(T) \underset{h_S(T)}{\times} h_Y(T).$$

Show that $h_X \times_{h_S} h_Y$ is represented by $X \times_S Y$.

Exercise 3.8 (Suerjections versus epimorphisms)**.** (1) Let $f : X \to Y$ be a map of sets. Show that f is surjective if and only if for each set T, the map

$$f \circ (-) : \mathrm{Map}(T, X) \longrightarrow \mathrm{Map}(T, Y),$$

is *injective*.

(2) A morphism schemes is called surjective if it gives rise to a surjective morphism of *topological spaces*. A morphism of schemes $f : X \to Y$ is called an *epimorphism* if for each $T \in \mathbf{Sch}_k$, the arrow

$$f \circ (-) : h_X(T) \longrightarrow h_Y(T)$$

is injective. Give an example of a surjective morphism which is not an epimorphism. (Hint: Work with rings and study nilpotent elements.)

Exercise 3.9 (Properness)**.** Explain the valuative criterion of properness in terms of functors of points.

Let me end with a word of terminology and notation. Because of Yoneda's Lemma, *we shall make no more distinction between a scheme and its functor of points*. That is, for a scheme, $X(S) = h_X(S) = \mathrm{Mor}_k(S, X)$.

4 Group functors and group schemes

Definition 4.1. A functor $\mathcal{G} : \mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Grp}$ shall be called a *group functor*. A scheme G such that $h_G : \mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Set}$ factors as a group-functor $\mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Set}$ and the "inclusion" $\mathbf{Grp} \to \mathbf{Set}$ is called a *group scheme.*

Example 4.2 (The additive group)**.** Define $\mathbf{G}_a(T) := (\mathcal{O}(T), +)$. This is represented by $\mathbf{A}^1$, and is hence a group-scheme.

Example 4.3 (The multiplicative group)**.** Define $\mathbf{G}_m(T) := (\mathcal{O}(T)^\times, \cdot)$. This is represented by $\mathbf{A}_k^1 \smallsetminus \{0\}$, and is hence a group-scheme. (Make sure you understand why this is represented by $\mathbf{A}_k^1$.)

Example 4.4. Let V be a vector space. Define $V_{\mathrm{a}}(T) = (\mathcal{O}(T) \otimes_k V, +)$. This is a group functor. If $\dim V = n \Rightarrow V_{\mathrm{a}}$ is representable by $\mathbf{A}^n$. In fact, let $k[V^*]$ be the *symmetric algebra* on the vector space V^*, see [La02, XVI, §8], or `https://en.wikipedia.org/wiki/Symmetric_algebra`. Its fundamental property is that for a k-algbera R, we have

$$\mathrm{Hom}_{k-\mathrm{vect}}\left(V^*, R\right) = \mathrm{Hom}_{k-\mathrm{alg}}\left(k[V^*], R\right).$$

Then $\mathrm{Spec}\, k[V^*]$ represents V_{a}. Indeed,

$$\mathrm{Mor}_k(T, \mathrm{Spec}\, k[V^*]) = \mathrm{Hom}_k\left(V^*, \mathcal{O}(T)\right) = \mathcal{O}(T) \otimes V.$$

Example 4.5. Define

$$\mathbf{GL}_n(T) = \left(\mathbf{GL}_n(\mathcal{O}(T)), \text{usual multiplication}\right).$$

This is a group scheme, which is representable by an open subscheme of $\mathbf{A}^{n^2}$ obtained by inverting the determinant function.

More generally: Let V be a vector space. We define the group functor $\mathbf{GL}(V)$ by

$$\mathbf{GL}_V(T) \;=\; \left(\{\mathcal{O}(T)\text{-linear isos. of } V \otimes \mathcal{O}(T)\}, \text{composition}\right).$$

If V is of finite dimension, then $\mathbf{GL}_V$ is representable.

Example 4.6. So far, we've only encountered group schemes which are available from "usual group theory". Here is a different one. Let $\mathrm{char.}\, k = p > 0$. Define $\boldsymbol{\alpha}_p(T) = \{f \in \mathcal{O}(T) : f^p = 0\}$. Then $\boldsymbol{\alpha}_p$ is represented by $\mathrm{Spec}\, k[x]/(x^p)$; the scheme $\boldsymbol{\alpha}_p$ is *not* reduced and its topological space is simply a point.

Lecture 4

(9 Novembre 2021).

Let G be group scheme. We have natural transformations defined by

$$\mu_T : h_G(T) \times h_G(T) \longrightarrow h_G(T), \qquad (g, g') \longmapsto gg',$$

$$\iota : G(T) \longrightarrow G(T), \qquad g \longmapsto g^{-1}$$

and

$$e : (\operatorname{Spec} k)(T) \longrightarrow G(T), \qquad * \longmapsto e_{G(T)}.$$

Using Yoneda and replacing the notation "h_G" by "G":

Lemma 4.7. *A group scheme structure on the scheme G is equivalent to the existence of arrows of algebraic schemes*

$$\mu : G \times G \longrightarrow G, \quad \iota : G \longrightarrow G, \quad e : \operatorname{Spec} k \longrightarrow G$$

such that the following diagrams commute:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mathrm{id}\times\mu} & G \times G \\ {\scriptstyle \mu\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times G & \xrightarrow[\mu]{} & G \end{array} \tag{Assoc}$$

$$\begin{array}{ccc} \operatorname{Spec} k \times G & \xrightarrow{e\times\mathrm{id}} & G \times G \\ & \searrow^{\sim} & \downarrow{\scriptstyle \mu} \\ & & G \end{array} \tag{Unit}$$

and

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle (\mathrm{id},\iota)}\uparrow & & \uparrow{\scriptstyle e} \\ G & \xrightarrow[\text{structural}]{} & \operatorname{Spec} k \end{array} \tag{Inverse}$$

One usually calls μ the multiplication, $\iota : G \to G$ the inversion and $e : \operatorname{Spec} k \to G$ the unit.

There are group schemes which are affine schemes. There are group schemes which are projective (elliptic curves), there are groups which are neither (you shall see them in Prof. Brion's lecture). The most evident are the affine ones.

5 Affine group schemes [Wa78]

Say $G = \operatorname{Spec} A$ is a group schemes.

