A survey on algebraic dilatations

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Abstract. In this text, we wish to provide the reader with a short guide to recent works on the theory of dilatations in Commutative Algebra and Algebraic Geometry. These works fall naturally into two categories: one emphasises *foundational and theoretical* aspects and the other *applications* to existing theories.

Contents

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Introduction

What is the concept of algebraic dilatations about ?

Dilatation of rings is a basic construction of commutative algebra, like localization or tensor product. It can be globalized so that it also make sense on schemes or algebraic spaces. In fact dilatations generalize localizations.

Let A be a ring and let S be a multiplicative subset of A. Recall that the localization $S^{-1}A$ is an A-algebra such that for any A-algebra $A \to B$ such that the image of s is invertible for any $s \in S$, then $A \to B$ factors through $A \to S^{-1}A$. Intuitively, $S^{-1}A$ is the A-algebra obtained from A adding all fractions $\frac{a}{s}$ with $a \in A$ and $s \in S$. Formally, $S^{-1}A$ is made of classes of fractions $\frac{a}{s}$ where $a \in A$ and $s \in S$ (two representative $\frac{a}{s}$ and $\frac{b}{t}$ are identified if $atr = bsr$ for some $r \in S$), addition and multiplication are given by usual formulas. Now let us give for any element $s \in S$ an ideal M_s of A containing s. The dilatation of A relatively to the data $S, \{M_s\}_{s \in S}$ is an A-algebra A' obtained intuitively by adding to A only the fractions $\frac{m}{s}$ with $s \in S$ and $m \in M_s$. The dilatation A' satisfies that for any $s \in S$, we have $sA' = M_s A'$ (intuitively any $m \in M_s$ belongs to sA', i.e. becomes a multiple of s, so that we have an element $\frac{m}{s}$ such that $m = s\frac{m}{s}$). As a consequence of the construction, the elements $s \in S$ become non-zero-divisor in A' so that $\frac{m}{s}$ is well-defined (i.e. unique). It turns out that it is convenient, with dilatations of schemes in mind, to make a bit more flexible the above framework, namely to remove the conditions that S is multiplicative and that $s \in M_s$, so we use the following definition.

Definition. Let A be a ring. Let I be an index set. A multi-center in A indexed by I is a set of pairs $\{[M_i, a_i]\}_{i \in I}$ where for each i, M_i is an ideal of A and a_i is an element of A.

To each multi-center $\{[M_i, a_i]\}_{i \in I}$, one has the dilatation $A[\{\frac{M_i}{a_i}\}_{i \in I}]$, it is an A-algebra. We will define and study in details dilatations of rings in Section 1, in particular we will state formally the universal property they enjoy. We will also see that $A[\{\frac{M_i}{a_i}\}_{i \in I}]$ is generated, as A-algebra, by $\{\frac{M_i}{a_i}\}_{i \in I}$. We will also see that if $M_i = A$ for all i, then $A[\{\frac{M_i}{a_i}\}_{i \in I}] = S^{-1}A$ where S is the multiplicative subset generated by $\{a_i\}_{i \in I}$. Reciprocally, we will see that any sub-A-algebra of a localization $S^{-1}A$ for a certain S is isomorphic to a dilatation of A.

Dilatations of schemes and algebraic spaces are obtained from dilatations of rings via glueing. We introduce the following definition.

Definition. Let X be a scheme. Let I be an index set. A multi-center in A indexed by I is a set of pairs $\{[Y_i, D_i]\}_{i \in I}$ such that Y_i and D_i are closed subschemes for each i and such that locally, all D_i are principal for $i \in I$.

Associated to each multi-center, one has the dilatation $\mathrm{Bl}\{{}^{D_i}_{Y_i}\}_{i \in I} X$, it is a scheme endowed with a canonical affine morphism $f : \mathrm{Bl}\{{}^{D_i}_{Y_i}\}_{i \in I} X \to X$. It satisfies, in a universal way, that $f^{-1}(D_i)$ is a cartier divisor (i.e. is locally defined by a non-zero-divisor) and that $f^{-1}(D_i) = f^{-1}(Y_i)$ for all $i \in I$. If $\#I = 1$, we use the terminology mono-centered dilatation. We will study several facets of this construction and show that it enjoys many wonderfull properties in Sections 2 and 3.

Where does this construction come from ?

As we saw in the previous section, dilatations are a basic construction which can be easily encountered in specific situations. As a consequence it was used for a very long time. As the reader may well know, the theory of dilatations has deep and distinguished roots, even though not formulated in the language which we use. Right from the start, we warn the reader that we do not mean to, and probably could not, present a comprehensive historical account. As soon as Cremona and Bertini started using quadratic transformations (or blowups) in the framework of algebraic geometry over fields, "substitutions" of the form $x' = x$ and $y' = y/x$ started being made by algebraic geometers, see for example equation (8) in [No1884, Section 11] and Noether's acknowledgement, at the start of [No1884, Section 12], that these manipulations come from Cremona's point of view. Examples of dilatations appear frequently in some works of Zariski and Abhyankar, cf. [AZ55, Definition, p. 86] and [Za43, p499 proof of Th.4, case (b)]. Other forerunner examples of dilatations play a central role in several independent and unrelated works later, cf. [Da67], [Ner64, Section 25] and [Ar69, Section 4]. As far as we know, the terminology dilatations emerged in [BLR90, §3.2], where a section is devoted to study dilatations of schemes over discrete valuation rings systematically. In the context of schemes over a discrete valuation ring, we draw the reader's attention to [Ana73], [WW80] and [PY06]. The paper [KZ99] studies dilatations (under the name affine modifications) systematically in the framework of algebraic geometry over fields. Over two-dimensional base schemes dilatations also appear in [PZ13, p. 175]. Set aside localizations, mono-centered dilatations have been the main focus of mathematicians in the past. However, in the context of group schemes over discrete valuation rings, examples of multi-centered dilatations of rings and schemes that are not localizations or mono-centered dilatations appeared and were used in [SGA3, Exp. VIB Ex. 13.3], [PY06] and [DHdS18]. In recent times, the authors of [Du05], [MRR20] and [Ma23d] have set out to accommodate all these constructions in a larger and unified frame, namely for arbitrary schemes and algebraic spaces and arbitrary multi-center. The paper [MRR20] introduces dilatations of arbitrary schemes in the mono-centered case and provides a systematic treatment of mono-centered dilatations of general schemes. An equivalent definition of mono-centered dilatations of general schemes, under the name affine modifications, was introduced earlier in [Du05, Définition 2.9] under few assumptions. The paper [Ma23d] introduces and studies dilatations of arbitrary rings, schemes and algebraic spaces for arbitrary multi-centers. Allowing multi-centers also leads naturally to the formulation of combinatorial isomorphisms on dilatations and gives birth to refined universal properties. Nevertheless, the mono-centered case remains a fundamental case that is frequently the 'atom' for some aspects of the theory. The first part (Sections 1-2-3) of this survey is devoted to theoretical and formal results on dilatations of rings, schemes and group schemes following [Du05], [MRR20] and [Ma23d]. Sections 4-5-6-7 of the second part will deal with several concrete situations were specific kind of dilatations play a role, also providing complementary inceptions on this construction. To finish, beyond rings and schemes, the concept of dilatations makes sense for other structures and geometric settings. Let us indicate some constructions already available. Some dilatation constructions in the framework of complex analytic spaces were introduced in [Ka94], these are used and discussed in Section 7. Dilatations also make sense for general algebraic spaces [Ma23d]. Similarly, for many other structures than rings, dilatations also make sense (e.g. categories, monoids and semirings) as noticed in [Ma23c]. It is possible that dilatations in other settings will be explored and find a significant role since, at the end, these are a basic mathematical concept.

Terminology

Recall that dilatations have distinguished roots, as a consequence, several other terminologies are used to call certains dilatations in literature. For examples the constructions named *affine* *blowups*, *affine modifications*, *automatic blowups*, *formal blowups*, *Kaliman modifications*, *localizations* and *Néron blowups* are examples of (eventually multi-centered) dilatations.

Some simple examples

We provided an intuition on dilatations of rings before. Let us now provide some simple examples of dilatations of schemes. If S is a scheme, we denote by e_S the trivial group scheme over S, as scheme it is isomorphic to S. If G is a separated group scheme over S, we denote by e_G the trivial closed group scheme, e_G is isomorphic to e_S as group schemes over S.

(i) We consider, once given a prime number p, the multiplicative group scheme

$$G = \mathbb{G}_{m,\mathbb{Z}_p}$$

over the ring $\mathbb{Z}_p$; its Hopf algebra is $A = \mathbb{Z}_p[x, x^{-1}]$ while the morphism $\Delta : A \to A \otimes_{\mathbb{Z}_p} A$ induced from multiplication $G \times_{\mathbb{Z}_p} G \to G$ is defined by $\Delta(x) = x \otimes x$. Now, consider the couple e_G and $G \times_{\mathbb{Z}_p} \mathbb{Z}_p/\mathfrak{p}^r$ of closed subschemes of G for any $r > 0$ ($\mathfrak{p}^r$ denotes $p^r\mathbb{Z}_p$). These are cut out, respectively, by the ideals $I = (x - 1)$ and (p^r) of A.

(a) For any $r > 0$, the dilatation A' of A centered at $[e_G, G \times_{\mathbb{Z}_p} \mathbb{Z}_p/\mathfrak{p}^r]$, or at $[I, (p^r)]$, is the sub-A-algebra of $A[1/p]$ generated by all the elements $p^{-r}f$, where $f \in I$. The dilatation $G' := \operatorname{Spec} A'$ is a group scheme of finite type over $\mathbb{Z}_p$. The base change $G' \times_{\mathbb{Z}_p} \mathbb{Z}_p/\mathfrak{p}^r$ is isomorphic to the *additive* group $\mathbb{G}_a$ over $\mathbb{Z}_p/\mathfrak{p}^r$, while $G' \times_{\mathbb{Z}_p} \mathbb{Q}_p$ is the *multiplicative* group $\mathbb{G}_m$ over $\mathbb{Q}_p$. Furthermore, on the level of points, $G'(\mathbb{Z}_p) = 1 + \mathfrak{p}^r$ is a *congruence subgroup.*

(b) The dilatation A' of A centered at $\{[e_G, G \times_{\mathbb{Z}_p} \mathbb{Z}_p/\mathfrak{p}^r]\}_{\{r>0\}}$, or at $\{[I, (p^r)]\}_{\{r>0\}}$, is the sub-A-algebra of $A[1/p]$ generated by all the elements $p^{-r}f$, where $f \in I$ and $r > 0$. The dilatation $G' := \operatorname{Spec} A'$ is a group scheme over $\mathbb{Z}_p$, it is not of finite type. The base change $G' \times_{\mathbb{Z}_p} \mathbb{Z}_p/\mathfrak{p}^r$ is isomorphic to the *trivial* group scheme over $\mathbb{Z}_p/\mathfrak{p}^r$, while $G' \times_{\mathbb{Z}_p} \mathbb{Q}_p$ is the *multiplicative* group $\mathbb{G}_m$ over $\mathbb{Q}_p$. Furthermore, on the level of points, $G'(\mathbb{Z}_p) = \{1\}$.

(ii) Let G be GL_3 over $\mathbb{Z}_p$ and let H be $GL_2 \times e_{\mathbb{Z}_p}$ diagonally inside G and let $e_G \cong (e_{\mathbb{Z}_p})^3$ be the trivial closed subgroup of G. For any $r > 0$, let $G \times_{\mathbb{Z}_p} \mathbb{Z}_p/\mathfrak{p}^r$ be the base change of G to $\mathbb{Z}/p^r\mathbb{Z}$. The dilatation $G' = \mathrm{Bl}\{{}^{G\times_{\mathbb{Z}_p}\mathbb{Z}_p/\mathfrak{p}^5}_{H}, {}^{G\otimes\mathbb{Z}_p/\mathfrak{p}^2}_{e_G}\}G$ is a group scheme over $\mathbb{Z}_p$. On the level of points, we have

$$\begin{aligned} GL_3(\mathbb{Z}_p) \supset G'(\mathbb{Z}_p) &= \begin{pmatrix} 1+\mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^5 \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 & \mathfrak{p}^5 \\ \mathfrak{p}^5 & \mathfrak{p}^5 & 1+\mathfrak{p}^5 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} 1+a & b & e \\ c & 1+d & f \\ g & h & 1+k \end{pmatrix} \middle| a,b,c,d \in \mathfrak{p}^2 \quad e,f,g,h,k \in \mathfrak{p}^5 \right\}. \end{aligned}$$

(iii) Let $X = \mathbb{A}^1 = \operatorname{Spec}(\mathbb{Z}[T])$ be the affine line over $\mathbb{Z}$, let $0 \subset \mathbb{A}^1$ be the closed subscheme defined by the ideal (T) and let $\emptyset \subset \mathbb{A}^1$ be the closed subscheme defined by the ideal $\mathbb{Z}[T]$. Then the dilatation $\mathrm{Bl}_{\emptyset}^{D}X$ identifies with the open subscheme $\mathbb{G}_m$ of $\mathbb{A}^1$, indeed $\mathbb{Z}[T][\frac{\mathbb{Z}[T]}{T}] \cong \mathbb{Z}[T, T^{-1}]$. This is an example of localization.

(iv) Let $X = \mathbb{A}^2 = \operatorname{Spec}(\mathbb{Z}[A, B])$ be the affine plane over $\mathbb{Z}$, let $D \subset \mathbb{A}^2$ be the line defined by the ideal (A) and let $0 \subset \mathbb{A}^2$ be the origin defined by the ideal (A, B). Then $\mathrm{Bl}_0^D X$ identifies with $\mathrm{Spec}(\mathbb{Z}[A,B,C]/(AC-B))$. Indeed, one has an isomorphism (e.g. by Proposition 1.4)

$$\mathbb{Z}[A,B][\frac{(A,B)}{A}] \cong \mathbb{Z}[A,B][\frac{(B)}{A}] \cong \mathbb{Z}[A,B,C]/AC = B.$$

The morphism $\mathrm{Bl}_0^D X \to X$ is given by $\mathbb{Z}[A,B] \to \mathbb{Z}[A,B,C]/(AC-B), A,B \mapsto A,B$. At the level of points $\big(\mathrm{Bl}_0^D X\big)(\mathbb{Z})$ is made of pairs $(a,b) \in \mathbb{Z}^2$ such that b is a multiple of a.

(v) More advanced examples of dilatations in contextual situations are available in the second part of this survey.

What is the aim of this survey ?

Recall that we wish to provide the reader with a short guide to recent works on the theory of dilatations. We do not mean to present a comprehensive account. We rather concentrate on the contributions that ourselves were responsible for [Du05, DF18, DHdS18, Ma19t, HdS21, MRR20, ADØ21, Ma23d] and those which were our starting points [Ar69, Ana73, And01, WW80, BLR90, KZ99, Yu15, PY06, PZ13, HKØ16].

Part I is devoted to an exposition of general definitions and results around the concept of algebraic dilatations introduced and proved in [MRR20] and [Ma23d]. Section 1 discusses dilatations of commutative unital rings. Section 2 summarizes general results on dilatations of schemes. Section 3 focuses on dilatations of group schemes.

