A NOTE ON STRATIFIED MODULES WITH FINITE INTEGRAL DIFFERENTIAL GALOIS GROUPS

JOÃO PEDRO P. DOS SANTOS

ABSTRACT. This note is a continuation of [dS07p]. As such it investigates the relation between the integral differential Galois group of an integral p-adic differential equation and the differential Galois group of the $\mathscr{D}$-module induced on the special fibre. We specifically concentrate on the case of finite differential Galois groups. We obtain, for every integral p-adic equation $\mathscr{M}$ whose differential Galois group is finite, a closely related integral p-adic equation $\mathscr{N}$ whose integral differential Galois group is etale. Furthermore we show that every such $\mathscr{M}$ comes from a representation of the etale fundamental group in positive characteristic.
2000 Mathematics Subject Classication: Primary 13N99, 13N10. Secondary 13B40, 14L15, 20C99

1. INTRODUCTION

The goal of this note is to show two (interrelated) properties concerning stratified modules in mixed characteristic (see §2.2.2 for the definition) whose differential Galois group is finite. The first such property is the existence of a closely related stratified module whose differential Galois group is in fact etale (Theorem 5). The second property states that such objects always arise as representations of the etale fundamental group in positive characteristic (Corollary 7).

The motivation behind all that follows is (the second part of) a conjecture made in [MvdP03] (Conjecture 8.5) concerning the reduction theory of stratified modules (or integral p-adic equations, or iterative differential modules, or $\mathscr{D}$-modules, etcetera). The conjecture raises the question of whether a stratified module with finite differential Galois group can be "desingularized" while keeping the stratified module on the generic fibre "equivalent". The meaning of desingularization being: "whose differential Galois group is smooth". In this note we show that it is possible to perform such a desingularization *without changing the Tannakian category on the generic fibre* proposing thereby an answer to the conjecture once the property in italics is taken to define equivalence. (This proposition is also in accordance with the examples following Conjecture 8.5 of [MvdP03]. See also Example 8 below.)

The main idea behind our most significant results (Theorem 5 and Corollary 7) is simple and is a mixture of two well-known mathematical phenomena: (1) linear differential equations have solutions in power series (Proposition 2); (2) rings of power series are completions and hence can be written as direct limits of smooth algebras (desingularization of Néron-Popescu [S99]). At the end we did not find a direct use for Néron-Popescu desingularization, but this great result was the fount of our inspiration.

We will now describe the organization of this note. In §2 we will introduce notations and standard properties of our objects of study. In §3 we make the observation that any stratified module over a ring of power series is trivial. This can be interpreted as saying that systems of differential equations arising from stratified modules (usually called integral p-adic differential equations, see the final remark in §2.2.2) always admit power series with *bounded* coefficients as solutions. In §4 we review some properties developed in [dS07p] concerning stratified modules with *finite* differential Galois group. In §5 we state and prove our main results: Theorem 5 and Corollary 7. The proof of Theorem 5 uses a desingularization lemma (Lemma 6), which is made possible due to the existence of power series solutions. The proof of Corollary 7 uses additionally a characterization, given in [dS07p], of the stratified modules which arise from representations of the fundamental group in positive characteristic.

Standard notation: Throughout this note, $\mathfrak{o}$ will denote a complete discrete valuation ring (DVR) of mixed characteristic $(0, p)$ with uniformizer ϖ, residue field k and fractions field K. The field k is assumed to be perfect. If D is an integral domain, $Q(D)$ will denote its field of fractions.

As it is already implicit in the examples proceeding [MvdP03, Conj. 8.5], "equivalent" there should not be understood as "isomorphic." I thank very much the referee for calling my attention to that and hence avoiding a confusion which was present in a previous version of this note.

Finally I thank Hélène Esnault (the editor) for teaching a young mathematician how to communicate better with his peers.

2. The objects of study

2.1. **The ambient ring.** In this work we will be interested in stratified modules defined over some affine $\mathfrak{o}$-adic ring. So let A be a flat and $\mathfrak{o}$-adic algebra.

Let
$$\mathfrak{o}\langle x_1, \ldots, x_n\rangle$$
be the ring of power series in $\mathfrak{o}[[x_1, \ldots, x_n]]$ whose radius of convergence is not smaller than 1; $\mathfrak{o}\langle x_1, \ldots, x_n\rangle$ can also be defined as the ϖ-adic completion of $\mathfrak{o}[x_1, \ldots, x_n]$. We assume that there are etale coordinated on Spf A: There exists a homomorphism

(1) $$\Psi : \mathfrak{o}\langle x_1, \ldots, x_n\rangle \longrightarrow A$$

of $\mathfrak{o}$-algebras such that $\Psi \otimes (\mathfrak{o}/\varpi^\mu)$ is *etale* for every $\mu \in \mathbb{N}$. In particular A is topologically of finite type over $\mathfrak{o}$ (see [BL1, Lemma 1.5]). This means that there exist variables $T_1, \ldots, T_m$ and a surjective homomorphism of $\mathfrak{o}$-algebras
$$\mathfrak{o}\langle T_1, \ldots, T_m\rangle \longrightarrow A.$$
As etale morphisms preserve regularity [SGA1, exp. I, Cor. 9.2], it follows that each ring $A \otimes_\mathfrak{o} (\mathfrak{o}/\varpi^\mu)$ is regular. It is not hard to see that this implies that A itself is regular. By base-changing Ψ, we have obtained that $A \otimes_\mathfrak{o} \mathfrak{o}'$ is regular for every finite extension of discrete valuation rings $\mathfrak{o}' \supseteq \mathfrak{o}$.

