

# FINITE TORSORS ON PROJECTIVE SCHEMES DEFINED OVER A DISCRETE VALUATION RING

PHÙNG HỒ HAI AND JOÃO PEDRO DOS SANTOS

ABSTRACT. Given a Henselian and Japanese discrete valuation ring  $A$  and a flat and projective  $A$ -scheme  $X$ , we follow the approach of [BdS11] to introduce a *full subcategory* of coherent modules on  $X$  which is then shown to be Tannakian. We then prove that, under normality of the generic fibre, the associated affine and flat group is pro-finite in a strong sense (so that its ring of functions is a Mittag-Leffler  $A$ -module) and that it classifies finite torsors  $Q \rightarrow X$ . This establishes an analogy to Nori's theory of the essentially finite fundamental group. In addition, we compare our theory with the ones recently developed by Mehta-Subramanian and Antei-Emsalem-Gasbarri. Using the comparison with the former, we show that any quasi-finite torsor  $Q \rightarrow X$  has a reduction of structure group to a finite one.

## 1. INTRODUCTION

Let  $A$  be a discrete valuation ring and  $X$  a projective flat  $A$ -scheme carrying an  $A$ -point  $x_0$ . There has been some interest in constructing an affine and flat group scheme  $\Pi(X, x_0)$  over  $A$  with the property that morphisms to finite and flat group schemes  $\Pi(X, x_0) \rightarrow G$  should canonically correspond to pointed  $G$ -torsors over  $X$ . See [Ga03], [AEG20] and [MS13]. This is an analogue Nori's theory [Nor76] (who deals with a base *field*) and hence can be developed through

- (SS) a Tannakian category of semi-stable vector bundles [Nor76, p. 37], through
- (F) the construction of fibre products of torsors [Nor82, II.1, Definition 2], or through
- (T) a Tannakian category of vector bundles trivialized by proper and surjective morphisms [BdS11].

The papers [Ga03] and [AEG20] follow (F), and [MS13] a variant of (SS). That is to say, [Ga03] and [AEG20] construct  $\Pi(X, x_0)$  by producing a universal  $\Pi(X, x_0)$ -torsor  $\tilde{X} \rightarrow X$ , and this is achieved by filtering the category of torsors under finite group schemes [AEG20, Theorem 4.2]. On the other hand, [MS13] looks at locally free  $\mathcal{O}_X$ -modules on  $X$  which lie in the categories considered by Nori when pulled-back to the fibres of  $X$  [MS13, top of p.346]. In the present work we focus on (T) to construct the affine and flat group scheme  $\Pi(X, x_0)$ . It is therefore useful to recall briefly the overall picture behind (T) in its original context. Given a smooth and projective variety  $V$  over a field, we look at those coherent sheaves which become trivial after being pulled-back along a projective and surjective morphism  $W \rightarrow V$ . Then, this category ends up *coinciding* with that of essentially finite vector bundles (used in (SS)) if  $V$  is at least normal, see [BdS11], [AM11], [TZ20].

Once the approach (T) is successfully implemented for  $X$  (details are to be reviewed further below) the category  $\text{Rep}_A(\Pi(X, x_0))$  is realised as a *full* abelian subcategory of  $\text{coh}(X)$  so that, not only we can construct the affine group scheme parametrising torsors, but we can naturally regard its category of representations in geometric terms. This facet

---

*Date:* May 31, 2022.

*2020 Mathematics Subject Classification.* 14F06, 14F35, 14L15, 14G20, 14L30 .

*Key words and phrases.* Discrete valuation rings, group schemes, Tannakian categories, coherent sheaves.

is missing in [AEG20] (since it is not understood what coherent modules are obtained by twisting representations via the fundamental torsor) as well as in [MS13] (since the authors there focus on representations on free  $A$ -modules).

Let us review the remaining sections of the paper in order to give the reader an idea of our most relevant results. Section 2 serves to gather some facts from algebraic geometry (Lemmas 2.1, 2.2 and 2.3), to fix notation for a class of morphisms which is used all over in the paper (see Definition 2.4) and to put forward criteria allowing to decide when a morphism of group schemes is faithfully flat (Lemma 2.8 and Lemma 2.9).

It is in Section 3 where our theory starts to take form. In it, we introduce the central notion of modules which are *relatively trivial* (Definition 3.1) and the category  $\mathfrak{T}_\varphi$  of modules which are *relatively trivialized* by a proper and surjective morphism  $\varphi$ . See Definition 3.4. The first useful properties of the category of such modules are also developed in Section 3: we show how to find more convenient trivializations (see Lemma 3.6, Definition 3.7 and Lemma 3.9) and how to make the first steps towards controlling kernels and cokernels of morphisms between relatively trivial modules (Proposition 3.10).

In Section 4, the work initiated in Section 3 is taken further ahead and we show that—under mild assumptions on the base scheme  $X$ —the category of modules which become relatively trivial after being pulled back along a suitable morphism  $\varphi : Y \rightarrow X$ , call it  $\mathfrak{T}_\varphi$ , is in fact abelian, see Theorem 4.3. In addition, concentrating on the objects in  $\mathfrak{T}_\varphi$  which can be “dominated” by locally free ones and using the fibre at an  $A$ -point  $x_0$  of  $X$ , we construct a neutral Tannakian category (terminology is that of [DuHa18, Definition 1.2.5]) over  $A$ . This category is called  $\mathfrak{T}_\varphi^{\text{tan}}$  in Definition 4.6-(iii). We consequently obtain an affine and flat group scheme  $\Pi(X, \varphi, x_0)$  (see [DuHa18, Theorem 1.2.6] or [Saa72, II.4.1.1]), whose category of representations is naturally a full subcategory of  $\mathbf{coh}(X)$ . This is the *first main result*.

If  $K$  stands for the field of fractions of  $A$  and  $X \otimes_A K$  is normal, Section 5 explains how to obtain information about  $\Pi(X, \varphi, x_0) \otimes_A K$  from the category of essentially finite vector bundles (as defined in [Nor76, Definition on pp 37-8]) on  $X \otimes_A K$ . See Theorem 5.3 and Corollary 5.4.

Section 6 studies  $\mathfrak{T}_\varphi$  (Definition 3.4) by means of an idea introduced in [HdS21]. In this work, we found a property of representations of a flat and affine group scheme  $G$  over a complete  $A$ , called prudence, which allows us to verify when the  $A$ -module of functions  $A[G]$  is free. This is useful here since prudence is roughly Grothendieck’s algebraization in geometry. This fact leads to our *second main result*, which is that the full Galois groups constructed from  $\mathfrak{T}_\varphi$  are in many cases *finite* group schemes, see Corollary 6.2. The findings here also serve to substantiate a claim made in Lemma 3.1 and Theorem 4.1 of [MS13]. (See also our summary of Section 11 below.)

Section 7 recalls the reduced fibre theorem and adapts it for future application, see Theorem 7.1. This theorem comes with the (mild) hypothesis that a certain generic fibre should be geometrically reduced; since we wish to profit from this result even in the absence of such an assumption, we benefit from this accessory section to explain how to get rid of schemes which fail to be geometrically reduced, see Lemma 7.2. It is perhaps worth pointing out that it is at this point that additional properties of  $A$ —that it be Henselian and Japanese—start being required.

Recall that in Section 3 and Section 4 we associated to a morphism  $\varphi : Y \rightarrow X$  (satisfying certain technical assumptions) a tensor category  $\mathfrak{T}_\varphi$  which, once polished, becomes neutral Tannakian. The task of putting the  $\mathfrak{T}_\varphi$  together for varying  $\varphi$  comes to the front. In Section 8 we address this issue. Our *third main result*, Theorem 8.1, provides the technical conditions for putting the  $\mathfrak{T}_\varphi$  together, while Corollary 8.2 expresses the consequence that the “system” of the  $\mathfrak{T}_\varphi$  is “filtered”. We are then able to construct a neutral Tannakian

category  $\mathfrak{T}_X$  which is a full subcategory of  $\mathbf{coh}(X)$ , see Definition 8.3. The group scheme associated to  $\mathfrak{T}_X$  via the fibre functor  $\bullet|_{x_0} : \mathfrak{T}_X \rightarrow A\text{-mod}$ , call it  $\Pi(X, x_0)$ , has the distinctive property that its category of representations  $\text{Rep}_A(\Pi(X, x_0))$  is equivalent, by construction, to the full subcategory  $\mathfrak{T}_X$  of  $\mathbf{coh}(X)$ . Under the assumption of normality of  $X$ , we are then able to show that  $\Pi(X, x_0)$  is *pro-quasi-finite* (in the sense of Definition 8.7), see Theorem 8.8. If  $A$  is in addition complete, the results of Section 6 apply directly and we obtain that  $\Pi(X, x_0)$  is *strictly* pro-finite, see Definition 8.7 and Theorem 8.8. Finally, an indirect argument allows us to deduce from this our *fourth main result*:  $\Pi(X, x_0)$  is *strictly* pro-finite even if  $A$  is simply Henselian and Japanese, see Theorem 8.10. As a consequence, the ring of functions of  $\Pi(X, x_0)$  is a Mittag-Leffler  $A$ -module (Corollary 8.11).

In Section 9 we make a brief digression to give examples and applications of the previous theory to invertible sheaves, see Proposition 9.1. These examples served as inspiration and may be useful to the reader, although they are not used anywhere else in the paper.

Section 10 gives conditions for a finite group scheme—appearing as the structure group of a torsor over  $X$ —to be a quotient of  $\Pi(X, x_0)$ , see Proposition 10.3 and Corollary 10.4.

In Section 11 we review parts of [MS13]. Let us explain the reason for this revision before summarising the contents of Section 11. If  $X_0$  stands for the special fibre of  $X$ , one of the main points of [MS13] is to use a technique in [DW05], see pages 574 to 576 there, to show, speaking colloquially, how to trivialise a vector bundle  $E \in \mathbf{VB}(X_0)$  on the special fibre of a *flat*  $A$ -scheme  $Y$  endowed with a finite morphism to  $X$ , see the *proof* of Lemma 3.1 in [MS13]. In our opinion, [MS13] only offers a sketch of how to implement this beautiful idea: in the proof of [MS13, Lemma 3.1] the necessary “base-change” to an extension of  $A$  is not mentioned, neither is [DW05, Theorem 17]. In addition, as far as we can see, the proof of Theorem 4.1 in [MS13]—which should extend Lemma 3.1 of op. cit.—offers solely a construction of a smooth curve inside a projective and smooth  $A$ -scheme. Given all these considerations, we set out to put the method of Deninger and Werner in a more robust form. In doing so we are able to avoid Liu’s theorem on the existence of semi-stable models of relative curves (a crucial point in [DW05] and hence in [MS13, Lemma 3.1]) by means of the reduced fibre theorem (Theorem 7.1). The advantage is that the latter result holds for more general schemes than just curves. This allows us to show Theorem 11.1 saying that an  $F$ -trivial vector bundle  $E$  (see the introduction of Section 11 for the definition) on  $X_0$  becomes relatively trivial (Definition 3.1) after pull back by a finite morphism  $Y \rightarrow X$  from a flat  $A$ -scheme. In the same vein, we offer Theorem 11.2 explaining how to trivialize any vector bundle  $\mathcal{E} \in \mathbf{VB}(X)$  with essentially finite fibres by a finite morphism  $Y \rightarrow X$  from a *flat*  $A$ -scheme. (It should be noted that the proof of this result is technically intricate, but the main idea comes straight from Theorem 11.1.) The section then ends with a direct comparison between the category  $\mathfrak{T}_X$  and the one studied by [MS13], see Corollary 11.4, where it is shown that vector bundles on  $\mathfrak{T}_X$  agree with those introduced in op. cit. This is our *fifth main finding*.

Section 12 gives further applications to the theory of torsors and shows our *sixth main result*, Theorem 12.1; it proves that torsors under quasi-finite group schemes actually come from finite ones. Another noteworthy point developed in this section is the comparison to the theory of [AEG20] and to the “classification” problem mentioned at the start of this introduction. By demonstrating that  $\Pi(X, x_0)$  “classifies” pointed quasi-finite torsors over  $X$ , which is our *seventh most relevant result* and Theorem 12.3, we show that  $\Pi(X, x_0)$  agrees with the pro-quasi-finite fundamental group, denoted  $\pi_1(X, x_0)^{\text{qf}}$ , in op.cit., see Theorem 12.4.

## NOTATIONS AND CONVENTIONS

### On the base ring.

- (1) We let  $A$  be a discrete valuation ring with uniformizer  $\pi$ , field of fractions  $K$  and residual field  $k$ . The quotient ring  $A/(\pi^{n+1})$  is denoted by  $A_n$ .
- (2) Given an object  $W$  over  $A$  (a scheme, a module, etc) and an  $A$ -algebra  $B$ , we find useful to write  $W_B$  instead of  $W \otimes_A B$ . If context prevents any misunderstanding, we also employ  $W_n$  instead of  $W_{A_n}$ .
- (3) The characteristic of a discrete valuation ring is the couple  $(r, p)$ , where  $r$  is the characteristic of the field of fractions and  $p$  that of the residue field.

### On general algebraic geometry.

- (1) A *vector bundle* over a scheme is a locally free sheaf of finite rank. A vector bundle is said to be trivial if it is isomorphic to a direct sum of a copies of the structure sheaf (in particular the rank is constant).
- (2) Let  $R$  be a noetherian ring and  $Y$  a proper  $R$ -scheme. We shall say that  $Y$  is  $H^0$ -flat over  $R$  if it is flat and cohomologically flat [BLR90, p.206] in degree zero over  $\text{Spec } R$ . In this work we shall employ constantly that if  $Y$  is  $R$ -flat, then (a)  $H^0$ -flatness amounts to exactness of  $M \mapsto H^0(\mathcal{O}_Y \otimes_R M)$  [Har77, III.12.5 and III.12.6] and (b) if the fibres of  $Y$  are geometrically reduced [EGA IV<sub>2</sub>, 4.6.2, p.68], then  $Y$  is  $H^0$ -flat over  $R$  [EGA III<sub>2</sub>, 7.8.6, p.206].
- (3) If  $R$ , respectively  $T$ , is an  $\mathbb{F}_p$ -algebra, respectively  $\mathbb{F}_p$ -scheme, we write  $F_R : R \rightarrow R$ , respectively  $F_T : T \rightarrow T$ , to denote the *Frobenius morphism*. If  $T$  is in addition a scheme over a perfect field, we adopt the notations of [J87, Part I, 9.1] with the exception that we write  $\text{Fr}^s : T^{(t)} \rightarrow T^{(s+t)}$  while Jantzen uses  $F_{T^{(t)}}^s : T^{(t)} \rightarrow T^{(t+s)}$ .
- (4) If  $R$  is a discrete valuation ring and  $y : \text{Spec } R \rightarrow Y$  is an  $R$ -point of a scheme  $Y$ , we shall write  $y_{\text{gen}}$  for the image of the generic point of  $\text{Spec } R$  in  $Y$ .
- (5) Let  $X \rightarrow S$  be a morphism of schemes possessing in addition a section  $x : S \rightarrow X$ . Given morphisms  $\varphi : X' \rightarrow X$  and  $f : S' \rightarrow S$ , any morphism of  $S$ -schemes  $x' : S' \rightarrow X'$  such that  $\varphi x' = x f$  shall be called an  $S'$ -point of  $X'$  above  $x$ . (That is, for the sake of economy we choose not to make base-changes to  $S'$ .)

### On group schemes.

- (1) To avoid repetitions, “*group scheme*” is a synonym for “*affine group scheme*.” If  $G$  is a group scheme over a ring  $R$ , and  $R'$  is an  $R$ -algebra, we write  $R'[G]$  instead of  $\mathcal{O}(R' \otimes_R G)$ .
- (2) The category of flat group schemes over a ring  $R$  is denoted by **(FGSch/ $R$ )**.
- (3) Let  $G$  be a flat group scheme over the noetherian ring  $R$ . When dealing with representations of  $G$  we follow the conventions of [J87, Part I, Ch. 2] with the exception that the word “representation” is reserved for  $G$ -modules which are of finite type over  $R$ . The category of representations is denoted by  $\text{Rep}_R(G)$ . The full subcategory of representations which underlie locally free  $R$ -modules is denoted by  $\text{Rep}_R^\circ(G)$ .
- (4) The *right-regular*, respectively *left-regular*,  $G$ -module [J87, 2.7] shall be denoted by  $R[G]_{\text{right}}$ , respectively  $R[G]_{\text{left}}$ .
- (5) If  $f : G \rightarrow H$  is an arrow of **(FGSch/ $R$ )**, we let  $f^\# : \text{Rep}_R(H) \rightarrow \text{Rep}_R(G)$  be the restriction functor.

### On torsors.

- (1) Let  $R$  be a ring,  $X$  an  $R$ -scheme,  $G$  and  $H$  group schemes over  $R$ ,  $P \rightarrow X$  a  $G$ -torsor and  $Q \rightarrow X$  an  $H$ -torsor. A *generalized morphism* from  $P$  to  $Q$  is a couple  $(f, \rho)$  consisting of an arrow of  $X$ -schemes  $f : P \rightarrow Q$  and a morphism of group schemes  $\rho : G \rightarrow H$  such that, for points with value on arbitrary  $R$ -algebras, we have  $f(yg) = f(y)\rho(g)$ . In this case, we say that the generalized morphism *covers*

- the morphism  $\rho$ . If  $G = H$ , then a morphism of torsors from  $P$  to  $Q$  is simply a  $G$ -equivariant morphism of  $X$ -schemes or a generalized morphism covering the identity.
- (2) If  $R$  is a ring,  $\rho : G \rightarrow H$  is an arrow of  $(\mathbf{FGSch}/R)$  and  $P \rightarrow X$  is a  $G$ -torsor, we let  $P \times^G H$  or  $P \times^\rho H$  be the associated  $H$ -torsor (see [DG70, III.4.3.2, p.368] or [J87, Part I, 5.14]).
  - (3) If  $G$  is a flat group scheme over a ring  $R$ ,  $P \rightarrow X$  is a  $G$ -torsor and  $M$  is a representation of  $G$ , then we let  $\theta_P(M)$  stand for the coherent sheaf constructed by twisting  $P$  by  $M$ , see [J87, Part 1, 5.8–9], where it is denoted by  $\mathcal{L}_{P/G}(M)$ .

### Miscellaneous.

- (1) All tensor categories and functors are to be taken in the sense of [DM82, §1]. Let  $(\mathfrak{C}, \otimes)$  be a rigid tensor category [DM82, Definition 1.7]. If  $\mathfrak{C}$  is additive, then, for  $a = (a_1, \dots, a_m)$  and  $b = (b_1, \dots, b_m)$  in  $\mathbb{N}^m$ , and  $E \in \mathfrak{C}$ , we write  $\mathbf{T}^{a,b}E$  to denote the object  $\bigoplus_i E^{\otimes a_i} \otimes \check{E}^{\otimes b_i}$ . If  $\mathfrak{C}$  is in addition abelian, we let  $\langle E; \mathfrak{C} \rangle_\otimes$  stand for the full subcategory of  $\mathfrak{C}$  whose objects are subquotients of some  $\mathbf{T}^{a,b}E$ .
- (2) If  $X$  is a proper and reduced scheme over a field, we let  $\mathbf{EF}(X)$  denote the category of essentially finite vector bundles on  $X$ . See [Nor76, Definition, p.37-8].

**Acknowledgements.** The research of PHH is supported by the Vietnam National Foundation for Science and Technology Development (Nafosted) grant number 101.04-2019.315, funded by Vingroup Joint Stock Company and supported by Vingroup Innovation Foundation (VinIF) under the project code VINIF.2021.DA00030. Parts of his research on this work were carried out during a visit to the Vietnam Institute for Advanced Study in Mathematics (VIASM) and several visits to France supported by the project ‘‘International Associated Laboratory FORMATH VIETNAM’’ of VAST, the Institute Henri Poincaré, Paris, and a ‘‘Poste Rouge’’ grant from the CNRS, France. JPdS thanks A. Ducros for pointing out [Tem10] and for useful discussions, Q. Liu for replying conscientiously to his e-mails concerning cohomological flatness, and I. Biswas as well as M. Romagny for answering some of his questions. The referees for this journal are also thanked for remarks which improved the exposition and helped in clarifying some arguments.

## 2. SUBSIDIARY MATERIAL

We collect here some simple facts which are useful in developing our arguments.