Part II provides an overview on some applications of dilatations in various mathematical contexts. In Section 4, we explain some recent applications of dilatations to the theory of affine group schemes and their representation categories in the case where objects are defined over a discrete valuation ring R. These were developed mainly in order to advance the study of Tannakian categories defined over R and appearing in geometry, such as the case of group schemes associated to $\mathcal{D}$-modules [And01, dS09, DH18, DHdS18, HdS21] and principal bundles with finite structure groups [HdS23]. After a brief introduction to Tannakian categories over R (Section 4.1), we go on to explain how to filter these categories by smaller ones and produce in this way the "Galois-Tannaka" group schemes 4.2. We show why Néron blowups are a fundamental tool for studying these groups and explain what has been done so far in order to exhaust Galois-Tannaka groups by means of Néron blowups, both in the case of mono-centered and multi-centered Néron blowups (cf. Section 4.2 and Section 4.3). In Section 5, *congruent isomorphisms*, formulated and stated using the language of dilatations, are discussed in relation with *the Moy-Prasad isomorphism*, *Bruhat-Tits buildings* and *representations of p-adic groups*. Section 6 shows that many level structures on *moduli stacks of G-bundles* are encoded in torsors under dilatations and that this can be used to obtain integral models of *shtukas*. Section 7 discusses dilatations in *affine geometry* and related progress in $\mathbb{A}^1$-*homotopy theory*.

All results stated in this paper are proved in indicated references. This survey is an expository text and does not contain any new mathematical result. What is perhaps new is to summarize some aspects of several independent works involving dilatations in a single text. We hope this could be a source of inspiration for future works.

Some conventions and notations

(1) All rings are unital and commutative, unless otherwise mentioned.

(2) Let $(M,+)$ be a monoid. A submonoid F is a face of M if whenever $x+y \in F$, then both x and y belong to F.

(3) If R is a discrete valuation ring with field of fractions K, then, for each R-scheme, we call $X \otimes_R K$ the generic fibre of X.

(4) If A is a ring, then A-**mod** is the category of finitely presented A-modules.

(5) If G is a group scheme over a noetherian ring R, or an abstract group, we denote by $\mathrm{Rep}_R(G)$ the category of all R–modules of finite type affording a representation of G as explained in [Ja03].

Part I. Algebraic dilatations

In this part, we introduce formally dilatations of rings and schemes. Locally, dilatations of schemes will be studied through dilatations of rings.

1. Dilatations of rings

We summarily present basic results on dilatations of rings following the more general path given in [Ma23d]. Let A be a ring. A *center in* A is a pair $[M, a]$ consisting of an ideal $M \subset A$ and an element $a \in A$. A *multi-center* is a family of center indexed by some set. Let I be an index set and let $\{[M_i, a_i]\}_{i \in I}$ be a multi-center. For $i \in I$, we put $L_i = M_i + (a_i)$, an ideal of A. Let $\mathbb{N}_I$ be the monoid $\bigoplus_{i \in I} \mathbb{N}$. If $\nu = (\nu_1, \ldots, \nu_i, \ldots) \in \mathbb{N}_I$ we put $L^\nu = L_1^{\nu_1} \cdots L_i^{\nu_i} \cdots$ (product of ideals of A) and $a^\nu = a_1^{\nu_1} \cdots a_i^{\nu_i} \cdots$ (product of elements of A). We also put $a^{\mathbb{N}_I} = \{a^\nu | \nu \in \mathbb{N}_I\}$.

Definition and Proposition 1.1 [Ma23d]. *The dilatation of A with multi-center $\{[M_i, a_i]\}_{i \in I}$ is the unital commutative ring $A[\{\frac{M_i}{a_i}\}_{i \in I}]$ defined as follows:*

- *The underlying set of $A[\{\frac{M_i}{a_i}\}_{i \in I}]$ is the set of equivalence classes of symbols $\frac{m}{a^\nu}$ where $\nu \in \mathbb{N}_I$ and $m \in L^\nu$ under the equivalence relation*

$$\frac{m}{a^\nu} \equiv \frac{p}{a^\lambda} \Leftrightarrow \exists \beta \in \mathbb{N}_I \text{ such that } m a^{\beta+\lambda} = p a^{\beta+\nu} \text{ in } A.$$

From now on, we abuse notation and denote a class by any of its representative $\frac{m}{a^\nu}$ if no confusion is likely.

- *The addition law is given by $\frac{m}{a^\nu} + \frac{p}{a^\beta} = \frac{m a^\beta + p a^\nu}{a^{\beta+\nu}}$.*
- *The multiplication law is given by $\frac{m}{a^\nu} \times \frac{p}{a^\beta} = \frac{mp}{a^{\nu+\beta}}$.*
- *The additive neutral element is $\frac{0}{1}$ and the multiplicative neutral element is $\frac{1}{1}$.*

From now on, we also use the notation $A[\frac{M}{a}]$ to denote $A[\{\frac{M_i}{a_i}\}_{i \in I}]$. We have a canonical morphism of rings $A \to A[\frac{M}{a}]$ given by $a \mapsto \frac{a}{1}$.

The element $\frac{a}{1}$ of $A[\frac{M}{a}]$ will be denoted by a if no confusion is likely.

Fact 1.2 [Ma23d]. *(i) Let $\{N_i\}_{i \in I}$ be ideals in A such that $N_i + (a_i) = L_i$ for all $i \in I$. Then we have identifications of A-algebras $A[\{\frac{M_i}{a_i}\}_{i \in I}] = A[\{\frac{N_i}{a_i}\}_{i \in I}] = A[\{\frac{L_i}{a_i}\}_{i \in I}]$.*

(ii) Dilatations of rings generalize entirely localizations of rings. Indeed, let A be a ring and let S be a multiplicative subset of A. Then $S^{-1}A = A[\{\frac{A}{s}\}_{s \in S}]$.

(iii) Any sub-A-algebra of a localization $S^{-1}A$ for a subset $S \subset A$ can be obtained as a multi-centered dilatation.

(iv) *Note that we did not used substraction to define dilatations of rings. In fact Definition 1.1 makes sense for arbitrary unital commutative semirings, cf. [Ma23d, §2] or more generally for categories (e.g. monoids) cf. [Ma23c].*

This construction enjoys the following properties, cf. [Ma23d, §2]. If $\#I = 1$, most results appear in [StP, Tag 052P].

Proposition 1.3 [Ma23d]. *The following assertions hold.*

(i) *As A-algebra, $A[\frac{M}{a}]$ is generated by $\{\frac{L_i}{a_i}\}_{i\in I}$. Since $L_i = M_i + (a_i)$, this implies that $A[\frac{M}{a}]$ is generated by $\{\frac{M_i}{a_i}\}_{i\in I}$.*

(ii) *If A is a domain and $a_i \neq 0$ for all i, then $A[\frac{M}{a}]$ is a domain.*

(iii) *If A is reduced, then $A[\frac{M}{a}]$ is reduced.*

(iv) *The following assertions are equivalent.*
  (a) *There exists $\nu \in \mathbb{N}_I$ such that $a^\nu = 0$ in A.*
  (b) *The ring $A[\frac{M}{a}]$ equals to the zero ring.*

(v) *Let ν be in $\mathbb{N}_I$. The image of a^ν in $A[\frac{M}{a}]$ is a non-zero-divisor.*

(vi) *Let $f : A \to B$ be a morphism of rings. Let $\{[N_i, b_i]\}_{i\in I}$ be centers of B such that $f(M_i) \subset N_i$ and $f(a_i) = b_i$ for all $i \in I$. Then we have a canonical morphism of A-algebras*
$$\phi : A[\{\tfrac{M_i}{a_i}\}_{i\in I}] \to B[\{\tfrac{N_i}{b_i}\}_{i\in I}] \ .$$

(vii) *Let c be a non-zero-divisor element in A. Then $\frac{c}{1}$ is a non-zero-divisor in $A[\frac{M}{a}]$.*

(viii) *Let $K \subset I$ put $J = I \setminus K$. Then we have a canonical morphism of A-algebras*
$$\varphi : A[\{\tfrac{M_i}{a_i}\}_{i\in K}] \longrightarrow A[\{\tfrac{M_i}{a_i}\}_{i\in I}].$$
Moreover
  (a) *if $M_i \subset (a_i)$ for all $i \in J$, then φ is surjective, and*
  (b) *if a_i is a non-zero-divisor in A for all $i \in J$, then φ is injective.*

(ix) *Let $K \subset I$. Then we have a canonical isomorphism of $A[\{\frac{M_i}{a_i}\}_{i\in K}]$-algebras*
$$A[\{\tfrac{M_i}{a_i}\}_{i\in I}] = A[\{\tfrac{M_i}{a_i}\}_{i\in K}][\{\frac{A[\{\frac{M_i}{a_i}\}_{i\in I}]\frac{M_j}{1}}{\frac{a_j}{1}}\}_{j\in I\setminus K}],$$
where $A[\{\frac{M_i}{a_i}\}_{i\in I}]\frac{M_j}{1}$ is the ideal of $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ generated by $\frac{M_j}{1} \subset A[\{\frac{M_i}{a_i}\}_{i\in I}]$.

(x) *Assume that $a_i = a_j =: b$ for all $i, j \in I$, then*
$$A[\{\tfrac{M_i}{a_i}\}_{i\in I}] = A[\tfrac{\sum_{i\in I} M_i}{b}]$$

(xi) *Let $\nu \in \mathbb{N}_I$. We have $L^\nu A[\frac{M}{a}] = a^\nu A[\frac{M}{a}]$.*

(xii) *(Universal property) If $\chi : A \to B$ is a morphism of rings such that $\chi(a_i)$ is a non-zero-divisor and generates $\chi(L_i)B$ for all $i \in I$, then there exists a unique morphism χ' of A-algebras $A[\{\frac{M_i}{a_i}\}_{i\in I}] \to B$. The morphism χ' sends $\frac{l}{a^\nu}$ ($\nu \in \mathbb{N}_I, l \in L^\nu$) to the unique element $b \in B$ such that $\chi(a^\nu)b = \chi(l)$.*

(xiii) *Assume that $I = \{1, \dots, k\}$ is finite. Then we have a canonical identification of A-algebras*
$$A[\{\tfrac{M_i}{a_i}\}_{i\in I}] = A[\tfrac{\sum_{i\in I}(M_i \cdot \prod_{j\in I\setminus\{i\}} a_j)}{a_1\cdots a_k}].$$

(xiv) *Write $I = \mathrm{colim}_{J\subset I} J$ as a filtered colimit of sets. We have a canonical identification of A-algebras*
$$A[\{\tfrac{M_i}{a_i}\}_{i\in I}] = \mathrm{colim}_{J\subset I} A[\{\tfrac{M_j}{a_i}\}_{i\in J}].$$

(xv) *Let $f : A \to B$ be an A-algebra. Put $N_i = f(M_i)B$ and $b_i = f(a_i)$ for $i \in I$. Then $B[\{\frac{N_i}{b_i}\}_{i\in I}]$ is the quotient of $B \otimes_A A[\{\frac{M_i}{a_i}\}_{i\in I}]$ by the ideal T_b of elements annihilated by some element in $b^{\mathbb{N}_I} := \{b^\nu | \nu \in \mathbb{N}_I\}$. If moreover $f : A \to B$ is flat, then $T_b = 0$ and we have a canonical isomorphism*

$$B[\{\tfrac{N_i}{b_i}\}_{i\in I}] = B \otimes_A A[\{\tfrac{M_i}{a_i}\}_{i\in I}].$$

(xvi) *Let $f : R \to A$ be a morphism of rings and let $\{r_i\}_{i\in I} \subset R$. Let $R' = R[\{\frac{R}{r_i}\}_{i\in I}]$; this is a localization of R and hence $R \to R'$ is a flat morphism. Let $A' = A \otimes_R R'$, $M_i' = M_i \otimes_R R' \subset A'$. Then, if $a_i := f(r_i)$, the dilatation $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ is isomorphic to the A-subalgebra of $A' = A \otimes_R R'$ generated by $\{M_i \otimes r_i^{-1}\}_{i\in I}$ and A.*

We finish with an important description of dilatations in a particular case, cf. [Ma23d, Proposition 5.5] and [StP, Tag 0BIQ].

Proposition 1.4. *Let A be a ring. Let $a, g_1, \ldots, g_n$ be a H_1-regular sequence in A (cf. [StP, Tag 062E] for H_1-regularity). Let $d_1, \ldots, d_n$ be positive integers. The dilatation algebra identifies with a quotient of a polynomial algebra as follows*

$$A[\tfrac{(g_1)}{a^{d_1}}, \ldots, \tfrac{(g_n)}{a^{d_n}}] = A[x_1, \ldots, x_n]/(g_1 - a^{d_1}x_1, \ldots, g_n - a^{d_n}x_n).$$

2. Dilatations of schemes

This section is an introduction to dilatations of schemes, the main references are [MRR20] and [Ma23d]. Dilatations of schemes involve operations on closed subschemes that we recall at the beginning of this section. We suggest readers to be familiar with §2.1 before reading other subsections of Section 2. Note that [Ma23d] deals with general algebraic spaces, in fact most results of Section 2 extend to this setting.

2.1 Definitions

Let X be a scheme. Let $Clo(X)$ be the set of closed subschemes of X. Recall that $Clo(X)$ corresponds to quasi-coherent ideals of $\mathcal{O}_X$. Let $IQCoh(\mathcal{O}_X)$ denote the set of quasi-coherent ideals of $\mathcal{O}_X$. It is clear that $(IQCoh(\mathcal{O}_X), +, \times, 0, \mathcal{O}_X)$ is a semiring. So we obtain a semiring structure on $Clo(X)$, usually denoted by $(Clo(X), \cap, +, X, \emptyset)$. For clarity, we now recall directly operations on $Clo(X)$. Given two closed subschemes Y_1, Y_2 given by ideals $\mathcal{J}_1, \mathcal{J}_2$, their sum $Y_1 + Y_2$ is defined as the closed subscheme given by the ideal $\mathcal{J}_1\mathcal{J}_2$. Moreover, if $n \in \mathbb{N}$, we denote by nY_1 the n-th multiple of Y_1. The set of locally principal closed subschemes of X (cf. [StP, Tag 01WR]), denoted $Pri(X)$, forms a submonoid of $(Clo(X), +)$. Effective Cartier divisors of X [StP, Tag 01WR], denoted $Car(X)$, form a submonoid of $(Pri(X), +)$. Note that $Car(X)$ is a face of $Pri(X)$. We have an other monoid structure on $Clo(X)$ given by intersection, this law is denoted $\cap$. The operation $\cap$ corresponds to the sum of quasi-coherent sheaves of ideals . The set $Clo(X)$ endowed with $\cap, +$ is a semiring whose neutral element for $+$ is $\emptyset$ and whose neutral element for $\cap$ is X. Let $C \in Car(X)$, a non-zero-divisor (for $+$) in the semiring $Clo(X)$. Let $Y, Y' \in Clo(X)$. If $C + Y$ is a closed subscheme of $C + Y'$, then Y is a closed subscheme of Y'. Moreover if $C + Y = C + Y'$, then $Y = Y'$. Let $f : X' \to X$ be a morphism of schemes, then f induces a morphism of semirings $Clo(f) : Clo(X) \to Clo(X'), Y \mapsto Y \times_X X'$, moreover $Clo(f)$ restricted to $(Pri(X), +)$ factors through $(Pri(X'), +)$, this morphism of monoids is denoted $Pri(f)$. In general the image of the map $Pri(f)|_{Car(X)}$ is not included in $Car(X')$. Let $Y_1, Y_2 \in Clo(X)$, we write $Y_1 \subset Y_2$ if Y_1 is a closed subscheme of Y_2. We obtain a poset $(Clo(X), \subset)$.

Let $Y_1, Y_2, Y_3 \in Clo(X)$, if $Y_1 \subset Y_2$ and $Y_1 \subset Y_3$ then $Y_1 \subset Y_2 \cap Y_3$. Let $Y_1, Y_2 \in Clo(X)$, then $(Y_1 \cap Y_2) \subset Y_1$ and $Y_1 \subset (Y_1 + Y_2)$. Finally, if $Y = \{Y_e\}_{e\in E}$ is a subset of $Clo(X)$ and if $\nu \in \mathbb{N}^E$, we put $Y^\nu = \{\nu_e Y_e\}_{e\in E}$ and if moreover $\nu \in \mathbb{N}_E$, we put $\nu Y = \sum_{e\in E} \nu_e Y_e$.