Another key property we will need from A is excellence (information on excellent rings can be found in [M80, Ch. 13]). From the fact that A is topologically of finite type over $\mathfrak{o}$, this is a consequence of the work of Valabrega.

Theorem 1 ([V76])**.** *Let D be an excellent Dedekind domain of characteristic zero, L a D-algebra of finite type over D and $\mathfrak{a}$ an arbitrary ideal of L. Then the $\mathfrak{a}$-adic completion of L is also excellent. In particular, any $\mathfrak{o}$-algebra which is topologically of finite type over $\mathfrak{o}$ is excellent.*

We will require another property from A. This is listed as P3 in the enumeration below.

  P1 A is an excellent ring.
  P2 For every finite extension of discrete valuation rings $\mathfrak{o}' \supseteq \mathfrak{o}$, the ring $A \otimes_\mathfrak{o} \mathfrak{o}'$ is regular.
  P3 If k' denotes the residue field of $\mathfrak{o}'$, then $A \otimes_\mathfrak{o} k'$ is a regular domain. (In particular Spec $(A \otimes_\mathfrak{o} k')$ is connected. Note that this is equivalent to P3 in the presence of the morphism Ψ.)

Since A is assumed to be complete for the ϖ-adic topology, the fact that Spec $(A \otimes_\mathfrak{o} k')$ is connected implies that $A \otimes_\mathfrak{o} \mathfrak{o}'$ is a domain (otherwise its spectrum would be disconnected).

2.2. **Stratified modules over A.** These are our main object of interest and go by other names in the literature.

2.2.1. *Differential operators.* Due to the main property of the morphism Ψ in (1), the natural homomorphism between modules of principal parts (for the definition of these the reader can consult EGA IV$_4$, 16.3, p. 14ff, but here we should work with $\hat{\otimes}_{\mathfrak{o}}$ instead of $\otimes_{\mathfrak{o}}$)

$$A \otimes P^{\nu}(\Psi) : A \otimes_{A_0} P^{\nu}_{A_0/\mathfrak{o}} \longrightarrow P^{\nu}_{A/\mathfrak{o}} \qquad (A_0 = \mathfrak{o}\langle x_1, \ldots, x_n \rangle)$$

is an isomorphism for each $\nu \in \mathbb{N}$ (see [dS07p, 4.1] for more information on this isomorphism). In particular, the ring of $\mathfrak{o}$-linear (continuous) differential operators on A,

$$\mathscr{D} = \mathscr{D}_{A/\mathfrak{o}} = \varinjlim_{\nu} \operatorname{Hom}_A(P^{\nu}_{A/\mathfrak{o}}, A),$$

is generated, as an A-algebra, by differential operators

$$\partial_q, \qquad (q \in \mathbb{N}^n, q \neq 0)$$

(the order of $\partial_{(q_1,\ldots,q_n)}$ being $|(q_1,\ldots,q_n)| = \Sigma q_i$). The restriction of ∂_q to $\mathfrak{o}\langle x_1, \ldots, x_n \rangle$ is given by the usual formula

$$\partial_{(q_1,\ldots,q_n)} \prod_j x_j^{a_j} = \prod_j \binom{a_j}{q_j} x_j^{a_j-q_j}.$$

(The binomial symbol is zero if the upper term is smaller than the lower one.) More details are in [dS07p, 4.1] or EGA IV$_4$, 16.11, p. 53ff.

2.2.2. *Stratified A-modules.* Let $\mathscr{M}$ be an A-module endowed with a compatible action of the ring $\mathscr{D}$:

$$\nabla : \mathscr{D} \longrightarrow \operatorname{End}_{\mathfrak{o}}(\mathscr{M}).$$

Compatibility here means that the above map is a *homomorphism of A-algebras*. Such a structure is called an $\mathfrak{o}$-*linear stratification on $\mathscr{M}$* and $\mathscr{M}$ is said to be a *stratified A-module* or simply a $\mathscr{D}$-*module*. Another way to define an $\mathfrak{o}$-linear stratification on an A-module $\mathscr{M}$ is to postulate an isomorphism (plus a descent data) between the two pull-backs $\operatorname{pr}_1^* \mathscr{M}$ and $\operatorname{pr}_2^* \mathscr{M}$, where

$$\operatorname{pr}_i : \text{completion of } \operatorname{Spf} A \hat{\otimes}_{\mathfrak{o}} A \text{ along the diagonal} \longrightarrow \operatorname{Spf} A$$

are the projections [BO78, Ch. 2].

The A-modules endowed with an $\mathfrak{o}$-linear stratification are objects of a category

$$\mathbf{Str} = \mathbf{Str}(A/\mathfrak{o}),$$

whose arrows are simply homomorphisms of $\mathscr{D}$-modules. The full subcategory of $\mathbf{Str}$ determined by the class of objects which are *finite A-modules* is denoted by

$$\mathbf{str} = \mathbf{str}(A/\mathfrak{o}).$$

The category $\mathbf{Str}$ possesses a "tensor product"

$$\otimes : \mathbf{Str} \times \mathbf{Str} \longrightarrow \mathbf{Str}$$

which is $\mathfrak{o}$-bilinear; when regarded as a functor in a category of A-modules, $\otimes$ is just the natural tensor product over A. The functor $\otimes$ gives **Str** a structure of monoidal category. Hence we can talk about algebras (monoids in the terminology of MacLane's *Categories for the working mathematician*) in **Str** and in **str**.