**Lemma 2.1.** *Let  $Y$  be a proper, reduced and connected scheme over the field  $K$ . If  $Y$  has a  $K$ -point then  $K = H^0(\mathcal{O}_Y)$ .*

*Proof.* We know that  $H^0(\mathcal{O}_Y) = R_1 \times \dots \times R_m$ , where each  $R_i$  is a local Artin algebra. Since  $Y$  is connected, we must have  $m = 1$ . So  $H^0(\mathcal{O}_Y)$  is a local Artin algebra. Since  $Y$  is reduced,  $H^0(\mathcal{O}_Y)$  is reduced, so  $H^0(\mathcal{O}_Y)$  is a finite field extension of  $K$ . The existence of a  $K$ -rational point determines a morphism of  $K$ -algebras  $H^0(\mathcal{O}_Y) \rightarrow K$ , and this forces  $H^0(\mathcal{O}_Y)$  to be  $K$ .  $\square$

**Lemma 2.2.** *Let  $Y$  be a flat and proper  $A$ -scheme. If  $H^0(\mathcal{O}_{Y \otimes_A K}) = K$ , then  $H^0(\mathcal{O}_Y) = A$ . In particular, if  $Y \otimes_A K$  is reduced and connected, and has a  $K$ -point, then  $H^0(\mathcal{O}_Y) = A$ .*

*Proof.* We know that  $H^0(\mathcal{O}_Y)$  is a finite  $A$ -algebra and that  $H^0(\mathcal{O}_Y) \otimes K = K$  by flat base-change [Har77, III.9.3, p. 255] together with the assumption. Then,  $A \subset H^0(\mathcal{O}_Y) \subset K$ . Since  $A$  is normal, we must have  $A = H^0(\mathcal{O}_Y)$ . The verification of the last claim follows immediately from Lemma 2.1.  $\square$

**Lemma 2.3.** *Let  $Y$  be a proper and  $H^0$ -flat  $A$ -scheme. Let  $\mathcal{F}$  be a vector bundle on  $Y$  whose base-changes  $\mathcal{F}_K$  and  $\mathcal{F}_k$  are trivial. Then  $\mathcal{F}$  is trivial.*

*Proof.* It is not difficult to see that the function  $y \mapsto \text{rank } \mathcal{F}_y$  has to be constant and equal to  $n$ , say. From the equalities

$$\begin{aligned} \dim_k H^0(Y_k, \mathcal{F}_k) &= \dim_k H^0(Y_k, \mathcal{O}_{Y_k})^{\oplus n} \\ &= \dim_K H^0(Y_K, \mathcal{O}_{Y_K})^{\oplus n} \\ &= \dim_K H^0(Y_K, \mathcal{F}_K), \end{aligned}$$

and Corollary 2 of [Mu70, §5], we conclude that the canonical morphism

$$H^0(Y, \mathcal{F}) \longrightarrow H^0(Y_k, \mathcal{F}_k)$$

is surjective. Pick sections  $s_1, \dots, s_n \in H^0(Y, \mathcal{F})$  such that  $\{s_i|_{Y_k}\}_{i=1}^n$  is a basis of  $\mathcal{F}_k$ . Then, for each point  $y$  of the closed fibre, Nakayama's Lemma shows that  $\{s_{i,y}\}_{i=1}^n$  is a basis of the free  $\mathcal{O}_y$ -module  $\mathcal{F}_y$  so that  $\text{Supp}(\mathcal{F} / \sum \mathcal{O}_Y s_i)$  is a closed subset of  $Y$  disjoint from the special fibre. Such a property is only possible if  $\mathcal{F} = \sum \mathcal{O}_Y s_i$ . The proof is then complete once we note that, for a  $y \in Y$ , the  $n$  generators  $\{s_{i,y}\}_{i=1}^n$  of  $\mathcal{F}_y \simeq \mathcal{O}_y^{\oplus n}$  do not admit any non-trivial relation.  $\square$

In organising our findings, we shall make repeated use of a certain class of morphisms of proper schemes. In order to avoid repetitions and to serve as a reference for the reader, we put forward:

**Definition 2.4.** Let  $X$  be a connected, proper and flat  $A$ -scheme carrying an  $A$ -point  $x_0$ . Let  $\mathfrak{S}(X, x_0)$  (or simply  $\mathfrak{S}$  if context prevents any misunderstanding) be the set of all  $X$ -schemes  $\varphi : Y \rightarrow X$  such that:

- Ⓐ1.  $\varphi$  is proper and surjective.
- Ⓐ2.  $Y$  is  $H^0$ -flat over  $A$ .
- Ⓐ3. The canonical arrow  $Y \rightarrow \text{Spec } H^0(\mathcal{O}_Y)$  admits a section  $y_0$  such that  $\varphi y_0$  extends  $x_0$ . (There exists an  $H^0(\mathcal{O}_Y)$ -point in  $Y$  above  $x_0$ .)

We shall denote by  $\mathfrak{S}^+(X, x_0)$  (or simply  $\mathfrak{S}^+$ ) the subset of morphisms  $\varphi : Y \rightarrow X$  in  $\mathfrak{S}$  which, in addition, satisfy

- Ⓐ4. The canonical morphism  $Y \rightarrow \text{Spec } H^0(\mathcal{O}_Y)$  is flat.

*Remark 2.5.* Let  $\varphi : Y \rightarrow X$  belong to  $\mathfrak{S}(X, x_0)$  as in Definition 2.4; it turns out that in this case, connectedness of  $X$  is automatic because of Ⓐ3. Indeed, let  $y_0$  be the alluded  $H^0(\mathcal{O}_Y)$ -point of  $Y$  above  $x_0$ . Suppose that  $e \in H^0(\mathcal{O}_X)$  is idempotent and that  $x_0^\#(e) = 0$ . Then,  $\varphi^\#(e) \in H^0(\mathcal{O}_Y)$  is such that  $y_0^\#(\varphi^\#(e)) = 0$  and this implies that  $\varphi^\#(e) = 0$ . Since  $\varphi$  is surjective, we conclude that  $e = 0$  (since  $e$  has to vanish on each local ring of  $X$ ). Hence, the only connected component of  $X$  must be the one containing the image of  $x_0$ .

We should now gather material on affine group schemes.

**Definition 2.6.** Let  $G \in (\mathbf{FGSch}/A)$ . We say that  $G$  is pseudo-finite if both its fibres are finite group schemes over the respective residue fields.

*Remark 2.7.* We have no examples of pseudo-finite group schemes which are not quasi-finite to offer.

As Lemma 2.8 below recalls, one advantage of finite group schemes over a field is that the standard criterion for verifying when a morphism is faithfully flat in terms of representation categories [DM82, Proposition 2.21, p.139] admits a considerable simplification. This is then transmitted to pseudo-finite group schemes as argued by Lemma 2.9.

**Lemma 2.8** ([Nor82, pp. 87-8]). *Let  $u : H \rightarrow G$  be a morphism of group schemes over a field and let*

$$H \xrightarrow{q} I \xrightarrow{i} G$$

be its factorisation into a faithfully flat morphism  $q$  and a closed immersion  $i$ . Assuming that  $I$  is finite (which is the case if either  $G$  or  $H$  is finite) then a necessary and sufficient condition for  $u$  to be faithfully flat is that  $\dim \mathcal{O}(G)^H = 1$ .

*Proof.* We leave the proof of necessity in the statement to the reader and from now on assume that  $\dim \mathcal{O}(G)^H = 1$ . Clearly, the equality  $I = G$  is equivalent to faithful flatness of  $u$ . Since  $\mathcal{O}(G)^H = \mathcal{O}(G)^I$ , the assumptions translate into  $\dim \mathcal{O}(G)^I = 1$ . Now, we know that  $\mathcal{O}(G)$  is a projective  $\mathcal{O}(G)^I$ -module whose rank equals  $\dim \mathcal{O}(I)$  (see [Mu70, §11, Theorem 1(B), p.111] or III.2.4 of [DG70]). Then, since  $\dim \mathcal{O}(G)^I = 1$ , we conclude that  $\dim \mathcal{O}(I) = \dim \mathcal{O}(G)$ , which means that  $I = G$ .  $\square$

**Lemma 2.9.** *Let  $u : H \rightarrow G$  be an arrow in  $(\mathbf{FGSch}/A)$  and assume that  $G$  is pseudo-finite. Then the following conditions are equivalent.*

(1) *Both equalities*

$$A \cdot 1 = (A[G]_{\text{right}})^H \quad \text{and} \quad k \cdot 1 = (k[G]_{\text{right}})^H ,$$

*are true.*

(2) *The morphism  $u$  is faithfully flat.*

(3) *Let*

$$\mathfrak{s}(G) = \left\{ \begin{array}{l} V \subset A[G]_{\text{right}} \text{ is } G\text{-invariant, is finitely generated} \\ \text{as an } A\text{-module and contains the constants} \end{array} \right\}$$

*and*

$$\mathfrak{s}_0(G) = \left\{ \begin{array}{l} M \subset k[G]_{\text{right}} \text{ is } G\text{-invariant, is finitely generated} \\ \text{as a } k\text{-space and contains the constants} \end{array} \right\} .$$

*Then, for each  $V \in \mathfrak{s}(G)$  and  $M \in \mathfrak{s}_0(G)$  we have*

$$A \cdot 1 = V^H \quad \text{and} \quad k \cdot 1 = M^H .$$

*Proof.* (1)  $\Rightarrow$  (2). The assumptions show that

$$(K[G]_{\text{right}})^{H_K} = K \quad \text{and} \quad (k[G]_{\text{right}})^{H_k} = k .$$

By Lemma 2.8 we conclude that  $u_K : H_K \rightarrow G_K$  and  $u_k : H_k \rightarrow G_k$  are faithfully flat. Because of [DuHa18, Theorem 4.1.1],  $u$  is faithfully flat.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). Any  $a \in A[G]_{\text{right}}$ , respectively  $b \in k[G]_{\text{right}}$ , belongs to a certain  $V \in \mathfrak{s}(G)$ , respectively  $M \in \mathfrak{s}_0(G)$  because of ‘‘local finiteness’’ [Se68, 1.5, Corollary, p.40]. The conclusion then follows.  $\square$

### 3. MODULES TRIVIALIZED BY A PROPER AND $H^0$ -FLAT SCHEME

In this section we introduce the category of coherent sheaves on which all further developments hinge: the category of sheaves which became ‘‘trivial’’ after a pull-back by a proper morphisms. Since we wish to work with schemes over a d.v.r., the notion of triviality of a coherent module becomes more subtle than the one over a field, and we need to account for modules coming from the base-ring. This is a source of difficulty specially because the ‘‘base’’ of the scheme effectuating the trivialisation might grow.

**Definition 3.1.** Let  $Y$  be any  $A$ -scheme. We say that  $\mathcal{F} \in \mathbf{coh}(Y)$  is trivial relatively to  $A$  if there exists a coherent sheaf (a finite  $A$ -module)  $F$  such that  $\mathcal{F} \cong \mathcal{O}_Y \otimes_A F$ .

*Remark 3.2.* Note that, if  $A$  is a field, then a coherent sheaf on  $Y$  is trivial relatively to  $A$  if and only if it is trivial in according to our conventions in Section 1.

In the above definition, it is to be expected that several different choices concerning the “descended” module are possible. The next lemma, which we state here for future use, explains how to be more canonical.

**Lemma 3.3.** *Let  $Y$  be proper and  $H^0$ -flat over  $A$ . Write  $B = H^0(\mathcal{O}_Y)$  and let  $F$  be a finite  $A$ -module. Then the canonical morphism*

$$\mathcal{O}_Y \otimes_B H^0(\mathcal{O}_Y \otimes_A F) \longrightarrow \mathcal{O}_Y \otimes_A F$$

*is an isomorphism.*

*Proof.* By definition of  $H^0$ -flatness, the canonical arrow

$$\sigma : B \otimes_A F \longrightarrow H^0(\mathcal{O}_Y \otimes_A F)$$

is bijective. We then consider the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_Y \otimes_B H^0(\mathcal{O}_Y \otimes_A F) & \longrightarrow & \mathcal{O}_Y \otimes_A F \\ \uparrow \sim \text{id} \otimes \sigma & \nearrow \sim & \\ \mathcal{O}_Y \otimes_B (B \otimes_A F) & & \end{array}$$

where the horizontal arrow is the one of the statement.  $\square$

Here is the central definition of this section.

**Definition 3.4.** If  $\varphi : Y \rightarrow X$  is a morphism of  $A$ -schemes, we let  $\mathfrak{T}_\varphi$  stand for the full subcategory of  $\mathbf{coh}(X)$  consisting of those coherent sheaves  $\mathcal{E}$  such that the coherent  $\mathcal{O}_Y$ -module  $\varphi^*(\mathcal{E})$  is trivial relatively to  $A$ .

Let  $X$  be a connected, proper and flat  $A$ -scheme, and  $x_0$  be an  $A$ -point of  $X$ . Let  $\varphi : Y \rightarrow X$  an object of  $\mathfrak{S}(X, x_0)$  (see Definition 2.4). If  $B = H^0(\mathcal{O}_Y)$ , we let  $y_0$  stand for the  $B$ -point of  $Y$  above  $x_0$ . We note that the  $A$ -module  $B$  is finite and torsion-free, hence free. The following will be useful further ahead.

**Lemma 3.5.** *In the above notation, a coherent module  $\mathcal{F} \in \mathbf{coh}(X)$  belongs to  $\mathfrak{T}_\varphi$  if and only if  $\varphi^*(\mathcal{F})$  is trivial relatively to  $B$ .*

*Proof.* We only show the “if” clause. In this case,  $\varphi^*(\mathcal{F}) \simeq \mathcal{O}_Y \otimes_B F$  and using the point  $y_0$ , we conclude that  $F \simeq \varphi^*(\mathcal{F})|_{y_0}$ . Now, since  $y_0$  is taken to the  $A$ -point  $x_0$ , we have that  $\varphi^*(\mathcal{F})|_{y_0} \simeq B \otimes_A (\mathcal{F}|_{x_0})$  so that  $\varphi^*(\mathcal{F}) \simeq \mathcal{O}_Y \otimes_A (\mathcal{F}|_{x_0})$ .  $\square$

We now give ourselves an object  $\mathcal{E}$  of  $\mathfrak{T}_\varphi$ : this means that  $\varphi^*(\mathcal{E})$  is isomorphic to the pull-back of a certain  $A$ -module to  $Y$ . We shall notice that  $\mathcal{E}|_{x_0}$  is a possible choice.

**Lemma 3.6.** *Let  $E := \mathcal{E}|_{x_0}$ . Then there exists an isomorphism  $\alpha : \mathcal{O}_Y \otimes_A E \xrightarrow{\sim} \varphi^*(\mathcal{E})$ . In addition, if  $\bar{E}$  is an  $A$ -module such that  $\varphi^*(\mathcal{E}) \simeq \mathcal{O}_Y \otimes_A \bar{E}$ , then  $\bar{E}$  must in fact be isomorphic to  $E$ .*

*Proof.* Write  $\varphi^*(\mathcal{E}) \simeq \mathcal{O}_Y \otimes_A \bar{E}$  for some other  $A$ -module  $\bar{E}$ . Using  $y_0$ , we conclude that  $B \otimes_A E \simeq B \otimes_A \bar{E}$ . Now this shows that  $\mathcal{O}_Y \otimes_A E \simeq \mathcal{O}_Y \otimes_A \bar{E}$  because  $\mathcal{O}_Y \otimes_B (B \otimes_A (-)) = \mathcal{O}_Y \otimes_A (-)$ . To handle the final statement, the isomorphism  $B \otimes_A E \simeq B \otimes_A \bar{E}$  is sufficient. Here are the details. Agreeing to write  $A(\ell) = A/(\pi^\ell)$ , we have

$$E \simeq A^r \oplus A(\delta_1) \oplus \cdots \oplus A(\delta_m) \quad \text{and} \quad \bar{E} \simeq A^{\bar{r}} \oplus A(\bar{\delta}_1) \oplus \cdots \oplus A(\bar{\delta}_m);$$



here  $m$  is either a positive integer, in which case  $\delta_1 \geq \dots \geq \delta_m$  are also positive integers, or  $m = 0$  and the factors  $A(\delta_i)$  are to be dropped, and analogous considerations are in force for  $\bar{E}$ . As  $B$  is free over  $A$ , of rank  $s \geq 1$  say, we have, as  $A$ -modules,

$$B \otimes_A E \simeq A^{rs} \oplus \underbrace{A(\delta_1) \oplus \dots \oplus A(\delta_1)}_s \oplus \dots \oplus \underbrace{A(\delta_m) \oplus \dots \oplus A(\delta_m)}_s$$

and

$$B \otimes_A \bar{E} \simeq A^{\bar{r}s} \oplus \underbrace{A(\bar{\delta}_1) \oplus \dots \oplus A(\bar{\delta}_1)}_s \oplus \dots \oplus \underbrace{A(\bar{\delta}_m) \oplus \dots \oplus A(\bar{\delta}_m)}_s,$$

so that the isomorphism  $B \otimes_A E \simeq B \otimes_A \bar{E}$  implies that  $r = \bar{r}$ ,  $m = \bar{m}$  and  $\delta_i = \bar{\delta}_i$ .  $\square$

We observe that the latter part of the above Lemma shall not be employed in the rest of this text. Now, we note that while an  $A$ -module such that  $\varphi^*(\mathcal{E}) \simeq \mathcal{O}_Y \otimes_A E$  is uniquely determined, one can find many isomorphisms  $\mathcal{O}_Y \otimes_A E \simeq \varphi^*(\mathcal{E})$  and we now single out a *special class* of such so to achieve a proof of Proposition 3.10 below. This Proposition is a key step in endowing the category  $\mathfrak{T}_\varphi$  with kernels and cokernels (see Section 4).

From now on we write

$$\boxed{E := \mathcal{E}|_{x_0} \quad \text{and} \quad F = \varphi^*(\mathcal{E})|_{y_0}.}$$

Using  $y_0$  and the fact that it “extends”  $x_0$ , we arrive at canonical isomorphisms

$$(1) \quad \iota : F \xrightarrow{\sim} B \otimes_A E \quad \text{and} \quad I : \mathcal{O}_Y \otimes_B F \xrightarrow{\sim} \mathcal{O}_Y \otimes_A E.$$

**Definition 3.7.** An isomorphism  $\beta : \mathcal{O}_Y \otimes_B F \rightarrow \varphi^*(\mathcal{E})$  is called *adapted to  $y_0$*  if the composition

$$F \xrightarrow{\text{canonical}} (\mathcal{O}_Y \otimes_B F)|_{y_0} \xrightarrow{\beta} \varphi^*(\mathcal{E})|_{y_0}$$

is the identity.

We shall show that adapted isomorphisms exist. For that and future applications, we need:

**Lemma 3.8.** *Let  $\bar{\mathcal{E}}$  be another object of  $\mathfrak{T}_\varphi$  and write  $\bar{F}$  for the  $B$ -module  $\varphi^*(\bar{\mathcal{E}})|_{y_0}$ . Then the pull-back morphism*

$$\text{id} \otimes (-) : \text{Hom}_B(F, \bar{F}) \longrightarrow \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y \otimes_B F, \mathcal{O}_Y \otimes_B \bar{F})$$

is bijective with inverse

$$(-)|_{y_0} : \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y \otimes_B F, \mathcal{O}_Y \otimes_B \bar{F}) \longrightarrow \text{Hom}_B(F, \bar{F}).$$

If  $\beta : \mathcal{O}_Y \otimes_B F \xrightarrow{\sim} \varphi^*(\mathcal{E})$  and  $\bar{\beta} : \mathcal{O}_Y \otimes_B \bar{F} \xrightarrow{\sim} \varphi^*(\bar{\mathcal{E}})$  are adapted, then the composition

$$\text{Hom}_{\mathcal{O}_Y}(\varphi^*(\mathcal{E}), \varphi^*(\bar{\mathcal{E}})) \xrightarrow{\bar{\beta}^{-1}(-)\beta} \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y \otimes_B F, \mathcal{O}_Y \otimes_B \bar{F}) \xrightarrow{(-)|_{y_0}} \text{Hom}_B(F, \bar{F})$$

is  $(-)|_{y_0}$ .