Definition 2.1 [MRR20, §2.3] [Ma23d]. Let $D = \{D_i\}_{i\in I}$ be a subset of $Clo(X)$. Let $\mathrm{Sch}_X^{D\text{-reg}}$ be the full subcategory of schemes $f : T \to X$ over X such that $T \times_X D_i$ is an effective Cartier divisor of T for each i.

If $T' \to T$ is flat and $T \to X$ is an object in $\mathrm{Sch}_X^{D\text{-reg}}$, so is the composition $T' \to T \to X$. In particular, the category $\mathrm{Sch}_X^{D\text{-reg}}$ can be equipped with the fpqc/fppf/étale/Zariski Grothendieck topology so that the notion of sheaves is well-defined.

Fact 2.2 [Ma23d]. *Let $D = \{D_i\}_{i\in I}$ be a subset of $Clo(X)$.*

(i) Let $f : T \to X$ be an object in $\mathrm{Sch}_X^{D\text{-reg}}$. Then for any $\nu \in \mathbb{N}_I$, the scheme $T \times_X \nu D$ is an effective Cartier divisor of T, namely $\nu(T \times_X D)$.

(ii) Assume that $\#I$ is finite, then $\mathrm{Sch}_X^{D\text{-reg}}$ equals $\mathrm{Sch}_X^{\sum_{i\in I} D_i}$.

Definition 2.3 [Ma23d]. A multi-center in X is a set $\{[Y_i, D_i]\}_{i\in I}$ such that

(i) Y_i and D_i belong to $Clo(X)$,

(ii) there exists an affine open covering $\{U_\gamma \to X\}_{\gamma\in\Gamma}$ of X such that $D_i|_{U_\gamma}$ is principal for all $i \in I$ and $\gamma \in \Gamma$ (in particular D_i belongs to $Pri(X)$ for all i).

In other words a multi-center $\{[Y_i, D_i]\}_{i\in I}$ is a set of pairs of closed subschemes such that locally each D_i is principal.

Remark 2.4. Let $\{Y_i, D_i\}_{i\in I}$ such that $Y_i \in Clo(X)$ and $D_i \in Pri(X)$ for any $i \in I$. Assume that I is finite, then $\{[Y_i, D_i]\}_{i\in I}$ is a multi-center in X, i.e. the second condition in Definition 2.3 is satisfied.

We now fix a multi-center $\{[Y_i, D_i]\}_{i\in I}$ in X. Denote by $\mathcal{M}_i$, respectively $\mathcal{J}_i$, the quasi-coherent sheaf of ideals of $\mathcal{O}_X$ defining Y_i, respectively D_i. We put $Z_i = Y_i \cap D_i$ and $\mathcal{L}_i = \mathcal{M}_i + \mathcal{J}_i$ so that Z_i is defined by $\mathcal{L}_i$. We put $Y = \{Y_i\}_{i\in I}$, $D = \{D_i\}_{i\in I}$ and $Z = \{Z_i\}_{i\in I}$. We now introduce dilatations $\mathcal{O}_X$-algebras by glueing.

Definition and Proposition 2.5. *The dilatation of $\mathcal{O}_X$ with multi-center $\{[\mathcal{M}_i, \mathcal{J}_i]\}_{i\in I}$ is the quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{O}_X\Big[\Big\{\frac{\mathcal{M}_i}{\mathcal{J}_i}\Big\}_{i\in I}\Big]$ obtained by glueing as follows. The quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{O}_X\Big[\Big\{\frac{\mathcal{M}_i}{\mathcal{J}_i}\Big\}_{i\in I}\Big]$ is characterized by the fact that its restriction, on any open subscheme $U \subset X$ such that U is an affine scheme and each D_i is principal on U and generated by a_{iU}, is given by*

$$\Big(\mathcal{O}_X\Big[\Big\{\frac{\mathcal{M}_i}{\mathcal{J}_i}\Big\}_{i\in I}\Big]\Big)\Big|_U = \widetilde{\Gamma(U, \mathcal{O}_X)\Big[\Big\{\frac{\Gamma(U, \mathcal{M}_i)}{a_{iU}}\Big\}_{i\in I}\Big]}$$

where $\widetilde{}$ is the associated sheaf of algebras on U.

Definition 2.6 [Ma23d]. The *dilatation* of X with multi-center $\{[Y_i, D_i]\}_{i\in I}$ is the X-affine scheme

$$\mathrm{Bl}_Y^D X \overset{\mathrm{def}}{=} \mathrm{Spec}_X\Big(\mathcal{O}_X\Big[\Big\{\frac{\mathcal{M}_i}{\mathcal{J}_i}\Big\}_{i\in I}\Big]\Big).$$

The terminologies *affine blowups* and *affine modifications* are also used.

Remark 2.7. In the mono-centered case, this definition is the one of [MRR20]. If moreover D is a Cartier divisor, one has another equivalent definition (cf. Proposition 2.14) that goes back to [KZ99] and [Du05].

Remark 2.8. We always have $\mathrm{Bl}_Y^D X = \mathrm{Bl}_Z^D X$.

Notation 2.9. *We will also use the notation* $\mathrm{Bl}\left\{{}^{D_i}_{Y_i}\right\}_{i\in I} X$ *and* $\mathrm{Bl}_{\{Y_i\}_{i\in I}}^{\{D_i\}_{i\in I}} X$ *to denote* $\mathrm{Bl}_Y^D X$. *If* $I=\{i\}$ *is a singleton we also use the notation* $\mathrm{Bl}_{Y_i}^{D_i} X$. *If* $K \subset I$, *we sometimes use the notation* $\mathrm{Bl}_{\{Y_i\}_{i\in K},\{Y_i\}_{i\in I\setminus K}}^{\{D_i\}_{i\in K},\{D_i\}_{i\in I\setminus K}} X$. *If* $I=\{1,\ldots,k\}$, *we use the notation* $\mathrm{Bl}_{Y_1,\ldots,Y_k}^{D_1,\ldots,D_k} X$. *Etc.*

Definition 2.10. We say that a morphism $f : X' \to X$ is a dilatation morphism if f is equal to $\mathrm{Bl}\left\{{}^{D_i}_{Y_i}\right\}_{i\in I} X \to X$ for some multi-center $\{[Y_i, D_i]\}_{i\in I}$.

Fact 2.11. *[Du05, Ma23d] The dilatation morphism* $\mathrm{Bl}\left\{{}^{D_i}_{\emptyset}\right\}_{i\in I} X \to X$ *is an open immersion. In other words, if* Y_i *is the empty closed subscheme defined by the ideal* $\mathcal{O}_X$ *for all* $i \in I$, *then* $\mathrm{Bl}_Y^D X$ *identifies with an open subscheme of* X. *In this case, we say that* $\mathrm{Bl}\left\{{}^{D_i}_{\emptyset}\right\}_{i\in I} X \to X$ *is a localization.*

2.2 Exceptional divisors

We proceed with the notation from § 2.1.

Proposition 2.12 [MRR20, Ma23d]. *As closed subschemes of* $\mathrm{Bl}_Y^D X$, *one has, for all* $\nu \in \mathbb{N}_I$,

$$\mathrm{Bl}_Y^D X \times_X \nu Z = \mathrm{Bl}_Y^D X \times_X \nu D,$$

which is an effective Cartier divisor on $\mathrm{Bl}_Y^D X$.

2.3 Relation to affine projecting cone

We proceed with the notation from § 2.1 and assume that $\{D_i\}_{i\in I}$ belong to $Car(X)$. In this case, we can also realize $\mathrm{Bl}_Y^D X$ as a closed subscheme of the multi-centered affine projecting cone associated to X, Z and D.

Definition 2.13. The *affine projecting cone* $\mathcal{O}_X$*-algebra* with multi-center $\{[Z_i = V(\mathcal{L}_i), D_i = V(\mathcal{J}_i)]\}_{i\in I}$ is

$$\mathrm{C}_{\mathcal{L}}^{\mathcal{J}} \mathcal{O}_X \overset{\mathrm{def}}{=} \bigoplus_{\nu\in\mathbb{N}_I} \mathcal{L}^{\nu} \otimes \mathcal{J}^{-\nu}.$$

The *affine projecting cone* of X with multi-center $\{[Z_i, D_i]\}_{i\in I}$ is

$$\mathrm{C}_Z^D X \overset{\mathrm{def}}{=} \mathrm{Spec}\big(\mathrm{C}_{\mathcal{L}}^{\mathcal{J}} \mathcal{O}_X\big).$$

Proposition 2.14 [KZ99, Du05, MRR20, Ma23d]. *The dilatation* $\mathrm{Bl}_Z^D X$ *is the closed subscheme of the affine projecting cone* $\mathrm{C}_Z^D X$ *defined by the equations* $\{\varrho_i - 1\}_{i\in I}$, *where for all* $i \in I$, $\varrho_i \in \mathrm{C}_{\mathcal{L}}^{\mathcal{J}} \mathcal{O}_X$ *is the image of* $1 \in \mathcal{O}_X$ *under the map*

$$\mathcal{O}_X \cong \mathcal{J}_i \otimes \mathcal{J}_i^{-1} \subset \mathcal{L}_i \otimes \mathcal{J}_i^{-1} \subset \mathrm{C}_{\mathcal{L}}^{\mathcal{J}} \mathcal{O}_X.$$

2.4 Description of the exceptional divisor in the mono-centered case

We proceed with the notation from § 2.1 and assume $I = \{i\}$ is a singleton and we ommit the subscripts i in notation. We saw in Lemma 2.12 that the preimage of the center $\mathrm{Bl}_Z^D X \times_X Z = \mathrm{Bl}_Z^D X \times_X D$ is an effective Cartier divisor in $\mathrm{Bl}_Z^D X$. In order to describe it following [MRR20], as before we denote by $\mathcal{L}$ and $\mathcal{J}$ the sheaves of ideals of Z and D in $\mathcal{O}_X$. Also we let $\mathcal{C}_{Z/D} = \mathcal{L}/(\mathcal{L}^2 + \mathcal{J})$ and $\mathcal{N}_{Z/D} = \mathcal{C}_{Z/D}^\vee$ be the conormal and normal sheaves of Z in D.

Proposition 2.15 [MRR20, Proposition 2.9]. *Assume that X is a scheme. Assume that $D \subset X$ is an effective Cartier divisor, and $Z \subset D$ is a regular immersion. Write $\mathcal{J}_Z := \mathcal{J}|_Z$.*

(1) The exceptional divisor $\mathrm{Bl}_Z^D X \times_X Z \to Z$ is an affine bundle (i.e. a torsor under a vector bundle), Zariski locally over Z isomorphic to $\mathbb{V}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}) \to Z$.

(2) If $H^1(Z, \mathcal{N}_{Z/D} \otimes \mathcal{J}_Z) = 0$ (for example if Z is affine), then $\mathrm{Bl}_Z^D X \times_X Z \to Z$ is globally *isomorphic to $\mathbb{V}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}) \to Z$.*

(3) If Z is a transversal intersection in the sense that there is a cartesian square of closed subschemes whose vertical maps are regular immersions

$$\begin{tikzcd} W \arrow[r, hook] \arrow[dr, phantom, "\square"] & X \\ Z \arrow[u, hook] \arrow[r, hook] & D \arrow[u, hook] \end{tikzcd}$$

then $\mathrm{Bl}_Z^D X \times_X Z \to Z$ is globally and canonically *isomorphic to $\mathbb{V}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}) \to Z$.*

2.5 Universal property

We proceed with the notation from § 2.1. As $\mathrm{Bl}_Y^D X \to X$ defines an object in $\mathrm{Sch}_X^{D\text{-reg}}$ by Proposition 2.12, the contravariant functor

$$\mathrm{Sch}_X^{D\text{-reg}} \to \mathrm{Sets}, \quad (T \to X) \mapsto \mathrm{Hom}_{X\text{-Schemes}}\big(T, \mathrm{Bl}_Y^D X\big) \tag{2.1}$$

together with $\mathrm{id}_{\mathrm{Bl}_Y^D X}$ determines $\mathrm{Bl}_Y^D X \to X$ uniquely up to unique isomorphism. The next proposition gives the universal property of dilatations.

Proposition 2.16 [MRR20, Ma23d]. *The dilatation $\mathrm{Bl}_Y^D X \to X$ represents the contravariant functor $\mathrm{Sch}_X^{D\text{-reg}} \to \mathrm{Sets}$ given by*

$$(f\colon T \to X) \longmapsto \begin{cases} \{*\}, & \text{if } f|_{T \times_X D_i} \text{ factors through } Y_i \subset X \text{ for } i \in I; \\ \varnothing, & \text{else.} \end{cases} \tag{2.2}$$

Proposition 2.17 [Ma23d]. *Put $f : \mathrm{Bl}_Y^D X \to X$. Then the morphism of monoids $Clo(f)|_{Car(X)}$ factors through $Car(\mathrm{Bl}_Y^D X)$. In other words, any effective Cartier divisor $C \subset X$ is defined for f, i.e. the fiber product $C \times_X \mathrm{Bl}_Y^D X \subset \mathrm{Bl}_Y^D X$ is an effective cartier divisor.*

2.6 Dilatations or affine blowups

We proceed with the notation from § 2.1 and assume $I = \{i\}$ is a singleton and we ommit the subscripts i in notation.

Proposition 2.18 [MRR20]. *The dilatation $\mathrm{Bl}_Z^D X$ is the open subscheme of the blowup $\mathrm{Bl}_Z X =$ $\mathrm{Proj}(\mathrm{Bl}_{\mathcal{I}} \mathcal{O}_X)$ defined by the complement of $V_+(\overline{\mathcal{J}})$ where $\overline{\mathcal{J}}$ is the sheaf of ideals generated by $\mathcal{J} \subset \mathcal{I}$, where $\mathcal{I}$ is the degree 1 part of $\mathrm{Bl}_{\mathcal{I}} \mathcal{O}_X$.*

For this reason dilatations are also called affine blowups. A similar description holds in the multi-centered case, cf [Ma23d].

2.7 Combinatorial and arithmetic relations

We proceed with the notation from § 2.1.