Notation: The category of algebras in **Str** (resp. **str**) will be denoted by **AStr** (resp. **Astr**). (We remark that this terminology is a bit in disagreement with the terminology of Deligne (also employed in [dS07]) since we do not consider the categories of inductive limits. On the other hand we are faithful to MacLane's definitions.) In detail, an object $R \in \mathbf{AStr}$ consists of an object $R \in \mathbf{Str}$ together with a arrows of **Str** called multiplication $m : R \otimes_A R \longrightarrow R$ and identity $1_R : A \longrightarrow R$. Furthermore, this data is supplemented by the obvious compatibilities which imitate the multiplication and identity of an A-algebra.

Finally we will find convenient to denote by

$$\mathbf{str}^{\#}$$

the full sub-category of **str** whose objects are flat A-modules. *This category coincides with the full subcategory of stratified A-modules which are $\mathfrak{o}$-flat or which admit a dual in* **str** [dS07p, Lemma 19]. Analogous notations are in force for the other categories: $\mathbf{AStr}^{\#}$ and $\mathbf{Astr}^{\#}$.

Remark: Let $n = 1$. A p-adic differential equation is a linear system of equations

$$\frac{dy_i}{dx} = \sum_{j=1}^{r} a_{ij} y_j, \qquad i = 1, \ldots, r;$$

where $a_{ij} \in A \otimes K$. Such a system induces a differential ∇ on the free $A \otimes K$-module $\mathcal{M} = (A \otimes K)^{\oplus r}$. If the following integrality condition

$$\frac{1}{q!} \nabla^q (A^{\oplus r}) \subseteq A^{\oplus r}$$

holds, then we obtain a stratified A-module structure on $A^{\oplus r}$. Therefore stratified A-modules can also be called *integral p-adic differential equations*.

3. FORMAL SOLUTIONS

For the rest of this work, we will assume that the structural inclusion $\mathfrak{o} \subseteq A$ admits a section

$$\xi : A \longrightarrow \mathfrak{o}$$

such that $\xi(x_i) = 0$ for each i, i.e. the formal scheme $\mathrm{Spf}\, A$ has an $\mathfrak{o}$-point above the origin of $\hat{\mathbb{A}}^n_{\mathfrak{o}}$. We let

$$R = \ker(\xi)\text{-adic completion of } A.$$

(R is also the $(\ker(\xi), \varpi)$-completion.) It is not hard to show that

$$R \cong \mathfrak{o}[[x_1, \ldots, x_n]].$$

From this explicit description it follows that the operators ∂_q act on R in the obvious way. Furthermore, multiplication $R \otimes_A R \longrightarrow R$ is a homomorphism of $\mathscr{D}$-modules. Using proper terminology from §2.2.2: $R \in \mathbf{AStr}$. The following proposition is an algebraic statement–crafted by Cartier–of Cauchy's theorem on the existence of power series solutions to linear differential equations.

Proposition 2. *Let (B, ∇) be an object of $\mathbf{Astr}^{\#}$. Then the $\mathfrak{o}$-algebra of horizontal elements in $B \otimes_A R$,*

$$(B \otimes_A R)^{\mathrm{horizontal}} = \{t \in B \otimes_A R;\ \nabla(\partial_q)t = 0\ \ \forall q \neq 0\},$$

is flat over $\mathfrak{o}$ and the natural homomorphism

$$(B \otimes_A R)^{\mathrm{horizontal}} \otimes_{\mathfrak{o}} R \longrightarrow B \otimes_A R$$

is an isomorphism of $\mathscr{D}$-modules.

Proof. All that is necessary is to copy the proof of Proposition 8.9 of [Ka70]. □

4. Stratified modules with finite integral differential Galois group

We keep the assumptions and notations of §3. The existence of the $\mathfrak{o}$-rational point ξ in $\mathrm{Spf}\, A$ produces an $\mathfrak{o}$-linear functor (taking the fibre at the point ξ)

$$\xi^* : \mathbf{str} \longrightarrow \mathfrak{o}\text{-mod}, \qquad \mathscr{M} \mapsto \mathscr{M} \otimes_{A,\xi} \mathfrak{o}$$

which is exact, faithful and monoidal [dS07p, 4.2].

The theory developed in [dS07p] associates to each $\mathscr{M} \in \mathbf{str}^{\#}$ a faithfully flat algebraic $\mathfrak{o}$-group scheme $\Pi = \Pi(\mathscr{M}, \xi)$, called the *integral differential Galois group* or the *monodromy group* of $\mathscr{M}$ at ξ. The category of finite rank representations (in the sense of [J87, Ch. 2]) $\mathrm{Rep}_{\mathfrak{o}}(\Pi)$ fits into a commutative (up to monoidal isomorphism) diagram

(2)
$$\begin{array}{ccc} & \mathrm{Rep}_{\mathfrak{o}}(\Pi) & \\ {\scriptstyle S}\big\downarrow & & {\scriptstyle \mathrm{forget}}\searrow \\ \mathbf{str} & \xrightarrow[\ \xi^*\ ]{} & \mathfrak{o}\text{-mod}; \end{array}$$

the functor S is $\mathfrak{o}$-linear, exact, monoidal and faithful. We are interested in the particular case where the group Π is finite.

Assumption: From now on we assume that $\Pi = \Pi(\mathcal{M}, \xi)$ is finite.