*Proof.* Let  $\bar{\iota} : \bar{F} \xrightarrow{\sim} B \otimes_A \bar{F}$  and  $\bar{I} : \mathcal{O}_Y \otimes_B \bar{F} \xrightarrow{\sim} \mathcal{O}_Y \otimes_A \bar{E}$  be the isomorphisms analogous to those in (1). It follows easily that

$$\begin{array}{ccc}
 \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y \otimes_B F, \mathcal{O}_Y \otimes_B \bar{F}) & \xrightarrow{\sim} & \mathrm{Hom}_B(F, H^0(\mathcal{O}_Y \otimes_B \bar{F})) & \xrightarrow[\text{via } \bar{I}]{\sim} & \mathrm{Hom}_B(F, H^0(\mathcal{O}_Y \otimes_A \bar{E})) \\
 & & \searrow \text{id} \otimes (-) & & \uparrow \sim \\
 & & & & \mathrm{Hom}_B(F, B \otimes_A \bar{E}) \\
 & & & & \uparrow \sim \\
 & & & & \mathrm{Hom}(F, \bar{F})
 \end{array}$$

commutes; this is enough to justify all claims of the first statement. Those of the second are very simple.  $\square$

**Lemma 3.9.** *Isomorphisms adapted to  $y_0$  always exist.*

*Proof.* We choose any isomorphism  $\beta : \mathcal{O}_Y \otimes_B F \xrightarrow{\sim} \varphi^*(\mathcal{E})$  and let  $b$  be the composition

$$F \xrightarrow{\text{canonical}} (\mathcal{O}_Y \otimes_B F)|_{y_0} \xrightarrow{\beta|_{y_0}} \varphi^*(\mathcal{E})|_{y_0}.$$

Let  $\gamma : \mathcal{O}_Y \otimes_B F \xrightarrow{\sim} \mathcal{O}_Y \otimes_B F$  induce  $b^{-1}$  via  $(-)|_{y_0}$  (which exists due to Lemma 3.8). Then  $\beta\gamma$  is adapted to  $y_0$ .  $\square$

Granted these preparations, we can now have a better control on kernels and cokernels in  $\mathfrak{T}_\varphi$ .

**Proposition 3.10.** *Let  $\bar{\mathcal{E}}$  be another object of  $\mathfrak{T}_\varphi$  and  $u : \mathcal{E} \rightarrow \bar{\mathcal{E}}$  a morphism of  $\mathcal{O}_X$ -modules. Write  $\bar{F}$  for  $\varphi^*(\bar{\mathcal{E}})|_{y_0}$ . Let  $\beta : \mathcal{O}_Y \otimes_B F \xrightarrow{\sim} \varphi^*(\mathcal{E})$  and  $\bar{\beta} : \mathcal{O}_Y \otimes_B \bar{F} \xrightarrow{\sim} \varphi^*(\bar{\mathcal{E}})$  be adapted to  $y_0$ . The following claims hold true.*

(1) *If  $v_0 := \varphi^*(u)|_{y_0} : F \rightarrow \bar{F}$ , then*

$$\varphi^*(u) = \bar{\beta} \circ (\mathrm{id} \otimes_B v_0) \circ \beta^{-1}.$$

*Put differently,  $\varphi^*(u)$  “descends” to  $B$ .*

(2) *Write  $\bar{E}$  for the  $A$ -module  $\bar{\mathcal{E}}|_{x_0}$ . Let  $\bar{\iota} : \bar{F} \xrightarrow{\sim} B \otimes_A \bar{E}$  and  $\bar{I} : \mathcal{O}_Y \otimes_B \bar{F} \xrightarrow{\sim} \mathcal{O}_Y \otimes_A \bar{E}$  be the isomorphisms analogous to those in (1). Define  $\alpha = \beta I^{-1}$  and  $\bar{\alpha} = \bar{\beta} \bar{I}^{-1}$ . Then, the morphism of  $A$ -modules  $u_0 := u|_{y_0} : E \rightarrow \bar{E}$  renders*

$$\begin{array}{ccc}
 \varphi^* \mathcal{E} & \xrightarrow{\varphi^* u} & \varphi^* \bar{\mathcal{E}} \\
 \alpha \uparrow & & \uparrow \bar{\alpha} \\
 \mathcal{O}_Y \otimes_A E & \xrightarrow[\text{id} \otimes u_0]{} & \mathcal{O}_Y \otimes_A \bar{E}
 \end{array}$$

*commutative. Put differently,  $\varphi^*(u)$  “descends” to  $A$ .*

(3) *The coherent sheaves  $\mathrm{Ker}(\varphi^* u)$  and  $\mathrm{Coker}(\varphi^* u)$  are trivial relatively to  $A$ .*

*Proof.* Claim (1) is a direct consequence of Lemma 3.8. Claim (2) is a simple consequence of (1). Claim (3) follows from (2) together with the fact that  $Y$  is flat over  $A$ .  $\square$

4. FURTHER PROPERTIES OF THE CATEGORY OF COHERENT MODULES TRIVIALIZED BY A PROPER MORPHISM

In this section, we let  $X$  be a proper and flat  $A$ -scheme with *reduced fibres*, and  $x_0$  an  $A$ -point of  $X$ . Let  $\varphi : Y \rightarrow X$  an object of  $\mathfrak{S}(X, x_0)$ . (Recall that in such a situation  $X$  is connected, see Remark 2.5.) If  $B = H^0(\mathcal{O}_Y)$ , we let  $y_0$  stand for the  $B$ -point of  $Y$  above  $x_0$ . Let us gather some simple properties concerning the category  $\mathfrak{T}_\varphi$  of Definition 3.4.

**Lemma 4.1.** *Let  $u : \mathcal{E} \rightarrow \mathcal{F}$  be arrow of  $\mathfrak{T}_\varphi$ . Then  $\mathcal{C} = \text{Coker}(u)$  belongs to  $\mathfrak{T}_\varphi$ .*

*Proof.* As  $\text{Coker}(\varphi^*u) \simeq \varphi^*(\text{Coker}(u))$  this is a straightforward consequence of Proposition 3.10-(3).  $\square$

**Lemma 4.2.** *If  $\mathcal{E} \in \mathfrak{T}_\varphi$  is  $A$ -flat, then  $\mathcal{E}$  is a vector bundle.*

*Proof.* By assumption,  $X_k$  and  $X_K$  are reduced schemes. It follows from [BdS11, Remarks (a), p.226] that  $\mathcal{E}_k$  and  $\mathcal{E}_K$  are *vector bundles*. (We note that the context in op.cit. presupposes the ground field to be algebraically closed, but this is not necessary for the proof to work.) Using the ‘‘fibre-by-fibre’’ flatness criterion [EGA IV<sub>3</sub>, 11.3.10, p.138], we are done.  $\square$

We are now ready to state the main structure theorem concerning  $\mathfrak{T}_\varphi$ .

**Theorem 4.3.** *The category  $\mathfrak{T}_\varphi$  is abelian, and the inclusion functor  $\mathfrak{T}_\varphi \rightarrow \mathbf{Coh}(X)$  is exact.*

The main point of the proof of Theorem 4.3, given Lemma 4.1, is to show that the kernel of an arrow of  $\mathfrak{T}_\varphi$  is also in  $\mathfrak{T}_\varphi$  and, by Proposition 3.10, all we need to do is to relate  $\varphi^*(\text{Ker}(u))$  and  $\text{Ker}(\varphi^*(u))$  for an arrow  $u$ . The argument takes up the ensuing lines.

First, recall that an  $\mathcal{O}_X$ -module  $\mathcal{M}$  is said to be  $\varphi^*$ -acyclic if the left derived functors  $L_i\varphi^*(\mathcal{M})$  vanish for  $i > 0$ . (These functors are obtained by means of resolutions by flat  $\mathcal{O}$ -modules, see [Har66, 99ff]. In addition, note that from [Har66, Proposition 4.4, p.99], if  $\mathcal{M}$  is coherent, then  $L_i\varphi^*(\mathcal{M})$  is also coherent.)

**Proposition 4.4.** *Any  $\mathcal{E}$  in  $\mathfrak{T}_\varphi$  is  $\varphi^*$ -acyclic.*

*Proof.* Let us first assume that  $\pi\mathcal{E} = 0$  so that  $\mathcal{E}$  is by assumption an  $\mathcal{O}_{X_k}$ -module which becomes trivial after being pulled-back to  $Y_k$ . Then, we know that  $\mathcal{E}$  is locally isomorphic to  $\mathcal{O}_{X_k}^r$  [BdS11, Remarks (a), p.226]. Hence, there exists a faithfully flat morphism  $\alpha : \overline{X} \rightarrow X$  (given by a disjoint union of open subsets of  $X$ ) such that  $\alpha^*(\mathcal{E}) \simeq \mathcal{O}_{\overline{X}_k}^r$ .

Employing the notations introduced in the following cartesian diagram

$$\begin{array}{ccc} \overline{Y} & \xrightarrow{\beta} & Y \\ \overline{\varphi} \downarrow & \square & \downarrow \varphi \\ \overline{X} & \xrightarrow{\alpha} & X \end{array}$$

we now prove that  $\mathcal{O}_{\overline{X}_k}^r$  is  $\overline{\varphi}^*$ -acyclic. The exact sequence

$$(2) \quad 0 \longrightarrow \mathcal{O}_{\overline{X}}^r \xrightarrow{\pi} \mathcal{O}_{\overline{X}}^r \longrightarrow \mathcal{O}_{\overline{X}_k}^r \longrightarrow 0,$$

gives rise to an exact sequence

$$0 \longrightarrow L_1\overline{\varphi}^*(\mathcal{O}_{\overline{X}_k}^r) \longrightarrow \overline{\varphi}^*(\mathcal{O}_{\overline{X}}^r) \xrightarrow{\pi} \overline{\varphi}^*(\mathcal{O}_{\overline{X}}^r) \longrightarrow \overline{\varphi}^*(\mathcal{O}_{\overline{X}_k}^r) \longrightarrow 0,$$

and this proves that  $L_1\overline{\varphi}^*(\mathcal{O}_{\overline{X}_k}^r) = 0$  as  $\overline{\varphi}^*(\mathcal{O}_{\overline{X}}^r) \simeq \mathcal{O}_{\overline{Y}}^r$  is  $A$ -flat. From sequence (2), we have, for any  $i \geq 1$ , another exact sequence

$$0 = L_{i+1}\overline{\varphi}^*(\mathcal{O}_{\overline{X}}^r) \longrightarrow L_{i+1}\overline{\varphi}^*(\mathcal{O}_{\overline{X}_k}^r) \longrightarrow L_i\overline{\varphi}^*(\mathcal{O}_{\overline{X}}^r) = 0;$$

this allows us to conclude that  $\mathcal{O}_{\overline{X}_k}^r$  is  $\overline{\varphi}^*$ -acyclic. As  $\alpha^*$  and  $\beta^*$  are exact, we know that  $\beta^* \circ L_i \varphi^* \simeq L_i \overline{\varphi}^* \circ \alpha^*$ , and therefore we can say that, for each  $i > 0$ ,

$$\begin{aligned} \beta^* \circ L_i \varphi^*(\mathcal{E}) &= L_i \overline{\varphi}^* \circ \alpha^*(\mathcal{E}) \\ &\simeq L_i \overline{\varphi}^*(\mathcal{O}_{\overline{X}_k}^r) \\ &= 0. \end{aligned}$$

But  $L_i \varphi^*(\mathcal{E})$  is coherent and  $\beta$  is faithfully flat, so  $\beta^*(L_i \varphi^*(\mathcal{E})) = 0$  implies  $L_i \varphi^*(\mathcal{E}) = 0$  and we have finished the verification that  $\mathcal{E}$  is  $\varphi^*$ -acyclic.

For any  $\mathcal{M} \in \mathbf{Coh}(X)$ , let us agree to write

$$\mathrm{Tors}(\mathcal{M}) = \bigcup_{m \geq 1} \mathrm{Ker} \pi^m : \mathcal{M} \rightarrow \mathcal{M}$$

for the sheaf of sections annihilated by some power of  $\pi$  and define

$$t(\mathcal{M}) := \min\{m \in \mathbb{N} : \pi^m \cdot \mathrm{Tors}(\mathcal{M}) = 0\}.$$

We shall show by induction on  $t(\mathcal{E})$  that  $\mathcal{E}$  is  $\varphi^*$ -acyclic. If  $t(\mathcal{M}) = 0$ , then  $\mathrm{Tors}(\mathcal{M}) = 0$ , so that, due to Lemma 4.2,  $\mathcal{E}$  is a vector bundle and a fortiori  $\varphi^*$ -acyclic.

Now, suppose that  $t(\mathcal{E}) \geq 1$  and that for all  $\mathcal{F} \in \mathfrak{T}_\varphi$  with  $t(\mathcal{F}) < t(\mathcal{E})$ , the  $\mathcal{O}_X$ -module  $\mathcal{F}$  is  $\varphi^*$ -acyclic.

Write  $\mathcal{E}' := \pi \mathcal{E}$ . Then, if  $e' \in \mathrm{Tors}(\mathcal{E}')(U)$  over some affine  $U$ , it follows that  $e' = \pi e$  where  $e \in \mathrm{Tors}(\mathcal{E})(U)$ . This being so, we conclude that  $\pi^{t(\mathcal{E})-1} e' = 0$  and hence that  $t(\mathcal{E}') < t(\mathcal{E})$ . Next, we consider the exact sequence

$$(3) \quad 0 \longrightarrow \mathcal{E}' \xrightarrow{\rho} \mathcal{E} \xrightarrow{\sigma} \mathcal{E}'' \longrightarrow 0,$$

where  $\mathcal{E}'' = \mathcal{E}/\pi \mathcal{E}$ . Since,

$$\mathcal{E}'' = \mathrm{Coker} \mathcal{E} \xrightarrow{\pi} \mathcal{E},$$

it follows that  $\mathcal{E}'' \in \mathfrak{T}_\varphi$  (there is no need to apply Lemma 4.1 here). Note that,  $\pi \mathcal{E}'' = 0$  and hence  $\mathcal{E}''$  is  $\varphi^*$ -acyclic by the first step in the proof. Consequently, we have the exact sequence

$$0 \longrightarrow \varphi^*(\mathcal{E}') \longrightarrow \varphi^*(\mathcal{E}) \xrightarrow{\varphi^*(\sigma)} \varphi^*(\mathcal{E}'') \longrightarrow 0,$$

which says that  $\varphi^*(\mathcal{E}') = \mathrm{Ker}(\varphi^*(\sigma))$ . But both  $\mathcal{E}$  and  $\mathcal{E}''$  are in  $\mathfrak{T}_\varphi$  so that 3.10-(3) guarantees that  $\mathcal{E}'$  lies in  $\mathfrak{T}_\varphi$ . As  $t(\mathcal{E}') < t(\mathcal{E})$ , we can say that  $\mathcal{E}'$  is  $\varphi^*$ -acyclic and hence, using the exact sequence (3) above, it follows that  $\mathcal{E}$  is equally  $\varphi^*$ -acyclic.  $\square$

We can now present our

*Proof of Theorem 4.3.* Let  $u : \mathcal{E} \rightarrow \mathcal{F}$  be an arrow of  $\mathfrak{T}_\varphi$ . If  $\mathcal{K} = \mathrm{Ker}(u)$ ,  $\mathcal{C} = \mathrm{Coker}(u)$  and  $\mathcal{I} = \mathrm{Im}(u)$ , then we have exact sequences

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{F} \longrightarrow \mathcal{C} \longrightarrow 0.$$

Note that  $\mathcal{C} \in \mathfrak{T}_\varphi$  by Lemma 4.1, so that Proposition 4.4 tells us that  $\mathcal{C}$  is  $\varphi^*$ -acyclic. This implies that the sequence

$$0 \longrightarrow \varphi^*(\mathcal{I}) \longrightarrow \varphi^*(\mathcal{F}) \longrightarrow \varphi^*(\mathcal{C}) \longrightarrow 0$$

is exact and Proposition 3.10 assures that  $\varphi^*(\mathcal{I})$  is trivial relatively to  $A$ . Otherwise said,  $\mathcal{I}$  belongs to  $\mathfrak{T}_\varphi$ . Proposition 4.4 then proves that  $\mathcal{I}$  is  $\varphi^*$ -acyclic. Applying the same reasoning, we conclude that  $\mathcal{K} \in \mathfrak{T}_\varphi$ .  $\square$

**Corollary 4.5.** *The functor  $\bullet|_{x_0} : \mathfrak{T}_\varphi \rightarrow A\text{-mod}$  is exact and faithful.*

*Proof.* We know that  $\bullet|_{x_0}$  is *right* exact, so that we only need to show that  $\bullet|_{x_0}$  preserves monomorphisms. Let  $u : \mathcal{E} \rightarrow \overline{\mathcal{E}}$  be such a monomorphism in  $\mathfrak{T}_\varphi$  and denote  $\mathcal{E}|_{x_0}$ , respectively  $\overline{\mathcal{E}}|_{x_0}$ , by  $E$ , respectively  $\overline{E}$ . Let  $\alpha : \mathcal{O}_Y \otimes_A E \xrightarrow{\sim} \varphi^*(\mathcal{E})$  and  $\overline{\alpha} : \mathcal{O}_Y \otimes_A \overline{E} \rightarrow \varphi^*(\overline{\mathcal{E}})$  be adapted to  $y_0$  (see Definition 3.7 and Lemma 3.9). Using Proposition 3.10, we see that  $\overline{\alpha}^{-1}\varphi^*(u)\alpha = \text{id}_{\mathcal{O}_Y \otimes_A u_0}$  for some  $u_0 : E \rightarrow \overline{E}$ . Because  $\varphi^*u$  is a monomorphism (due to Lemma 4.1 and Proposition 4.4), we conclude that  $u_0$  has to be a monomorphism: otherwise the pull-back functor  $A\text{-mod} \rightarrow \mathbf{Coh}(Y)$  would fail to be exact and faithful. This implies that  $\varphi^*(u)|_{y_0} : \varphi^*(\mathcal{E})|_{y_0} \rightarrow \varphi^*(\overline{\mathcal{E}})|_{y_0}$  is a monomorphism, which shows that  $B \otimes_A u|_{x_0}$  is a monomorphism. Since  $B$  is a faithfully flat  $A$ -algebra, we conclude that  $u|_{x_0}$  is also a monomorphism.

We now verify that  $\bullet|_{x_0}$  is faithful by showing that if  $\mathcal{E} \in \mathfrak{T}_\varphi$  is such that  $\mathcal{E}|_{x_0} = 0$ , then  $\mathcal{E} = 0$ . Under this assumption, we see that  $\varphi^*(\mathcal{E})|_{y_0} = 0$  which, together with an isomorphism  $\mathcal{O}_Y \otimes_A E \simeq \varphi^*(\mathcal{E})$ , proves the equality  $\varphi^*(\mathcal{E}) = 0$ . Now, given any point  $\xi \in X$ , one easily sees using the surjectivity of  $\varphi$  that  $\mathcal{E}(\xi) = 0$ . Because of [Har77, II, Exercise 5.8(c), p.125] and the fact that  $X$  is reduced [Mat89, Corollary 23.9, p.184] we conclude that  $\mathcal{E} = 0$ .  $\square$

In possession of Theorem 4.3 and Corollary 4.5, we can now apply the theory of Tannakian categories over  $A$  to define certain fundamental group schemes.

- Definition 4.6** (The fundamental and Galois groups). i) We shall let  $\mathfrak{T}_\varphi^\circ$  denote the full subcategory of  $\mathfrak{T}_\varphi$  whose objects are vector bundles.  
 ii) Given  $\mathcal{E} \in \mathfrak{T}_\varphi^\circ$ , we denote by  $\langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes$  the full subcategory of  $\mathfrak{T}_\varphi$  having as objects subquotients of  $\mathbf{T}^{a,b}\mathcal{E}$  for varying multi-indexes  $a, b$ .  
 iii) Following [HdS21], by  $\mathfrak{T}_\varphi^{\text{tan}}$  we mean the full subcategory of  $\mathfrak{T}_\varphi$  whose objects are

$$\{ \mathcal{V} \in \mathfrak{T}_\varphi : \mathcal{V} \text{ is a quotient of some } \mathcal{E} \in \mathfrak{T}_\varphi^\circ \} = \bigcup_{\mathcal{E} \in \mathfrak{T}_\varphi^\circ} \text{Ob} \langle \mathcal{E}, \mathfrak{T}_\varphi \rangle_\otimes.$$

- iv) Given  $\mathcal{E} \in \mathfrak{T}_\varphi^\circ$ , we let  $\text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi, x_0)$ , or simply  $\text{Gal}'(\mathcal{E})$  if context prevents any misunderstanding, stand for the group scheme obtained from  $\langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes$  via Tannakian duality (see [DuHa18, Theorem 1.2.6] or [Saa72, II.4.1.1]) by the functor  $\bullet|_{x_0} : \langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes \rightarrow A\text{-mod}$ .  
 v) We let  $\Pi(X, \varphi, x_0)$  be the flat group scheme obtained from  $\mathfrak{T}_\varphi^{\text{tan}}$  via Tannakian duality by the functor  $\bullet|_{x_0} : \mathfrak{T}_\varphi^{\text{tan}} \rightarrow A\text{-mod}$

To end, we raise a point which was naturally suggested in the elaboration of the arguments in this section. (It should be compared to Exercise of [Har77, Exercise 5.8, p.125].)