Multiplication gives rise to *co-multiplication*

$$\mu^\# = \Delta : A \longrightarrow A \otimes_k A$$

the inversion gives

$$\iota^{\#} = \sigma : A \longrightarrow A,$$

the *antipode*, and the identity gives

$$e^{\#} = \varepsilon : A \longrightarrow k.$$

the *co-unity* or *co-identity*. Note that Δ, ε and α are morphisms of k-algebras. In addition, the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xleftarrow{\mathrm{id}\otimes\Delta} & A \otimes A \\ \uparrow{\scriptstyle \Delta\otimes\mathrm{id}} & & \uparrow{\scriptstyle \Delta} \\ A \otimes A & \xleftarrow[\Delta]{} & A \end{array} \qquad \text{(Assoc)}$$

$$\begin{array}{ccc} A \otimes A & \xleftarrow{\Delta} & A \\ \downarrow{\scriptstyle \mathrm{id}\cdot\varepsilon} & \diagup\!\!\diagup & \\ A & & \end{array} \qquad \text{(Unit)}$$

and

$$\begin{array}{ccc} A \otimes A & \xleftarrow{\Delta} & A \\ \downarrow{\scriptstyle \mathrm{id}\cdot\sigma} & & \downarrow{\scriptstyle \varepsilon} \\ A & \xleftarrow[\text{inclusion}]{} & k \end{array} \qquad \text{(Inverse)}$$

Definition 5.1. The triple consisting of a k-algebra and morphisms of k-algebras Δ, σ and ε as above is called a *Hopf algebra*.

Example 5.2. Let us render explicit the previous arrows for $\mathbf{G}_a = \operatorname{Spec} k[x]$. Let T be affine and let $t_1, t_2 : T \to \mathbf{G}_a$ be morphisms. Write $t_i^{\#}(x) = x_i$; these are in $\mathcal{O}(T)$. The element $t_1 + t_2 \in \mathbf{G}_a(T) = \mathcal{O}(T)$ is just $x_1 + x_2$. On the other hand, by definition of the multiplication $\mu : \mathbf{G}_a \times \mathbf{G}_a \to \mathbf{G}_a$ (check that you understand this!) we have also

$$t_1 + t_2 = \ T \xrightarrow{(t_1,t_2)} \mathbf{G}_a \times \mathbf{G}_a \xrightarrow{\mu} \mathbf{G}_a.$$

Hence, $x_1 + x_2$ is the image of x under

$$k[x] \xrightarrow{\Delta} k[x] \otimes k[x] \xrightarrow{t_1^{\#} \bullet t_2^{\#}} \mathcal{O}(T)$$

Hence, the element $\Delta x = \sum c_{ij} x^i \otimes x^j$ is such that $\sum c_{ij} t_1^i t_2^j = t_1 + t_2$. It must be that $\Delta x = 1 \otimes x + x \otimes 1$.

Exercise 5.3. Let $\mathbf{G}_m = \operatorname{Spec} k[x, x^{-1}]$. Show that $\Delta x = x \otimes x$. More generally, write $\mathbf{GL}_n = \operatorname{Spec} k[x_{ij}, 1/\det(x_{ij})]$ and show that $\Delta x_{ij} = \sum_\nu x_{i\nu} \otimes x_{\nu j}$.

A word about terminology: Mathematicians usually talk about *linear algebraic groups* to mean affine group schemes which are represented by the spectrum of *reduced k-algebras of finite type*. Hence, $\boldsymbol{\alpha}_p$ from the above example is an affine algebraic group scheme, but not a *linear* algebraic group...

Exercise 5.4. Let $V = k\vec{e}_0 \oplus k\vec{e}_1 \oplus \cdots$ be a vector space with a countable basis. Show that V_{a} is *not* representable by a k-scheme. Here is a possible way to show this.

(1) Let X be an affine k-scheme. Write $T_n = \operatorname{Spec} k[t]/(t^{n+1})$ and $T = \operatorname{Spec} k[\![t]\!]$. We denote by θ_n the evident closed immersion $\theta_n : T_n \to T$. Show that the natural map

$$\operatorname{Mor}_k(T, X) \longrightarrow \varprojlim_n \operatorname{Mor}_k(T_n, X) \tag{*}$$

is *bijective*.

(2) Generalise: take out the assumption that X is affine.

(3) Using the elements $f_n := \sum_{i=0}^n t^i \otimes \vec{e}_i \in V_{\mathrm{a}}(T_n)$, show that (*) cannot be satisfied.

6 Representations, actions and comodules [Wa78]

6.1 Definitions

Now that we have introduced the concept of group scheme, we may study actions. Because of Yoneda's Lemma, we shall make *no distinction* between a scheme and its functor of points.

Definition 6.1. Let $G : \mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Grp}$ be a group functor and $X : \mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Set}$ a functor. An action of G on X (on the *left*) is a morphism

$$\alpha : G \times X \longrightarrow X$$

such that for each T, the map $\alpha(T)$ defines an action of $G(T)$ on the left of $G(T)$.

If V is a k vector space, a *linear action* of G on V is an action $G \times V_{\mathrm{a}} \to V_{\mathrm{a}}$ such that $G(T)$ acts on $\mathcal{O}(T) \otimes V$ $\mathcal{O}(T)$-linearly. Linear actions are also called *representations*, or *G-modules.*

If is not hard to see that a linear action α of G on V is equivalent to a *natural transformation of group functors* $G \to \mathrm{GL}(V)$. In particular, when $V = k^n$ so that $\mathbf{GL}_V = \mathbf{GL}_n$, the representation si determined by a matrix $(\rho_{ij}) \in \mathbf{GL}_n(\mathcal{O}(G))$. Another important characterisation of G-modules (=linear actions) in the case G is affine follows.

6.2 Comodules

In dealing with representations of an abstract group Γ, a key role is played by the fact that a representation is a module over the group algebra $k[\Gamma]$ [La02, XVIII.1]. In case of general group schemes, there is no simple replacement for the group algebra, but the problem can be taken in a different: we work with *comodules* instead of modules.