The left regular representation $\mathscr{R} := (\mathscr{O}(\Pi), \rho_l)$ is an algebra in the monoidal category $\mathrm{Rep}_{\mathfrak{o}}(\Pi)$. Endowing $\mathscr{O}(\Pi)$ with the *trivial* action of Π we obtain a Hopf-algebra $\mathscr{H}$ in $\mathrm{Rep}_{\mathfrak{o}}(\Pi)$. Furthermore, there is a co-action

$$\mathscr{R} \longrightarrow \mathscr{R} \otimes \mathscr{H}$$

of $\mathscr{H}$ on $\mathscr{R}$ causing it to be an $\mathscr{H}$-torsor algebra: the obvious arrow

(3) $$\mathscr{R} \otimes \mathscr{R} \longrightarrow \mathscr{R} \otimes \mathscr{H}$$

is an isomorphism (more information on this formalism–due to M. Nori–can be found in [dS07, 2.3.2]). All this structure is transported to **str** via the functor S:

(1) The algebra $\mathscr{R}$ in $\mathrm{Rep}_{\mathfrak{o}}(\Pi)$ is taken to an algebra $B_{\mathcal{M}} \in \mathbf{Astr}^{\#}$.
(2) $\mathscr{H}$ is taken to the Hopf-algebra of regular functions $\mathscr{O}(\Pi \otimes_{\mathfrak{o}} A)$ on the A-group scheme $\Pi \otimes_{\mathfrak{o}} A$; the stratification on $\mathscr{O}(\Pi \otimes_{\mathfrak{o}} A)$ is the trivial one.

Proposition 3. *Notations as before.*

(a) $\mathrm{Spec}(B_{\mathcal{M}}) \longrightarrow \mathrm{Spec}(A)$ *is a Π-torsor;*
(b) $A \otimes K \longrightarrow B_{\mathcal{M}} \otimes K$ *is etale;*
(c) *there exists an $\mathfrak{o}$-point $\eta : B_{\mathcal{M}} \longrightarrow \mathfrak{o}$ inducing the $\mathfrak{o}$-point $\xi : A \longrightarrow \mathfrak{o}$;*
(d) *if R is any object of $\mathbf{AStr}^{\#}$ and $\sigma : B_{\mathcal{M}} \longrightarrow R$ is an arrow of $\mathbf{AStr}$, then σ is injective.*

Proof. We only need to prove items (c) and (d); (a) and (b) follow from the discussion above and the fact that group schemes in characterisitc zero are smooth. To simplify notation let us write B for $B_{\mathcal{M}}$.

Since the functor $\xi^* \circ S$ is naturally isomorphic as a monoidal functor to the forgetful functor $\mathrm{Rep}_{\mathfrak{o}}(\Pi) \longrightarrow (\mathfrak{o}\text{-mod})$, it follows that the algebra $\xi^* B = \xi^* S(\mathscr{O}(\Pi), \rho_l)$ is naturally isomorphic to the algebra $\mathscr{O}(\Pi)$; this algebra is endowed with a $\mathfrak{o}$-homomorphism onto $\mathfrak{o}$ (co-identity). This proves (c). The proof of (d) will be a consequence of the theory developed in [dS07p, 4.3.1]. Diagram (2) can be extended to a commutative diagram

(4)

$$
\begin{array}{ccc}
 & \mathrm{Rep}_{\mathfrak{o}}(\Pi) & \\
 \overset{\overline{\xi^*}}{\nearrow} \quad \downarrow \quad \overset{\text{forget}}{\searrow} & & \\
\mathscr{C} \xrightarrow[\text{inclusion}]{} \mathbf{str} \xrightarrow[\xi^*]{} (\mathfrak{o}\text{-mod}) &
\end{array}
$$

where $\mathscr{C}$ is an $\mathfrak{o}$-linear monoidal full sub-category of $\mathbf{str}^{\#}$ containing $\underline{\mathcal{M}}$ and B (see also the discussion preceding Corollary 7). The functor $\overline{\xi^*}$ is a monoidal equivalence between $\mathscr{C}$ and the category of objects in $\mathrm{Rep}_{\mathfrak{o}}(\Pi)$ which have a dual (are free as $\mathfrak{o}$-modules) and S restricted to the latter

category is an equivalence inverse to $\overline{\xi^*}$. (It may also be of use for the reader to point out that $\overline{\xi}^*\mathscr{M}$ is a faithful representation of Π.) Furthermore, even though $\mathscr{C}$ is not abelian, it possesses the following property. If

$$0 \longrightarrow \mathscr{M}' \longrightarrow B \longrightarrow \mathscr{M}'' \longrightarrow 0$$

is an exact sequence in **str** with $\mathscr{M}'$ and $\mathscr{M}''$ $\mathfrak{o}$-flat, then $\mathscr{M}', \mathscr{M}'' \in \mathscr{C}$. Now the rest of the proof is just formalism. As R is flat over $\mathfrak{o}$, the kernel $\mathscr{I}$ of σ will be an object of $\mathscr{C}$. As σ is an arrow of **AStr**, $\mathscr{I}$ is stable by multiplication

$$\mathscr{I} \otimes_A B \longrightarrow B$$

(that is, $\mathscr{I}$ is an ideal!). Hence $\overline{\xi^*}(\mathscr{I})$ is either 0 or $\mathscr{O}(\Pi)$—there are no non-trivial Π-equivariant (for the left regular action) ideals in the algebra $\mathscr{O}(\Pi)$. It follows, from the basic properties of ξ^* stated in the beginning, that σ is an injective homomorphism of A-algebras. $\qquad\square$

Corollary 4. *Assume that $\xi(x_i) = 0$ for all i and let R denote the $\ker(\xi)$-adic completion of A. Then there exists an inclusion (monomorphism)*

$$B_{\mathscr{M}} \hookrightarrow R$$

in the category **AStr**.