**Question 4.7.** Let  $Z$  be a flat, reduced and noetherian  $A$ -scheme. Call a dvr-point of  $Z$  any morphism  $z : \text{Spec } R \rightarrow Z$  of  $A$ -schemes where  $R$  is a dvr. Now, for  $\mathcal{F} \in \mathbf{coh}(Z)$  and  $z : \text{Spec } R \rightarrow Z$  a dvr-point of  $Z$ , define its “type” as the invariants of  $\mathcal{F}|_z$ . Assume now that the type of  $\mathcal{F}$  is the same for all  $z$ . Is it true that  $\mathcal{F}$  trivial relatively to  $A$ ?

## 5. THE GENERIC FIBRE OF THE GROUP SCHEME $\Pi$

In this section, we let  $X$  be a proper and flat  $A$ -scheme with *reduced fibres*, and  $x_0$  an  $A$ -point of  $X$ . Let  $\varphi : Y \rightarrow X$  an object of  $\mathfrak{S}(X, x_0)$ .

Let  $\mathfrak{T}_{\varphi_K}$  stand for the full subcategory of  $\mathbf{coh}(X_K)$  whose set of objects is

$$\{ E \in \mathbf{coh}(X_K) : \varphi_K^*E \text{ is trivial} \}.$$

We note that  $\mathfrak{T}_{\varphi_K}$  is a Tannakian category over  $K$  which is neutralized by the functor  $\bullet|_{x_{0K}} : \mathfrak{T}_{\varphi_K} \rightarrow K\text{-mod}$ . (The reader is right to object that we have not proved this last

claim since in Section 4 we work with a discrete valuation ring, but the translation is evident and in fact much simpler.)

**Theorem 5.1.** *Suppose that  $X_K$  is normal and let  $E \in \mathfrak{T}_{\varphi_K}$ . Then,  $E$  is essentially finite and  $\langle E; \mathfrak{T}_{\varphi_K} \rangle_{\otimes} = \langle E; \mathbf{EF}(X_K) \rangle_{\otimes}$ . In particular, the  $K$ -group scheme associated to  $\langle E; \mathfrak{T}_{\varphi_K} \rangle_{\otimes}$  via the fibre functor  $\bullet|_{x_{0K}}$  is finite.*

*Proof.* Let  $\mathfrak{C}$  be the full subcategory of all vector bundles over  $X_K$  which become trivial after being pulled back via some proper and surjective morphism. Then [AM11, Theorem 1] or [TZ20, Theorem I] guarantee that  $\mathfrak{C} = \mathbf{EF}(X_K)$  and in particular  $\mathfrak{T}_{\varphi_K}$  is a full subcategory of  $\mathbf{EF}(X_K)$ . Following [BdS11, Corollary 2.3], let us show that  $\langle E; \mathfrak{C} \rangle_{\otimes} = \langle E; \mathfrak{T}_{\varphi_K} \rangle_{\otimes}$ . Clearly, each object in  $\langle E; \mathfrak{T}_{\varphi_K} \rangle_{\otimes}$  is also an object of  $\langle E; \mathfrak{C} \rangle_{\otimes}$ . Let then  $T$  be some tensor power of  $E$  and  $q : T \rightarrow Q$  be a quotient morphism with  $Q \in \mathfrak{C}$ . Then,  $\varphi_K^*(q)$  is a quotient morphism from  $\varphi_K^*(T) \simeq \mathcal{O}_{Y_K}^{\oplus r}$  to  $\varphi_K^*Q$ . From Lemma 5.2, we see that  $\varphi_K^*Q$  is trivial, which means that  $Q \in \mathfrak{T}_{\varphi_K}$ . Using duality of vector bundles, we then conclude that  $\langle E; \mathfrak{C} \rangle_{\otimes} = \langle E; \mathfrak{T}_{\varphi_K} \rangle_{\otimes}$ .  $\square$

The following result was employed in the proof of Theorem 5.1.

**Lemma 5.2.** *Let  $V$  be a proper scheme over a field and  $\mathcal{O}_V^{\oplus r} \rightarrow \mathcal{Q}$  a quotient morphism to a vector bundle  $\mathcal{Q}$  of rank  $d$ . Suppose that for a certain surjective and proper morphism  $f : V' \rightarrow V$ , the pull-back  $f^*(\det \mathcal{Q})$  is trivial. Then  $\mathcal{Q}$  is trivial.*

*Proof.* We let  $\text{Grass}(r, d)$  be the Grassmann scheme as described in [Ni05, 5.1.5(2), 110ff] and denote by  $\mathcal{U}$  the universal quotient of  $\mathcal{O}_{\text{Grass}(r, d)}^{\oplus r}$ . By construction of  $\text{Grass}(r, d)$ , there exists a morphism  $\gamma : V \rightarrow \text{Grass}(r, d)$  such that  $\mathcal{Q} = \gamma^*(\mathcal{U})$ . Consequently,  $f^*\gamma^*(\det \mathcal{U})$  is trivial. Let  $\delta : \text{Grass}(r, d) \rightarrow \mathbb{P}^n$  ( $n = \binom{r}{d} - 1$ ) be the determinant (Plücker) immersion so that  $\det(\mathcal{U}) = \delta^*\mathcal{O}(1)$  [Ni05, p. 114]. We then conclude that  $f^*\gamma^*\delta^*\mathcal{O}(1)$  is trivial. Using Chow's lemma [EGA II, 5.6.1, p.106], there is no loss of generality in supposing  $V'$  to be projective. Now, Exercise 8.1.7(a) on p. 331 of [Liu02] says that the closed schematic image of  $\delta\gamma f$  in  $\mathbb{P}^n$  is finite over the ground field. Hence, the image of  $\gamma$  is a closed subset with finitely many points so that  $\gamma^*(\mathcal{U})$ , which is  $\mathcal{Q}$ , is trivial.  $\square$

For  $E \in \mathfrak{T}_{\varphi_K}$ , let us denote the group scheme over  $K$  associated to  $\langle E; \mathfrak{T}_{\varphi_K} \rangle_{\otimes}$  via the functor  $\bullet|_{x_{0K}}$  by  $\text{Gal}(E, \mathfrak{T}_{\varphi_K}, x_{0K})$ .

**Theorem 5.3.** *Suppose that  $X_K$  is normal and let  $\mathcal{E} \in \mathfrak{T}_{\varphi}$ . If  $E$  stands for  $\mathcal{E}_K$ , then  $\text{Gal}'(\mathcal{E}, \mathfrak{T}_{\varphi}, x_0) \otimes K \simeq \text{Gal}(E, \mathfrak{T}_{\varphi_K}, x_{0K})$ . In particular,  $\text{Gal}'(\mathcal{E}, \mathfrak{T}_{\varphi}, x_0) \otimes K$  is finite.*

*Proof.* Let  $\sigma : \langle \mathcal{E}; \mathfrak{T}_{\varphi} \rangle_{\otimes} \rightarrow \langle E; \mathfrak{T}_{\varphi_K} \rangle_{\otimes}$  be the base change functor,  $G$  be the group scheme  $\text{Gal}'(\mathcal{E}, \mathfrak{T}_{\varphi}, x_0)$  and  $H$  be  $\text{Gal}(E, \mathfrak{T}_{\varphi_K}, x_{0K})$ . Let  $\theta : \text{Rep}_A(G) \rightarrow \langle \mathcal{E}; \mathfrak{T}_{\varphi} \rangle_{\otimes}$  be an inverse to  $\bullet|_{x_0} : \langle \mathcal{E}; \mathfrak{T}_{\varphi} \rangle_{\otimes} \rightarrow \text{Rep}_A(G)$  and denote by  $\tau : \text{Rep}_A(G) \rightarrow \text{Rep}_K(H)$  the composition

$$\text{Rep}_A(G) \xrightarrow{\theta} \langle \mathcal{E}; \mathfrak{T}_{\varphi} \rangle_{\otimes} \xrightarrow{\sigma} \langle E; \mathfrak{T}_{\varphi_K} \rangle_{\otimes} \xrightarrow{\bullet|_{x_{0K}}} \text{Rep}_K(H).$$

If  $V \subset A[G]_{\text{right}}$  contains the constants, the fact that  $\theta$  is fully faithful assures that  $H^0(X, \theta(V)) \simeq A$ . Hence,  $H^0(X_K, \sigma\theta(V)) \simeq K$ , by flat base-change. Consequently,

$$\tau(V)^H \simeq K.$$

Let  $i : \text{Rep}_A(G) \rightarrow \text{Rep}_K(G_K)$  be the extension of scalars. It is not difficult to see that there exists a morphism  $\xi : H \rightarrow G_K$  of group schemes such that

$$\begin{array}{ccc} \text{Rep}_A(G) & \xrightarrow{\tau} & \text{Rep}_K(H) \\ \downarrow i & \nearrow \xi\# & \\ \text{Rep}_K(G_K) & & \end{array}$$

is commutative (up to natural isomorphism of tensor functors).

If  $V \subset K[G]_{\text{right}}$ , there exists  $V^b \subset A[G]_{\text{right}}$  containing the constants and an injection  $V \rightarrow i(V^b)$ . Then,

$$\begin{aligned} \xi^\#(V)^H &\subset \tau(V^b)^H \\ &\simeq K. \end{aligned}$$

This implies that  $K \simeq K[G]_{\text{right}}^H$ , and we conclude, with the help of Lemma 2.8 and the fact that  $H$  is finite (Theorem 5.1), that  $\xi$  is faithfully flat. On the other hand, since  $E|_{x_0K}$  is a faithful representation of  $H$ , the standard criterion [DM82, 2.21, p.139] immediately shows that  $\xi$  is a closed immersion: it follows that  $\xi$  is an isomorphism.  $\square$

As a simple consequence we have

**Corollary 5.4.** *Suppose that  $X_K$  is normal. Then  $\Pi(X, \varphi, x_0) \otimes K$  is pro-finite.*  $\square$

*Remark 5.5.* We are unable to show that Theorem 5.3 holds without the finiteness assumption on  $\text{Gal}(E, \mathfrak{T}_{\varphi_K}, x_0)$ . This is because, once this assumption is removed, Lemma 2.8 cannot be applied so that, in its place, we need the standard criterion guaranteeing faithful flatness [DM82, 2.21, p.139].

## 6. PRUDENCE IN THE CATEGORY $\mathfrak{T}_*$

We shall assume that  $A$  is complete. In [HdS21, Section 6] we introduced the notion of *prudence* of an affine flat group scheme over  $A$ . As shown in op. cit. this concept turns out to be equivalent to the Mittag-Leffler property (or freeness) of the  $A$ -module of functions of the group. In this section we show that the  $A$ -module of functions of a Galois group (see Definition 4.6) is free by using prudence.

Let us briefly recall the concept of prudent affine group schemes over a complete d.v.r. Our reference is [HdS21, Section 6.2]. Let  $G$  be a flat affine group scheme over  $A$  and  $V$  be an object of  $\text{Rep}_A(G)$ , with the coaction  $\rho : V \rightarrow V \otimes A[G]$ . An element  $v \in V$  is said to be *semi-invariant* if  $\rho(v) = v \otimes \theta$  for some  $\theta \in A[G]$ . Assume furthermore that  $V$  is  $A$ -free. We say that  $V$  is prudent if an element  $\mathbf{v}$  whose reductions modulo  $\pi^{n+1}V$  are semi-invariant for all  $n$  must necessarily be semi-invariant. We say that  $G$  is prudent if each  $V \in \text{Rep}_A^\circ(G)$  is prudent.

Let us now fix the hypothesis on the ambient space in this section. Let  $X$  be a proper and flat  $A$ -scheme having reduced fibres. We give ourselves an  $A$ -point  $x_0$  and an arrow  $\varphi : Y \rightarrow X$  in  $\mathfrak{S}^+(X, x_0)$  (see Definition 2.4). If  $B = H^0(\mathcal{O}_Y)$ , we let  $y_0$  be the  $B$ -point of  $Y$  above  $x_0$  assured by the definition of  $\mathfrak{S}^+$ .

**Theorem 6.1.** *Let  $\mathcal{E} \in \mathfrak{T}_\varphi^\circ$ . Then  $\text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi, x_0)$ , as introduced in Definition 4.6, is prudent. In particular, its  $A$ -module of functions is free.*

Before starting the proof, let us obtain the

**Corollary 6.2.** *Suppose that  $X_K$  is normal. Then  $\text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi, x_0)$  is finite.*

*Proof.* We abbreviate  $G := \text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi, x_0)$ . By Theorem 5.3,  $G \otimes K$  is finite over  $K$ . On the other hand, Theorem 6.1 ensures that the  $A$ -module  $A[G]$  is free. Consequently it has to be finite.  $\square$

*Proof of Theorem 6.1.* Recall that by assumption  $Y$  is  $H^0$ -flat over  $A$  and flat over  $B = H^0(\mathcal{O}_Y)$ . We wish to show that  $\text{Gal}'(\mathcal{E})$  is prudent [HdS21, Section 6], and for that we introduce the following setting.

Let  $\mathcal{V} \in \langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes$  be a vector bundle and let

$$v_n : \mathcal{L}_n \rightarrow \mathcal{V}_n$$

be a compatible family of monomorphisms of  $\mathcal{O}_X$ -modules (as customary,  $\mathcal{V}_n = A_n \otimes_A \mathcal{V}$ ,  $B_n = A_n \otimes_A B$ , etc.) such that for each  $n \in \mathbb{N}$ , the  $\mathcal{O}_X$ -module  $\mathcal{L}_n$  belongs to  $\mathfrak{T}_\varphi$ . By Grothendieck's existence theorem [II06, Theorem 8.4.2], there exists an arrow of coherent  $\mathcal{O}_X$ -modules  $v : \mathcal{L} \rightarrow \mathcal{V}$  such that  $v_n$  is none other than  $v \otimes_A A_n$ .

Write  $\mathcal{M} := \varphi^*(\mathcal{L})$ ; our assumption on  $\mathcal{L}_n$  tells us that  $\mathcal{M}_n = \varphi^*(\mathcal{L}_n)$  is trivial relatively to  $A$ . Then, since we assume that  $Y$  is  $H^0$ -flat over  $A$ , Lemma 3.3 tells us that the canonical arrow

$$\mathcal{O}_Y \otimes_B H^0(\mathcal{M}_n) \xrightarrow{\theta_n} \mathcal{M}_n$$

is an *isomorphism*. We now require the

**Lemma 6.3.** *Under the above notations and assumption, the canonical morphism of  $\mathcal{O}_Y$ -modules*

$$\tau : \mathcal{O}_Y \otimes_B H^0(\mathcal{M}) \longrightarrow \mathcal{M}$$

*is an isomorphism. In particular,  $\mathcal{M}$  is trivial relatively to  $B$ .*

*Proof.* Consider the commutative diagram

$$\begin{array}{ccccccc} \mathcal{M}_{n+1} & \xrightarrow{\pi^{n+1}} & \mathcal{M}_{n+1} & \xrightarrow{q_n} & \mathcal{M}_n & \longrightarrow & 0 \\ \theta_{n+1} \uparrow & & \theta_{n+1} \uparrow & & \theta_n \uparrow & & \\ \mathcal{O}_Y \otimes_B H^0(\mathcal{M}_{n+1}) & \xrightarrow{\pi^{n+1}} & \mathcal{O}_Y \otimes_B H^0(\mathcal{M}_{n+1}) & \xrightarrow{\text{id} \otimes H^0(q_n)} & \mathcal{O}_Y \otimes_B H^0(\mathcal{M}_n) & \longrightarrow & 0, \end{array}$$

where  $q_n$  is the canonical arrow. By assumption, the vertical arrows are isomorphisms while the upper row is tautologically an exact sequence of  $\mathcal{O}_Y$ -modules. We conclude that the bottom row is exact and faithful flatness of  $Y$  over  $B$  (faithfulness is guaranteed by [EGA III<sub>1</sub>, 4.3.1, p.130]) shows that the sequence

$$H^0(\mathcal{M}_{n+1}) \xrightarrow{\pi^{n+1}} H^0(\mathcal{M}_{n+1}) \xrightarrow{H^0(q_n)} H^0(\mathcal{M}_n) \longrightarrow 0$$

is equally exact. Hence, if

$$\mathfrak{M} := \varprojlim \left( \cdots \longrightarrow H^0(\mathcal{M}_{n+1}) \xrightarrow{H^0(q_n)} H^0(\mathcal{M}_n) \longrightarrow \cdots \right),$$

it follows that  $\mathfrak{M}$  is a finitely generated  $B$ -module and that the projection

$$\mathfrak{M} \longrightarrow H^0(\mathcal{M}_n)$$

induces an isomorphism

$$B_n \otimes_B \mathfrak{M} \xrightarrow{\sim} H^0(\mathcal{M}_n).$$

(This follows from the fact that  $B$  is  $\pi$ -adically complete and [EGA 0<sub>I</sub>, 7.2.9].) More importantly, letting  $u_n : \mathcal{M} \rightarrow \mathcal{M}_n$  stand for the natural epimorphism, a direct application of the theorem of formal functions (see [II06, 8.2.4, p. 188] or [Har77, III.11.1, p.277]) guarantees that the obvious arrow

$$H^0(\mathcal{M}) \xrightarrow{\varprojlim_n H^0(u_n)} \mathfrak{M}$$

is a bijection. Consequently, for any given  $n \in \mathbb{N}$ , the arrow  $H^0(u_n) : H^0(\mathcal{M}) \rightarrow H^0(\mathcal{M}_n)$  induces an isomorphism

$$U_n : B_n \otimes_B H^0(\mathcal{M}) \xrightarrow{\sim} H^0(\mathcal{M}_n).$$



To show that  $\tau : \mathcal{O}_Y \otimes_B H^0(\mathcal{M}) \rightarrow \mathcal{M}$  is an isomorphism, by Grothendieck's algebraization theorem [II06, Theorem 8.2.9, p.192], we only need to show that for each  $n$ ,

$$B_n \otimes_B (\mathcal{O}_Y \otimes_B H^0(\mathcal{M})) \xrightarrow{\text{id} \otimes \tau} B_n \otimes_B \mathcal{M}$$

is an isomorphism. Now, we have the commutative diagram

$$\begin{array}{ccccc} B_n \otimes_B (\mathcal{O}_Y \otimes_B H^0(\mathcal{M})) & \xrightarrow{\text{id} \otimes \tau} & B_n \otimes_B \mathcal{M} & \xrightarrow[\sim]{\bar{u}_n} & \mathcal{M}_n \\ \sim \uparrow \text{canonic} & & & & \uparrow \theta_n \sim \\ \mathcal{O}_Y \otimes_B (B_n \otimes_B H^0(\mathcal{M})) & \xrightarrow{\text{id} \otimes U_n} & \mathcal{O}_Y \otimes_B H^0(\mathcal{M}_n), & & \end{array}$$

where  $\bar{u}_n$  is associated to  $u_n$ , and the conclusion follows.  $\square$

By virtue of the above Lemma,  $\mathcal{M}$  is trivial relatively to  $B$ ; using Lemma 3.5 we conclude that  $\mathcal{M}$  is trivial relatively to  $A$ . This shows that  $\mathcal{L} \in \mathfrak{T}_\varphi$ . As it is not difficult to show that  $v$  is a monomorphism by looking at the arrow  $\mathcal{L}|_{x_0} \rightarrow \mathcal{V}|_{x_0}$ , we see that  $\mathcal{L}$  does indeed belong to  $\langle \mathcal{E}, \mathfrak{T}_\varphi \rangle_\otimes$ .

We have then showed that the group scheme  $\text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi)$  is prudent, and hence that its ring of functions is free as an  $A$ -module [HdS21, Section 6].  $\square$

## 7. DIGRESSION ON THE REDUCED FIBRE THEOREM

For future applications, we shall state a version of the reduced fibre theorem (see [BLR95, Theorem 2.1'] and [Tem10]) suitable for us.