Let $A = \mathcal{O}(G)$ and denote by $\Delta : A \to A \otimes A$ be the co-multiplication of G. Let V be a G-module and denote by α the natural transformation of group functors $G \to \mathbf{GL}_V$. The element $\mathrm{id}_G \in G(G)$ gives a A-linear map

$$\alpha_{\mathrm{id}} : V \otimes A \longrightarrow V \otimes A.$$

(I write α_{id} instead of $\alpha_G(\mathrm{id}_G)\ldots$) In addition, for any scheme T and any arrow $t : T \to G$, we obtain a commutative diagram

$$\begin{array}{ccc} V \otimes A & \xrightarrow{\alpha_{\mathrm{id}}} & V \otimes A \\ \downarrow{\scriptstyle \mathrm{id}\otimes t^{\#}} & & \downarrow{\scriptstyle \mathrm{id}\otimes t^{\#}} \\ V \otimes \mathcal{O}(T) & \xrightarrow[\alpha_t]{} & V \otimes \mathcal{O}(T) \end{array} \tag{1}$$

because the element $t \in G(T)$ is just $G(t)(\mathrm{id})$. Let now

$$\boxed{\rho(v) := \alpha_{\mathrm{id}}(v \otimes 1) \in V \otimes A.} \tag{2}$$

Let me pick generators $\{v_i\}$ of V (no need to be a basis) and write

$$\rho v_j = \sum_i v_i \otimes \rho_{ij}.$$

With this notation, we can say that with the help of diagram (1) that

$$\alpha_t(v_j \otimes r) = \sum_i v_i \otimes t^{\#}(\rho_{ij}) \cdot r. \tag{3}$$

Pick now $u \in G(T)$. The fact that $\alpha_{ut} = \alpha_u \alpha_t$ has the following consequence:

$$\sum_h v_h \otimes (ut)^{\#}(\rho_{hj}) = \sum_h v_h \otimes \left(\sum_i u^{\#}(\rho_{hi}) \cdot t^{\#}(\rho_{ij}) \right). \tag{4}$$

We apply this in case $u : G \times G \to G$ is the first projection and t the second, in which case $ut : G \times G \to G$ is μ. Hence,

$$\sum_h v_h \otimes \Delta\rho_{hj} = \sum_h v_h \otimes \left(\sum_i \rho_{hi} \otimes \rho_{ij} \right). \tag{5}$$

This is *equivalent* to the commutativity of

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes A \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho\otimes\mathrm{id}_A} \\ V \otimes A & \xrightarrow[\mathrm{id}_V\otimes\Delta]{} & V \otimes A \otimes A. \end{array} \tag{Coass}$$

This is called the *co-associativity*

Lecture 5

(11 Novembre 2021).

Last time, we dealt with the following. Given is a G-module V; the natural transformation $G \Rightarrow \mathbf{GL}_V$ is denoted by α. Using $\mathrm{id}_G \in G(G)$, we then get an element $\alpha_{\mathrm{id}} \in \mathbf{GL}_V(A)$ and write $\rho : V \to V \otimes A$ for the k-linear arrow $v \mapsto \alpha_{\mathrm{id}}(v \otimes 1)$. We then fixed a set of generating elements $\{v_i\}$ of V and introduced elements $\rho_{ij} \in A$ by $\rho v_j = \sum_i v_i \otimes \rho_{ij}$. Using the fact that $\alpha_u \alpha_t = \alpha_{ut}$ for each couple $u, t \in G(T)$, we deduced eq. (4), which in turn led to the commutativity of

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes A \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho\otimes\mathrm{id}_A} \\ V \otimes A & \xrightarrow[\mathrm{id}_V \otimes \Delta]{} & V \otimes A \otimes A. \end{array} \qquad \text{(Coass)}$$

This commutativity was called *co-associativity* of ρ. Let us go farther.

Because $\alpha_e = \mathrm{id}_V$, applying equation (3) to $T = \operatorname{Spec} k$ and $t = e : \operatorname{Spec} k \to G$ we conclude that

$$v_j = \sum v_i \varepsilon(\rho_{ij})$$

which assures commutativity of

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes A \\ & {\scriptstyle\mathrm{id}}\searrow & \downarrow{\scriptstyle\mathrm{id}_V\otimes\varepsilon} \\ & & V \otimes k. \end{array} \qquad \text{(Coun)}$$

Definition 6.2. A *comodule* for the Hopf-algebra A is a k-linear map

$$\rho : V \longrightarrow V \otimes A$$

such that (Coass) and (Coun) commute. The map ρ is the coaction.

To summarise the above discussion: Staring with a G-module $G \to \mathbf{GL}_V$, we obtained an A-comodule $\rho : V \to A \otimes V$ by equation (2).

Example 6.3. Let $\mathbf{GL}_n$, which is the spectrum of $k[x_{ij}, 1/\det]$, act on $(k^n)_{\mathrm{a}}$ in the standard way. Then the coaction is

$$\rho \vec{e}_j = \sum_i \vec{e}_i \otimes x_{ij}.$$

Let us now start with a comodule $\rho : V \to V \otimes A$. Using the universal property of the tensor product, we get an arrow of A-modules:

$$\alpha_{\mathrm{id}} : V \otimes A \longrightarrow V \otimes A.$$

$$v \otimes a \longmapsto \rho(v) \cdot a.$$

For $t \in G(T)$, we *define* α_t by extension of scalars:

$$\begin{array}{ccc} (V \otimes A) \underset{A}{\otimes} \mathcal{O}(T) & \xrightarrow{\alpha_{\mathrm{id}} \otimes \mathrm{id}} & (V \otimes A) \underset{A}{\otimes} \mathcal{O}(T) \\ \| & & \| \\ V \otimes \mathcal{O}(T) & \xrightarrow[\alpha_t]{} & V \otimes \mathcal{O}(T) \end{array}$$

commutes. Explicitly: if $\rho(v_j) = \sum v_i \otimes \rho_{ij}$, then

$$\alpha_t(v_j \otimes f) = \sum v_i \otimes t^{\#}(\rho_{ij}) \cdot f.$$

This gives us our natural transformation

$$\alpha : G \Longrightarrow \mathrm{End}_V;$$

here End_V is simply the functor $T \mapsto \mathrm{End}_{\mathcal{O}(T)}(V \otimes \mathcal{O}(T))$.