Proposition 2.19 [Ma23d]. *Let J be a subset of I and put $K = I \setminus J$. Then*

$$\mathrm{Bl}\left\{\begin{smallmatrix} D_i \\ Y_i \end{smallmatrix}\right\}_{i\in I} X = \mathrm{Bl}\left\{\begin{smallmatrix} D_k \times_X \mathrm{Bl}\left\{\begin{smallmatrix} D_i \\ Y_i \end{smallmatrix}\right\}_{i\in J} X \\ Y_k \times_X \mathrm{Bl}\left\{\begin{smallmatrix} D_i \\ Y_i \end{smallmatrix}\right\}_{i\in J} X \end{smallmatrix}\right\}_{k\in K} \mathrm{Bl}\left\{\begin{smallmatrix} D_i \\ Y_i \end{smallmatrix}\right\}_{i\in J} X.$$

In particular, there is a unique X-morphism

$$\mathrm{Bl}\left\{\begin{smallmatrix} D_i \\ Y_i \end{smallmatrix}\right\}_{i\in I} X \to \mathrm{Bl}\left\{\begin{smallmatrix} D_i \\ Y_i \end{smallmatrix}\right\}_{i\in J} X.$$

Proposition 2.20 [Ma23d]. *Write $I = \mathrm{colim}_{J\subset I} J$ as a filtered colimit of sets where transition maps are given by inclusions of subsets. We have a canonical identification*

$$\mathrm{Bl}\left\{\begin{smallmatrix} D_i \\ Y_i \end{smallmatrix}\right\}_{i\in I} X = \lim_{J\subset I} \mathrm{Bl}\left\{\begin{smallmatrix} D_i \\ Y_i \end{smallmatrix}\right\}_{i\in J} X.$$

Proposition 2.21 [Ma23d]. *Assume that $\#I = k$ is finite. We fix an arbitrary bijection $I = \{1, \ldots, k\}$. We have a canonical isomorphism of X-schemes*

$$\mathrm{Bl}^{\{D_i\}_{i\in I}}_{\{Y_i\}_{i\in I}} X \cong \mathrm{Bl}^{(\mathrm{Bl}\cdots)\times_X D_k}_{(\mathrm{Bl}\cdots)\times_X Y_k}\left(\cdots \mathrm{Bl}^{(\mathrm{Bl}\cdots)\times_X D_3}_{(\mathrm{Bl}\cdots)\times_X Y_3}\left(\mathrm{Bl}^{(\mathrm{Bl}^{D_1}_{Y_1} X)\times_X D_2}_{(\mathrm{Bl}^{D_1}_{Y_1} X)\times_X Y_2}(\mathrm{Bl}^{D_1}_{Y_1} X)\right)\right).$$

Proposition 2.22 [Ma23d]. *Assume that $\#I = k$ is finite. We fix an arbitrary bijection $I = \{1, \ldots, k\}$. We have a canonical isomorphism of X-schemes*

$$\mathrm{Bl}^{\{D_i\}_{i\in I}}_{\{Y_i\}_{i\in I}} X \cong \mathrm{Bl}^{D_1+\ldots+D_k}_{\bigcap_{i\in I}(Y_i + D_1 + \ldots + D_{i-1} + D_{i+1} + \ldots + D_k)} X.$$

2.8 Functoriality

We proceed with the notation from § 2.1. Let X' and $\{[Y'_i, D'_i]\}_{i\in I}$ be another datum as in § 2.1. As usual, put $Z'_i = Y'_i \cap D'_i$. A morphism $f : X' \to X$ such that, for all $i \in I$, its restriction to D'_i (resp. Z'_i) factors through D_i (resp. Z_i), and such that $f^{-1}(D_i) = D'_i$, induces a unique morphism $\mathrm{Bl}^{D'}_{Y'} X' \to \mathrm{Bl}^{D}_{Y} X$ such that the following diagram of schemes commutes

$$\begin{tikzcd} \mathrm{Bl}^{D'}_{Y'} X' \arrow[r] \arrow[d] & \mathrm{Bl}^{D}_{Y} X \arrow[d] \\ X' \arrow[r] & X. \end{tikzcd}$$

2.9 Base change

We proceed with the notation from § 2.1. Let $X' \to X$ be a map of schemes, and denote by $Y'_i, Z'_i, D'_i \subset X'$ the preimage of $Y_i, Z_i, D_i \subset X$. Then $D'_i \subset X'$ is locally principal for any i so that the dilatation $\mathrm{Bl}^{D'}_{Y'} X' \to X'$ is well-defined. By § 2.8 there is a canonical morphism of X'-schemes

$$\mathrm{Bl}^{D'}_{Y'} X' \longrightarrow \mathrm{Bl}^{D}_{Y} X \times_X X'. \tag{2.3}$$

Lemma 2.23 [MRR20, Ma23d]. *If $\mathrm{Bl}^{D}_{Y} X \times_X X' \to X'$ is an object of $\mathrm{Sch}^{D\text{-reg}}_{X'}$, then (2.3) is an isomorphism.*

Corollary 2.24 [MRR20, Ma23d]. *If the morphism $X' \to X$ is flat and satisfies a property $\mathcal{P}$ which is stable under base change, then $\mathrm{Bl}^{D'}_{Y'} X' \to \mathrm{Bl}^{D}_{Y} X$ is flat and satisfies $\mathcal{P}$.*

2.10 Iterated multi-centered dilatations

We proceed with the notation from § 2.1. Let $\nu, \theta \in \mathbb{N}^I$ such that $\theta \leqslant \nu$, i.e. $\theta_i \leqslant \nu_i$ for all $i \in I$.

Proposition 2.25 [Ma23d]. *There is a unique X-morphism*

$$\varphi_{\nu,\theta} : \mathrm{Bl}_Y^{D^\nu} X \longrightarrow \mathrm{Bl}_Y^{D^\theta} X.$$

Assume now moreover that $\nu, \theta \in \mathbb{N}_I \subset \mathbb{N}^I$. We will prove that, under some assumptions, $\varphi_{\nu,\theta}$ is a dilatation morphism with explicit descriptions. We need the following observation.

Proposition 2.26 [MRR20, Ma23d]. *Assume that we have a commutative diagram of schemes*

$$\begin{tikzcd} B \arrow[rr, "f'"] \arrow[dr, "f"'] & & \mathrm{Bl}_Y^D X \arrow[dl] \\ & X & \end{tikzcd}$$

where the right-hand side morphism is the dilatation map. Assume that f is a closed immersion. Then f' is a closed immersion.

We now assume that $Z_i \subset Y_i$ is a Cartier divisor inclusion for all $i \in I$. Let $\mathcal{D}_i$ be the canonical diagram of closed immersions

$$\begin{tikzcd} Y_i \arrow[r] & \mathrm{Bl}_{Y_i}^{\nu_i D_i} X \\ Z_i \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & D_i \arrow[u] \end{tikzcd}$$

obtained by Proposition 2.26. Let f_i be the canonical morphism (e.g. cf. 2.19 or 2.25)

$$\mathrm{Bl}_Y^{D^\nu} X \to \mathrm{Bl}_{Y_i}^{\nu_i D_i} X.$$

We denote by $Y_i \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X$ the fiber product obtained via the arrows given by f_i and $\mathcal{D}_i$. We use similarly the notation $D_i \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X$.

Proposition 2.27 [PY06, §7.2] [MRR20, Ma23d]. *Recall that $\theta \leqslant \nu$. Put $\gamma = \nu - \theta$. Put $K = \{i \in I | \gamma_i > 0\}$. We have an identification*

$$\mathrm{Bl}_Y^{D^\nu} X = \mathrm{Bl}_{\{Y_i \times_{\mathrm{Bl}_{Y_i}^{\theta_i D_i} X} \mathrm{Bl}_Y^{D^\theta} X\}_{i \in K}}^{\{\gamma_i D_i \times_{\mathrm{Bl}_{Y_i}^{\theta_i D_i} X} \mathrm{Bl}_Y^{D^\theta} X\}_{i \in K}} \mathrm{Bl}_Y^{D^\theta} X.$$

In particular the unique X-morphism

$$\varphi_{\nu,\theta} : \mathrm{Bl}_Y^{D^\nu} X \longrightarrow \mathrm{Bl}_Y^{D^\theta} X$$

of Proposition 2.25 is a dilatation map.

It is now natural to introduce the following terminology.

Definition 2.28. For any $\nu \in \mathbb{N}^k$, let us consider

$$\mathrm{Bl}_Y^{D^\nu} X = \mathrm{Bl}\left\{{}_{Y_i}^{\nu_i D_i}\right\}_{i \in I} X$$

and call it the ν-th iterated dilatation of X with multi-center $\{Y_i, D_i\}_{i \in I}$.

2.11 Some flatness and smoothness results

We proceed with the notation from § 2.1 and assume $I = \{i\}$ is a singleton and we ommit the subscripts i in notation. We assume further that there exists a scheme S under X together with a locally principal closed subscheme $S_0 \subset S$ fitting into a commutative diagram of schemes

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & D \arrow[r] \arrow[d] & X \arrow[d] \\ & S_0 \arrow[r] & S, \end{tikzcd} \tag{2.4}$$

where the square is cartesian, that is $D \to X_0 := X \times_S S_0$ is an isomorphism.

Proposition 2.29 [MRR20]. *Assume that S_0 is an effective Cartier divisor on S.*

(1) If $Z \subset D$ is regular, then $\mathrm{Bl}_Z^D X \to X$ is of finite presentation.

(2) If $Z \subset D$ is regular, the fibers of $\mathrm{Bl}_Z^D X \times_S S_0 \to S_0$ are connected (resp. irreducible, geometrically connected, geometrically irreducible) if and only if the fibers of $Z \to S_0$ are.

(3) If $X \to S$ is flat and if moreover one of the following holds:

(i) *$Z \subset D$ is regular, $Z \to S_0$ is flat and S, X are locally noetherian,*

(ii) *$Z \subset D$ is regular, $Z \to S_0$ is flat and $X \to S$ is locally of finite presentation,*

(iii) *the local rings of S are valuation rings,*

then $\mathrm{Bl}_Z^D X \to S$ is flat.

(4) If both $X \to S$, $Z \to S_0$ are smooth, then $\mathrm{Bl}_Z^D X \to S$ is smooth.

Remark 2.30. Complementary smoothness and flatness results for multi-centered dilatations can be found in [Ma23d, §6].

2.12 Remarks

Dilatations commute with algebraic attractors [Ma23a, Proposition 13.1].

3. Dilatations of group schemes or Néron blowups

One of the key properties allowed by dilatations is that it preserves the structure of group schemes in many cases. Dilatations of group schemes are also called Néron blowups and we also often use this terminology.

3.1 Definitions of multi-centered Néron blowups

Let S be a scheme and $G \to S$ a group scheme. Let $C = \{C_i\}_{i \in I}$ be a set of locally principal closed subschemes of S. Put $G|_{C_i} = G \times_S C_i$ and $G|_C = \{G|_{C_i}\}$. Let $H_i \subset G|_{C_i}$ be a closed subgroup scheme over C_i for all $i \in I$ and let $H = \{H_i\}$. The multi-centered dilatation

$$\mathcal{G} := \mathrm{Bl}_H^{G|_C} G \longrightarrow G$$

is called the *Néron blowup* of G with multi-center $H, G|_C$. We also use the notation $\mathrm{Bl}_H^C G$ to denote $\mathcal{G}$. In the case I has a single element, we shall refer to $\mathrm{Bl}_H^C G$ as mono-centred Néron blowups By Proposition 2.12 the structural morphism $\mathcal{G} \to S$ defines an object in $\mathrm{Sch}_S^{C\text{-reg}}$.

Proposition 3.1 [MRR20, Ma23d]. *Let $\mathcal{G} \to S$ be the above multi-centered Néron blowup.*

(1) The S-scheme $\mathcal{G}$ represents the contravariant functor $\mathrm{Sch}_S^{C\text{-reg}} \to \mathrm{Sets}$

$$T \longmapsto \left\{ T \to G \ : \ \begin{array}{c} T|_{C_i} \to G|_{C_i} \text{ factors through} \\ H_i \subset G|_{C_i} \text{ for all } i \end{array} \right\}.$$

(2) Let $T \to S$ be an object in $\mathrm{Sch}_S^{C\text{-reg}}$, then as subsets of $G(T)$

$$\mathcal{G}(T) = \bigcap_{i \in I} \big(\mathrm{Bl}_{H_i}^{C_i} G\big)(T).$$

(3) The map $\mathcal{G} \to G$ is affine. Its restriction over C_i factors as $\mathcal{G}_i \to H_i \subset G|_{C_i}$ for all i

(4) If the Néron blowup $\mathcal{G} \to S$ is flat, then it is equipped with the structure of a group scheme such that $\mathcal{G} \to G$ is a morphism of S-group schemes.

Remark 3.2. We saw that in favorable cases, dilatations preserve group scheme structures. In fact dilatations preserve similarly monoid scheme structures and Lie algebra schemes structures, or more generally structures defined by products, cf. [Ma23d, §7] for details.

Remark 3.3. Dilatations commute with the formation of Lie algebra schemes in a natural sense

$$\mathbb{L}ie(\mathrm{Bl}_H^{G|_C} G) \cong \mathrm{Bl}\{{}^{\mathbb{L}ie(G)\times_S C_i}_{\mathbb{L}ie(H_i)}\}_{i\in I} \mathbb{L}ie(G)$$

cf. [Ma23d, §7] for precise flatness assumptions.

3.2 Mono-centered Néron blowups

We proceed with the notation from §3.1 and now deal with the mono-centered case, so now $k = 1$. We put $S_0 = C_1$ and $H = H_1$. We also put $G_0 = G \times_S S_0$, $H_0 = H \times_S S_0$ and $K_0 = K \times_S S_0$.

Proposition 3.4 [Ma23d]. *Assume that S_0 is a Cartier divisor in S and $G \to S$ is flat. Let $\eta : K \to G$ be a morphism of group schemes over S such that $K \to S$ is flat. Assume that $H \subset G$ is a closed subgroup scheme over S such that $H \to S$ is flat and $\mathrm{Bl}_H^{S_0} G \to S$ is flat (and in particular a group scheme). Assume that K_0 commutes with H_0 in the sense that the morphism $K_0 \times_{S_0} H_0 \to G_0$, $(k, h) \mapsto \eta(k) h \eta(k)^{-1}$ equals the composition morphism $K_0 \times_{S_0} H_0 \to H_0 \subset G_0$, $(k, h) \mapsto h$. Then K normalizes $\mathrm{Bl}_H^{S_0} G$, more precisely the solid composition map*

$$\begin{tikzcd} K \times_S \mathrm{Bl}_H^{S_0} G \arrow[r, "Id\times \mathrm{Bl}"] \arrow[dr, dashed] & K \times_S G \arrow[r, "k{,}g\mapsto \eta(k)g\eta(k)^{-1}"] & G \\ & \mathrm{Bl}_H^{S_0} G \arrow[ur, "\mathrm{Bl}"] & \end{tikzcd}$$

factors uniquely through $\mathrm{Bl}_H^{S_0} G$.

Theorem 3.5 [WW80, MRR20]. *Assume that $G \to S$ is flat, locally finitely presented and $H \to S_0$ is flat, regularly immersed in G_0. Let $\mathcal{G} \to G$ be the dilatation $\mathrm{Bl}_H^{S_0} G$ with exceptional divisor $\mathcal{G}_0 := \mathcal{G} \times_S S_0$. Let $\mathcal{J}$ be the ideal sheaf of G_0 in G and $\mathcal{J}_H := \mathcal{J}|_H$. Let V be the restriction of the normal bundle $\mathbb{V}(\mathcal{C}_{H/G_0} \otimes \mathcal{J}_H^{-1}) \to H$ along the unit section $e_0 \colon S_0 \to H$.*

(1) Locally over S_0, there is an exact sequence of S_0-group schemes $1 \to V \to \mathcal{G}_0 \to H \to 1$.

(2) Assume given a lifting of H to a flat S-subgroup scheme of G. Then there is globally an exact, canonically split sequence $1 \to V \to \mathcal{G}_0 \to H \to 1$.

(3) If $G \to S$ is smooth, separated and $\mathcal{G} \to G$ is the dilatation of the unit section of G, there is a canonical isomorphism of smooth S_0-group schemes $\mathcal{G}_0 \xrightarrow{\sim} \mathrm{Lie}(G_0/S_0) \otimes \mathrm{N}_{S_0/S}^{-1}$ where $\mathrm{N}_{S_0/S}$ is the normal bundle of S_0 in S.

Remark 3.6. In the situation of Theorem 3.5 (2), the group H acts by conjugation on $V = \mathbb{V}(e_0^*\mathcal{C}_{H/G_0} \otimes \mathcal{J}_{S_0}^{-1})$. It is expected that this additive action is linear, and is in fact none other than the "adjoint" representation of H on its normal bundle as in [SGA3, Exp. I, Prop. 6.8.6]. When the base scheme is the spectrum of a discrete valuation ring this is proved in [DHdS18, Prop. 2.7].

Assume now that $j\colon S_0 \hookrightarrow S$ is an effective Cartier divisor, that $G \to S$ is a flat, locally finitely presented group scheme and that $H \subset G_0 := G \times_S S_0$ is a flat, locally finitely presented closed S_0-subgroup scheme. In this context, there is another viewpoint on the dilatation $\mathcal{G}$ of G in H, namely as the kernel of a certain map of syntomic sheaves.