Proof. As in the proof of the proposition we write B for $B_{\mathscr{M}}$. Due to item (d) of the proposition, we only need to show that there exits a homomorphism of algebras $B \longrightarrow R$ which is an arrow of **AStr**. Let $\hat{\xi} : R \longrightarrow \mathfrak{o}$ denote the homomorphism induced by $\xi : A \longrightarrow \mathfrak{o}$. Then, by restricting $\eta \otimes_\xi \hat{\xi}$, we obtain a homomorphism of $\mathfrak{o}$-algebras

$$\zeta : (B \otimes_A R)^{\mathrm{horizontal}} \longrightarrow \mathfrak{o}.$$

This gives a homomorphism of stratified algebras

$$B \longrightarrow B \otimes_A R \longrightarrow (B \otimes_A R)^{\mathrm{horizontal}} \otimes_{\mathfrak{o}} R \longrightarrow R,$$

where the first arrow is the obvious one, the second is an inverse to the canonical map $(B \otimes_A R)^{\mathrm{horizontal}} \otimes_{\mathfrak{o}} R \longrightarrow B \otimes_A R$ (see Proposition 2) and the third is $\zeta \otimes_{\mathfrak{o}} \mathrm{id}_R$.

$$\square$$

5. Desingularization and the main results

We will keep the notations and assumptions of §4. In particular, the spectrum of the underling A-algebra

$$B_{\mathscr{M}} \in \mathbf{Astr}^{\#}$$

is a torsor over $\mathrm{Spec}\,(A)$ with structural group $\Pi = \Pi(\mathscr{M}, \xi)$ (Proposition 3).

In order to state our main result we still need to introduce some notation from the theory of Tannakian categories. Given an object M of $\mathbf{str}(A \otimes K/K)$, we let $\langle M \rangle_{\otimes}$ denote the smallest Tannakian sub-category of $\mathbf{str}(A \otimes K/K)$ containing M. That is, $\langle M \rangle_{\otimes}$ is full and its objects are subquotients of the generalized tensors $M^{\otimes a_1} \otimes (M^{\vee})^{\otimes b_1} \oplus \cdots \oplus M^{\otimes a_t} \otimes (M^{\vee})^{\otimes b_t}$.

Theorem 5. *Assume that k is algebraically closed. Let $\mathscr{M} \in \mathbf{str}^{\#}$ and let $\Pi = \Pi(\mathscr{M}, \xi)$ be its integral differential Galois at the $\mathfrak{o}$-point ξ, which we assume to be finite (over $\mathfrak{o}$). Then there exists $\mathscr{N} \in \mathbf{str}^{\#}$ with the following properties:*

(1) *The integral differential Galois group of $\mathscr{N}$, $\Pi(\mathscr{N}, \xi)$, is etale.*
(2) *The subcategories $\langle \mathscr{N} \otimes K \rangle_{\otimes}$ and $\langle \mathscr{M} \otimes K \rangle_{\otimes}$ of $\mathbf{str}(A \otimes K/K)$ are identical.*
(3) *The etale K-group-scheme $\Pi \otimes K$ is constant.*

As an A-module, $\mathscr{N}$ is the normalization of $B_{\mathscr{M}}$. This normalization is etale over A and the structure of stratified module arrises from this property.

The proof of this theorem relies on a desingularization result.

Lemma 6. *Let $f : A \subseteq B$ be a finite extension of integral domains and $I \subset A$ an ideal. Assume that:*

(i) *$f \otimes K : A \otimes K \longrightarrow B \otimes K$ is etale.*
(ii) *There exists a finite extension $\mathfrak{o}' \supseteq \mathfrak{o}$ such that $B \otimes_{\mathfrak{o}} \mathfrak{o}'$ is an integral domain whose fractions field is a Galois extension of the fractions field of $A \otimes_{\mathfrak{o}} \mathfrak{o}'$.*
(iii) *If R denotes the I-adic completion of A, then the ideals ϖR and $\varpi' R \otimes_{\mathfrak{o}} \mathfrak{o}'$ are prime.*
(iv) *There an injective homomorphism of A-algebras $B \hookrightarrow R$.*

Then the normalization of B is etale over A.

Proof. The proof will be divided in three parts.

(a) Let A', B', R' and C' denote $A \otimes_{\mathfrak{o}} \mathfrak{o}'$, $B \otimes_{\mathfrak{o}} \mathfrak{o}'$, $R \otimes_{\mathfrak{o}} \mathfrak{o}'$ and the normalization of $B \otimes_{\mathfrak{o}} \mathfrak{o}'$ respectively. Observe that since A enjoys properties P1–P3 of §2.1, so does A'. We claim that the finite [M89, Lemma 1, §33] A'-algebra C' is etale. Otherwise, by the Zariski-Nagata Purity Theorem [SGA1, Exp. X, Thm 3.1] there exists a prime $\mathfrak{q} \in \mathrm{Spec}\,(C')$ of height one such that the extension of DVRs $A'_{\mathfrak{q} \cap A'} \subseteq C'_{\mathfrak{q}}$ is ramified. Since $A' \otimes K \longrightarrow C' \otimes K$ is etale, $\mathfrak{q} \cap A' = \varpi' A'$ (by assumption $A \otimes k'$ is integral). But $\mathrm{Gal}(Q(B')/Q(A')) = \mathrm{Gal}(Q(C')/Q(A'))$ acts transitively [M89, Thm. 9.3] on the primes $\mathfrak{r} \in \mathrm{Spec}\,(C')$ above $\varpi' A'$, and we see that for *any* $\mathfrak{r} \in \mathrm{Spec}\,(C')$ above $\varpi' A'$, the extension $A'_{\varpi' A'} \subseteq C'_{\mathfrak{r}}$ is ramified. In particular we can take $\mathfrak{r} = \varpi' R' \cap$