Commutativity of (Coass) forces

$$\sum_h v_h \otimes \Delta\rho_{hj} = \sum_h v_h \otimes \left(\sum_i \rho_{hi} \otimes \rho_{ij} \right).$$

For each couple of points $u, t \in G(T)$ we get by applying $u^{\#} \bullet t^{\#} : A \otimes A \to \mathcal{O}(T)$ to the above equation:

$$\sum_h v_h \otimes \underbrace{u^{\#} \bullet t^{\#} \Delta}_{(ut)^{\#}}(\rho_{hj}) = \sum_h v_h \otimes \left(\sum_i u^{\#}(\rho_{hi}) \cdot t^{\#}(\rho_{ij}) \right).$$

This is (4), which means $\alpha_u \alpha_t = \alpha_{ut}$.

Finally, the commutativity of (Coun) above shows that the unit of $G(T)$ is taken to the unit of $\mathrm{End}_{\mathcal{O}(T)}(V \otimes \mathcal{O}(T))$ by $\alpha : G \Rightarrow \mathrm{End}_V$. Hence, α takes values on $\mathbf{GL}_V$.

I shall say this imprecisely, but suggestively.

Proposition 6.4. *$\mathcal{O}(G)$-comodules and G-modules are the same thing.*

6.3 Construction with G-modules

Let $G = \operatorname{Spec} A$ affine group scheme. One fundamental property of the category of G-modules is that we can perform numerous linear algebraic operations on their objects: tensor products, duals, alternating products, determinants, etc. At this point, the reader with some background knowledge on representation of abstract groups may just think that "everything works in the same way".

Tensor product Let V and W be G-modules. Write $\rho : V \to V \otimes A$ and $\sigma : W \to W \otimes A$ for the co-module maps. Then, $V \otimes W$ carries the structure of an A-comodule:

$$V \otimes W \xrightarrow{\rho \otimes \sigma} V \otimes A \otimes W \otimes A = V \otimes W \otimes A \otimes A \xrightarrow{\mathrm{id} \otimes \mathrm{mult}} V \otimes W \otimes A.$$

The resulting G-module is the tensor product G-module.

The contragredient. Let V be a G-module of finite dimension. For each $T \in \mathbf{Sch}_k$, we have an action of $G(T)$ on $\mathcal{O}(T) \otimes V$. Hence, a left action of $G(T)$ on $\mathrm{Hom}_{\mathcal{O}(T)}(\mathcal{O}(T) \otimes V, \mathcal{O}(T))$ by

$$g * \phi : \vec{v} \longmapsto \phi(g^{-1}\vec{v}).$$

Now, we note that

$$V^* \otimes \mathcal{O}(T) \longrightarrow \mathrm{Hom}_{\mathcal{O}(T)}(\mathcal{O}(T) \otimes V, \mathcal{O}(T))$$

$$a \otimes \phi \longmapsto (b \otimes \vec{v} \mapsto ab\phi(\vec{v}))$$

establishes an isomorphism

$$(V^*)_{\mathrm{a}}(T) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}(T)}(\mathcal{O}(T) \otimes V, \mathcal{O}(T)).$$

This representation is called the "dual" or "contragredient" representation. For the student who never before met these constructions, I suggest to first have a look at the "standard one" , say, in [La02, XVIII.1].

Lecture 6

(11 Novembre 2021).

Comodules obtained from actions on affine schemes

We *never assumed that the G-modules are finite dimensional.* This is because we want always to pay attention to the following kind of G-modules.

Let $G = \operatorname{Spec} A$ act on the affine scheme $X = \operatorname{Spec} R$. We then obtain an action on the *right* by means of $x \star g = g^{-1}x$. This gives a morphism of k-algebras $\alpha^{\#} : R \to R \otimes A$. It is a simple matter to see that $\alpha^{\#}$ defines on R the structure of an A-comodule. (Note that we needed a *right action.*)

Example 6.5. Let $\rho : G \to \mathbf{GL}_n$ be a representation defined by $(\rho_{ij}) \in \mathbf{GL}_n(A)$. Let $\alpha : G \times \mathbf{A}^n \to \mathbf{A}^n$ be the resulting action, where $\mathbf{A}^n = \operatorname{Spec} k[u_1, \ldots, u_n]$. Then

$$\begin{pmatrix} \rho_{11} & \cdots & \rho_{1n} \\ \vdots & \ddots & \vdots \\ \rho_{n1} & \cdots & \rho_{nn} \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \longmapsto \begin{pmatrix} \rho_{11}u_1 + \cdots + \rho_{1n}u_n \\ \vdots \\ \rho_{n1}u_1 + \cdots + \rho_{nn}u_n \end{pmatrix}.$$

To define the coaction of A on $k[\boldsymbol{u}]$ we form the action on the right:

$$\mathbf{A}^n \times G \xrightarrow{\mathrm{id}\times\iota} \mathbf{A}^n \times G \xrightarrow{\sim} G \times \mathbf{A}^n \xrightarrow{\alpha} \mathbf{A}^n.$$

Hence, in the above composition, the image of u_i is

$$\sum_j u_j \otimes \rho^{ij},$$

where ρ^{ij} is the (i,j) entry of the matrix $(\rho_{ij})^{-1}$. This is because $\iota(\rho_{ij}) = \rho^{ij}$.

Note that, in this case, the subspace $\sum k \cdot u_i \subset k[\boldsymbol{u}]$ is invariant under G and corresponds to the contragredient representation of G, which is determined by

$$G \xrightarrow{\rho} \mathbf{GL}_n \xrightarrow{\text{invert and transpose}} \mathbf{GL}_n.$$

Said differently, if V is a finite G-module, then the G-module structure on $\mathcal{O}(V_{\mathrm{a}}) = k[V^*]$ is simply the one induced by the contragredient structure.