To explain this, let $f\colon G_0 \to G_0/H$ be the morphism to the fppf quotient sheaf, which by Artin's theorem ([Ar74, Cor. 6.3] and [StP, 04S6]) is representable by an algebraic space. By the structure theorem for algebraic group schemes (see [SGA3, Exp. $\mathrm{VII_B}$, Cor. 5.5.1]) the morphisms $G \to S$ and $H \to S_0$ are syntomic. Since $f\colon G_0 \to G_0/H$ makes G_0 an H-torsor, it follows that f is syntomic also.

Proposition 3.7 [MRR20, Lemma 3.8]. *Let S_{syn} be the small syntomic site of S. Let $\eta\colon G \to j_*j^*G$ be the adjunction map in the category of sheaves on S_{syn} and consider the composition $v = (j_*f) \circ \eta$:*

$$G \xrightarrow{\ \eta\ } j_*j^*G = j_*G_0 \xrightarrow{\ j_*f\ } j_*(G_0/H).$$

Then the dilatation $\mathcal{G} \to G$ is the kernel of v. More precisely, we have an exact sequence of sheaves of pointed sets in S_{syn}:

$$1 \longrightarrow \mathcal{G} \longrightarrow G \xrightarrow{v} j_*(G_0/H) \longrightarrow 1.$$

If $G \to S$ and $H \to S_0$ are smooth, then the sequence is exact as a sequence of sheaves on the small étale site of S.

As a corollary, one has the useful and typical following result.

Corollary 3.8. *[MRR20] Let $\mathcal{O}$ be a ring and $\pi \subset \mathcal{O}$ an invertible ideal such that $(\mathcal{O}, \pi)$ is a henselian pair. Let G be a smooth, separated $\mathcal{O}$-group scheme and $\mathcal{G} \to G$ the dilatation of the trivial subgroup over $\mathcal{O}/\pi$. If either $\mathcal{O}$ is local or G is affine, then the exact sequence of Proposition 3.7 induces an exact sequence of groups:*

$$1 \longrightarrow \mathcal{G}(\mathcal{O}) \longrightarrow G(\mathcal{O}) \longrightarrow G(\mathcal{O}/\pi) \longrightarrow 1.$$

Part II. Some applications

4. Models of group schemes, representation categories and Tannakian groups

In several mathematical theories, one finds the structure of a category with a *tensor product*, and one of the main goals of categorical Tannakian theory is to realize the latter categories as representations of group schemes. If we deal with categories over a *field*, and this is a somewhat well-known area with [DM82] being a fundamental reference, dilatations have not played a role. In the case we deal with categories which are linear over a *discrete valuation* ring, a *Dedekind* *domain*, or more complicated rings, the outputs are much scarcer and the main reference is the beautiful, yet arid, monograph [S72]. But in this situation, dilatations have played a role.

Following [dS09], N. D. Duong and P. H. Hai [DH18] went into technical aspects of [S72] and produced a more contemporaneous text to study tensor categories over Dedekind domain. This prompted further study [DHdS18, HdS21]; in these papers, the authors begin to look at Néron blowups (in the sense of Section 3) and the resulting categories systematically. It is also useful to mention here the paper [CK10], where the idea of looking at representation categories of Néron blowups already appears.

In this section we fix a discrete valuation ring R with uniformizer π, residue field k and fraction field K. We put $S = \mathrm{Spec}(R)$ and $S_i = \mathrm{Spec}(R/(\pi^{i+1}))$ for $i \in \mathbb{N}$.

4.1 Group schemes from categories

Let $\mathcal{T}$ be a neutral Tannakian category over R in the sense of [DH18, Definition 1.2.5]. The reader having encountered only (neutral) Tannakian categories over fields [DM82, Section 2] should note that the distinctive property of $\mathcal{T}$ is a weakening of the existence of "duals" [DM82, Definition 1.7]. This is to be replaced by the property that every object is a quotient of an "object having a dual." That this property holds for representation categories of group schemes is [Se68, Proposition 3]. (For a higher dimensional bases, see [Th87, Lema 2.5].) But we face a non-trivial requirement: for example, $\mathrm{Rep}_{W(\overline{\mathbb{F}}_p)}(\overline{\mathbb{F}}_p)$ fails to satisfy it [HdS21, Example 4.7].

Once this definition of neutral Tannakian category is given, the main theorem of [DM82, Theorem 2.11] has his analogue in the present context: If $\omega : \mathcal{T} \to R$-**mod** is a faithful, R–linear and exact tensor functor, then there exists an affine and *flat* group scheme $\Pi_{\mathcal{T}}$ over R and an equivalence

$$\overline{\omega} : \mathcal{T} \longrightarrow \mathrm{Rep}_R(\Pi_{\mathcal{T}})$$

such that composing $\overline{\omega}$ with the forgetful functor $\mathrm{Rep}_R(\Pi_{\mathcal{T}}) \to R$-**mod** renders us ω back. See [S72, II.4.1.1] and [DH18, Theorem 1.2.2].

Let us present some examples of categories to which the theory can be applied.

Example 4.1. Let Γ be an abstract group and suppose that $R = k[\![\pi]\!]$. Then, the category of $R[\Gamma]$-modules which are of finite type over R together with the forgetful functor is a neutral Tannakian category [HdS21, 4.1].

Example 4.2. Let X be a smooth and connected scheme over R, $\mathcal{D}_{X/R}$ the ring of differential operators [EGA, IV.16.8], and $\mathcal{T}^+$ the category of $\mathcal{D}_{X/R}$-modules which, as $\mathcal{O}_X$–modules, are coherent. Using the fibre-by-fibre flatness criterion and [BO78], one proves that an object $E \in \mathcal{T}^+$ is locally free if and only if it is R-flat.

Let now $\mathcal{T}$ be the full subcategory of $\mathcal{T}^+$ having

$$\left\{ M \in \mathcal{T}^+ : \begin{array}{c} \text{There exists } E \in \mathcal{T}^+ \text{ which} \\ \text{is } R\text{-flat and a surjection } E \to M \end{array} \right\}$$

as objects. Once we give ourselves an R-point $x_0 \in X(R)$, it follows that

$$\mathcal{T} \longrightarrow R\text{-}\mathbf{mod}, \qquad E \longmapsto (\text{global sections of})\ x_0^*(E)$$

defines a neutral Tannakian category. For more details, see [And01] and [DHdS18].

Example 4.3. We assume that R is Henselian and Japanese, e.g. R is complete. Let X be an irreducible, proper and flat R-scheme with geometrically reduced fibres. Let $x_0 \in X(R)$. Given a coherent sheaf E on X, we say that E is *trivialized by a proper morphism* if there exists a surjective and proper morphism $\psi : Y \to X$ such that ψ^*E "comes from $S = \operatorname{Spec} R$", by which we mean that ψ^*E is the pull-back of a module via the structural morphism $Y \to S$. Let $\mathcal{T}^+$ be the full subcategory of the category of coherent modules on X having as objects those sheaves which are trivialized by a proper morphism. Proceeding along the lines of Example 4.2, it is possible to construct a smaller full subcategory $\mathcal{T}$ of $\mathcal{T}^+$ such that, endowing $\mathcal{T}$ with the tensor product of sheaves, the functor

$$\mathcal{T} \longrightarrow R\text{-}\mathbf{mod}, \qquad E \longmapsto \text{(global sections of) } x_0^*(E)$$

defines a neutral Tannakian category. Details are in [HdS23]. This is the analogue theory of Nori's theory for the fundamental group scheme [Nor76] in the relative setting, and one objective is to show that the group scheme associated to $\mathcal{T}$ is *pro-finite*. See [HdS23, Theorem 8.8].

4.2 Galois-Tannaka group schemes

One obvious strategy to study Tannakian categories is to filter them by categories "generated" by a single object, just as in studying Galois groups it is fundamental to study finite extensions. Let $\omega : \mathcal{T} \to R\text{-}\mathbf{mod}$ be as in the previous section so that $\mathcal{T}$ is equivalent to $\operatorname{Rep}_R(\Pi)$ for some affine and flat group scheme Π. We shall take this equivalence as an equality, but we warn the reader that the structure of Π should be considered as being very complicated (just as is that of an absolute Galois group).

Definition 4.4. Let $M \in \mathcal{T}$ be an object possessing a dual $M^\vee$ and for each couple of non-negative integers a, b, define $\mathbf{T}^{a,b}M$ as $M^{\otimes a} \otimes M^{\vee \otimes b}$. Then, $\langle M \rangle_\otimes$ is the full subcategory of $\mathcal{T}$ having as objects those which are quotients of subobjects of elements of the form

$$\mathbf{T}^{a_1,b_1}M \oplus \cdots \oplus \mathbf{T}^{a_r,b_r}M,$$

for varying r, $a_1, \ldots, a_r$, $b_1, \ldots, b_r$. The Tannakian group scheme associated to $\langle M \rangle_\otimes$ via ω shall will be called here the *(full) Galois-Tannaka group (scheme)* of M.

As we concentrate on a neutral Tannakian category, it is instructive to note that the splicing of $\mathcal{T}$ by various $\langle M \rangle_\otimes$ amounts to looking at various "images" of Π. Before entering this topic, recall that, given a base field F and a morphism $\varphi : G' \to G$ of affine group schemes over F, the closed image $\operatorname{Im}_\varphi$ [EGA, I.9.5] is a *closed subgroup scheme* of G such that the natural morphism $G' \to \operatorname{Im}_\varphi$ is *faithfully flat* [Wa79, Theorem on 15.1]. In this case, $\operatorname{Im}_\varphi$ enjoys both "desirable properties" of and image.

Definition 4.5. Let $\rho : \Pi \to G$ be a morphism of flat and affine group schemes over R. Define the *restricted* image of ρ, denoted Im_ρ, as the affine scheme associated to the algebra

$$B_\rho = \text{Image of } \mathcal{O}(G) \to \mathcal{O}(\Pi).$$

(In other words, Im_ρ is the "closed" image of ρ [EGA, I.9.5].) Define its *full* image Im'_ρ as being the affine scheme associated to

$$B'_\rho = \{f \in K \otimes \mathcal{O}(\Pi) : \ \pi^m f \in B_\rho, \text{ for some } m \geqslant 0\}. \tag{§}$$

It is not difficult to see that Im_ρ and Im'_ρ are affine group schemes. With these definitions, ρ factors as

$$\Pi \xrightarrow{\psi} \operatorname{Im}'_\rho \xrightarrow{u} \operatorname{Im}_\rho \xrightarrow{\iota} G, \tag{†}$$

where ι is a *closed immersion* and u induces an isomorphism between generic fibres. A fundamental result [DH18, Theorem 4.1.1] now assures that ψ is faithfully flat, so that the the terms "images" are justified and the factorization in (†) is called the *diptych* of ρ.

In addition, if

$$\rho_K : \Pi \otimes K \longrightarrow G \otimes K$$

stands for the morphism obtained from ρ by base-change to K, we have

$$\mathrm{Im}'_\rho \otimes K = \mathrm{Im}_\rho \otimes K = \mathrm{Im}(\rho_K).$$

Proposition 4.6 [DHdS18, Proposition 4.10]. *Let M be a finite and free R-module affording a representation of Π and let $\rho : \Pi \to \mathrm{GL}(M)$ be the associated homomorphism. Then the obvious functor $\mathrm{Rep}_R(\mathrm{Im}'_\rho) \to \mathrm{Rep}_R(\Pi)$ defines an equivalence between $\mathrm{Rep}_R(\mathrm{Im}'_\rho)$ and $\langle M\rangle_\otimes$. Put differently, Im'_ρ is the Galois-Tannaka group of M (in $\mathrm{Rep}_R(\Pi)$).*

Remark 4.7. Let $\mathrm{Rep}^\circ_R(\mathrm{Im}_\rho)$ be the full subcategory of $\mathrm{Rep}_R(\mathrm{Im}_\rho)$ consisting of objects having a dual; it is possible to show that $\mathrm{Rep}^\circ_R(\mathrm{Im}_\rho)$ is equivalent to a full subcategory of $\langle M\rangle_\otimes$. On the other hand, the functor $\mathrm{Rep}_R(\mathrm{Im}_\rho) \to \mathrm{Rep}_R(\Pi)$ may easily fail to be full.

From now on, we give ourselves a representation $\rho : \Pi \to \mathrm{GL}(M)$ as in Proposition 4.6. It is at this point that the theory over R parts from the theory over a field in a significant way. Indeed, in the case of a base-field, Galois-Tannaka group schemes are known to be of finite type [DM82, Proposition 2.20]. *This is not unconditionally true over R* since in order to construct Im'_ρ, it was required to "saturate" the ring B_ρ in (§). On the other hand, the morphism $\mathrm{Im}_\rho \to \mathrm{GL}(M)$ is a closed immersion and Im_ρ is of finite type.

Definition 4.8. A *model* of a group scheme of finite type G over K is a flat group scheme $\mathbb{G}$ over R such that $\mathbb{G} \otimes_R K \cong G$, as K-group schemes. We often identify G and the generic fibre $\mathbb{G} \otimes_R K$. A morphism of models $\mathbb{G} \to \mathbb{G}'$ of G is a morphism $\mathbb{G} \to \mathbb{G}'$ of group schemes over R which induces the identity on G once unravelled the proper identifications.

Remark 4.9. The definition of model used here differs from the one used in [StP, Tag 0C2R] and [WW80] namely, we do not assume our models to be of finite type over R.

With this terminology, Im_ρ and Im'_ρ are models of $\mathrm{Im}(\rho_K)$. A well-known result of Waterhouse-Weisfeler about the relations between models is the following.

Theorem 4.10 [WW80, Theorem 1.4], [DHdS18, Theorem 2.11]. *Let $v : G' \to G$ be a morphism of flat S-group schemes such that v is an isomorphism on generic fibres. Then v is a composite of mono-centered Néron blowups (along the divisor defined by π). In other words, a morphism of models of finite type is a composite of mono-centered Néron blowups. If G and G' are of finite type, then the number of Néron blowups is finite.*

More precisely: Define $v_0 = v$ and $G_0 = G$. Suppose that $v_n : G' \to G_n$ is obtained and put

$$G_{n+1} = \mathrm{Bl}^{G_n \otimes k}_{\mathrm{Im}_{v_n \otimes k}}(G_n).$$

(Recall that k is the residue field.) Letting $v_{n+1} : G' \to G_{n+1}$ be the morphism deduced from the universal property of $\mathrm{Bl}^{G_n \otimes k}_{\mathrm{Im}_{v_n \otimes k}}(G_n)$ (cf. Proposition 3.1), then

$$\varprojlim_n v_n : G' \longrightarrow \varprojlim_n G_n$$

is an isomorphism. In particular, if for some $n \in \mathbb{N}$ the homomorphism $v_n \otimes k$ is faithfully flat, then $G' \simeq G_n$.

As was mentioned before, it is possible that Im'_ρ fails to be of finite type and hence the number of Néron blowups proposed by Theorem 4.10 to describe $u : \mathrm{Im}'_\rho \to \mathrm{Im}_\rho$ may be infinite. But in some cases, it does happen that the number of Néron blowups is finite and a condition for this situation is described in Theorem 4.10. At this point, we remind the reader that in the situations we have in mind, the group scheme Π is usually extremely complicated and the determination of the image of a morphism $\Pi\otimes k \to G\otimes k$, so that it is possible to apply the last claim in Theorem 4.10, can only be achieved on the side of $\mathrm{Rep}_R(\Pi)$.

It then becomes relevant to determine faithful representations of Néron blowups. (Here, we say that a representation is *faithful* if the morphism to the associated general linear group is a closed immersion. This is not universally adopted.) The next result explains how to proceed in certain cases.