C'. (Since R' is the I-adic completion of A', the natural homomorphism $A' \longrightarrow R'$ is regular and R' is inherits the normality of A' [M80, 33.I, p. 258]. This allows us to extend the canonical embedding $B' \hookrightarrow R'$ to an embedding $C' \hookrightarrow R'$. That $\varpi'R' \cap A' = \varpi'A'$ follows from the fact that the going-down holds between A' and R' [M89, Thm. 9.5, p. 68] and thus $\mathrm{ht}(\varpi'R' \cap A') \leq \mathrm{ht}(\varpi'R') = 1$ [M89, ex. 9.9, p. 70].) Using the regularity of $A' \longrightarrow R'$ [M80, 33.I, p. 258] we see that $A'_{\varpi'A'} \subseteq R'_{\varpi'R'}$ is formally unramified (this means simply that a uniformizer is taken to a uniformizer and that the extension of residue fields is separable). Because $C'_{\mathfrak{q}}$ is a DVR in between this last unramified extension, it follows that $A_{\varpi'A'} \subseteq C'_{\mathfrak{q}}$ is also unramified. This violates the Zariski-Nagata Purity Theorem and we obtain that $A' \longrightarrow C'$ is etale.

(b) Let C denote the normalization of B. We want to show that

$$\mathrm{Spec}\,(C/\varpi C)$$

is irreducible. We know from item (a) that $A \otimes_{\mathfrak{o}} \mathfrak{o}' \longrightarrow C'$ is etale. Since C' is complete for the ϖ'-adic topology (use Lemma 1, §33 of [M89]) and is a *domain*, it follows that $\mathrm{Spec}\,(C'/\varpi'C')$ is connected (generalized Hensel's Lemma [J87, 3.15, p. 51]). Because

$$\mathrm{Spec}\,(C'/\varpi'C') \longrightarrow \mathrm{Spec}\,(A \otimes_{\mathfrak{o}} k')$$

is etale, the normality of $A \otimes_{\mathfrak{o}} k'$ implies the normality of $C'/\varpi'C'$ [SGA1, exp. I, Thm. 9.5]. If $\mathrm{Spec}\,(C'/\varpi'C')$ was reducible, then it would be disconnected [M89, Ex. 9.11]; this is impossible by what was said earlier.

Now let us prove that $\mathrm{Spec}\,(C/\varpi C)$ is irreducible. As the extension $C \subseteq C'$ is finite it follows that the correponding morphism of spectra is surjective [M89, Thm. 9.3, p. 66]. As a consequence

$$\mathrm{Spec}\,(C'/\varpi C') \longrightarrow \mathrm{Spec}\,(C/\varpi C)$$

is also surjective. Because the topological spaces $\mathrm{Spec}\,(C'/\varpi C')$ and $\mathrm{Spec}\,(C'/\varpi'C')$ coincide it follows that $\mathrm{Spec}\,(C/\varpi C)$ is irreducible.

(c) Let C be the normalization of B; it is the integral closure of B in $B \otimes K$. In particular $C \otimes K = B \otimes K$. Assume that $A \subseteq C$ is *not* etale. The argument will be the same as that in item (a), but we will subsitute the assumption that the field extension is Galois. By the Zariski-Nagata Purity Theorem, there exists a prime $\mathfrak{q} \in C$ of height one such that the extension of DVRs $A_{\mathfrak{q} \cap A} \subseteq C_{\mathfrak{q}}$ is ramified. Since $\mathfrak{q} \supseteq \varpi C$ and $\mathrm{ht}(\mathfrak{q}) = 1$, $\mathfrak{q}$ is a minimal prime of ϖC. From item (b) it follows that $\mathfrak{q} = \sqrt{\varpi C}$. But $\varpi R \cap C$ is also a prime of height one containing ϖC so that we must have $\mathfrak{q} = \varpi R \cap C$. As in item (a), the DVR $C_{\mathfrak{q}}$ is squeezed between the formally unramified extension $A_{\varpi A} \subseteq R_{\varpi R}$ and hence $A_{\varpi A} \subseteq C_{\mathfrak{q}}$ has to be unramified. This is a contradiction. □

Proof of Thm. 5. Let $B = B_{\mathscr{M}}$. We want to show, using Lemma 6, that the normalization C of B is etale over A. So we take $I = \ker(\xi)$ and set out to verify the validity of hypothesis (i)–(iv) of Lemma 6. Hypothesis (i) is part of Proposition 3 and (iv) is Corollary 4. Since hypothesis (iii) is readily satisfied we only need to check (ii). Let $K' \supseteq K$ be a finite extension and let $\mathfrak{o}'$ be the ring of integers of K'. Let us denote $A \otimes \mathfrak{o}'$ (resp. $B \otimes \mathfrak{o}'$) by A' (resp. B'). Using Corollary 4 it is clear that these are all domains. If we choose K' such that

$$\#\Pi(\mathfrak{o}') = \#\Pi(K') = \mathrm{rank}_{\mathfrak{o}} \mathscr{O}(\Pi),$$

then the injection $\Pi(\mathfrak{o}') \hookrightarrow \mathrm{Aut}(B'/A')$ gives

$$\#\mathrm{Aut}(B'/A') = \mathrm{rank}_{A'}(B') = [Q(B') : Q(A')],$$

which implies that $Q(B')/Q(A')$ is Galois. Hence (ii) holds and it follows that $A \longrightarrow C$ is etale.