**Theorem 7.1** (The reduced fibre theorem). *Let  $A$  be Henselian and  $Y$  be a flat  $A$ -scheme of finite type whose fibre  $Y_K$  is geometrically reduced. Then, there exists a finite extension of discrete valuation rings  $B \supset A$  and a commutative diagram*

$$\begin{array}{ccc} & Z & \\ & \downarrow \varepsilon & \\ Y & \longleftarrow Y_B & \\ \downarrow & \square & \downarrow \\ \text{Spec } A & \longleftarrow & \text{Spec } B \end{array}$$

where:

- RFT1)** *The  $B$ -scheme  $Z$  is flat and has geometrically reduced fibres.*
- RFT2)** *The morphism  $\varepsilon$  is finite and surjective.*
- RFT3)** *If  $L = \text{Frac}(B)$ , then*

$$\varepsilon \otimes_B L : Z \otimes_B L \longrightarrow Y_B \otimes_B L$$

*is an isomorphism.*

*In particular, the composition  $Z \rightarrow Y_B \rightarrow Y$  is finite and surjective.*

*Proof.* We show how to arrive at the conclusion starting from [Tem10, Theorem 3.5.5]. By this theorem (see also the notation on p. 619 of op. cit.), we obtain the existence of an integral scheme  $S$  and a separable alteration (proper, dominant and inducing a finite and separable extension of function fields)

$$S \longrightarrow \text{Spec } A,$$

and a commutative diagram with cartesian square

$$\begin{array}{ccc}
 & & \tilde{Y} \\
 & & \downarrow \delta \\
 Y & \longleftarrow & Y_S \\
 \downarrow & \square & \downarrow \\
 \text{Spec } A & \longleftarrow & S,
 \end{array}$$

such that:

- The  $S$ -scheme  $\tilde{Y}$  is flat and has geometrically reduced fibres.
- The morphism  $\delta$  is finite.
- The base-change

$$\tilde{Y} \times_S S_K \xrightarrow{\delta \times_S S_K} Y_S \times_S S_K$$

is an isomorphism.

As  $Y_S \rightarrow S$  is an open morphism [EGA IV<sub>2</sub>, 2.4.6] and  $S_K$  is dense in  $S$ , we can say that  $Y_S \times_S S_K$  is dense in  $Y_S$ ; consequently,

- The morphism  $\delta$  is *surjective*.

If  $L$  is the function field of  $S$  and  $B$  is the integral closure of  $A$  in  $L$  (necessarily a d.v.r. and a finite  $A$ -module [Mat80, 31.B, p.232]), properness of  $S$  gives the existence of a morphism of  $A$ -schemes  $\text{Spec } B \rightarrow S$  extending  $\text{Spec } L \rightarrow S$ . We then arrive at a commutative diagram with cartesian squares

$$\begin{array}{ccccc}
 & & \tilde{Y} & \longleftarrow & Z \\
 & & \downarrow \delta & \square & \downarrow \varepsilon \\
 Y & \longleftarrow & Y_S & \longleftarrow & Y_B \\
 \downarrow & \square & \downarrow & \square & \downarrow \\
 \text{Spec } A & \longleftarrow & S & \longleftarrow & \text{Spec } B
 \end{array}$$

where

- The  $B$ -scheme  $Z$  is flat and has geometrically reduced fibres (for the last condition, see [EGA IV<sub>2</sub>, 4.6.1, p.68]).
- The morphism  $\varepsilon$  is finite and surjective (for surjectivity, see [EGA I, 3.5.2, p.115]).
- The base-change morphism

$$\varepsilon \otimes_B L : Z \otimes_B L \longrightarrow Y_B \otimes_B L$$

is an isomorphism.

The proof of the last claim is quite simple and we omit it.  $\square$

The ensuing result will prove useful in order to apply Theorem 7.1. In it, we employ the notion of Japanese discrete valuation ring [EGA 0<sub>IV</sub>, 23.1.1, p.213]. (The reader should recall that all complete d.v.r.'s are Japanese, as are those whose field of fractions has characteristic zero. See [EGA 0<sub>IV</sub>, 23.1.5] and [EGA 0<sub>IV</sub>, 23.1.2].)

**Lemma 7.2.** *Let  $A$  be Japanese and  $S$  a flat  $A$ -scheme of finite type. There exists a finite purely inseparable extension  $\tilde{K}$  of  $K$  such that the following property holds. If  $\tilde{A}$  is*

the integral closure of  $A$  in  $\tilde{K}$ , then the  $\tilde{A}$ -scheme  $\tilde{S} := (S \otimes_A \tilde{A})_{\text{red}}$  has a geometrically reduced generic fibre and is flat .

*Proof.* Let  $\tilde{K}/K$  be a finite and purely inseparable extension such that  $(S \otimes_A \tilde{K})_{\text{red}}$  is geometrically reduced over  $\tilde{K}$  [EGA IV<sub>2</sub>, 4.6.6, p.69]. Now, in the notation of the statement,  $\tilde{S}[1/\pi] = (S \otimes_A \tilde{K})_{\text{red}}$ , and we are done. To prove that  $\tilde{S}$  is flat over  $\tilde{A}$ , let  $\tilde{\pi}$  be a uniformizer of  $\tilde{A}$ . Let  $\tilde{f}$  be a function on some unspecified affine open subset of  $S \otimes_A \tilde{A}$  such that  $\tilde{\pi}\tilde{f}$  is nilpotent, say  $\tilde{\pi}^m\tilde{f}^m = 0$ . Then, the fact that  $S \otimes_A \tilde{A}$  is  $\tilde{A}$ -flat says that  $\tilde{f}$  is nilpotent, and this shows that  $(S \otimes_A \tilde{A})_{\text{red}}$  is  $\tilde{A}$ -flat.  $\square$

## 8. THE FUNDAMENTAL GROUP SCHEME

Let  $X$  be an irreducible, proper and flat  $A$ -scheme with geometrically reduced fibres, and  $x_0$  an  $A$ -point of  $X$ . We now wish to assemble the categories  $\mathfrak{T}_\varphi$  (see Definition 3.4 and Theorem 4.3) for varying  $\varphi$  in a single one, and for that we need the following:

**Theorem 8.1.** *We suppose that  $A$  is Henselian and Japanese. Let  $\varphi : Y \rightarrow X$  be a proper and surjective morphism. Then, there exists a commutative diagram of schemes*

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & \searrow \psi & \downarrow \varphi \\ & & X \end{array}$$

enjoying the following properties.

- (1) The morphism  $\psi$  is surjective and proper.
- (2) The ring  $C := H^0(\mathcal{O}_Z)$  is a discrete valuation ring and is a finite extension of  $A$ .
- (3) The canonical morphism  $Z \rightarrow \text{Spec } C$  is flat and has geometrically reduced fibres.
- (4) The  $C$ -scheme  $Z$  has a  $C$ -point above  $x_0$ .

In particular,  $\psi : Z \rightarrow X$  belongs to the category  $\mathfrak{S}^+(X, x_0)$  introduced in Definition 2.4.

*Proof.* The construction requires several steps.

*Step 1.* Because  $X$  is irreducible, there exists an irreducible component  $Y'$  of  $Y$  such that  $\varphi : Y' \rightarrow X$  is surjective.

*Step 2.* Let  $j : Y'' \rightarrow Y$  be reduced closed subscheme underlying  $Y'$ . Note that  $Y''$  is integral and that  $\varphi'' := \varphi \circ j : Y'' \rightarrow X$  is proper and surjective.

*Step 3.* Let now  $\nu : Y^\dagger \rightarrow Y''$  be the normalization [EGA II, p.119]. Since  $A$  is universally Japanese [EGA IV<sub>2</sub>, 7.7.2, p.212], we conclude that  $\nu$  is finite, and hence  $\varphi^\dagger := \varphi'' \circ \nu$  is proper and surjective. In addition,  $H^0(\mathcal{O}_{Y^\dagger})$  is a normal integral domain (see [Liu02, 2.4.17, p.65] and [Liu02, 4.1.5, p.116]). The morphism  $Y^\dagger \rightarrow \text{Spec } A$  is surjective and hence flat [Liu02, 4.3.10, p.137]; it then follows that  $H^0(\mathcal{O}_{Y^\dagger})$  is a finite and flat  $A$ -module. As such, it must in addition be a local ring, since  $A$  is Henselian. Consequently,  $H^0(\mathcal{O}_{Y^\dagger})$  is a discrete valuation ring. In hindsight, the canonical morphism  $Y^\dagger \rightarrow \text{Spec } H^0(\mathcal{O}_{Y^\dagger})$  is surjective and flat [Liu02, 4.3.10, p.137].

*Step 4.* Let  $B^\dagger := H^0(\mathcal{O}_{Y^\dagger})$  and write  $L^\dagger$  for its field of fractions. Let  $\tilde{L}$  be a purely inseparable extension of  $L^\dagger$  such that, denoting by  $\tilde{B}$  the integral closure of  $B^\dagger$  in  $\tilde{L}$ , the generic fibre of

$$(Y^\dagger \otimes_{B^\dagger} \tilde{B})_{\text{red}} \longrightarrow \text{Spec } \tilde{B}$$

is geometrically reduced (we employ Lemma 7.2). Writing  $\tilde{Y} := (Y^\dagger \otimes_{B^\dagger} \tilde{B})_{\text{red}}$ , we note that the composition

$$\tilde{Y} \longrightarrow Y^\dagger \otimes_{B^\dagger} \tilde{B} \xrightarrow{\text{pr}} Y^\dagger$$

induces a homeomorphism between the underlying topological spaces. From this, it follows that  $\tilde{Y}$  is integral and that the obvious morphism  $\tilde{\varphi} : \tilde{Y} \rightarrow X$  is surjective. Since  $\tilde{B}$  is a finite  $B^\dagger$ -module—recall that  $A$  is Japanese—, we conclude that  $\tilde{\varphi}$  is proper. Since  $\tilde{B}$  is a Dedekind domain, the fact that  $\tilde{Y} \rightarrow \text{Spec } \tilde{B}$  is surjective (note that  $Y^\dagger \otimes_{B^\dagger} \tilde{B} \rightarrow \text{Spec } \tilde{B}$  is surjective by [EGA I, 3.5.2, p.115]) shows that it is in addition flat.

For future usage, we also remark that

$$\begin{aligned} H^0(\mathcal{O}_{\tilde{Y}}) &= H^0\left(\mathcal{O}_{Y^\dagger \otimes_{B^\dagger} \tilde{B}}\right)_{\text{red}} && \text{(see [Liu02, 2.4.2(c), p.60])} \\ &= \left(H^0(\mathcal{O}_{Y^\dagger}) \otimes_{B^\dagger} \tilde{B}\right)_{\text{red}} && \text{(flat base-change [Liu02, 3.1.24. p.85])} \\ &= \left(B^\dagger \otimes_{B^\dagger} \tilde{B}\right)_{\text{red}} && \text{(definition of } B^\dagger) \\ &= \tilde{B}. \end{aligned}$$

*Step 5.* By the reduced fibre theorem (Theorem 7.1), there exists a finite extension  $B^\natural \supset \tilde{B}$  of discrete valuation rings and a commutative diagram

$$\begin{array}{ccc} & & Y^\natural \\ & & \downarrow \varepsilon \\ \tilde{Y} & \longleftarrow & \tilde{Y} \otimes_{\tilde{B}} B^\natural \\ \downarrow & & \downarrow \\ \text{Spec } \tilde{B} & \longleftarrow & \text{Spec } B^\natural \end{array}$$

such that

- The  $B^\natural$ -scheme  $Y^\natural$  is flat, proper and has geometrically reduced fibres.
- The morphism  $\varepsilon$  is finite and surjective.
- If  $L^\natural = \text{Frac}(B^\natural)$ , then

$$\varepsilon \otimes_{B^\natural} L^\natural : Y^\natural \otimes_{B^\natural} L^\natural \longrightarrow (\tilde{Y} \otimes_{\tilde{B}} B^\natural) \otimes_{B^\natural} L^\natural$$

is an isomorphism.

Furthermore,

$$\begin{aligned} H^0\left(\mathcal{O}_{Y^\natural \otimes_{B^\natural} L^\natural}\right) &\simeq H^0(\mathcal{O}_{\tilde{Y} \otimes_{\tilde{B}} L^\natural}) && \text{(via } \varepsilon \otimes_{B^\natural} L^\natural) \\ &\simeq H^0(\mathcal{O}_{\tilde{Y}}) \otimes_{\tilde{B}} L^\natural && \text{(by flat base-change)} \\ &\simeq L^\natural && \text{(by Step 4).} \end{aligned}$$

Let  $\varphi^\natural : Y^\natural \rightarrow X$  be the composition

$$Y^\natural \xrightarrow{\varepsilon} \tilde{Y} \otimes_{\tilde{B}} B^\natural \xrightarrow{\text{pr}} \tilde{Y} \xrightarrow{\tilde{\varphi}} X;$$

as  $\tilde{\varphi}$  is surjective and proper, it is clear that  $\varphi^\natural$  is surjective and proper.

*Step 6.* Let now  $C \supset B^\natural$  be a finite extension of d.v.r.'s such that  $Y^\natural$  has a  $C$ -point above the  $A$ -point  $x_0$  (here we apply the valuative criterion of properness for the  $A$ -scheme  $Y^\natural|_{x_0}$  to obtain a finite extension  $A \subset C$ ). We then define  $Z = Y^\natural \otimes_{B^\natural} C$  and note that the following claims hold true:

- The composition  $Z \xrightarrow{\text{pr}} Y^{\natural} \xrightarrow{\varphi^{\natural}} X$  is surjective and proper.
- The  $C$ -scheme  $Z$  is flat, proper and has geometrically reduced fibres.
- Let  $M = \text{Frac}(C)$ . Then,  $H^0(\mathcal{O}_{Z \otimes_C M}) = M$  (because of  $H^0(\mathcal{O}_{Y^{\natural} \otimes_{B^{\natural}} L^{\natural}}) = L^{\natural}$  and flat base-change).
- The ring of global functions of  $Z$  is  $C$  (because of the previous claims and Lemma 2.2.)
- There is a  $C$ -point in  $Z$  above the  $A$ -point  $x_0$ .

This proves all conclusions in the statement.  $\square$

**Corollary 8.2.** *We suppose that  $A$  is Henselian and Japanese. Let  $\varphi : Y \rightarrow X$  and  $\varphi' : Y' \rightarrow X$  be surjective and proper morphisms. Then there exists  $\psi : Z \rightarrow X$  in  $\mathfrak{S}^+(X, x_0)$  and a commutative diagram*

$$\begin{array}{ccccc}
 & & Z & & \\
 & \swarrow & | & \searrow & \\
 Y & & \psi & & Y' \\
 & \searrow & | & \swarrow & \\
 & & X & & 
 \end{array}$$

$\varphi$                        $\varphi'$

In addition, it is possible to find  $Z$  such that the extra conditions hold:

- (1) The ring  $H^0(\mathcal{O}_Z)$  is a d.v.r. which is a finite extension of  $A$ , and
- (2) the canonical morphism  $Z \rightarrow \text{Spec } H^0(\mathcal{O}_Z)$  is flat with geometrically reduced fibres.

$\square$

**Definition 8.3.** We let  $\mathfrak{T}_X$ , stand for the full subcategory of  $\mathbf{coh}(X)$  whose objects are

$$\bigcup_{\varphi \in \mathfrak{S}(X, x_0)} \text{Ob } \mathfrak{T}_{\varphi}^{\text{tan}}.$$

In addition,  $\mathfrak{T}_X^{\circ}$  is the full subcategory of  $\mathfrak{T}_X$  having objects which are in addition vector bundles.

In the terminology of the definition we can say:

**Corollary 8.4.** *If  $A$  is Henselian and Japanese, then the category  $\mathfrak{T}_X$  is a full abelian subcategory of  $\mathbf{coh}(X)$  which is in addition stable under tensor products. With this structure and with the functor  $\bullet|_{x_0} : \mathfrak{T}_X \rightarrow A\text{-mod}$ ,  $\mathfrak{T}_X$  becomes a neutral Tannakian category over  $A$  in the sense of [DuHa18, Definition 1.2.5, p.1109].*

*Proof.* Let  $\varphi : Y \rightarrow X$  and  $\varphi' : Y' \rightarrow X$  belong to  $\mathfrak{S}(X, x_0)$  and  $\psi : Z \rightarrow X$  be as in Corollary 8.2. Then,  $\mathfrak{T}_{\varphi}^{\text{tan}}$  and  $\mathfrak{T}_{\varphi'}^{\text{tan}}$  are full subcategories of  $\mathfrak{T}_{\psi}^{\text{tan}}$  and all the claims made in the statement follow from Theorem 4.3 and Corollary 4.5.  $\square$

**Definition 8.5.** The fundamental group scheme of  $X$  at the point  $x_0$  is the affine and flat group scheme obtained from  $\mathfrak{T}_X$  and the functor  $\bullet|_{x_0} : \mathfrak{T}_X \rightarrow A\text{-mod}$  via Tannakian duality [Saa72, Theorem II.4.1.1]. It shall be denoted by  $\Pi(X, x_0)$ .

If  $\mathcal{E} \in \mathfrak{T}_X^{\circ}$ , we let  $\text{Gal}'(\mathcal{E}, \mathfrak{T}_X, x_0)$  be the flat group scheme over  $A$  defined by the category  $\langle \mathcal{E} \rangle_{\otimes}$  and the functor  $\bullet|_{x_0}$ .

The next result clarifies the relation between  $\mathfrak{T}_X$  and its constituents  $\mathfrak{T}_{\varphi}$ .

**Proposition 8.6.** *Let  $\varphi : Y \rightarrow X$  be an arrow of  $\mathfrak{S}(X, x_0)$  and  $\mathcal{E} \in \mathfrak{T}_{\varphi}^{\circ}$ . Then, the natural morphism  $\nu : \Pi(X, x_0) \rightarrow \Pi(X, \varphi, x_0)$  is faithfully flat, while  $\text{Gal}'(\mathcal{E}, \mathfrak{T}_X, x_0) \rightarrow \text{Gal}'(\mathcal{E}, \mathfrak{T}_{\varphi}, x_0)$  is an isomorphism.*

*Proof.* Let us write  $G := \Pi(X, x_0)$  and  $G^\varphi := \Pi(X, \varphi, x_0)$ . In addition, given any  $A$ -linear category  $\mathfrak{C}$ , we denote by  $\mathfrak{C}_{(k)}$  the full subcategory of  $\mathfrak{C}$  whose objects are “annihilated by  $\pi$ ”, meaning that multiplication by  $\pi$  coincides with 0. We shall prove that

- i) For any  $V \in \text{Rep}_A^\circ(G^\varphi)$  and any quotient morphism  $q : \nu^\#(V) \rightarrow Q$  with  $Q \in \text{Rep}_A^\circ(G)$ , there exists  $Q^\varphi \in \text{Rep}_A^\circ(G^\varphi)$  such that  $\nu^\#(Q^\varphi) = Q$ .
- ii) For any  $M \in \text{Rep}_A(G^\varphi)_{(k)}$  and any quotient morphism  $q : \nu^\#(M) \rightarrow Q$  with  $Q \in \text{Rep}_A(G)_{(k)}$ , there exists  $Q^\varphi \in \text{Rep}_A(G^\varphi)$  such that  $\nu^\#(Q^\varphi) = Q$ .

If these two conditions are verified, then the evident morphisms  $A[G^\varphi] \rightarrow A[G]$  and  $k[G^\varphi] \rightarrow k[G]$  are injective (as follows from the “dual statements” in [DuHa18, 3.2.1] and [DM82, 2.21, p.139]) and Theorem 4.1.1 in [DuHa18] proves that  $\nu$  is faithfully flat.

To verify (i), we give ourselves a morphism  $\psi : Z \rightarrow X$  in  $\mathfrak{S}(X, x_0)$ , an object  $\mathcal{Q}$  from  $\mathfrak{T}_\psi^\circ$ , an object  $\mathcal{V}$  of  $\mathfrak{T}_\varphi^\circ$  and an epimorphism  $q : \mathcal{V} \rightarrow \mathcal{Q}$ , and aim at showing that  $\mathcal{Q}$  lies in  $\mathfrak{T}_\varphi^\circ$ . By Lemma 2.3, we only need to show that  $\varphi^*(\mathcal{Q})|_{Y_k}$  and  $\varphi^*(\mathcal{Q})|_{Y_K}$  are both trivial. But this is a direct consequence of Lemma 5.2.