Here is a fundamental result about comodules:

Theorem 6.6 (Local finiteness)**.** *Let V be a G-module and let ρ co-action. Then any element $v \in V$ is contained in a G-submodule which is of finite dimension.*

Proof. Let $\{a_i\}_{i\in I}$ be basis of A. Write

$$\rho v = \sum_{i\in I} v_i \otimes a_i,$$

with $v_i = 0$ except for finitely many i. Let

$$V_0 = \sum_{i \in I} k v_i.$$

Since $(\rho \otimes \mathrm{id}_A)\rho = (\mathrm{id}_V \otimes \Delta) \circ \rho$, we have

$$\sum_{i \in I} \rho v_i \otimes a_i = \sum_i v_i \otimes \Delta a_i.$$

If

$$\Delta a_i = \sum_{r,s \in I} c^i_{rs} \quad a_r \otimes a_s,$$

we have

$$\begin{aligned}
\sum_{i \in I} \rho v_i \otimes a_i &= \sum_{i,r,s \in I} c^i_{rs} \quad v_i \otimes a_r \otimes a_s \\
&= \sum_{i,r,s \in I} c^s_{ri} \quad v_s \otimes a_r \otimes a_i \\
&\Rightarrow \\
\rho v_i &= \sum_{r,s} c^s_{ri} \quad v_s \otimes a_r \\
&\in V_0 \otimes A.
\end{aligned}$$

Finally, $\sum_i \varepsilon(a_i) v_i = v$ by the axiom. So V_0 is a sub-comodule and $v \in V_0$. $\square$

One usually says that any A-comodule is "locally finite". This is a remarkable algebraic property.

Example 6.7. Local finiteness is not always true. For example, the action of $\mathbf{Z}$ act on $V = k(x)$ defined by $m * p(x) = x^m p(x)$ is not locally finite.

Corollary 6.8. *Let G be a group scheme. Let X be an affine and algebraic k-scheme with an action of G. Then there exists a G-module W, a closed and G–equivariant immersion $\theta : X \to W_{\mathrm{a}}$.*

Proof. Let $R = \mathcal{O}(X)$. Let $V \subset R$ be a G-invariant subspace containing algebra generators of R. By the universal property of the symmetric algebra [La02, XVI.8,] we have a surjection $k[V] \to R$. In addition, this is a map of A-comodules. This implies that the induced arrow $\operatorname{Spec} R \to \operatorname{Spec} k[V]$ is G-equivariant. Now, we take $W = V^*$ and note that the action of G on $k[W^*] = k[V]$ is the one obtained by the representation W. $\square$

Corollary 6.9. *Suppose A is finitely generated. Then there is a closed immersion $G \to \mathbf{GL}_n$ which is also a morphism of groups. Said differently, algebraic implies "matrix group".*

Proof. We let G act on G on the *right* in the obvious way; in this case, the comodule map $A \to A \otimes A$ is Δ. (We can let G act on the left, but then the comodule map is different...) Let $V \subset \mathcal{O}(G)$ be a subcomodule containing algebra generators. Let $\{v_i\}$ be a basis and write $\Delta v_j = \sum_i a_{ij} \otimes v_i$; this is possible since V is an A-subcomodule. Consider now the representation $\rho : G \to \mathbf{GL}_n$ defined by $[a_{ij}]$: on the level of rings it is given by $\rho^{\#} : x_{ij} \mapsto a_{ij}$, so $k[x_{ij}, 1/\det] \to R$ is surjective. This shows that ρ is closed immersion. $\square$

7 Lie algebras and smoothness [Wa78], [MAV]

Let $G = \operatorname{Spec} A$ be an affine group scheme. In what follows, $\mathfrak{a}$ is the kernel of the co-unit $\varepsilon : A \to k$; it is a maximal ideal.

Recall that a vector field on G is simply a k-derivation $X : A \to A$, i.e. it satisfies Leibniz's rule $X(a_1 a_2) = a_1 X(a_2) + a_2 X(a_1)$. I shall write $\mathfrak{X}(G)$ for the vector fields.

Let $g : \operatorname{Spec} k \to G$ be a $\operatorname{Spec} k$-point of G; it gives us a map of k-algebras $g : A \to k$. To g we associate an automorphism of G: left-translation ℓ_g. This is given by

$$G \xrightarrow{(g,\mathrm{id})} G \times G \xrightarrow{\mu} G.$$

On the algebra level of algebras we have[†]

$$\ell_g^{\#} = (g^{\#} \bullet \mathrm{id}_A) \circ \Delta.$$

Hence,

$$X^g := (\ell_g^{\#})^{-1} X \ell_g^{\#}$$

becomes a derivation. We say that X is "$G(k)$-left invariant" if $X^g = X$ for all possible g. In this case, we must have

$$\ell_g^{\#} \circ X = X \circ \ell_g^{\#}, \qquad \forall g \in G(\operatorname{Spec} k).$$

Using $\ell_g^{\#} = (g^{\#} \bullet \mathrm{id}_A) \circ \Delta$, we conclude that X is invariant if and only if

$$\begin{aligned}(g^{\#} \bullet \mathrm{id}_A) \circ \Delta \circ X &= X \circ (g^{\#} \bullet \mathrm{id}_A) \circ \Delta \\ &= (g^{\#} \bullet X) \circ \Delta.\end{aligned}$$

Note that, if

$$\boxed{\Delta X = (\mathrm{id}_A \otimes X) \circ \Delta,}$$

then $G(k)$-invariance holds. Hence:

Definition 7.1. X is left invariant if the above equation holds.

It is not difficult to see that the set of all left-invariant vector fields is a vector space. In addition, it is also not difficult to see that if X and Y are invariant vector fields, then $[X, Y] = XY - YX$ is also an invariant vector field.

[†]Here, $g^{\#} \bullet \mathrm{id}_A : A \otimes A \to A$ is defined by $a_1 \otimes a_2 \mapsto g^{\#}(a_1) a_2$.

Definition 7.2. We define:

$$\mathrm{Lie}(G) = \text{left invariant vector fields} \ .$$

With the above mentioned bracket, it is a Lie algebra.

Lecture 7

(12 Novembre 2021).

Let $\mathfrak{X}_e(G)$ stand for the vector space of all ε-derivations, that is, the linear maps $\xi : A \to k$ s.t.

$$\xi(ab) = \varepsilon(a)\xi(b) + \varepsilon(b)\xi(a).$$

Since $A = k1 \oplus \mathfrak{a}$, it is not difficult to see that

$$\mathfrak{X}_e(G) \longrightarrow (\mathfrak{a}/\mathfrak{a}^2)^*$$

$$\xi \longmapsto \xi|_{\mathfrak{a}}$$

is an isomorphism. As you've learned in Prof. Hai's course, the vector space $(\mathfrak{a}/\mathfrak{a}^2)^*$ is called the Zariski tangent space of G at the point e and is denoted by $T_e(G)$.