Theorem 4.11 [DHdS18, Corollary 3.6]. *Let G be an affine and flat group scheme of finite type over S. Let M be a finite and free R-module affording a faithful representation of G. Given $m \in M$, let*

$$H_0 = \textit{stabilizer of } m\otimes 1 \in M\otimes k$$

in $G\otimes k$. Let $G' = \mathrm{Bl}_{H_0}^{G\otimes k}(G)$. Then, letting $\mathbb{1} = R$ stand for the trivial representation of G', the obvious map $\mathbb{1} \to M\otimes k$ determined by $1 \mapsto v\otimes 1$ is G'-equivariant and the fibered product

$$M' := M \underset{M\otimes k}{\times} \mathbb{1} \qquad (¶)$$

now affords a faithful representation of G'.

Let us illustrate the above result with a simple example showing how to compute a Galois-Tannaka group.

Example 4.12. Let k be of characteristic zero and $\mathcal{T}$ be the category of representations of the abstract group $\mathbb{Z}$ on finite R-modules. It is not difficult to see that $\mathcal{T}$ is neutral Tannakian [HdS21, Corollary 4.5]. Let $\mathbb{Z}$ act on $M = R$ by $\gamma\cdot r = (1+\pi)^\gamma r$ and write $\rho : \Pi \to \mathrm{GL}(M)(\simeq \mathbb{G}_{m,R})$ for the associated morphism of group schemes. It is not difficult to see that $\mathrm{Im}_\rho = \mathbb{G}_{m,R}$ and we wish to compute Im'_ρ. As mentioned above, the construction (§) is of little use. On the other hand, we know that Π will act trivially on $M\otimes k$ because $\mathbb{Z}$ does. We then need to perform the "dilatation" M' of M as in (¶), which is a faithful representation of the Néron blowup $\mathrm{Bl}_{\{e\}}^{\mathbb{G}_m\otimes k}(\mathbb{G}_m)$. The elements $m_1 := (\pi, 0)$ and $m_2 := (1,1)$ obviously form a basis for M' and hence the resulting representation of $\mathbb{Z}$ is defined by

$$\gamma \longmapsto \begin{pmatrix} 1+\pi & 1 \\ 0 & 1\end{pmatrix}^\gamma .$$

If $\rho' : \Pi \to \mathrm{GL}(M')(\simeq \mathrm{GL}_2)$ stands for the associated representation of Π, we can say that $\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix} \in \mathrm{GL}_2(k)$ belongs to the image of $\rho'\otimes k$ and therefore $\mathrm{Im}'_\rho \simeq \mathrm{Bl}_{\{e\}}^{\mathbb{G}_m\otimes k}(\mathbb{G}_m)$ because $\mathrm{Bl}_{\{e\}}^{\mathbb{G}_m\otimes k}(\mathbb{G}_m)\otimes \simeq \mathbb{G}_{a,k}$, and any element of $k\setminus\{0\}$ generates a dense subgroup.

On the other hand, when the number of Néron blowups envisaged by Theorem 4.10 is infinite, a general principle behind [DHdS18, HdS21] is that the Galois-Tannaka groups can be obtained from group schemes of finite type via certain special types of (what we now call) *multi-centered* Néron blowups. This is treated in the next section.

4.3 Néron blowups of formal subgroup schemes

Multi-centered dilatations having divisors which are supported on the same space have been studied more closely. For an affine group scheme G over R, we shall write $\widehat{G}$ for the completion $G_{/G_0}$ of G along its closed fiber [EGA, I.10].

DEFINITION 4.13 [DHdS18, Definition 5.6]. Let $G \to S$ be an affine flat group scheme of finite type. For each $i \in \mathbb{N}$, let G_i be the S_i-group scheme $G \times_S S_i$, and let $H_i \to S_i$ be a closed, S_i–flat, subgroup-scheme of G_i. Assume, in addition, that the natural base-change morphism

$$H_{i+1} \times_{S_{i+1}} S_i \longrightarrow G_{i+1} \times_{S_{i+1}} S_i = G_i$$

defines an isomorphism $H_{i+1} \times_{S_{i+1}} S_i \simeq H_i$ of group schemes. Said differently, the family $\{H_i\}$ induces a *formal closed subgroup scheme* $\mathfrak{H}$ *of* $\widehat{G}$. We define the *Néron blowup* of G along $\mathfrak{H}$, call it $\mathrm{Bl}_{\mathfrak{H}}^{\widehat{G}} G$, as being $\mathrm{Bl}_{\{H_i\}}^{\{G_i\}} G \to G$.

REMARK 4.14. If the formal scheme $\mathfrak{H}$ is "algebraizable", meaning that it comes from a closed and flat subgroup scheme $H \subset G$, this is mentioned in [PY06, § 7.2].

EXAMPLE 4.15. Let p be a prime number, $R = \mathbb{Z}_p$ and $G = \mathbb{G}_{a,R}$. It then follows that the completion of G along its closed fibre is $\mathrm{Spf}\,\mathbb{Z}_p\langle x\rangle$, where $\mathbb{Z}_p\langle x\rangle$ is the subring of $\mathbb{Z}_p[\![x]\!]$ consisting of power series $\sum_n a_n x^n$ such that $\lim a_n = 0$. Let $\mathfrak{H}$ be the closed formal subscheme of $\widehat{G}$ determined by the ideal $(x) \subset \mathbb{Z}_p\langle x\rangle$. Then, it is not difficult to see that $\mathrm{Bl}_{\mathfrak{H}}^{\widehat{G}} G$ is the group scheme determined by the Hopf subalgbra $A = \{P \in \mathbb{Q}_p[x] \,:\, P(0) \in \mathbb{Z}_p\}$. Note that $\mathrm{Bl}_{\mathfrak{H}}^{\widehat{G}} G \otimes \mathbb{F}_p$ is the trivial group scheme, while $\mathrm{Bl}_{\mathfrak{H}}^{\widehat{G}} G \otimes \mathbb{Q}_p$ is $\mathbb{G}_{a,\mathbb{Q}_p}$. In particular, the dimension of the generic and special fibres is distinct, even though $\mathrm{Bl}_{\mathfrak{H}}^{\widehat{G}} G$ is itself flat over $\mathbb{Z}_p$. Note, on the other hand, that the $\mathbb{Z}_p$-module A contains a copy of $\mathbb{Q}_p$ and hence fails to be *projective* over $\mathbb{Z}_p$. This seemingly harmless property is the cause of complications in the category of representations [HdS21, Proposition 6.19] as the *inexistence* of intersections of subrepresentations.

EXAMPLE 4.16 [HdS21, 4.3]. Let $R = k[\![\pi]\!]$, where k is a field of characteristic zero and let $G = \mathbb{G}_{a,R} \times_R \mathbb{G}_{m,R}$. Letting x stand for "the" coordinate of $\mathbb{G}_{a,R}$ and y for "the" coordinate of $\mathbb{G}_{m,R}$, we define

$$e^{\pi x} = \sum_{i=0}^{\infty} \frac{\pi^i}{i!} x^i;$$

this is an element of $\widehat{\mathcal{O}(G)}$. It is not difficult to see that $y - e^{\pi x}$ cuts out a closed and formal subgroup scheme of $\widehat{G}$, call it $\mathfrak{H}$, and hence we obtain a model $\mathrm{Bl}_{\mathfrak{H}}^{\widehat{G}} \to G$. Note that $\mathfrak{H}$ is not algebraizable. Differently from the situation in Exemple 4.15, the R-module $\mathcal{O}(\mathrm{Bl}_{\mathfrak{H}}^{\widehat{G}})$ is *projective.*

One important consequence of the procedure of taking formal blowups is the following. It says that, in some contexts, all the information concerning a model of a group scheme can be encoded in a formal Néron blowup (Theorem 4.17).

THEOREM 4.17 [HdS21, Corollary 3.3]. *Suppose that the R is complete and of residual characteristic zero. Let $\mathcal{G} \to G$ be a morphism of affine and flat R-group schemes inducing an isomorphism on the generic fibres, and suppose in addition that G is of finite type. Then, there exists a group scheme G' over R, flat and of finite type, and a morphism of group schemes $G' \to G$ which is an isomorphism on generic fibres, a closed and formal subgroup scheme $\mathfrak{H}'$ of $\widehat{G}'$, and an isomorphism*

$$\mathcal{G} \xrightarrow{\sim} \mathrm{Bl}_{\mathfrak{H}'}^{\widehat{G}'} G'.$$

Remark 4.18. Under the assumptions of Theorem 4.17, Theorems 4.10 and 4.17 together say that any morphism of models $G' \to G$ with G of finite type over R is obtained as a composite of multi-centered Néron blowups, and more precisely as a formal Néron blowup composed by several mono-centered Néron blowups.

5. Congruent isomorphisms and relations with Bruhat-Tits buildings, the Moy-Prasad isomorphism and admissible representations of p-adic groups

In this section we report on congruent isomorphisms. Let $(\mathcal{O}, \pi)$ be a henselian pair where $\pi \subset \mathcal{O}$ is an invertible ideal. Let us start with the following result proved in [MRR20].

Theorem 5.1 [MRR20]. *(Congruent isomorphism) Let r, s be integers such that $0 \leqslant r/2 \leqslant s \leqslant r$. Let G be a smooth, separated $\mathcal{O}$-group scheme. Let G_r be the r-th iterated dilatation of the unit section (i.e. $G_r = \mathrm{Bl}_{e_G}^{\mathcal{O}/\pi^r} G$) and $\mathfrak{g}_r$ be its Lie algebra. If $\mathcal{O}$ is local or G is affine, there is a canonical and functorial isomorphism of groups:*

$$G_s(\mathcal{O})/G_r(\mathcal{O}) \xrightarrow{\sim} \mathfrak{g}_s(\mathcal{O})/\mathfrak{g}_r(\mathcal{O}). \qquad (\star)$$

Remark 5.2. We comment on works prior to Theorem 5.1.

(i) In the case of an affine, smooth group scheme over a discrete valuation ring, the isomorphism of Theorem 5.1 appears without proof in [Yu15, proof of Lemma 2.8].

(ii) The proof of Theorem 5.1 relies on Proposition 2.15 and Theorem 3.5 whose proofs (given in [MRR20, Prop. 2.9] and [MRR20, Th. 3.5]) basically consist in playing and computing with quasi-coherent ideals. These computations on quasi-coherent ideals in [MRR20] were partly motivated by related computations on ideals done in the affine case in [Ma19t, Appendix A] to understand the congruent isomorphism. The statement of [MRR20, Th. 3.5] is moreover partly inspired by [WW80, Th. 1.5, Th. 1.7].

(iii) If $G = \mathbb{G}_m/\mathbb{Z}_p$, isomorphism $(\star)$ follows from the multiplicative structure of $\mathbb{Z}_p$ cf. e.g. [He1913], [Ha50] and [Ha80, Chap. 15]. Similar isomorphisms for matrix groups over non-Archimedean local fields were used in [Ho77, p. 442 line 1], [Mo91, 2.13], [BK93, p. 22], [Sec04, p. 337] and many other references to study admissible representations of p-adic classical groups. In the matrix case, the filtrations involved are defined using matrix theoretic descriptions and avoiding scheme theoretic tools. For general reductive groups over non-Archimedean local fields, such kind of isomorphisms were introduced and used in [PR84, § 2], [MP94, § 2], [MP96], [Ad98, § 1], [Yu01, § 1] to study admissible representations. In the reductive case, the filtrations involved are the Moy-Prasad filtrations [MP94], [MP96] and the isomorphism is called the Moy-Prasad isomorphism. These filtrations are defined for points in the Bruhat-Tits building using the associated valued root datum [BT72] [BT84]. The Moy-Prasad isomorphim in these references was defined using somehow ad hoc formulas and the valued root datum, in particular avoiding the congruent isomorphism. However it is known that one has to modify the original Moy-Prasad filtrations to ensure the validity of the Moy-Prasad isomorphism in full generality, cf. [Yu15, §0.3] and [KP22, §13].

If $G = \mathrm{GL}_2/\mathbb{Z}_p$, $G_n(\mathbb{Z}_p) = \begin{pmatrix} 1+\mathfrak{p}^n & \mathfrak{p}^n \\ \mathfrak{p}^n & 1+\mathfrak{p}^n \end{pmatrix} \subset \mathrm{GL}_2(\mathbb{Z}_p)$ and $\mathfrak{g}_n(\mathbb{Z}_p) = \begin{pmatrix} \mathfrak{p}^n & \mathfrak{p}^n \\ \mathfrak{p}^n & \mathfrak{p}^n \end{pmatrix} \subset M_2(\mathbb{Z}_p)$ for any $n > 0$. The isomorphism $(\star)$ gives us, for pairs (r, s) such that $0 < \frac{r}{2} \leqslant s \leqslant r$, isomorphisms

$$\begin{pmatrix} 1+\mathfrak{p}^s & \mathfrak{p}^s \\ \mathfrak{p}^s & 1+\mathfrak{p}^s \end{pmatrix} / \begin{pmatrix} 1+\mathfrak{p}^r & \mathfrak{p}^r \\ \mathfrak{p}^r & 1+\mathfrak{p}^r \end{pmatrix} \cong \begin{pmatrix} \mathfrak{p}^s & \mathfrak{p}^s \\ \mathfrak{p}^s & \mathfrak{p}^s \end{pmatrix} / \begin{pmatrix} \mathfrak{p}^r & \mathfrak{p}^r \\ \mathfrak{p}^r & \mathfrak{p}^r \end{pmatrix}. \tag{$*$}$$

These maps are given by $[1+M] \mapsto [M]$. Using the formula $[1+M] \mapsto [M]$, it is elementary to check that we have other isomorphisms of abstract groups

$$\begin{pmatrix} 1+\mathfrak{p}^3 & \mathfrak{p}^3 \\ \mathfrak{p}^3 & 1+\mathfrak{p}^3 \end{pmatrix} / \begin{pmatrix} 1+\mathfrak{p}^5 & \mathfrak{p}^6 \\ \mathfrak{p}^6 & 1+\mathfrak{p}^5 \end{pmatrix} \cong \begin{pmatrix} \mathfrak{p}^3 & \mathfrak{p}^3 \\ \mathfrak{p}^3 & \mathfrak{p}^3 \end{pmatrix} / \begin{pmatrix} \mathfrak{p}^5 & \mathfrak{p}^6 \\ \mathfrak{p}^6 & \mathfrak{p}^5 \end{pmatrix}, \tag{$**$}$$

$$\begin{pmatrix} 1+\mathfrak{p}^3 & \mathfrak{p}^9 \\ \mathfrak{p}^3 & 1+\mathfrak{p}^3 \end{pmatrix} / \begin{pmatrix} 1+\mathfrak{p}^6 & \mathfrak{p}^9 \\ \mathfrak{p}^6 & 1+\mathfrak{p}^6 \end{pmatrix} \cong \begin{pmatrix} \mathfrak{p}^3 & \mathfrak{p}^9 \\ \mathfrak{p}^3 & \mathfrak{p}^3 \end{pmatrix} / \begin{pmatrix} \mathfrak{p}^6 & \mathfrak{p}^9 \\ \mathfrak{p}^6 & \mathfrak{p}^6 \end{pmatrix}. \tag{$***$}$$

These isomorphisms are obtained as follows from the point of view of dilatations.

Theorem 5.3 [Ma23d]. *(Multi-centered congruent isomorphism) Let G be a separated and smooth group scheme over S. Let $H_0 \subset H_1 \subset \ldots \subset H_k$ be closed subgroup schemes of G such that H_i is smooth over S for $0 \leqslant i \leqslant d$ and $H_0 = e_G$. Let $s_0, s_1, \ldots, s_k$ and $r_0, r_1, \ldots, r_k$ be in $\mathbb{N}$ such that*

(i) $s_i \geqslant s_0$ and $r_i \geqslant r_0$ for all $i \in \{0, \ldots, k\}$,

(ii) $r_i \geqslant s_i$ and $r_i - s_i \leqslant s_0$ for all $i \in \{0, \ldots, k\}$.