An etale A-algebra has a unique structure of stratified algebra (this is elaborated in [dS07p, 6.2.1], the reason is that etale morphisms preserve differential structures). So C possesses such a structure. By the very same reason, the equality $C \otimes K = B \otimes K$ of $A \otimes K$-algebras is also an equality of stratified $A \otimes K$-algebras. Since $\langle \mathscr{M} \otimes K \rangle_{\otimes} = \langle B \otimes K \rangle_{\otimes}$, by choosing $C = \mathscr{N}$ we obtain (2).

We now prove (3). The group scheme $\Pi \otimes K$ is constant if and only if $\#\Pi(K) = \mathrm{rank}_{\mathfrak{o}}(\Pi) = \mathrm{rank}_{A \otimes K}(C \otimes K)$. Since $\mathrm{Spec}\,(C \otimes K)$ has a K-point, (3) is a consequence of the

Claim: The etale extension $A \longrightarrow C$ is Galois.
Proof: This follows easily from Grothendieck's "remarkable equivalence" since we already know that for some finite DVR $\mathfrak{o}' \supseteq \mathfrak{o}$, the etale covering $A \otimes_{\mathfrak{o}} \mathfrak{o}' \longrightarrow C \otimes_{\mathfrak{o}} \mathfrak{o}'$ is Galois. Indeed, if $\mathrm{Et}(?)$ denotes the category of finite etale extensions of an algebra ? and k' denotes the residue field of $\mathfrak{o}'$, then there exists a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Et}(A) & \xrightarrow{\;\otimes \mathfrak{o}'\;} & \mathrm{Et}(A \otimes \mathfrak{o}') \\
{\scriptstyle \otimes k}\big\downarrow & & \big\downarrow{\scriptstyle \otimes k'} \\
\mathrm{Et}(A \otimes k) & \xrightarrow[\;\otimes_k k'\;]{} & \mathrm{Et}(A \otimes_{\mathfrak{o}} k'),
\end{array}
$$

where both vertical arrows are equivalences (EGA IV$_4$, 18.1, p. 109). Since $k \longrightarrow k'$ is an isomorphism, the lower horizontal arrow is also an equivalence and the claim follows.

The claim also proves that

> The differential Galois group $\Pi(\mathscr{N})$ of $\mathscr{N} = C$ is etale and equal to $\mathrm{Gal}(Q(C)/Q(A))$.

Perhaps the most convenient way of seeing this is as follows. The differential Galois group G of $\mathscr{N} \otimes k$ is etale and equal to $\mathrm{Gal}(Q(C)/Q(A))$, and there exists a closed embedding $\iota : G \hookrightarrow \Pi(\mathscr{N}) \otimes k$ [dS07p, Thm. 27]. Since "$\otimes K$ commutes with $\Pi(?, \xi)$" [dS07p, Cor. 26], we have $\mathrm{rank}_k \Pi(\mathscr{N}) \otimes k = \mathrm{rank}_K \Pi(\mathscr{M}) \otimes K = \#\mathrm{Gal}(Q(C)/Q(A))$. Hence the closed embedding ι is an isomorphism. $\qquad\square$

The next step is to verify that the stratified A-modules with finite differential Galois group arise from representations of the etale fundamental group of $\mathrm{Spec}\,(A \otimes k)$. We keep the assumptions and notations from Theorem 5 and its proof, e.g. $B_{\mathscr{M}} = B$ and C is the normalization of B. We also remind the reader that the integral differential Galois group of $C = \mathscr{N}$ is the constant group scheme $G = \mathrm{Gal}(Q(B)/Q(A))$ over $\mathfrak{o}$.

Let V be a finite and free $\mathfrak{o}$-module which affords a representation of G. By taking G-invariants on $C \otimes_{\mathfrak{o}} V$, we obtain an object

$$L_C(V) \in \mathbf{str}^{\#}(A/\mathfrak{o})$$

and this construction defines a monoidal functor from the category of representations of G on finite and free $\mathfrak{o}$-modules to the category $\mathbf{str}^{\#}$. (More details of this standard construction are in [dS07p, 6.2.2].) From [dS07p, Prp. 34] we can easily obtain that the essential image

$$\mathrm{im}(L_C)$$

of L_C is the full sub-category of $\mathbf{str}$ whose objects $\mathscr{E}$ satisfy the following properties (an analogous category–not the same though–already appears, under the symbol $\mathscr{C}$, in diagram (4)):

(*) $\mathscr{E}$ is $\mathfrak{o}$-flat.
(**) There exist a surjection $\alpha : \mathscr{E}' \longrightarrow \mathscr{E}$ and a monomorphism $\beta : \mathscr{E}' \longrightarrow \mathscr{E}''$ in $\mathbf{str}$, where (a) the cokernel of β is $\mathfrak{o}$-flat and (b) $\mathscr{E}''$ is of the form
$$\bigoplus_{i=1}^{t} C^{\otimes a_i} \otimes (C^{\vee})^{\otimes b_i} \qquad (a_i, b_i \in \mathbb{N}).$$

(The upper-script "$\vee$" means we are taking the dual in $\mathbf{str}^{\#}$).

Mildly abusing the terminology of [dS07p] we will say that the objects of $\mathbf{str}$ having the previous properties are *special sub-quotients of C*. Since an A-module endowed with an $\mathfrak{o}$-linear stratification is A-flat if and only if it is $\mathfrak{o}$-flat, it is not hard to show that the relation "X is a special subquotient of Y " is transitive.