Finally, it is not hard to see that

$$\mathfrak{X}^{\mathrm{inv}}(G) \longrightarrow \mathfrak{X}_e(G)$$

$$X \longmapsto \varepsilon X$$

is a bijection, the inverse being

$$\xi \longmapsto (\mathrm{id} \otimes \xi) \circ \Delta.$$

This allows us to show the following.

Theorem 7.3 (Cartier's theorem). *Suppose that k has characteristic zero and that G is algebraic. Then, for any closed point $g \in G$, the completion $\widehat{\mathcal{O}}_{G,g}$ of the local ring $\mathcal{O}_{G,g}$ is isomorphic to the power series ring $k[\![t_1, \dots, t_n]\!]$, where $n = \dim T_e G$.*

Proof. This result requires a bit more of commutative algebra than before: We shall need the notion of the completion of a local ring. Let $(R, \mathfrak{m})$ be a noetherian local ring and let $(\widehat{R}, \widehat{\mathfrak{m}})$ be its completion [AM69, Ch.10] . The arrow $R \to \widehat{R}$ is injective adn $\mathfrak{m}\widehat{R} = \widehat{\mathfrak{m}}$

Using the fact that $\mathcal{O}_g \simeq \mathcal{O}_e$ by means of left translations, we only need to consider the case $g = e$. Let now $t_1, \dots, t_n \in \mathfrak{a}$ be such that their images in $\mathfrak{a}/\mathfrak{a}^2$ is a basis. (Recall that $\mathfrak{a}/\mathfrak{a}^2 \simeq \mathfrak{a} \otimes_A (A/\mathfrak{a})$.) We also note that,

$$\mathcal{O}_e t_1 + \cdots + \mathcal{O}_e t_n = \mathfrak{a},$$

by Nakayama. We choose a dual basis in $(\mathfrak{a}/\mathfrak{a}^2)^*$ and let $\{\xi_i\}$ be the basis of $\mathfrak{X}_e(G)$ corresponding to it via $\mathfrak{X}_e \simeq (\mathfrak{a}/\mathfrak{a}^2)^*$; this means that $\xi_i(t_j) = \delta_{ij}$. Let now $X_i \in \mathfrak{X}^{\mathrm{inv}}$ be such that $\varepsilon X_i = \xi_i$. It then follows that

$$X_i(t_j) \equiv \delta_{ij} \mod \mathfrak{a}.$$

Hence, $X_i(t_j)$ is *invertible* in $\mathcal{O}_e$. Let (c_{ij}) be its inverse and define the derivations $D_i = \sum_j c_{ij} X_j$, so that $D_i(t_j)$ is now δ_{ij}.

Since $D_i(t_1^{r_1}\cdots t_n^{r_n}) = r_i t_1^{r_1}\cdots t_i^{r_i-1}\cdots t_n^{r_n}$, we see that $D_i(\mathfrak{a}^p)\subset\mathfrak{a}^{p-1}$. We can then extend:

$$D_i : \widehat{\mathcal{O}}_e \longrightarrow \widehat{\mathcal{O}}_e.$$

We now consider the natural morphism of k-algebras

$$\alpha : k[\![T_1,\dots,T_n]\!] \longrightarrow \widehat{\mathcal{O}}_e,$$

$$T_i \longmapsto t_i.$$

It is an exercise in the theory of complete local rings that α si in fact *surjective* because its image contains generators of the maximal ideal.

Let us now consider the Taylor series:

$$\tau : \widehat{\mathcal{O}}_e \longrightarrow k[\![T_1,\dots,T_n]\!]$$

$$f \longmapsto \sum_{q\in\mathbf{N}^n} \frac{D^q(f)}{q!}(e)\cdot T^q.$$

(Here, for an element $\varphi\in\widehat{\mathcal{O}}_e$, we write $\varphi(e)$ for its image in the residue field..) As usual in Analysis, for $q=(q_1,\dots,q_n)$, we've put $q! = \prod_j q_j!$, $D^q = D_1^{q_1}\cdots D_n^{q_n}$, etc. Now, it is not difficult to show, by induction on $|q| = q_1+\cdots+q_n$, that

$$\frac{D^q}{q!}(ab) = \sum_{0\le r\le q}\frac{D^r}{r!}(a)\cdot\frac{D^{q-r}}{(q-r)!}(b).$$

It then turns out that τ is a homomorphism of local rings. We show α *is injective*: indeed,

$$\tau\alpha\left(\sum_q a_q T^q\right) = \tau\left(\sum_q a_q t^q\right).$$

Now $\tau(t_i) = T_i$ and hence $\tau(t^q) = T^q$, so that $\tau\alpha\left(\sum_q a_q T^q\right) = \tau\left(\sum_q a_q t^q\right) = \sum_q a_q T^q$. If $\alpha(F)=0$, then $F=0$. □

Exercise 7.4. Let $\mathbf{G}_a = \operatorname{Spec} k[x]$. Show that $\operatorname{Lie}(\mathbf{G}_a)\simeq k\frac{d}{dx}$. Show that the bracket $[-,-]$ is allover zero on it.

Exercise 7.5. Compute the Lie algebra of $\mathbf{GL}_2$ and show that it is isomorphic to the Lie algebra of 2×2 matrices having the usual bracket.

Lecture 8

(12 November 2021).

8 The affine quotients [MAV], [Ne11]

In lecture 1, we saw that a possible way to construct moduli spaces relied on **I** and **II**. Condition **I** is usually very dependent on the given problem. Condition **II** is more generic.

Let G be a group acting on a set X and let $F(X, \mathbf{C})$ be the ring of functions $X \to \mathbf{C}$. Write Y for the set of G-orbits of X and $\pi : X \to Y$ for the obvious function. Recall that $F(X, \mathbf{C})$ becomes a G-module, and it is a simple matter to prove that

$$(-) \circ \pi : F(Y, \mathbf{C}) \longrightarrow F(X, \mathbf{C}), \qquad f \longmapsto f \circ \pi$$

gives an isomorphism between $F(Y, \mathbf{C})$ and $F(X, \mathbf{C})^G$. We then need the notion of invariants.

Let now G be an affine group scheme with ring of functions A.

Definition 8.1. Let V be a G-module with coaction ρ. We define V^G as $\{v \in V : \rho v = v \otimes 1\}$.