Assume that G is affine or $\mathcal{O}$ is local. Then we have a canonical isomorphism of groups

$$\mathrm{Bl}^{s_0,\, s_1,\, \ldots, s_k}_{H_0, H_1, \ldots, H_k} G / \mathrm{Bl}^{r_0,\, r_1,\, \ldots, r_k}_{H_0, H_1, \ldots, H_k} G \cong \mathrm{Lie}(\mathrm{Bl}^{s_0,\, s_1,\, \ldots, s_k}_{H_0, H_1, \ldots, H_k} G) / \mathrm{Lie}(\mathrm{Bl}^{r_0,\, r_1,\, \ldots, r_k}_{H_0, H_1, \ldots, H_k} G)$$

where $\mathrm{Bl}^{t_0,\, \ldots, t_k}_{H_0, \ldots, H_k} G$ denotes $\mathrm{Bl}^{\mathcal{O}/\pi^{t_0}, \ldots, \mathcal{O}/\pi^{t_k}}_{H_0,\, \ldots\, , H_k} G$ for any $t_0, \ldots, t_k \in \mathbb{N}$.

Now let G be $GL_2/\mathbb{Z}_p$. Let $e_G \subset G$ be the trivial subgroup. Let T be the diagonal split torus in G. Let B be the lower triangular Borel in G over $\mathbb{Z}_p$.

(i) The isomorphism ($**$) above is given by Theorem 5.3 with $(\mathcal{O}, \pi) = (\mathbb{Z}_p, \mathfrak{p})$, $H_0 = e_G$, $H_1 = T$, $s_0 = 3, s_1 = 3, r_0 = 5$ and $r_1 = 6$.

(ii) The isomorphism ($***$) above is given by Theorem 5.3 with $(\mathcal{O}, \pi) = (\mathbb{Z}_p, \mathfrak{p})$, $H_0 = e_G$, $H_1 = B$, $s_0 = 3, s_1 = 9, r_0 = 6$ and $r_1 = 9$.

Remark 5.4. We comment Theorem 5.3.

(i) Theorem 5.3 corresponds to [Ma23d, Corollary 8.3], a slightly more general result is given by [Ma23d, Theorem 8.1].

(ii) The proof of Theorem 5.3 given in [Ma23d] relies on Theorem 5.1 and the study of multi-centered dilatations.

(iii) Note that [Yu01, Lemma 1.3] provides a comparable "multi-centered" isomorphism, in the framework of reductive groups over non-Archimedean local field.

Recall that dilatations of schemes over discrete valuation rings are used in Yu's approach [Yu15] on Bruhat-Tits theory for reductive groups over henselian discrete valuation field with perfect residue field. We refer to the monograph by Kaletha and Prasad [KP22] that include among other things a detailled exposition of [Yu15]. The congruent isomorphism (Theorem 5.1) and its proof (relying on several results of [MRR20]) are now used as foundation to prove the Moy-Prasad isomorphism for reductive groups mentioned in Remark 5.2, cf. [KP22, Theorem 13.5.1 and its proof, Proposition A.5.19 (3) and its proof]. As a consequence, dilatations and congruent isomorphisms are now part of the foundation to study admissible representations of reductive p-adic groups. Furthermore, other connections between dilatations and groups used to study admissible representations can be found in [Yu15, §10] and [Ma23d, Example 1.4]. Reciprocally, the problem of constructing supercuspidal representations of p-adic groups (cf. e.g. [Ma19o, Remark/Conclusion], [Ma19t, MY22]), or more generally types in the sense of [BK98], could continue to be a source of inspiration to expend the theory of dilatations.

Remark 5.5. As we explained before, the book [KP22] provides a carefully written new approach to Bruhat-Tits theory in the case of discrete valuations. This beautiful monograph uses the theory of dilatations to deal with integral models whereas the original Bruhat-Tits theory [BT84] did not. Let us quote [KP22, Introduction]:

> "*Next we turn to the construction of integral models [...]. Instead of using the approach of Bruhat–Tits via schematic root data, we employ a simpler and more direct method due to Jiu-Kang Yu [Yu15], based on the systematic use of Néron dilatations.*"

The book [KP22] offers an appendix on dilatations. Though [KP22, Appendix A.5] takes into account the treatment of dilatations in [MRR20], it restricts to the framework of discrete valuations. Originally, Bruhat-Tits theory [BT84] deals also with non discrete valuations, it is natural to ask whether the modern and general approach to dilatations of schemes initiated in [MRR20] could help to provide a more conceptual treatment (in the spirit of [Yu15] and [KP22]) of some parts of [BT84]. Bruhat-Tits theory and dilatations over non discrete valuations were used in [RTW10] and [Ma22] to study Berkovich's point of view [Be90, Chap. 5] on Bruhat-Tits buildings of reductive groups over discrete and non-discrete valuations (cf. e.g. [RTW10, 1.3.4] for precise assumptions).

6. Torsors, level structures and shtukas

In this subsection, we explain that many level structures on moduli stacks of G-bundles are encoded in torsors under Néron blowups of G following [MRR20]. Assume that X is a smooth, projective, geometrically irreducible curve over a field k with a Cartier divisor $N \subset X$, that $G \to X$ is a smooth, affine group scheme and that $H \to N$ is a smooth closed subgroup scheme of $G|_N$. In this case, the Néron blowup $\mathcal{G} \to X$ is a smooth, affine group scheme. Let Bun_G (resp. $\mathrm{Bun}_{\mathcal{G}}$) denote the moduli stack of G-torsors (resp. $\mathcal{G}$-torsors) on X. This is a quasi-separated, smooth algebraic stack locally of finite type over k (cf. e.g. [He10, Prop. 1] or [AH19, Thm. 2.5]). Pushforward of torsors along $\mathcal{G} \to G$ induces a morphism $\mathrm{Bun}_{\mathcal{G}} \to \mathrm{Bun}_G$, $\mathcal{E} \mapsto \mathcal{E} \times^{\mathcal{G}} G$. We also consider the stack $\mathrm{Bun}_{(G,H,N)}$ of G-torsors on X with level-(H,N)-structures, cf. [MRR20, Definition 4.5]. Its k-points parametrize pairs $(\mathcal{E}, \beta)$ consisting of a G-torsor $\mathcal{E} \to X$ and a section β of the fppf quotient $(\mathcal{E}|_N/H) \to N$, i.e., β is a reduction of $\mathcal{E}|_N$ to an H-torsor.

Proposition 6.1 [MRR20]. *There is an equivalence of k-stacks*

$$\mathrm{Bun}_{\mathcal{G}} \xrightarrow{\cong} \mathrm{Bun}_{(G,H,N)}, \quad \mathcal{E} \longmapsto (\mathcal{E} \times^{\mathcal{G}} G,\, \beta_{\mathrm{can}}),$$

where β_{can} denotes the canonical reduction induced from the factorization $\mathcal{G}|_N \to H \subset G|_N$.

Thus, many level structures are encoded in torsors under Néron blowups. This construction is also compatible with the adelic viewpoint as follows. Let $|X| \subset X$ be the set of closed points, and let $\eta \in X$ be the generic point. We denote by $F = \kappa(\eta)$ the function field of X. For each $x \in |X|$, we let $\mathcal{O}_x$ be the completed local ring at x with fraction field F_x and residue field $\kappa(x) = \mathcal{O}_x/\mathfrak{m}_x$. Let $\mathbb{A} := \prod'_{x\in|X|} F_x$ be the ring of adeles with subring of integral elements $\mathbb{O} = \prod_{x\in|X|} \mathcal{O}_x$.

PROPOSITION 6.2. *Assume either that k is a finite field and $G \to X$ has connected fibers, or that k is a separably closed field. The Néron blowup $\mathcal{G} \to X$ is smooth, affine with connected fibers, and there is a commutative diagram of groupoids*

$$\begin{array}{ccc} \mathrm{Bun}_{\mathcal{G}}(k) & \xrightarrow{\simeq} & \coprod_\gamma G_\gamma(F)\backslash\big(G_\gamma(\mathbb{A})/\mathcal{G}(\mathbb{O})\big) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_G(k) & \xrightarrow{\simeq} & \coprod_\gamma G_\gamma(F)\backslash\big(G_\gamma(\mathbb{A})/G(\mathbb{O})\big), \end{array}$$

identifying the vertical maps as the level maps.

Now assume that k is a finite field. As a consequence of Proposition 6.1 one naturally obtains integral models for moduli stacks of G-shtukas on X with level structures over N via an isomorphism $\mathrm{Sht}_{\mathcal{G},I_\bullet} \xrightarrow{\cong} \mathrm{Sht}_{(G,H,N),I_\bullet}$ (cf. [MRR20, §4.2.2] for precise definitions and details).

7. The “topology”of dilatations of affine schemes

7.1 Constructing smooth complex affine varieties with controlled topology

Dilatations have played an important role in complex affine algebraic geometry during the nineties in connection to the construction and study of exotic complex affine spaces [KZ99, Za00], that is, smooth algebraic $\mathbb{C}$-varieties X of dimension n whose analytifications X^{an} are homeomorphic to the Euclidean space$\mathbb{R}^{2n}$ endowed with its standard structure of topological manifold but which are not isomorphic to the affine space $\mathbb{A}^n_{\mathbb{C}}$ as $\mathbb{C}$-varieties.

In this context, dilatations appeared under the name *affine modifications* and were used as a powerful tool to produce from a given smooth complex affine variety X a new smooth complex affine variety $X' = \mathrm{Bl}_Z^D X$ for which the homology or homotopy type of the underlying topological manifold of the analytification of X' can determined under suitable hypotheses in terms of those of X and of the center $\{[Z, D]\}$ of the dilatation.

The study of the strong topology of affine modifications was initiated in this context mainly by Kaliman through an analytic counter part of the notion of dilatation:

DEFINITION 7.1 [Ka94]. Given a triple (M, H, C) consisting of a complex analytic manifold M, a closed submanifold C of M of codimension at least 2 and a complex analytic hypersurface H of M containing C in its smooth locus, the *Kaliman modification M along H with center at C* is the complex analytic manifold defined as the complement M' of the proper transform H' of H in the blow-up $\sigma_C : \hat{M} \to M$ of M with center at C.

In the case where (M, H, C) is the analytification of a triple (X, D, Z) consisting of a smooth algebraic $\mathbb{C}$-variety X, a smooth algebraic sub-variety Y of X of codimenion at least two and of a reduced effective Cartier divisor D on X containing Z in its regular locus, the Kaliman modification of (M, H, C) coincides with the analytification of the dilatation $X' = \mathrm{Bl}_Z^D X$ of X with center $\{[Z, D]\}$ of Section 2.

Kaliman and Zaidenberg [KZ99] developed a series of tools to describe the topology of the analytications of affine modifications of smooth affine $\mathbb{C}$-varieties along principal divisors D with non-necessarily smooth centers. One of these provides in particular a control on the preservation of the topology of the analytification under affine modifications:

Theorem 7.2 [KZ99, Proposition 3.1 and Theorem 3.1]. *Let X be a smooth affine $\mathbb{C}$-variety and let $\{[Z,D]\}$ be a center on X consisting of closed sub-scheme Z of codimension 2 and of a principal effective divisor D containing Z as a closed subscheme. Let $\sigma : \tilde{X} = \mathrm{Bl}_Z^D(X) \to X$ be the dilatation of X with center $\{[Z,D]\}$ and let E be the exceptional divisor of σ.*

Assume that the following conditions are satisfied:

(i) The $\mathbb{C}$-variety $\tilde{X} = \mathrm{Bl}_Z^D(X)$ is smooth;

(ii) The divisors E' and D are irreducible, $E' = \sigma^* D$, and the analytifications of E'_{red} and D_{red} are topological manifolds.

Then the following properties hold:

(a) The homomorphism $\sigma_*^{\mathrm{an}} : \pi_1(\tilde{X}^{\mathrm{an}}) \to \pi_1(X^{\mathrm{an}})$ induced by σ^{an} is an isomorphism;

(b) The homomorphism $\sigma_*^{\mathrm{an}} : H_*(\tilde{X}^{\mathrm{an}};\mathbb{Z}) \to H_*(X^{\mathrm{an}};\mathbb{Z})$ induced by σ^{an} is an isomorphism if and only if the homomorphism $\sigma|_{E,*}^{\mathrm{an}} : H_*(E_{\mathrm{red}}^{\mathrm{an}};\mathbb{Z}) \to H_*(D_{\mathrm{red}}^{\mathrm{an}},\mathbb{Z})$ is.

Corollary 7.3. *In the setting of Theorem 7.2, assume further that X^{an} is a contractible smooth manifold, that $Z_{\mathrm{red}}^{\mathrm{an}}$ is a topological manifold and that the homomorphism $j_*^{\mathrm{an}} : H_*(Z_{\mathrm{red}}^{\mathrm{an}};\mathbb{Z}) \to H_*(D_{\mathrm{red}}^{\mathrm{an}};\mathbb{Z})$ induced by the closed immersion $j : Z \hookrightarrow D$ is an isomorphisms.*

Then the analytification of $\tilde{X} = \mathrm{Bl}_Z^D(X)$ is a contractible smooth manifold.

Having the flexibility to use as centers or divisors of modifications schemes which are either non-reduced or whose analytifications are not necessarily smooth manifolds but only topological manifolds is particularly relevant for applications to the construction smooth $\mathbb{C}$-varieties with contractible analytifications, as illustrated by the following examples.

Example 7.4 The tom Dieck - Petrie surfaces. . Let $p,q \geqslant 2$ be a pair of relatively prime integers and let $C_{p,q} \subset \mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x,y])$ be an irreducible rational cuspidal curve with equation $x^p - y^q = 0$. The underlying topological space of $C_{p,q}^{\mathrm{an}}$ is a contractible real topological surface, and hence, Corollary 7.3 applies to conclude that the analytification of the dilatation $S_{p,q} = \mathrm{Bl}_{(1,1)}^{C_{p,q}}\mathbb{A}^2_{\mathbb{C}}$ of $\mathbb{A}^2_{\mathbb{C}}$ along the principal Cartier divisor $D = C_{p,q}$ with center at the closed point $Z = (1,1) \in C_{p,q}$ is a smooth contractible real 4-manifold. The smooth affine surface $S_{p,q}$, which can be described explicitly as the hypersurface in $\mathbb{A}^3_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x,y,z])$ with equation

$$\frac{(xz+1)^p - (yz+1)^q}{z} = 1,$$

is not isomorphic to $\mathbb{A}^2_{\mathbb{C}}$ since, for instance, it has non negative logarithmic Kodaira dimension [Za00, Example 2.4]. Moreover, the underlying real 4-manifold of $S_{p,q}^{\mathrm{an}}$ is an example of a contractible 4-manifold with non-trivial fundamental group at infinity, hence non-homeomorphic to the standard euclidean space $\mathbb{R}^4$.

Example 7.5 Some Koras-Russell threefolds. Let again $p,q \geqslant 2$ be a pair of relative prime integers and consider for every $n \geqslant 2$ the smooth hypersurface $X_{p,q,n}$ in $\mathbb{A}^4_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x,y,z,w])$ with equation

$$x^n y + z^p + w^q + x = 0.$$

The restriction $\sigma_{p,q,n} : X_{p,q,n} \to \mathbb{A}^3_{\mathbb{C}}$ of the projection to the coordinates x, z and w expresses $X_{p,q,n}$ as the dilatation of $\mathbb{A}^3_k$ along the principal divisor $D_n = \mathrm{div} x^n$ and with center at the non-reduced condimension closed sub-scheme $Z_{p,q,n}$ with defining ideal $I_{p,q,n} = (z^p + w^q + x, x^n) \subset \mathbb{C}[x,z,w]$. The analytification of Z_{red} is a topological manifold homeomorphic to the underlying topological space of the curve $C_{p,q}^{\mathrm{an}}$ of the previous example. Thus, Corollary 7.3 applies to conclude that $X_{p,q,n}^{\mathrm{an}}$ is a contractible real 6-manifold, hence, by a result of Dimca–Ramanujam, is diffeomorphic to the standard euclidean space $\mathbb{R}^6$, see [Za00, Theorem 3.2].