To verify (ii) we give ourselves an object  $\mathcal{M}$  of  $(\mathfrak{T}_\varphi^{\text{tan}})_{(k)}$ , a morphism  $\psi : Z \rightarrow X$  in  $\mathfrak{S}(X, x_0)$ , an object  $\mathcal{Q}$  of  $(\mathfrak{T}_\psi^{\text{tan}})_{(k)}$  and an epimorphism  $q : \mathcal{M} \rightarrow \mathcal{Q}$ . Since  $\mathcal{M}$  and  $\mathcal{Q}$  are locally free  $\mathcal{O}_{X_k}$ -modules [BdS11, Remarks (a), p.226], the exact same argument as before proves that  $\mathcal{Q}$  lies in  $\mathfrak{T}_{\varphi, (k)}$ . Being a quotient of  $\mathcal{M}$ , which belongs to  $(\mathfrak{T}_\varphi^{\text{tan}})_{(k)}$ ,  $\mathcal{Q}$  must be in  $(\mathfrak{T}_\varphi^{\text{tan}})_{(k)}$ .

The natural arrow  $\text{Gal}'(\mathcal{E}, \mathfrak{T}, x_0) \rightarrow \text{Gal}'(\mathcal{E}, \mathfrak{T}_\varphi, x_0)$  is induced by the obvious fully faithful functor  $\langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes \rightarrow \langle \mathcal{E}; \mathfrak{T} \rangle_\otimes$ . Now, any  $\mathcal{V}$  in  $\langle \mathcal{E}; \mathfrak{T} \rangle_\otimes$  is of the form  $\mathcal{V}'/\mathcal{V}''$ , where  $\mathcal{V}'' \subset \mathcal{V}' \subset \mathbf{T}^{a,b}\mathcal{E}$ . Since  $\nu$  is faithfully flat, Theorem 4.1.2 of [DuHa18] shows that  $\mathcal{V}'$  and  $\mathcal{V}''$  belong to  $\mathfrak{T}_\varphi^\circ$ ; consequently  $\mathcal{V}$  is an object of  $\langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes$  and the fully faithful functor  $\langle \mathcal{E}; \mathfrak{T}_\varphi \rangle_\otimes \rightarrow \langle \mathcal{E}; \mathfrak{T} \rangle_\otimes$  is an equivalence.  $\square$

In order to make some properties of  $\Pi(X, x_0)$  conspicuous, let us make the following definitions.

**Definition 8.7.** Let  $G \in \mathbf{FGSch}/A$  be given. Let  $\mathbf{P}$  be one of the adjectives “finite”, “quasi-finite” or “pseudo-finite”. We say that  $G$  is pro- $\mathbf{P}$  (respectively strictly pro- $\mathbf{P}$ ) if there exists a directed set  $I$  and a diagram  $\{\nu_{ij} : G_j \rightarrow G_i : i \leq j \in I\}$  in  $\mathbf{FGSch}/A$  where each  $G_i$  is  $\mathbf{P}$  (respectively each  $G_i$  is  $\mathbf{P}$  and each  $\nu_{ij}$  is faithfully flat) such that  $G \simeq \varprojlim_i G_i$ . (For the definition of quasi-finite, the reader is directed to [EGA II, 6.2.3, p.115].)

**Theorem 8.8.** *We suppose that  $A$  is Henselian and Japanese, and moreover assume that  $X_K$  is normal.*

- (1) *The group scheme  $\Pi(X, x_0)$  is pro-quasi-finite.*
- (1') *The group scheme  $\Pi(X, x_0)$  is strictly pro-pseudo-finite.*
- (2) *If in addition  $A$  is complete, then, for each  $\mathcal{E} \in \mathfrak{T}_X^\circ$ , the group scheme  $\text{Gal}'(\mathcal{E}, \mathfrak{T}_X, x_0)$  is finite and  $\Pi(X, x_0)$  is strictly pro-finite.*

We note that this result shall be improved below—see Theorem 8.10—by a careful application of (2) and Lemma 2.9, but not to make the argument overly involved, we opt for less generality now.

*Proof.* We consider the set of isomorphism classes  $I$  of objects in  $\mathfrak{T}_X^\circ$  and order it by decreasing that  $\mathcal{E} \leq \mathcal{F}$  if and only if  $\mathcal{E} \in \langle \mathcal{F} \rangle_\otimes$ . Putting  $G_\mathcal{E} := \text{Gal}'(\mathcal{E}, \mathfrak{T}_X, x_0)$ , we obtain a system of group schemes  $\{\nu_{\mathcal{E}\mathcal{F}} : G_\mathcal{F} \rightarrow G_\mathcal{E}\}$  whose limit is  $\Pi(X, x_0)$ .

(1) and (1'). We know that for each  $\mathcal{E} \in \mathfrak{T}_X^\circ$  the group  $G_\mathcal{E} \otimes K$  is finite (by Proposition 8.6 and Theorem 5.3). Hence, the proof follows from the

**Lemma 8.9.** *Let  $G \in \mathbf{FGSch}/A$  be such that  $G \otimes K$  is a pro-finite group scheme. Then  $G$  is (a) strictly pro-pseudo-finite, and (b) is pro-quasi-finite.*

*Proof.* Following [DHdS18, Theorem 2.17, p. 989], we write  $G = \varprojlim_{\alpha} G_{\alpha}$  where each  $G_{\alpha}$  lies in  $(\mathbf{FGSch}/A)$ , the transition morphisms are faithfully flat and each  $G_{\alpha} \otimes K$  is of finite type over  $K$ . Since  $G \otimes K$  is pro-finite,  $G_{\alpha} \otimes K$  is in fact finite. But for a flat  $A$ -module  $M$ , the inequality  $\dim_K M \otimes K < \infty$  entails  $\dim_k M \otimes k < \infty$ , as we see by lifting a linearly independent set in  $M \otimes k$  to  $M$ . We conclude that  $G_{\alpha}$  is pseudo-finite. In addition, by the same Theorem in [DHdS18], we know that  $G_{\alpha}$  is a projective limit  $\varprojlim_i G_{\alpha,i}$  where  $G_{\alpha,i}$  is flat and of finite type over  $A$ , and the transition morphisms  $G_{\alpha,j} \rightarrow G_{\alpha,i}$  induce isomorphisms on the generic fibres. In particular,  $G_{\alpha,i} \otimes K \simeq G_{\alpha} \otimes K$ , and hence  $G_{\alpha,i} \otimes K$  is again finite. By the same argument as before  $G_{\alpha}$  is, being of finite type, quasi-finite.  $\square$

(2) In view of Corollary 6.2 and Proposition 8.6, only the final statement needs proof. Due to [DuHa18, Theorem 4.1.2] each arrow  $\nu_{\mathcal{E}\mathcal{F}} : G_{\mathcal{F}} \rightarrow G_{\mathcal{E}}$  is faithfully flat, which is enough argument.  $\square$

We now preset the amplification of Theorem 8.8 already mentioned before.

**Theorem 8.10.** *We suppose that  $A$  is Henselian and Japanese, and moreover assume that  $X_K$  is normal. Then  $\Pi(X, x_0)$  is strictly pro-finite. In particular, if  $\mathcal{E} \in \mathfrak{T}_X^{\circ}$ , then  $\text{Gal}'(\mathcal{E}, \mathfrak{T}, x_0)$  is finite.*

*Proof.* Theorem 8.8 tells us that  $\Pi(X, x_0)$  is strictly pro-pseudo-finite. To show that it is strictly pro-finite, we let

$$\Pi(X, x_0) \longrightarrow G$$

be a faithfully flat morphism to a pseudo-finite flat group scheme over  $A$ , and show that  $G$  is indeed finite over  $A$ .

Define

$$\theta : \text{Rep}_A(G) \longrightarrow \mathfrak{T}_X$$

as the composition of the natural functor  $\text{Rep}_A(G) \rightarrow \text{Rep}_A(\Pi(X, x_0))$  with a tensor inverse to  $\bullet|_{x_0} : \mathfrak{T}_X \rightarrow \text{Rep}_A(\Pi(X, x_0))$ . (That such an inverse exists is proved in [Saa72, I.4.4.2, p.69].) Since  $\Pi(X, x_0) \rightarrow G$  is faithfully flat, we conclude  $\text{Rep}_A(G) \rightarrow \text{Rep}_A(\Pi(X, x_0))$  is fully faithful (see [DuHa18, Proposition 3.2.1(ii)], for example) and hence that  $\theta$  is fully faithful.

Let  $\widehat{A}$  be the completion of  $A$  and write  $\widehat{X}$  for the base-change  $X \otimes_A \widehat{A}$ ; note that  $\widehat{X}$  is a flat and proper  $\widehat{A}$ -scheme with geometrically reduced fibres and that  $x_0$  induces an  $\widehat{A}$ -point  $\widehat{x}_0$  on it. In addition, since  $A$  is assumed Japanese, we can say that  $\widehat{K}$ , the field of fractions of  $\widehat{A}$ , is a separable extension of  $K$  [EGA IV<sub>2</sub>, 7.6.6, p.211]. Consequently,  $\widehat{X} \otimes_{\widehat{A}} \widehat{K}$  is also a normal scheme [EGA IV<sub>2</sub>, 6.7.4, p.146] and it is then a simple matter to deduce that  $\widehat{X}$  is also irreducible so that all properties imposed on the morphism  $X \rightarrow \text{Spec } A$  in the beginning of this section are valid for  $\widehat{X} \rightarrow \text{Spec } \widehat{A}$ .

Using the base-change functor  $\sigma : \mathfrak{T}_X \rightarrow \mathfrak{T}_{\widehat{X}}$  and the equivalence

$$\bullet|_{\widehat{x}_0} : \mathfrak{T}_{\widehat{X}} \longrightarrow \text{Rep}_{\widehat{A}}(\Pi(\widehat{X}, \widehat{x}_0)),$$

we derive a tensor functor

$$\tau : \text{Rep}_A(G) \longrightarrow \text{Rep}_{\widehat{A}}(\Pi(\widehat{X}, \widehat{x}_0))$$

preserving forgetful functors (up to tensor natural isomorphism). Now, if  $i : \text{Rep}_A(G) \rightarrow \text{Rep}_{\widehat{A}}(G_{\widehat{A}})$  stands for the base-extension functor, then  $\tau$  can be prolonged to a tensor functor

$$\xi : \text{Rep}_{\widehat{A}}(G_{\widehat{A}}) \longrightarrow \text{Rep}_{\widehat{A}}(\Pi(\widehat{X}, \widehat{x}_0))$$

rendering

$$\begin{array}{ccc} \mathrm{Rep}_A(G) & \xrightarrow{\tau} & \mathrm{Rep}_{\widehat{A}}(\Pi(\widehat{X}, \widehat{x}_0)) \\ \downarrow i & \nearrow \xi & \\ \mathrm{Rep}_{\widehat{A}}(G_{\widehat{A}}) & & \end{array}$$

commutative up to natural isomorphism of tensor functors. In addition,  $\xi$  preserves the forgetful functors. We contend that the morphism of group schemes induced by  $\xi$  is faithfully flat, and for that we rely on Lemma 2.9, whose notations are from now on in force.

Let  $V \in \mathfrak{s}(G)$ . Since  $\theta$  is full, we conclude that  $H^0(X, \theta(V)) \simeq A$ . Hence, flat base-change gives  $H^0(\widehat{X}, \sigma\theta(V)) \simeq \widehat{A}$ . This implies that

$$\tau(V)^{\Pi(\widehat{X}, \widehat{x}_0)} \simeq \widehat{A}.$$

Likewise, if  $M \in \mathfrak{s}_0(G)$ , we conclude that

$$\tau(M)^{\Pi(\widehat{X}, \widehat{x}_0)} \simeq k.$$

Now, as we learn from 1.4 and Proposition 2 of 1.5 in [Se68], for any  $V \in \mathfrak{s}(G_{\widehat{A}})$ , respectively  $M \in \mathfrak{s}_0(G_{\widehat{A}})$ , there exists  $V^b \in \mathfrak{s}(G)$ , respectively  $M^b \in \mathfrak{s}_0(G)$ , and an injection  $V \rightarrow i(V^b)$ , respectively an injection  $M \rightarrow i(M^b)$ . Then,

$$\begin{aligned} \xi(V)^{\Pi(\widehat{X}, \widehat{x}_0)} &\subset \tau(V^b)^{\Pi(\widehat{X}, \widehat{x}_0)} \\ &\simeq \widehat{A}, \end{aligned}$$

and

$$\begin{aligned} \xi(M)^{\Pi(\widehat{X}, \widehat{x}_0)} &\subset \tau(M^b)^{\Pi(\widehat{X}, \widehat{x}_0)} \\ &\simeq k. \end{aligned}$$

Consequently, Lemma 2.9 guarantees that the morphism

$$\Pi(\widehat{X}, \widehat{x}_0) \longrightarrow G_{\widehat{A}}$$

deduced from  $\xi$  is faithfully flat. Now, since  $\Pi(\widehat{X}, \widehat{x}_0)$  is strictly pro-finite (Theorem 8.8) it is not hard to see that  $G_{\widehat{A}}$  is actually finite, so that  $G$  must then be finite [BAC, I.3.6, Proposition 11, p.52].  $\square$

Using [SP, Tag 0AS7], we have:

**Corollary 8.11.** *Let us adopt the hypothesis of Theorem 8.10. Then, the ring of functions of  $\Pi(X, x_0)$  is a Mittag-Leffler  $A$ -module.*  $\square$

## 9. AN APPLICATION TO THE THEORY OF TORSION POINTS ON THE PICARD SCHEME

We assume that  $A$  is Henselian and Japanese. Let  $X$  be an irreducible, proper and flat  $A$ -scheme with geometrically reduced fibres, and  $x_0$  an  $A$ -point of  $X$ . Assume in addition that  $X_K$  is normal. The following result connects Theorem 6.1 with the theory of torsion points on abelian schemes.

**Proposition 9.1.** *Suppose that  $A$  has characteristic  $(0, p)$  and absolute ramification index  $e$ . We give ourselves a positive integer  $r$  and an invertible sheaf  $\mathcal{L}$  on  $X$ .*

- (1) *If  $\mathcal{L} \in \mathrm{Pic}(X)$  has order  $p^r$  and is taken upon reduction to the identity of  $\mathrm{Pic}(X_k)$ , then  $p^r - p^{r-1} \leq e$ .*
- (2) *If  $Y$  is an abelian scheme over  $A$  and  $y \in Y(A)$  is a point of order  $p^r$  which reduces to the identity in  $Y(k)$ , then  $p^r - p^{r-1} \leq e$ .*



The proof requires:

**Lemma 9.2.** *Let  $A$  have characteristic  $(0, p)$  and absolute ramification index  $e$ . Let  $r$  be a positive integer. Then, the Neron blowup [WW80, Section 1]  $\tilde{\mu}_{p^r}$  of  $\mu_{p^r}$  at the origin in the special fibre is finite if and only if*

$$p^r \leq er + \min_{0 \leq i < r} \{p^i - ei\}.$$

*Proof.* Let us write  $\mu_{p^r} = \text{Spec } \mathcal{O}$ , where  $\mathcal{O} = A[t]/(t^{p^r} - 1)$ . Putting  $t = 1 + s$ , we have  $\mathcal{O} = A[s]/(\varphi)$ , where  $\varphi(s) = s^{p^r} + \sum_{n=1}^{p^r-1} \binom{p^r}{n} s^n$ . According to [Liu02, 8.1.2(e), p. 318] the ring of the Neron blowup  $\tilde{\mathcal{O}}$  is the quotient of the polynomial ring  $\mathcal{O}[\tilde{s}]$  by the ideal

$$\{P \in \mathcal{O}[\tilde{s}] : \pi^d P \in (\pi\tilde{s} - s) \text{ for some } d \geq 0\}.$$

Therefore, if we agree to write  $\tilde{\varphi}(\tilde{s}) = \varphi(\pi\tilde{s})$ , then

$$\begin{aligned} \tilde{\mathcal{O}} &= \frac{A[\tilde{s}]/(\tilde{\varphi})}{(\text{elements annihilated by some power of } \pi)} \\ &= \frac{A[\tilde{s}]}{\{P : \pi^d P \in (\tilde{\varphi}) \text{ for some } d \geq 0\}}. \end{aligned}$$

Let  $\nu : A[\tilde{s}] \rightarrow \mathbb{N} \cup \{\infty\}$  be the ‘‘Gauss valuation’’; it extends the normalized valuation of  $A$  and satisfies  $\nu(\tilde{s}) = 0$ . Write  $\tilde{\varphi} = \pi^{\nu(\tilde{\varphi})} \tilde{\varphi}_0$ , where  $\nu(\tilde{\varphi}_0) = 0$ . We note that the natural surjection  $A[\tilde{s}]/(\tilde{\varphi}_0) \rightarrow \tilde{\mathcal{O}}$  is an isomorphism.

If  $\text{ord}_p : \mathbb{Z} \setminus \{0\} \rightarrow \mathbb{N}$  denotes the  $p$ -adic valuation, then Legendre’s formula shows that

$$\text{ord}_p \binom{p^r}{n} = r - \text{ord}_p(n),$$

so that

$$\begin{aligned} \tilde{\varphi} &= \pi^{p^r} \tilde{s}^{p^r} + \sum_{n=1}^{p^r-1} \pi^{n+er-e \cdot \text{ord}_p(n)} u_n \tilde{s}^n \\ &= \pi^{p^r} \tilde{s}^{p^r} + \sum_{n=1}^{p^r-1} \pi^{\alpha(n)} u_n \tilde{s}^n, \end{aligned}$$

with  $u_n \in A^\times$  and  $\alpha(n) := n + er - e \cdot \text{ord}_p(n)$ . Now, if  $\text{ord}_p(n) = i$ , then  $\alpha(n) = n + er - ei$ , so that  $\alpha(p^i) \leq \alpha(n)$  in this case. It follows that,

$$\begin{aligned} \min_{1 \leq n < p^r} \alpha(n) &= \min_{0 \leq i < r} \alpha(p^i) \\ &= er + \min_{0 \leq i < r} \{p^i - ei\}. \end{aligned}$$

Hence,  $\nu(\tilde{\varphi})$  is the minimum between  $er + \min_{0 \leq i < r} \{p^i - ei\}$  and  $p^r$ . The Lemma is now proved once we note that  $\tilde{\mathcal{O}}$  is finite if and only if  $\tilde{\varphi}_0$  above has an invertible leading coefficient.  $\square$

*Proof of Proposition 9.1.* Let us deal first with the case  $r = 1$ . Using the fact that  $\mathcal{L}$  becomes trivial on the  $\mu_p$ -torsor associated to it and Theorem 8.1, we see that  $\mathcal{L} \in \mathfrak{T}_X$ . Let  $G = \text{Gal}'(\mathcal{L})$  and let  $\lambda : G \rightarrow \mathbb{G}_m$  be the associated representation. Then,  $\lambda$  factors through  $\mu_p \subset \mathbb{G}_m$  and  $\lambda \otimes K$  induces an isomorphism  $G \otimes K \rightarrow \mu_p \otimes K$ . Since  $\lambda_k$  is trivial, the morphism  $\lambda$  factors through an arrow  $\tilde{\lambda} : G \rightarrow \tilde{\mu}_p$ , where  $\tilde{\mu}_p$  is the Neron blowup of  $\mu_p$  at the identity of the special fibre. Now, because  $G \rightarrow \tilde{\mu}_p$  induces an isomorphism between generic fibres and a fortiori an injection among rings of functions, finiteness of  $G$  (by Theorem 8.10) implies that of  $\tilde{\mu}_p$  and Lemma 9.2 finishes the proof.

Let us now assume that  $r \geq 2$  and that the claim is true for all  $i < r$ . Since  $\mathcal{L}^{\otimes p^{r-i}}$  has order  $p^i$ , we conclude that

$$(4) \quad p^i \leq e + p^{i-1}$$

for  $i \in \{1, \dots, r-1\}$ . Using the fact that  $\mathcal{L}$  becomes trivial on the  $\mu_{p^r}$ -torsor associated to it and Theorem 8.1, we see that  $\mathcal{L} \in \mathfrak{T}_X$ . Let  $G = \text{Gal}'(\mathcal{L})$  and let  $\lambda : G \rightarrow \mathbb{G}_m$  be the associated representation. Just as for the particular case, we conclude that the blowup of  $\mu_{p^r}$  at the identity of the special fibre is finite. Hence, by Lemma 9.2,

$$p^r \leq er + \min_{0 \leq i < r} \{p^i - ei\}.$$

But using (4), we see that

$$1 \geq p - e \geq p^2 - 2e \geq \dots \geq p^{r-1} - (r-1) \cdot e.$$

Consequently,  $\min_{0 \leq i < r} \{p^i - ei\} = p^{r-1} - (r-1) \cdot e$ , so that

$$p^r \leq er + p^{r-1} - (r-1) \cdot e,$$

and this is what we wanted.