Corollary 7. *There exists a representation*
$$\rho : G \longrightarrow \mathrm{GL}(V),$$

with V finite and free over $\mathfrak{o}$, such that the stratified module $\mathscr{M}$ is isomorphic to $L_C(V)$.

Proof. We will first show that $B = B_{\mathscr{M}} \in \mathrm{im}(L_C)$. To obtain this, it is sufficient check property (**). We claim that there is an isomorphism

$$C \otimes_A B \cong C^{\oplus r}$$

in **str**. This will prove that B satisfies (**) because $A \longrightarrow C$ is faithfully flat and hence the A-module C/A is $\mathfrak{o}$-flat. By construction of $B = B_{\mathscr{M}}$, there exists an isomorphism in **str**

$$\lambda : B \otimes_A B \longrightarrow B^{\oplus r}, \qquad b_1 \otimes b_2 \mapsto b_1 \mu(b_2),$$

which arrises from the isomorphism (3) in $\mathrm{Rep}_{\mathfrak{o}}(\Pi)$. As the natural inclusion $\iota : B \subseteq C$ is an arrow of **str**, it is obvious that

$$C \otimes_B \alpha : C \otimes_B (B \otimes_A B) \longrightarrow C^{\oplus r}$$

(the B-algebra structure on $B \otimes_A B$ is via the first factor) is an isomorphism of **str**. Thus we have proved that $B \in \mathrm{im}(L_C)$. For any representation $\theta : \Pi \longrightarrow \mathrm{GL}(W)$ ($W \cong \mathfrak{o}^m$) we have an isomorphism in $\mathrm{Rep}_{\mathfrak{o}}(\Pi)$:

$$W \otimes (\mathscr{O}(\Pi), \rho_l) \xrightarrow{\cong} (R, \rho_l)^{\oplus m},$$

which takes $f : \Pi \longrightarrow W \in W \otimes (\mathscr{O}(\Pi), \rho_l)$ to $x \mapsto \theta(x^{-1})f(x) \in (R, \rho_l)^{\oplus m}$. Applying S we obtain an isomorphism in **str** $\mathscr{M} \otimes B \cong B^{\oplus r}$. It follows that $\mathscr{M}$ is a special subquotient of B and we are done by a simple transitivity property of special subquotients. $\qquad\square$

Example 8. Let $p = 3$ and let $\mathfrak{o} = W(\overline{\mathbb{F}}_p)[\varpi]$ where $\varpi^3 + 3\varpi = 0$. Take $A = \mathfrak{o}\langle x \rangle$ and $B = A[Y]/(Y^3 - e)$ where

$$e(x) = \exp(3\varpi x) = \sum_{j=0}^{\infty} \frac{(3\varpi)^j}{j!} x^j \in \mathfrak{o}\langle x \rangle.$$

The equation $Y^3 - e$ has a solution in $\mathfrak{o}[[x]]$:

$$y(x) = \exp(\varpi x) = \sum_{j=0}^{\infty} \frac{\varpi^j}{j!} x^j,$$

so that we get an injective homomorphism $\sigma : B \longrightarrow \mathfrak{o}[[x]]$. Let

$$z := \frac{y - 1}{\varpi} \in \mathfrak{o}[[x]].$$

Using the equation $y^3 - e = 0$ we get the equation

$$F(z) = z^3 - \varpi z^2 - z + \frac{e - 1}{3\varpi} = 0,$$

so that we have inclusions of rings $A \subseteq B \subseteq C = A[Z]/F(Z)$. Note that the A-algebra C is etale because the special fibre is the coordinate ring of an Artin-Schreier covering of the affine line $\mathrm{Spec}\,(k[x])$. In particular C is the normalization of B.

References

[BO78] P. Berthelot and A. Ogus, Notes on crystalline cohomology, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[BL1] S. Bosch and W. Lütkebohmert. Formal and rigid geometry. Math. Ann. 295 (1993), 291–317.

[dS07] J. P. P. dos Santos. Fundamental groups for stratified sheaves. Journal of Algebra, Volume 317, Issue 2, 691–713 (2007).

[dS07p] J. P. P. dos Santos. The behaviour of the monodromy on the generic and special fibres. Available at:
http://people.math.jussieu.fr/~dos-santos. (Accepted for publication in J. reine angew. Math.)

[SGA1] A. Grothendieck. Revêtements étales et groupe fondamental. Lecture Notes in Mathematics 224. Springer-Verlag, Berlin-New York, 1971.

[J87] J. C. Jantzen, Representations of algebraic groups, Pure and Applied Mathematics, 131. Academic Press, Inc., Boston, MA, 1987.

[Ka70] N. M. Katz, Nilpotent connections and the monodromy theorem: applications of a result of Turrittin, Publ. Math. IHES, t. 39 (1970) 175–232.

[M80] H. Matsumura. Commutative Algebra. Second edition. Mathematics Lecture Note Series, 56. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980

[M89] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1989.

[MvdP03] B. H. Matzat and M. van der Put, Iterative differential equations and the Abhyankar conjecture. J. reine angew. Math. 557 (2003), 1–52.

[S99] M. Spivakovsky, A new proof of D. Popescu's Theorem on smoothing of ring homomorphisms. Jour. of the American Math. Soc. Volume 12, Number 2, 1999, pp. 381–444.

[V76] P. Valabrega. A few theorems on completion of excellent rings. Nagoya Math. Jour. 61 (1976), 127–133.