Exercise 8.2. In case A is a *domain* of finite type over k, show that $V^G = \bigcap_{g \in G(\operatorname{Spec} k)} \{v \in V : gv = v\}$. Give a counterexample to this in case G is not reduced.

In algebraic geometry, it then becomes natural to put:

Definition 8.3. Let X be an affine scheme having an action of and G. The affine quotient is $\operatorname{Spec} \mathcal{O}(X)^G$.

Now, this suggests the fundamental

Question 8.4. Let G act on $X = \operatorname{Spec} R$. Suppose that G and X are algebraic. Is the affine quotient $\operatorname{Spec} R^G$ also algebraic?

This is a very important problem, which was solved in several levels of generality by several influential mathematicians. We shall give a detailed answer to Question 8.4 in case G is *finite and constant* and comment on the solution in case G is linearly reductive. The notion of *linear reductivity* mentioned above is closely connected with the notion of reductivity, as in Prof. Ngô's lecture.

The case of finite constant group schemes

Definition 8.5. A group scheme G is finite if it is affine and the vector space $\mathcal{O}(G)$ is has finite dimension. The *rank* of G is $\dim_k \mathcal{O}(G)$.

Example 8.6. Define $\boldsymbol{\mu}_n(T) = \{f \in \mathcal{O}(T) : f^n = 1\}$. This is represented by $\operatorname{Spec} k[x]/(x^n - 1)$ and has rank n.

Example 8.7. Let G be a finite group. We associate to it a finite group scheme, called the *constant group scheme*.

Define $F(G)$ as the algebra of functions $G \to k$. As a vector space, it has an obvious basis given by the "Dirac" functions:

$$\delta_g(h) = \begin{cases} 0, & \text{if } h \neq g \\ 1, & \text{if } h = g. \end{cases}$$

Note that $F(G \times G) = F(G) \otimes F(G)$ (use the basis). Multiplication on G gives co-multiplication on $F(G) \to F(G) \otimes F(G)$. Evaluation at e gives thje co-identity $F(G) \to k$, and inversion gives the antipode $F(G) \to F(G)$. Then $F(G)$ is a Hopf algebra and

$$\underline{G} = \operatorname{Spec} F(G)$$

is the associated fintie group scheme; the *constant* group scheme associated to G.

Note taht: $\Delta(\delta_g) = \sum_{g'g''=g} \delta_{g'} \otimes \delta_{g''}$.

Exercise 8.8. Let G be a finite group. Show that $\underline{G}(\operatorname{Spec} K) = G$ for any field K. Is ti always the case that $\underline{G}(T) \simeq G$? (The answer is no!)

Let G be a finite group. It is a simple matter to show that an action $\underline{G} \times X \to X$ is just an action of the abstract group G, that is, a group homomorphism $G \to \operatorname{Aut}(X)$. Also, if $X = \operatorname{Spec} R$, then

$$R^G = R^{\underline{G}}.$$

Theorem 8.9. *The k-algebra R^G is of finite type and the extension $R^G \subset R$ is finite.*

Proof. Let $R = k[x_1, \ldots, x_n]$ and let r be the order of G. Let $E_1(f), \ldots, E_r(f)$ be the elementary symmetric functions on $(gf)_{g \in G}$; that is, $E_1(f) = \sum_g g(f)$, etc. Let

$$R_0 = k[E_1(x_1), \ldots, E_1(x_n), \ldots, E_r(x_n), \ldots, E_r(x_n)].$$

Obviously $R_0 \subset R^G$. Note that

$$P_j(T) := \prod_{g \in G} (T - g(x_j))$$

is $T^r - E_1(x_j)T^{r-1} + \cdots + (-1)^r E_r(x_j)$ so that, since $P_j(x_j) = 0$, x_j is integral over R_0. Then $R_1 = R_0[x_1]$ is a finite R_0–module, $R_2 = R[x_1, x_2]$ is a finite R_1–module, etc $\Rightarrow R$ is a finite R_0-module. Since R^G is an R_0–subomdule of R *and* R_0 is Noetherian $\Rightarrow R^G$ a finite R_0–module $\Rightarrow R$ finitely generated over k. □

Now, recall something from topology. Let $G \backslash X$ be the quotient topological space [Ke, 94ff] and $\varphi : X \to G \backslash X$ the natural map. By definition, $G \backslash X$ is just the set of *orbits of* G and φ is $x \mapsto Gx$. Now we give $G \backslash X$ the quotient topology: $V \subset G \backslash X$ is open $\Leftrightarrow \varphi^{-1}(V)$ is open. The next result says that $\operatorname{Spec} R^G$ is just the quotient topological space.

The next result is based on [MAV, Section 7] with some extra details worked out following [BA, Chapter 5, §2, no. 2, Theorem 8].

Corollary 8.10. *Let $Y = \operatorname{Spec} R^G$ and let $\pi : X \to Y$ be the morphism derived from the inclusion $R^G \subset R$. Then*

(1) π is closed and surjective.

(2) Y has the quotient topology.

(3) For each $g \in G$, we have $\pi \circ g = \pi$.

(4) Any two not-necessarily closed points $\mathfrak{p}$ and $\mathfrak{q}$ of X lie on the same orbit of G if and only if $\pi(\mathfrak{p}) = \pi(\mathfrak{q})$.

(5) For each open subset V of Y, we have

$$\mathcal{O}_Y(V) = \mathcal{O}_X(\pi^{-1}(V))^G.$$

In other words, $\mathcal{O}_Y \simeq (\pi_* \mathcal{O}_X)^G$.

Conditions (2)–(4) show that Y is the quotient topological space [Ke, 94ff].

Proof. (1) You should know how to prove this using what you've learned from commutative algebra and the fact that $R^G \subset R$ is a finite extension. (Going-down, going-up, etc.)

(2) This is general topology [Ke, Theorem 8, p.95].

(3) Let $\mathfrak{p}$ and $\mathfrak{q}$ primes of A s.t. $\mathfrak{p} = g(\mathfrak{q})$. Let $f \in \mathfrak{q} \cap R^G$. Then $g(f) \in \mathfrak{p}$, but $g(f) = f$ and hence $f \in \mathfrak{p} \cap R^G \Rightarrow \pi(\mathfrak{q}) \subset \pi(\mathfrak{p})$. As $g^{-1}(\mathfrak{p}) = \mathfrak{q}$, we conclude that $\pi(\mathfrak{p}) \subset \pi(\mathfrak{q})$. Working with the equality $\mathfrak{q} = g^{-1}(\mathfrak{p})$ we get $\pi(\mathfrak{q}) \subset \pi(\mathfrak{p})$.