(2) One takes  $X$  to be an abelian scheme such that  $Y$  is its dual and applies part (1).  $\square$

*Remark 9.3.* Corollary 9.1(2) can be deduced from ‘‘Cassels’ Theorem’’ on formal groups. The one dimensional case is classical [Sil86, IV.6.1, p.123], while the higher dimensional can be found in [Gr13, p.966].

## 10. APPLICATION TO THE THEORY OF TORSORS

We suppose that  $A$  is Henselian and Japanese. Let  $X$  be an irreducible, proper and flat  $A$ -scheme with geometrically reduced fibres, and  $x_0$  an  $A$ -point of  $X$ .

**Proposition 10.1.** *Let  $G \in (\mathbf{FGSch}/A)$  be finite and  $\varphi : Q \rightarrow X$  be a  $G$ -torsor. Then, there exists a  $\psi : Z \rightarrow X$  in  $\mathfrak{S}^+(X, x_0)$  such that  $\theta_Q : \text{Rep}_A(G) \rightarrow \mathbf{coh}(X)$  takes values in  $\mathfrak{T}_\psi^{\text{tan}}$ .*

*Proof.* According to Theorem 8.1, there exists  $\psi : Z \rightarrow X$  in  $\mathfrak{S}^+(X, x_0)$  and a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & Q \\ & \searrow \psi & \downarrow \varphi \\ & & X. \end{array}$$

Hence, for each  $M \in \text{Rep}_A(G)$  we conclude that  $\psi^*(\theta_Q(M))$  is trivial relatively to  $A$  since  $\varphi^*\theta_Q(M) \simeq \mathcal{O}_Q \otimes_A M$  is trivial relatively to  $A$ . Finally, since each  $M \in \text{Rep}_A(G)$  is a quotient of some object in  $\text{Rep}_A^\circ(G)$  [Se68, Proposition 3, p.41], it follows that  $\theta_Q$  takes values in  $\mathfrak{T}_\psi^{\text{tan}}$ .  $\square$

In [Nor82, 87ff], Nori defines the notion of ‘‘reduced torsor’’ in order to understand which group schemes do come as quotients of his fundamental group. We follow the same idea here, but instead of starting off with Nori’s definition, we prefer to use an equivalent characterization [Nor82, Proposition 3, p.87].

Before reading the definition to come, the reader might profit to recall that, for any  $G \in (\mathbf{FGSch}/A)$  and any  $G$ -torsor  $\varphi : Q \rightarrow X$  above  $X$ , the functor  $\theta_Q : \text{Rep}_A(G) \rightarrow \mathbf{coh}(X)$  is exact and faithful since  $\varphi^*\theta_Q$  is naturally isomorphic to  $M \mapsto \mathcal{O}_Q \otimes_A M$  (see for example the proof of (a) in [J87, Part 1, Proposition 5.9]).

**Definition 10.2.** Let  $G \in (\mathbf{FGSch}/A)$  be finite and  $Q \rightarrow X$  be a  $G$ -torsor having an  $A$ -point  $q_0$  above  $x_0$ . We say that the data  $(Q, G, q_0)$  defines a Nori-reduced torsor if  $\theta_Q : \mathrm{Rep}_A(G) \rightarrow \mathbf{coh}(X)$  is a fully faithful functor.

**Proposition 10.3** (compare to [MS13, Theorem 7.1]). *Let  $G \in (\mathbf{FGSch}/A)$  be finite and  $Q \rightarrow X$  be a  $G$ -torsor having an  $A$ -point  $q_0$  above  $x_0$ . Then, the following conditions are equivalent:*

- (i) *The triple  $(Q, G, q_0)$  is Nori-reduced.*
- (ii) *The ring of global functions of  $Q_k$  is  $k$ .*
- (ii') *The triple  $(Q_k, G_k, q_{0,k})$  is reduced in the sense of [Nor82, Definition 3, p.87].*
- (iii) *The  $A$ -scheme  $Q$  is  $H^0$ -flat and  $A = H^0(Q, \mathcal{O}_Q)$ .*

*Proof.* The proof relies on the fact that  $A[G]_{\mathrm{left}}$  is an algebra in the category  $\mathrm{Rep}_A(G)$ , and that the corresponding  $\mathcal{O}_X$ -algebra  $\theta_Q(A[G]_{\mathrm{left}})$  is simply  $\mathcal{O}_Q$ .

(i)  $\Rightarrow$  (ii). Because the functor  $\theta_Q$  is fully faithful and  $\theta_Q(k[G]_{\mathrm{left}}) \simeq \mathcal{O}_Q \otimes k$ , we conclude that  $k = H^0(X, \mathcal{O}_Q \otimes k)$ . But  $H^0(X, \mathcal{O}_Q \otimes k) = H^0(Q, \mathcal{O}_Q \otimes k)$ , and hence the only regular functions on the  $k$ -scheme  $Q_k$  are the constants.

(ii)  $\Leftrightarrow$  (ii'). This is [Nor82, II, Proposition 3].

(ii)  $\Rightarrow$  (iii). Recall that  $Q$  is flat over  $X$  and a fortiori over  $A$ . Now, because  $k = H^0(Q, \mathcal{O}_Q \otimes_A k)$ , we can employ Proposition 12.10 of Chapter III in [Har77] to conclude that  $Q$  is cohomologically flat of degree zero over  $A$ . Since  $H^0(\mathcal{O}_Q)$  is a finite and flat  $A$ -module, the isomorphism  $k \simeq k \otimes_A H^0(\mathcal{O}_Q)$  proves that  $A \simeq H^0(\mathcal{O}_Q)$ .

(iii)  $\Rightarrow$  (i). Let  $\varphi$  denote the structural morphism  $Q \rightarrow X$ ; by assumption it belongs to  $\mathfrak{S}^+(X, x_0)$ . If we agree to write  $H := \Pi(X, \varphi, x_0)$  (see Definition 4.6), the existence of the point  $q_0$  gives an isomorphism between  $(\bullet|_{x_0}) \circ \theta_Q$  and the forgetful functor  $\mathrm{Rep}_A(G) \rightarrow A\text{-mod}$  and hence a morphism of group schemes

$$\rho : H \longrightarrow G$$

together with a commutative diagram

$$(5) \quad \begin{array}{ccc} \mathrm{Rep}_A(G) & \xrightarrow{\theta_Q} & \mathfrak{T}_\varphi \\ \rho^\# \downarrow & \swarrow \sim & \bullet|_{x_0} \\ \mathrm{Rep}_A(H) & & \end{array}$$

Now,

$$\begin{aligned} k &= H^0(X, \mathcal{O}_{Q_k}) \\ &\simeq H^0(X, \theta_Q(k[G]_{\mathrm{left}})) \\ &\simeq (k[G]_{\mathrm{left}})^H \\ &\simeq (k[G]_{\mathrm{left}})^{H_k}. \end{aligned}$$

According to Lemma 2.8, this is only possible when  $\rho_k : H_k \rightarrow G_k$  is faithfully flat. Analogously, we have that  $(A[G]_{\mathrm{left}})^H \simeq A$ . This implies that  $(K[G]_{\mathrm{left}})^{H_K} \simeq K$ , and hence  $\rho_K$  is faithfully flat according to Lemma 2.8. In conclusion,  $\rho_k$  and  $\rho_K$  are faithfully flat, and hence  $\rho$  must be faithfully flat [DuHa18, 4.1.1, p. 1124]. Together with [DuHa18, 3.2.1(ii), p. 1121], we conclude that  $\rho^\#$  is fully faithful, so that diagram (5) secures fully faithfulness of  $\theta_Q$ .  $\square$

We shall now keep the notation and assumptions of Proposition 10.3 and offer other properties equivalent to the ones in its statement. This will allow us, in passing, to render the connection with [MS13, Theorem 7.1] and to [Nor82, Definition 3, p.87] more transparent. First we develop some preliminaries.

Let us abbreviate  $\Pi = \Pi(X, x_0)$ . Similarly to [Nor76] (see §2 and the argument on p.39), there exists a  $\Pi$ -torsor

$$\tilde{X} \longrightarrow X$$

with an  $A$ -point  $\tilde{x}_0$  above  $x_0$  such that  $\theta_{\tilde{X}} \circ (\bullet|_{x_0}) \simeq \text{id}$  and  $(\bullet|_{x_0}) \circ \theta_{\tilde{X}} \simeq \text{id}$  as tensor functors. The quasi-coherent  $\mathcal{O}_X$ -algebra of  $\tilde{X}$  is a direct limits of coherent modules belonging to  $\mathfrak{T}_{\tilde{X}}$  and corresponds, in  $\text{Ind Rep}_A(\Pi)$ , to  $A[\Pi]_{\text{left}}$ , see [Nor76, Definition, p. 32]. The torsor  $\tilde{X}$  is called the *universal pointed torsor*.

Let  $\rho : G' \rightarrow G$  be an arrow of  $(\mathbf{FGSch}/A)$ . We say that  $Q$  has a *reduction of structure group to  $\rho$ , or to  $G'$* , if there exists a  $G'$ -torsor  $Q' \rightarrow X$  together with an isomorphism  $Q' \times^{\rho} G \rightarrow Q$ . In addition, if  $Q'$  can be picked to come with an  $A$ -point  $q'_0$  such that  $(q'_0, e)$  corresponds to  $q_0$  under the aforementioned isomorphism, then the reduction is called *pointed*. Note that, we do not assume  $\rho$  to be a closed embedding.

**Corollary 10.4.** *The equivalent properties appearing in Proposition 10.3 are also equivalent to each one of the following conditions:*

- (a) *If  $Q$  has a pointed reduction to  $\rho : G' \rightarrow G$ , then  $\rho$  is faithfully flat.*
- (b) *There exists a faithfully flat morphism  $\rho : \Pi \rightarrow G$  and an isomorphism of pointed  $G$ -torsors  $\tilde{X} \times^{\rho} G \xrightarrow{\sim} Q$ . (That is,  $\tilde{X}$  defines a pointed reduction of  $Q$ .)*

*Proof.* Proposition 10.3-(i)  $\Rightarrow$  (a). Let  $\rho : G' \rightarrow G$  define a pointed reduction  $Q' \rightarrow X$ ; it then follows that  $\theta_{Q'} \circ \rho^{\#}$  is isomorphic to  $\theta_Q$ . Since  $\theta_{Q'}$  is faithful, we conclude that  $\rho^{\#} : \text{Rep}_A(G) \rightarrow \text{Rep}_A(G')$  is full and faithful. By Lemma 2.9,  $\rho$  is faithfully flat.

(a)  $\Rightarrow$  (b). According to Proposition 10.1,  $\theta_Q$  takes values in  $\mathfrak{T}_X$ ; the existence of the point  $q_0$  allows us to say that  $\bullet|_{x_0} \circ \theta_Q$  is isomorphic to the forgetful functor  $\text{Rep}_A(G) \rightarrow A\text{-mod}$  which gives us an arrow  $\rho : \Pi \rightarrow G$  such that  $\rho^{\#} \simeq \bullet|_{x_0} \circ \theta_Q$  (as tensor functors). Hence,  $\theta_{\tilde{X}} \circ \rho^{\#} \simeq \theta_Q$ , and we conclude, as in [Nor76, Proposition 2.9(c)], that  $Q$  has a pointed reduction to  $\rho$ . But (a) forces  $\rho$  to be faithfully flat.

(b)  $\Rightarrow$  Proposition 10.3-(i). We know that  $\theta_Q \simeq \theta_{\tilde{X}} \circ \rho^{\#}$  in this case; but, by construction,  $\theta_{\tilde{X}}$  is fully faithful as is  $\rho^{\#}$  (by [DuHa18, 3.2.1(ii), p.1121], say). Therefore,  $\theta_Q$  is fully faithful.  $\square$

*Remark 10.5.* Let  $Q \rightarrow X$  be as in the statement of Proposition 10.3. The condition that  $\theta_Q : \text{Rep}_A^{\circ}(G) \rightarrow \mathfrak{T}_X^{\circ}$  be full is not enough to assure that  $G$  is a faithfully flat quotient of  $\Pi$  (so that this is missing in [MS13, Theorem 7.1]).

## 11. ESSENTIALLY FINITE VECTOR BUNDLES ON THE FIBRES: REVIEWING A THEORY OF MEHTA AND SUBRAMANIAN

Let  $X$  be an irreducible, projective and flat  $A$ -scheme with geometrically reduced fibres, and  $x_0$  an  $A$ -point of  $X$ . (Recall that over a perfect field, an algebraic scheme is geometrically reduced if and only if it is reduced [EGA IV<sub>2</sub>, 4.6.1, p.68]. Note also that  $X$  must be reduced.)

The following result, which is one of the main points in [MS13], is essentially a consequence of the method employed by Deninger and Werner in proving [DW05, Theorem 17, p.573] plus Section 8. Before putting forth its statement, let us recall the notion of an  $F$ -trivial vector bundle.

If  $M$  is a proper scheme over an unspecified *perfect* field of positive characteristic, a vector bundle  $E$  on  $M$  is called  $F$ -trivial [MS02, Section 2, p. 144] if for a certain  $s \in \mathbb{N}$ , the pull-back of  $E$  by a geometric Frobenius morphism  $\mathrm{Fr}^s : M^{(-s)} \rightarrow M$  is trivial.

**Theorem 11.1.** *Suppose that  $A$  is Henselian and Japanese, and that  $k$  is perfect of characteristic  $p > 0$ . Let  $E$  be an  $F$ -trivial vector bundle on  $X_k$ . Then, there exists a proper and surjective morphism  $\psi : Z \rightarrow X$  such that:*

- (1) *The ring  $B := H^0(\mathcal{O}_Z)$  is a discrete valuation ring and is a finite extension of  $A$ .*
- (2) *The canonical morphism  $Z \rightarrow \mathrm{Spec} B$  is flat and has geometrically reduced fibres.*
- (3) *The  $B$ -scheme  $Z$  has a  $B$ -point above  $x_0$ .*
- (4) *Write  $\ell$  for the residue field of  $B$  and denote by*

$$\psi_0 : Z \otimes_B \ell \longrightarrow X_k$$

*the morphism of  $k$ -schemes naturally induced by  $\psi$ . Then  $\psi_0^*(E)$  is trivial.*

*Proof.* *The case of characteristic  $(0, p)$ .* We assume that  $E$  is trivialized by  $\mathrm{Fr}^s : X_k^{(-s)} \rightarrow X_k$ . Let  $X \rightarrow \mathbb{P}_A^n$  be a closed immersion. Write  $\Phi : \mathbb{P}_A^n \rightarrow \mathbb{P}_A^n$  for the evident  $A$ -morphism lifting the  $k$ -linear Frobenius morphisms  $\mathrm{Fr}^s : \mathbb{P}_k^n \rightarrow \mathbb{P}_k^n$ , and consider the cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \downarrow & \square & \downarrow \\ \mathbb{P}_A^n & \xrightarrow{\Phi} & \mathbb{P}_A^n \end{array}$$

We note that  $\Phi$  is a finite, flat and surjective morphism, so that  $\varphi : Y \rightarrow X$  is likewise; in particular this implies that  $Y$  is  $A$ -flat.

Base-changing by means of  $A \rightarrow k$  we get the cartesian diagram

$$\begin{array}{ccc} Y_k & \xrightarrow{\varphi_k} & X_k \\ \downarrow & \square & \downarrow \\ \mathbb{P}_k^n & \xrightarrow{\mathrm{Fr}^s} & \mathbb{P}_k^n \end{array}$$

so that, since  $X_k^{(-s)}$  is reduced, there exists a closed embedding

$$j : X_k^{(-s)} \longrightarrow Y_k$$

which identifies  $X_k^{(-s)}$  with  $Y_{k, \mathrm{red}}$  and, in addition, produces a factorisation of  $\mathrm{Fr}^s : X_k^{(-s)} \rightarrow X_k$  like so

$$\begin{array}{ccc} X_k^{(-s)} & \xrightarrow{\mathrm{Fr}^s} & X_k \\ \searrow j & & \downarrow \\ Y_k & \xrightarrow{\varphi_k} & X_k \\ \downarrow & \square & \downarrow \\ \mathbb{P}_k^n & \xrightarrow{\mathrm{Fr}^s} & \mathbb{P}_k^n \end{array}$$

See Lemma 19 in [DW05]. Consequently, if  $V$  is any *reduced* scheme and  $\alpha : V \rightarrow Y_k$  is a  $\mathbb{Z}$ -morphism, we conclude that  $\alpha^* \varphi_k^* E$  is trivial since  $\alpha$  factors as

$$V \longrightarrow X_k^{(-s)} \xrightarrow{j} Y_k.$$

A direct application of Theorem 8.1 now gives us a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & \searrow \psi & \downarrow \varphi \\ & & X \end{array}$$

such that

- (i) the morphism  $\psi$  is surjective and proper.
- (ii) The ring  $B := H^0(\mathcal{O}_Z)$  is a discrete valuation rings and is a finite extension of  $A$ .
- (iii) The canonical morphism  $Z \rightarrow \text{Spec } B$  is flat and has geometrically reduced fibres.
- (iv) The  $B$ -scheme  $Z$  has a  $B$ -point above  $x_0$ .

In this situation, the proof is concluded by the observation preceding it. Indeed, if  $\ell$  is the residue field of  $B$  and  $\psi_0 : Z \otimes_B \ell \rightarrow X_k$  is the arrow induced by  $\psi$ , we conclude that  $\psi_0^*(E)$  is trivial because  $Z \otimes_B \ell$  is reduced so that  $\psi_0$  factors through  $Z \otimes_B \ell \rightarrow Y_k$ .

*Proof in the case of characteristic  $(p, p)$ .* The idea behind the proof is much simpler, but notation and technicalities hinder its handling.

Suppose that  $E \in \mathbf{VB}(X_k)$  becomes trivial after being pulled back by  $F_{X_k}^s : X_k \rightarrow X_k$ . Employing the commutative diagram of  $\mathbb{F}_p$ -schemes

$$\begin{array}{ccc} X & \xrightarrow{F_X^s} & X \\ \downarrow & & \downarrow \\ \text{Spec } A & \xrightarrow{F_A^s} & \text{Spec } A, \end{array}$$

we see that the morphism  $(F_X^s)_0 : X_k \rightarrow X_k$  induced on special fibres is none other than  $F_{X_k}^s$ . Hence, if  $F_A^s$  is a finite morphism, the choice  $Z = X$  and  $\Psi = F_X^s$  is sufficient to fulfill all but condition (3) of the statement. But finiteness of  $F_A$  is not always assured, and we choose to argue as in [EGA IV<sub>3</sub>, §8].

Let

$$\Lambda = \left\{ \begin{array}{l} B \text{ is a d.v.r. dominated by } A \text{ and dominating } A^{p^s}, \\ \text{and such that } \text{Frac } B \text{ is a finite extension of } K^{p^s} \end{array} \right\},$$

and endow it with the partial order defined by domination of d.v.r.'s. As  $A$  is Japanese, for any  $B \in \Lambda$ , the  $A^{p^s}$ -module  $B$  is finite, and any element in  $A$  belongs to some  $B \in \Lambda$  (see Theorem 10.2 and Exercise 11.2 in [Mat89]). Consequently, the limit  $\varprojlim_{B \in \Lambda} \text{Spec } B$  in the category of  $A$ -schemes is simply  $F_A^s : \text{Spec } A \rightarrow \text{Spec } A$ . Employing [EGA IV<sub>3</sub>, 8.8.2-ii, p.28], there exists  $\mathfrak{o} \in \Lambda$  and a  $\mathfrak{o}$ -scheme of *finite type*  $Y$  fitting into a cartesian commutative diagram

$$(6) \quad \begin{array}{ccc} X & \xrightarrow{u} & Y \\ \downarrow & \square & \downarrow \\ \text{Spec } A & \longrightarrow & \text{Spec } \mathfrak{o}. \end{array}$$

In addition, if  $u_B : X \rightarrow Y \otimes_{\mathfrak{o}} B$  stands for the canonical morphism, an application of [EGA IV<sub>3</sub>, 8.2.5,p.9] shows that

$$(u_B) : X \longrightarrow \varprojlim_{B \geq \mathfrak{o}} \begin{array}{c} Y \otimes B \\ \mathfrak{o} \end{array}$$

is in fact an isomorphism of  $\mathfrak{o}$ -schemes. Also by loc.cit., the canonical morphism

$$X \otimes_{A, F_A^s} A \longrightarrow \varprojlim_{B \geq \mathfrak{o}} X \otimes_{A, F_A^s} B$$

is also an isomorphism. The relative Frobenius morphism

$$\mathfrak{f} : X \longrightarrow X \otimes_{A, F_A^s} A$$

now gives rise, via [EGA IV<sub>3</sub>, 8.8.2-i, p.28], to a  $B \geq \mathfrak{o}$  and a morphism of  $B$ -schemes

$$f : Y \otimes_{\mathfrak{o}} B \longrightarrow X \otimes_{A, F_A^s} B,$$

such that  $f \otimes_B A$  corresponds to  $\mathfrak{f}$ . Hence, if  $\psi$  stands for the composition of  $f$  with the projection  $X \otimes_{A, F_A^s} B \rightarrow X$ , we arrive at a commutative diagram

$$(7) \quad \begin{array}{ccc} Y \otimes B & \xrightarrow{\psi} & X \\ \downarrow & & \downarrow \\ \text{Spec } B & \longrightarrow & \text{Spec } A. \end{array}$$

In addition,

$$\psi \circ u_B = F_X^s.$$

Let us agree to write  $Z = Y \otimes_{\mathfrak{o}} B$ . Then, paralleling diagram (6), we have

$$(8) \quad \begin{array}{ccc} X & \xrightarrow{u_B} & Z \\ \downarrow & \square & \downarrow \\ \text{Spec } A & \longrightarrow & \text{Spec } B. \end{array}$$

*Claim.* The following statements are true.