The interest in these affine threefolds $X_{p,q,n}$ was motivated in the nineties by their appearance in the course of the study of the linearization problem for $\mathbb{G}_{m,\mathbb{C}}$ on $\mathbb{A}^3_{\mathbb{C}}$ by Koras and Russell [KR97]. One crucial question at that time was to decide whether these threefolds were isomorphic to $\mathbb{A}^3_{\mathbb{C}}$ or not.

The fact that none of them is isomorphic to $\mathbb{A}^3_{\mathbb{C}}$ was finally established by Makar-Limanov [ML96] by commutative algebra techniques. An interesting by-product of his proof is that the dilatation morphism $\sigma_{p,q,n} : X_{p,q,n} \to \mathbb{A}^3_{\mathbb{C}}$ is equivariant with respect to the natural action of the group of $\mathbb{C}$-automorphism of $X_{p,q,n}$, more precisely, $\sigma_{p,q,n}$ induces an isomorphism

$$\sigma_{p,q,n}^* : \mathrm{Aut}_{\mathbb{C}}(\mathbb{A}^3_{\mathbb{C}}, \{[Z_{p,q,n}, D_n]\}) \to \mathrm{Aut}_{\mathbb{C}}(X_{p,q,n})$$

between the subgroup $\mathrm{Aut}_{\mathbb{C}}(\mathbb{A}^3_{\mathbb{C}}, \{[Z_{p,q,n}, D_n]\})$ of $\mathrm{Aut}_{\mathbb{C}}(\mathbb{A}^3_{\mathbb{C}})$ consisting of $\mathbb{C}$-automorphisms preserving the divisor and the center of the dilatation $\sigma_{p,q,n}$ and the group $\mathrm{Aut}_{\mathbb{C}}(X_{p,q,n})$, see [MJ11].

7.2 Deformation to the normal cone

A very natural class of dilatations which plays a fundamental role in intersection theory is given by the affine version of the deformation space $D(X,Y)$ of a closed immersion $Y \hookrightarrow X$ of schemes of finite type over a fixed base scheme S to its normal cone, [Fu98], [Ro96, § 10]. Indeed, $D(X,Y)$ is simply the dilatation of $X \times_S \mathbb{A}^1_S$ with divisor $D = X \times_S \{0\}_S$, where $\{0\}_S$ denotes the zero section, and center $Z = Y \times_S \{0\}$. In the affine setting, say $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(A/I)$ for some ideal $I \subset A$, $D(X,Y)$ is the spectrum of the sub-algebra $A[t]$-algebra

$$A[\frac{(I,t)}{t}] \cong \sum_n I^n t^{-n} \subset A[t,t^{-1}].$$

The composition $f : D(X,Y) \to \mathbb{A}^1_S$ of the dilatation morphism $\sigma : D(X,Y) \to X \times_S \mathbb{A}^1_S$ with the projection $\mathrm{p}_2 : X \times_S \mathbb{A}^1_S \to \mathbb{A}^1_S$ is a flat morphism restricting to the trivial bundle $X \times_S (\mathbb{A}^1_S \setminus \{0\}_S)$ over $\mathbb{A}^1_S \setminus \{0\}_S$ and whose fiber over $\{0\}_S$ equals the normal cone $N_{Y/X} = \mathrm{Spec}(\bigoplus_{n \geqslant 0} I^n/I^{n+1})$ of the closed embedding $Y \hookrightarrow X$ (see Proposition 2.15). For regular immersions $Y \hookrightarrow X$ between smooth schemes of dimension n and m over a field k, the deformation space $D(X,Y)$ étale locally looks like the deformation space

$$\mathrm{Spec}(k[x_1, \ldots, x_m][t][u_1, \ldots, u_{m-n}]/(tu_i - x_i)_{i=1,\ldots,m-n}) \cong \mathbb{A}^{m+1}_k$$

of the immersion of $\mathbb{A}^n_k$ as the linear subspace $\{x_1 = \ldots = x_{m-n} = 0\}$ of $\mathbb{A}^m_k = \mathrm{Spec}(k[x_1, \ldots, x_m])$.

Deformation spaces of closed immersions between smooth affine $\mathbb{C}$-varieties provide an endless source of smooth affine $\mathbb{C}$-varieties whose analytifications are contractible smooth manifolds:

Example 7.6. Given a smooth affine $\mathbb{C}$-variety X such that X^{an} is contractible and a smooth subvariety $Y \subset X$ such that the induced inclusion $Y^{\mathrm{an}} \subset X^{\mathrm{an}}$ is a topological homotopy equivalence, Theorem 7.2 implies that the analytification of the deformation space $D(X,Y)$ is a contractible smooth manifold.

For instance, the deformation spaces $D(\mathbb{A}^3_{\mathbb{C}}, S_{p,q}) \subset \mathbb{A}^5_{\mathbb{C}}$ of the tom Dieck - Petrie surfaces $S_{p,q}$ of Example 7.4 are smooth affine $\mathbb{C}$-varieties of dimension 4 whose analytifications are all diffeomorphic to $\mathbb{R}^8$. In the same way, for every Koras-Russell threefold $X_{p,q,n} \subset \mathbb{A}^4_{\mathbb{C}}$ in Example

7.5, the deformation space

$$D(\mathbb{A}^4_{\mathbb{C}}, X_{p,q,n}) \cong \{tu = x^n y + z^p + w^q + x\} \subset \mathbb{A}^6_{\mathbb{C}}$$

is a smooth affine $\mathbb{C}$-variety whose analytification is diffeomorphic to $\mathbb{R}^{10}$. It is not known whether these deformation spaces are algebraically isomorphic to affine spaces.

More general versions of Kaliman and Zaidenberg techniques allow to fully describe the singular homology of the analytification of the deformation space $D(\mathbb{A}^n_{\mathbb{C}}, Z)$ of a smooth hypersurface $Z = Z(p) = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n]/(p))$ of $\mathbb{A}^n_{\mathbb{C}}$ in terms of that of the analytification of Z, namely

Proposition 7.7 [KZ99, Proposition 4.1]. *For a smooth hypersurface $Z \subset \mathbb{A}^n_{\mathbb{C}}$, $D(\mathbb{A}^n_{\mathbb{C}}, Z)^{\mathrm{an}}$ is simply connected and the inclusion $N_{Z/\mathbb{A}^n_{\mathbb{C}}} \hookrightarrow D(\mathbb{A}^n_{\mathbb{C}}, Z)$ induces an isomorphism of reduced homology groups $\tilde{H}_*(D(\mathbb{A}^n_{\mathbb{C}}, Z)^{\mathrm{an}}; \mathbb{Z}) \cong \tilde{H}_{*-2}(Z^{\mathrm{an}}; \mathbb{Z})$.*

In particular, $D(\mathbb{A}^n_{\mathbb{C}}, Z)^{\mathrm{an}}$ has the reduced homology type of the S^2-suspension of Z^{an}.

7.3 Contractible affine varieties in motivic $\mathbb{A}^1$-homotopy theory

The possibility to import Kaliman and Zaidenberg techniques in the framewok of Morel-Voevodsky $\mathbb{A}^1$-homotopy theory of schemes [MV99] has focused quite a lot of attention recently, especially in the direction of the construction of $\mathbb{A}^1$-contractible smooth affine varieties, motivated in part by possible applications to the Zariski Cancellation Problem, see [AØ21] and the reference therein for a survey.

Very informally, one views in this context smooth schemes over a fixed base field k as analogous to topological manifolds, with the affine line $\mathbb{A}^1_k$ playing the role of the unit interval, and consider the corresponding homotopy category. More rigorously, the $\mathbb{A}^1$-homotopy category $\mathrm{H}_{\mathbb{A}^1}(k)$ of k-schemes is defined as the left Bousfield localization of the injective Nisnevich-local model structure on the category of simplicial presheaves of sets on the category Sm_k of smooth k-schemes, with respect to the class of maps generated by projections from the affine line $\mathcal{X} \times_k \mathbb{A}^1_k \to \mathcal{X}$. Isomorphisms in the homotopy category $\mathrm{H}_{\mathbb{A}^1}(k)$ are called $\mathbb{A}^1$-*weak equivalences*, and a smooth k-scheme X is called $\mathbb{A}^1$-*contractible* if the structure morphism $X \to \mathrm{Spec}(k)$ is an isomorphism in $\mathrm{H}_{\mathbb{A}^1}(k)$.

The affine space $\mathbb{A}^n_k$ is by definition $\mathbb{A}^1$-contractible. Since the analytification of an $\mathbb{A}^1$-contractible smooth $\mathbb{C}$-variety is a contractible smooth manifold, smooth algebraic $\mathbb{C}$-varieties with contractible analytifications provided conversely a first natural framework to seek for interesting $\mathbb{A}^1$-contractible affine varieties non isomorphic to affine spaces. A first step in this direction was accomplished by Hoyois Krishna and Østvær [HKØ16, Theorem 4.2] who used the underlying geometry associated to the dilatations morphisms $\sigma_{p,q,n} : X_{p,q,n} \to \mathbb{A}^3_{\mathbb{C}}$ to verify that the Koras-Russell threefolds of Example 7.5 were $\mathbb{A}^1$-contractible possibly up to a finite number of $\mathbb{P}^1$-suspension, in the sense that for some $n \geqslant 0$, the suspension $(X_{p,q,n}, o) \wedge (\mathbb{P}^1)^{\wedge n}$ is an $\mathbb{A}^1$-contractible object in $\mathrm{H}_{\mathbb{A}^1}(\mathbb{C})$, where here, $X_{p,q,n} \subset \mathbb{A}^4_{\mathbb{C}}$ is considered as a pointed smooth $\mathbb{C}$-scheme with distinguished point $o = (0,0,0,0)$.

These first constructions motivated a more systematic study to obtain $\mathbb{A}^1$-homotopic analogues of Kaliman and Zaidenberg's topological comparison results for affine modifications. The best counterparts of Theorem 7.2 and Corollary 7.3 available so far are the following:

Theorem 7.8 [DPØ19, Theorem 2.17]. *Let (X, D, Z) be a triple in Sm_k where D is a Cartier divisor on X and $Z \subset D$ is a closed subscheme and let $\sigma : \tilde{X} = \mathrm{Bl}^{\mathrm{Z}}_{\mathrm{D}}\mathrm{X} \to \mathrm{X}$ be the dilatation of X along D with center at Z. Assume that the following conditions are satisfies:*

(i) The supports of D and of the exceptional divisor E of σ are irreducible;

(ii) The regular closed immersion $Z \hookrightarrow D$ is an $\mathbb{A}^1$-weak equivalence.

Then there is a naturally induced $\mathbb{A}^1$-weak equivalence $\Sigma_s\sigma : \Sigma_s\tilde{X} \to \Sigma_s X$ between the simplicial 1-suspensions of $\tilde{X}$ and X respectively.

In particular, if X is $\mathbb{A}^1$-contractible then $\Sigma_s\tilde{X}$ is $\mathbb{A}^1$-contractible. Moreover, a stronger conclusion holds in the reverse direction: if $\tilde{X}$ is $\mathbb{A}^1$-contractible then X is $\mathbb{A}^1$-contractible.

Corollary 7.9. *Let $i : Y \hookrightarrow X$ be a closed immersion between $\mathbb{A}^1$-contractible smooth k-schemes. Then the simplicial 1-suspension $\Sigma_s D(X,Y)$ of the deformation space $D(Y,X)$ of Y in X is $\mathbb{A}^1$-contractible.*

In contrast with the results of subsection 7.1 which can be applied to possibly singular triples (X,D,Z), Theorem 7.8 and its corollary fundamentally depend on smoothness hypotheses. In particular, Theorem 7.8 is not applicable to tom Dieck -Petrie surfaces and Koras-Russell threefolds over $\mathbb{C}$ and their natural generalization over other fields. It was nevertheless verified in [DF18] by different geometric methods that over any base field k of characteristic zero, the Koras-Russell threefolds $X_{p,q,n} = \{x^n y + z^p + w^q + x = 0\}$ are indeed all $\mathbb{A}^1$-contractible. These provide in turn when combined with Theorem 7.8 and Corollary 7.9 the building blocks for the construction of many other new examples of smooth affine k-varieties whose simplicial 1-suspensions are $\mathbb{A}^1$-contractible, among which some can be further verified by additional methods to be genuinely $\mathbb{A}^1$-contractible, see [DPØ19, Section 4].

A more detailed re-reading of the notion of deformation to the normal cone of a closed immersion $Y \hookrightarrow X$ between smooth k-schemes gives rise to a notion of "parametrized" deformation space over a smooth base k-scheme W, which is defined as a dilatation of the scheme $Y \times_k W$ with appropriate center, see [ADØ21, Construction 2.1.2]. This leads to the following counterpart and extension of Proposition 7.7 in the $\mathbb{A}^1$-homotopic framework:

Theorem 7.10 [ADØ21, Theorem 2]. *Let X be a smooth k-scheme, let $\pi : X \to \mathbb{A}^n_k$ be a smooth morphism with a section s. Assume that $\pi|_{\pi^{-1}(\mathbb{A}^n_k \setminus \{0\})} : \pi^{-1}(\mathbb{A}^n_k \setminus \{0\}) \to \pi^{-1}(\mathbb{A}^n_k \setminus \{0\})$ is an $\mathbb{A}^1$-weak equivalence. Then there exists an induced pointed $\mathbb{A}^1$-weak equivalence*

$$(X, s(0)) \sim (\mathbb{P}^1)^{\wedge n} \wedge (\pi^{-1}(0), s(0)).$$

In particular, the deformation space $D(X,Y)$ of a closed immersion $(Y,\star) \hookrightarrow (X,\star)$ between pointed smooth k-schemes is $\mathbb{A}^1$-weakly equivalent to $\mathbb{P}^1 \wedge (Y,\star)$.

Example 7.11. Let $Q_{2n} \subset \mathbb{A}^{2n+1}_k = \mathrm{Spec}(k[u_1,\ldots,u_n,v_1,\ldots,v_n,z])$ be the smooth $2n$-dimensional split quadric with equation $\sum_{i=1}^n u_i v_i = z(z+1)$. The projection $\pi = \mathrm{pr}_{u_1,\ldots,u_n} : Q_{2n} \to \mathbb{A}^{n+1}_k$ is a smooth morphism restricting to a Zariski locally trivial $\mathbb{A}^n$-bundle over $\mathbb{A}^n_k \setminus \{0\}$, hence to an $\mathbb{A}^1$-weak equivalence over $\mathbb{A}^n_k \setminus \{0\}$, and having the morphism

$$s : \mathbb{A}^n_k \to Q_{2n},\ (u_1,\ldots,u_n) \mapsto (u_1,\ldots,u_n,0,\ldots,0,0)$$

as a natural section. On the other hand, $\pi^{-1}(0)$ is $\mathbb{A}^1$-weakly equivalent to the disjoint union of $s(0)$ and of the point $p = (0,\ldots,0\ldots,-1)$. Theorem 7.10 thus renders the conclusion that $(Q_{2n}, s(0))$ is $\mathbb{A}^1$-weakly equivalent to $(\mathbb{P}^1)^{\wedge n}(p \sqcup s(0)) \sim (\mathbb{P}^1)^{\wedge n}$. In particular Q_{2n} provides a smooth k-scheme model of the motivic sphere $(\mathbb{P}^1)^{\wedge n} = S^n \wedge \mathbb{G}_{m,k}^{\wedge n}$.

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