(4) Now, suppose that $\pi(\mathfrak{p}) = \pi(\mathfrak{q})$. Let $b \in \mathfrak{q}$. Then $\prod_g g(b) \in \mathfrak{q} \cap R^G = \pi(\mathfrak{q})$. Now $\pi(\mathfrak{q}) = \pi(\mathfrak{p}) \subset \mathfrak{p}$. Hence, $g_0(b) \in \mathfrak{p}$ for some $g_0 \in G \Rightarrow b \in g_0^{-1}(\mathfrak{p}) \Rightarrow \mathfrak{q} \subset \bigcup_g g(\mathfrak{p})$. A well-known result from commutative algebra [AM69, Proposition 1.11] shows that $\mathfrak{q} \subset g_1(\mathfrak{p})$ for some $g_1 \in G$. Since $\pi(\mathfrak{q}) = \pi(g_1(\mathfrak{p}))$, it must be that $\mathfrak{q} = g_1(\mathfrak{p})$, as a standard property of inclusions between ideals in finite extensions shows [AM69, Corollary 5.9].

(5) For each $g \in G$, we have automorphism of the sheaf $\alpha_g : \pi_* \mathcal{O}_X \xrightarrow{\sim} \pi_* \mathcal{O}_X$: for $V \subset Y$ define

$$\alpha_g : \mathcal{O}_X(\pi^{-1}(V)) \longrightarrow \mathcal{O}_X(\pi^{-1}(V))$$

by means of g. (I'll leave to the reader to fill in the details here.) Note that this is an isomorphism of $\mathcal{O}_Y$-modules. Let $\alpha'_g = \alpha_g - \mathrm{id}_{\mathcal{O}_Y}$. Then

$$(\pi_* \mathcal{O}_X)^G = \bigcap_{g \in G} \operatorname{Ker}(\alpha'_g).$$

Hence $(\pi_* \mathcal{O}_X)^G$ is coherent on the affine scheme Y. The inclusion $\mathcal{O}_Y \to (\pi_* \mathcal{O}_X)^G$ is isomorphism since is iso. on global sections. □

An important consequence of this construction is that we can now glue and take quotients in more general setting.[‡]

[‡]What follows was not explained in the lectures.

Theorem 8.11. *Let X be an algebraic k-scheme with action of G. Suppose that for each x, the orbit Gx is contained in an affine scheme and that X is separated. Then there exists finite morphism $\pi : X \to Y$ such that:*

(1) As a topological space, Y is the quotient for the action of G.

(2) For each $V \subset Y$ open, the set $U := \pi^{-1}(V)$ is open and invariant under G and $\mathcal{O}_Y(V) \simeq \mathcal{O}_X(U)^G$.

Proof. Let $\pi : X \to Y$ be the quotient topological space of X. We now endow Y with the sheaf of rings $\mathcal{B} := \pi_*(\mathcal{O}_X)^G$. Recall that this means taht for any open V of Y, we have

$$\mathcal{B}(V) = \mathcal{O}_X(\pi^{-1}(V))^G.$$

We need to prove that Y is an algebraic scheme. For that, we need to cover Y by a finite number of open subsets V such that $(V, \mathcal{B}|V)$ is an affine algebraic scheme.

For each $x \in X$, let U' be affine open neighbourhood of Gx. It follows that $U = \cap_g g(U')$ is affine open and *invariant*. Hence $V := \pi(U)$ is open. Note that $\pi^{-1}(V) = U$. Note that V is, as a topological space, the quotient of U by the action of G, and taht the sheaf on V is $(\pi_* \mathcal{O}_U)^G$. Hence this is $\operatorname{Spec} \mathcal{O}(U)^G$.

□

The case of geometric reductivity

Another case where Question 8.4 has an affirmative answer is when G is a *linearly reductive* group. In characteristic zero, all reductive linear algebraic groups are *linearly* reductive. If $\operatorname{char} k > 0$, this notion is too restrictive. To explain what a linearly reductive group means, I digress on semi-simple representations. Let $G = \operatorname{Spec} A$ be an affine group scheme, where A is a k-algebra of finite type.

Definition 8.12 (Semi-simplicity)**.** Let V be a finite dimensional G-module. We say that V is *simple* if the only subrepresentations are $\{0\}$ and V. We say that V is *semi-simple* if there exist simple sub-representations $\{V_i\}$ of V such that $V = \sum V_i$.

Definition 8.13. G is linearly reductive if each finite dimensional G-module is semi-simple.

Example 8.14. The group $\mathbf{G}_a = \operatorname{Spec} k[x]$ is *not* linearly reductive. Indeed, consider the representation $\rho : \mathbf{G}_a \to \mathbf{GL}_2$ defined by

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

Note that $k\vec{e}_1$ is invariant and hence ρ is not simple. Let then $V = k\vec{v} \oplus k\vec{w}$ with $k\vec{v}$ and $k\vec{w}$ $\mathbf{G}_a$-invariant. This means that $\vec{v}$ and $\vec{w}$ are eigenvalues for all matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. But this is impossible.

Exercise 8.15. Let G be a group scheme and V a finite dimensional G-module. Show that the following are equivalent. (In a different context, this is carefully explained in [La02, XVII.2].)

(1) V is semi-simple.

(2) There exist simple sub-representations $\{V_i\}_{i=1}^m$ of V such that $V = V_1 \oplus \cdots \oplus V_m$.

(3) For each G-submodule $W \subset V$, there exists a G-submodule $C \subset W$ such that $V = W \oplus C$.

Show that the following conditions are equivalent.

(1) Every finite dimensional G-module is semi-simple.

(2) If $V \to W$ is a surjective map of finite dimensional G-modules, then $V^G \to W^G$ is also surjecitve.

(3) For each finite dimensional G-module V and each $v \in V^G \setminus \{0\}$, there exists a linear form $F \in (V^*)^G$ such that $F(v) \neq 0$.

Theorem 8.16. *Let G be linearly reductive. Let $X = \operatorname{Spec} R$ be an affine algebraic scheme with an action of G. Then R^G is finitely generated.*

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