- (i) The morphism  $\psi$  is finite and surjective.
- (ii) As a  $B$ -scheme,  $Z$  is flat and proper.
- (iii) The geometric fibres of  $Z$  over  $B$  are reduced.
- (iv) The ring of global functions of  $Z$  is  $B$ .
- (v) Write  $\psi_0$  for the morphism induced from  $\psi$  between special fibres. Then  $\psi_0^*(E)$  is trivial.

*Proof.* (i) Surjectivity follows from  $F_X^s = \psi \circ u_B$ . Because  $\mathfrak{f}$  is finite and the inclusion  $B \rightarrow A$  is faithfully flat, we conclude that  $f$  is finite [EGA IV<sub>2</sub>, 2.7.1, p.29]. Hence,  $\psi$  is finite as  $F_A^s : A \rightarrow B$  is finite.

(ii) We note that the morphism  $\text{Spec } A \rightarrow \text{Spec } B$  in diagram (8) is faithfully flat. Consequently, the claim is proved by employing [EGA IV<sub>3</sub>, 2.5.1, p. 22], [EGA IV<sub>3</sub>, 2.7.1, p.29] and the fact that  $X$  is flat and proper over  $A$ .

(iii) This is a direct consequence of diagram (8) and the fact that being geometrically reduced is independent of the field extension [EGA IV<sub>2</sub>, 4.6.10, p.70].

(iv) This is a direct consequence of flat base-change applied to diagram (8) and  $A = H^0(\mathcal{O}_X)$  (which follows from Lemma 2.2, say).

(v) Let now  $\ell$  be the residue field of  $B$ ; it is clear that  $B \rightarrow A$  in fact induces an isomorphism  $\ell \xrightarrow{\sim} k$ . Since  $X = Z \otimes_B A$ , it follows that  $(u_B)_0 : X \otimes_A k \rightarrow Z \otimes_B \ell$  is also an isomorphism. Hence,  $\psi_0^*(E)$  is trivial because  $\psi u_B = F_X^s$  so that  $(u_B)_0^*(\psi_0^*(E))$  is trivial. The claim is proved.

To finish the proof, we note that  $\psi : Z \rightarrow X$  satisfies all the conditions in the statement of the Theorem except for the existence of a  $B$ -point above  $x_0$ . Now, the inverse image  $\psi^{-1}(x_0)$  comes with a finite and surjective morphism to  $\text{Spec } A$  (for surjectivity, see [EGA

I, 3.5.2,p.115]). Hence, it is possible to find a *finite* extension of d.v.r.'s  $B' \supset A$  and a point  $\text{Spec } B' \rightarrow \Psi^{-1}(x_0)$  which then gives a commutative diagram

$$\begin{array}{ccc} & & Z \\ & \nearrow^{z'_0} & \downarrow \\ \text{Spec } B' & \longrightarrow & \text{Spec } B \end{array}$$

such that  $z'_0$  is a  $B'$ -point of  $X$  above  $x_0$  and the induced arrow  $B \rightarrow B'$  is a finite extension. Consequently, letting  $Z'$  be  $Z \otimes_B B'$  and  $\psi' : Z' \rightarrow X$  the composition  $Z' \xrightarrow{\text{pr}} Z \xrightarrow{\psi} X$ , we see that  $\psi' : Z' \rightarrow X$  now satisfies all properties in the statement of the theorem.  $\square$

As a consequence of Theorem 11.1, we can give a simple alternative description of the vector bundles in  $\mathfrak{T}$  and, in doing so, connect our theory to that of [MS13]. See Corollary 11.4.

**Theorem 11.2** (Compare to [MS13, Lemma 3.1]). *Suppose that  $A$  is Henselian and Japanese, that  $k$  is perfect, and that  $X$  is in addition normal. Let  $\mathcal{E} \in \mathbf{VB}(X)$  be such that  $\mathcal{E}_k$  and  $\mathcal{E}_k$  are essentially finite. Then, there exists a proper and surjective morphism*

$$\zeta : X' \longrightarrow X$$

such that

- (1) The ring  $A' := H^0(\mathcal{O}_{X'})$  is a discrete valuation ring and a finite extension of  $A$ .
- (2) The canonical morphism  $X' \rightarrow \text{Spec } A'$  is flat and has geometrically reduced fibres.
- (3) The  $A'$ -scheme  $X'$  has an  $A'$ -point above  $x_0$ .
- (4) The vector bundle  $\zeta^*(\mathcal{E})$  is trivial.

*Proof.* As the vector bundle  $\mathcal{E}_k$  is essentially finite, it is possible to find a torsor under an étale group scheme  $f : Y \rightarrow X_k$ , and a fortiori an étale covering, such that  $f^*(\mathcal{E}_k)$  is  $F$ -trivial see [Nor76, §3]. In addition,  $Y$  can be chosen to possess a  $k$ -rational point  $y_0$  above  $x_{0,k}$  (cf. loc.cit) and to satisfy  $k = H^0(Y, \mathcal{O}_Y)$  [Nor82, II, Proposition 3, p. 87]. Now, as  $A$  is Henselian, Theorem 3.1 on p.30 of [A69] (the remarkable equivalence, Grothendieck's existence theorem and Artin approximation) allows us to find an étale covering  $\tilde{f} : \tilde{Y} \rightarrow X$  lifting  $Y \rightarrow X_k$ . Looking at the finite and étale  $A$ -scheme  $\tilde{f}^{-1}(x_0)$ , applying one of the main properties of a Henselian local ring [Ray70, VII.3, Proposition 3, p.76], and making use of the  $k$ -point  $y_0 : \text{Spec } k \rightarrow \tilde{f}^{-1}(x_0)$ , we can find an  $A$ -point  $\tilde{y}_0$  in  $\tilde{Y}$  above  $x_0$ . Note that  $\tilde{Y}$  inherits the following properties from  $X$ : it is flat, proper, and has geometrically reduced fibres over  $A$ , and it is normal. In addition,  $\tilde{Y}$  is connected and its normality then assures irreducibility. (Irreducibility might fail without normality.) Therefore,  $\tilde{Y}$  satisfies all hypothesis imposed on  $X$  in the beginning of the section and in the statement of the theorem. Note that, by construction, the restriction of  $\tilde{f}^*\mathcal{E}$  to  $\tilde{Y}_k$  is  $F$ -trivial.

Because of the previous paragraph, we suppose, so to lighten notation, that  $\mathcal{E}_k$  is  $F$ -trivial already on  $X_k$ . Let us apply Theorem 11.1 to the vector bundle  $E = \mathcal{E}_k$ . Then, keeping with the notations of this theorem, we conclude that

$$\psi^*(\mathcal{E})|_{Z \otimes_B \ell}$$

is trivial. Let  $L = \text{Frac}(B)$ . Since  $\psi^*(\mathcal{E})|_{Z \otimes_B L}$  is an essentially finite vector bundle, [Nor76, §3] assures that we can find a torsor with finite structural group

$$\lambda^\circ : Q^\circ \longrightarrow Z \otimes_B L$$

such that

$$\lambda^{\circ*}(\psi^*(\mathcal{E})|_{Z \otimes_B L})$$



is trivial. In addition,  $Q^\circ$  might be chosen to come with two extra properties, which are:

- Letting  $z_0 : \text{Spec } B \rightarrow Z$  be the point above  $x_0$  mentioned in Theorem 11.1,  $Q^\circ$  has an  $L$ -rational point  $q_0^\circ$  above  $z_{0,L}$ . (This is not used in what follows.)
- The ring of global functions of  $Q^\circ$  is  $L$ , see Proposition 3 of Chapter II, p.87, in [Nor82]. In particular  $Q^\circ$  is connected.

Let  $Q^\square \rightarrow Q^\circ$  be the associated reduced scheme and write

$$\lambda^\square : Q^\square \longrightarrow Z \otimes_B L$$

for the induced morphism. Clearly  $Q^\square$  is connected,  $\lambda^\square$  is surjective, finite and

$$\lambda^{\square*}(\psi^*(\mathcal{E})|_{Z \otimes_B L})$$

is trivial.

Let  $\mu : Z \otimes_B L \rightarrow Z$  be the natural immersion and write

$$\lambda : Q \longrightarrow Z$$

for the integral closure of the quasi-coherent  $\mathcal{O}_Z$ -algebra  $(\mu\lambda^\square)_*(\mathcal{O}_{Q^\square})$ , see [EGA II, 6.3, 116ff] or [SP, Tag 035H]. This means, according to [EGA II, 6.3.4, p.117], that for each affine open subset  $V \subset Z$ , the ring  $\mathcal{O}_Q(\lambda^{-1}V)$  is the integral closure of  $\mathcal{O}_Z(V)$  inside  $\mathcal{O}_{Q^\square}((\mu\lambda^\square)^{-1}(V))$ . In particular,  $Q$  is flat as a  $B$ -scheme and

$$Q \otimes_B L = Q^\square.$$

Since  $A$  is universally Japanese [EGA IV<sub>2</sub>, 7.7.2, p. 212], for each affine and open subset  $V$  of  $Z$ , the ring  $\mathcal{O}_Z(V)$  is universally Japanese and noetherian, and hence is a Nagata ring [SP, Tag 032R]. As a consequence of [SP, Tag 03GH] and the fact that  $Q^\square$  is reduced, we see that  $\lambda$  is a *finite* morphism. Because  $\lambda(Q)$  contains  $Z \otimes_B L$  (recall that  $\lambda^\square$  is surjective), we conclude that  $\lambda$  is surjective. Finally, both  $(\psi\lambda)^*(\mathcal{E})|_{Q \otimes_B L}$  and  $(\psi\lambda)^*(\mathcal{E})|_{Q \otimes_B L}$  are trivial.

Theorem 8.1 can be applied to  $\psi\lambda : Q \rightarrow X$ , and this allows us to find a commutative diagram of schemes

$$\begin{array}{ccc} R & \longrightarrow & Q \\ & \searrow \theta & \downarrow \psi\lambda \\ & & X, \end{array}$$

such that:

- The morphism  $\theta$  is proper and surjective.
- The ring  $C := H^0(\mathcal{O}_R)$  is a discrete valuation ring and a finite extension of  $A$ .
- The canonical morphism  $R \rightarrow \text{Spec } C$  is flat and has geometrically reduced fibres.
- The  $C$ -scheme  $R$  has a  $C$ -point above  $x_0$ .

As the natural arrow  $\text{Spec } C \rightarrow \text{Spec } B$  sends the generic, respectively special, point to the generic, respectively special, point, triviality of  $(\psi\lambda)^*(\mathcal{E})|_{Q \otimes_B L}$  and  $(\psi\lambda)^*(\mathcal{E})|_{Q \otimes_B L}$  allows us to conclude that the restrictions of  $\theta^*\mathcal{E}$  to the generic and special fibres of  $R$  over  $C$  are trivial. Because  $R$  is  $H^0$ -flat over  $C$  (it has reduced fibres), we conclude by employing Lemma 2.3 that  $\theta^*\mathcal{E}$  is trivial.  $\square$

For the sake of discussion, let us make the following:

**Definition 11.3.** The Mehta-Subramanian category of  $X$ , denote it  $\mathbf{MS}(X)$ , is the full subcategory of  $\mathbf{VB}(X)$  whose objects are

$$\{\mathcal{E} \in \mathbf{VB}(X) : \mathcal{E}_k \text{ and } \mathcal{E}_K \text{ are essentially finite}\}.$$

An immediate consequence of [AM11, Theorem 1] (or [TZ20, Theorem I]) and Theorem 11.2 is then:

**Corollary 11.4.** *Suppose that  $A$  is Henselian and Japanese, that  $k$  is perfect, and that in addition to the hypothesis in the beginning of the section,  $X_k$  and  $X_K$  are normal. Then  $\mathfrak{T}_X^\circ = \mathbf{MS}(X)$ .*

*Proof.* Let  $\mathcal{E} \in \mathbf{MS}(X)$ . Since  $X$  is flat over  $A$ , normality of  $X_k$  and  $X_K$  implies normality of  $X$  [EGA IV<sub>2</sub>, 6.5.4, p.143] and Theorem 11.2 may be applied. Consequently,  $\mathcal{E}$  belongs to  $\mathfrak{T}_X^\circ$  as  $\zeta$  in Theorem 11.2 lies in  $\mathfrak{S}^+(X, x_0)$ . Conversely, let  $\mathcal{E}$  be a vector bundle in  $\mathfrak{T}_X$ . Since  $X_k$  and  $X_K$  are normal, we know that  $\mathcal{E}_k$  and  $\mathcal{E}_K$  are essentially finite (due to [AM11, Theorem 1] or [TZ20, Theorem I]).  $\square$

*Remark 11.5.* Since a point  $x \in X$  above the generic fibre specializes to a point on the special fibre, normality of  $X$  is equivalent to normality of  $X$  on the points of  $X_k$ . Of course,  $X_k$  can easily fail to be normal even when  $X$  is regular.

## 12. FURTHER APPLICATIONS TO THE THEORY OF TORSORS

We assume that  $A$  is Henselian, Japanese and has a perfect residue field. Let  $X$  be an irreducible, projective and flat  $A$ -scheme with geometrically reduced fibres, and  $x_0$  an  $A$ -point of  $X$ .

**Theorem 12.1** (compare to [MS13, Corollary 3.2]). *Let us add to the assumptions made at the start of this section that  $X$  is normal. Let  $G \in (\mathbf{FGSch}/A)$  be quasi-finite over  $A$ ,*

$$Q \longrightarrow X$$

be a  $G$ -torsor, and  $q_0$  an  $A$ -point of  $Q$  above  $x_0$ .

- (1) *There exists  $\zeta : X' \rightarrow X$  in  $\mathfrak{S}^+(X, x_0)$  (see Definition 2.4) and  $\mathcal{E} \in \mathfrak{T}_\zeta^\circ$  such that  $\theta_Q : \mathrm{Rep}_A(G) \rightarrow \mathbf{coh}(X)$  takes values in  $\langle \mathcal{E}; \mathfrak{T}_\zeta \rangle_\otimes$  (and a fortiori in  $\mathfrak{T}_X$ ).*
- (2) *There exists a finite  $H \in (\mathbf{FGSch}/A)$ , a morphism  $\rho : H \rightarrow G$ , an  $H$ -torsor  $R \rightarrow X$  and an  $A$ -point  $r_0 : \mathrm{Spec} A \rightarrow R$  together with an isomorphism of torsors*

$$R \times^H G \xrightarrow{\sim} Q$$

*sending the  $A$ -point  $(r_0, e)$  of  $R \times^H G$  to  $q_0$ . In addition, it is possible to choose  $\rho$  to be a closed immersion. Put differently,  $Q$  has a reduction of structure group to a finite group scheme.*

- (3) *If  $H^0(Q, \mathcal{O}_Q) \simeq A$ , then  $G$  is in fact finite.*

*Proof.* As is well-known (by adapting the proofs in [Wa79, 3.3]) the facts that  $G$  is of finite type and  $A$  is a d.v.r. allow us to find  $E \in \mathrm{Rep}_A^\circ(G)$  such that the resulting morphism  $G \rightarrow \mathbf{GL}(E)$  is a closed immersion, or, in the terminology of [DHdS18, §3],  $E$  is a faithful representation. We then write  $\mathcal{E} = \theta_Q(E)$  and note that since  $Q_k \rightarrow X_k$  and  $Q_K \rightarrow X_K$  are finite principal bundles, the vector bundles  $\mathcal{E}_k = \theta_{Q_k}(E_k)$  and  $\mathcal{E}_K = \theta_{Q_K}(E_K)$  are in fact essentially finite [Nor76, Proposition 3.8, p.38]. Let  $\zeta : X' \rightarrow X$ ,  $A'$ , and  $x'_0$  be as in Theorem 11.2 when applied to  $\mathcal{E}$ . Note that  $\zeta : X' \rightarrow X$  is in  $\mathfrak{S}^+(X, x_0)$  and that  $\mathcal{E} \in \mathfrak{T}_\zeta$ .

- (1). We write  $Q'$  for the  $G$ -torsor  $X' \times_X Q$ . We know that for each  $M \in \mathrm{Rep}_A(G)$ , the coherent  $\mathcal{O}_{X'}$ -module  $\zeta^* \theta_Q(M)$  is isomorphic to  $\theta_{Q'}(M)$ . Because each  $\theta_Q(\mathbf{T}^{a,b}E)$  belongs to  $\langle \mathcal{E}; \mathfrak{T}_\zeta \rangle_\otimes$ , we conclude that each  $\theta_{Q'}(\mathbf{T}^{a,b}E)$  is trivial.

Let  $T \in \mathrm{Rep}_A^\circ(G)$  be such that  $\theta_Q(T)$  belongs to  $\langle \mathcal{E}; \mathfrak{T}_\zeta \rangle_\otimes$ . If  $V \in \mathrm{Rep}_A^\circ(G)$  is the target of an epimorphism  $T \rightarrow V$ , exactness of  $\theta_{Q'}$  produces an epimorphism

$$\mathcal{O}_{X'}^{\oplus r} \simeq \theta_{Q'}(T) \longrightarrow \theta_{Q'}(V).$$

Since  $\theta_{Q'}(V) \otimes_A k$  and  $\theta_{Q'}(V) \otimes_A K$  become trivial when pulled back via  $Q' \otimes_A k \rightarrow X' \otimes_A k$  and  $Q' \otimes_A K \rightarrow X' \otimes_A K$ , we may apply Lemma 5.2 to conclude that  $\theta_{Q'}(V) \otimes k$  and  $\theta_{Q'}(V) \otimes K$  are trivial. Because  $X'$  is  $H^0$ -flat over  $A$ , Lemma 2.3 says that  $\theta_{Q'}(V)$  is

equally trivial and hence that  $\theta_Q(V) \in \langle \mathcal{E}; \mathfrak{T}_\zeta \rangle_\otimes$ . By the same argument, now applied to  $\tilde{T}$ , we conclude that  $\theta_Q(W)$  belongs to  $\langle \mathcal{E}; \mathfrak{T}_\zeta \rangle_\otimes$  once  $W \rightarrow T$  is a special subobject (we employ [dS09, Definition 10]).

Now we know that for every  $U \in \text{Rep}_A^\circ(G)$ , there exists a special monomorphism  $V \rightarrow \mathbf{T}^{a,b}E$  and an epimorphism  $V \rightarrow U$  [dS09, Proposition 12]. From what was proved above,  $\theta_Q(U)$  belongs to  $\langle \mathcal{E}; \mathfrak{T}_\zeta \rangle_\otimes$ .

To end the proof, let  $M \in \text{Rep}_A(G)$  be arbitrary. Using [Se68, Corollary 2.2, p.41], we find an equivariant presentation

$$0 \longrightarrow U_1 \longrightarrow U_0 \longrightarrow M \longrightarrow 0$$

with  $U_0$  and  $U_1$  in  $\text{Rep}_A^\circ(G)$ . An application of Lemma 4.1 (and the exactness of the functor  $\theta_Q$ ) assures that  $\theta_Q(M)$  belongs to  $\mathfrak{T}_\zeta$ , and hence to  $\langle \mathcal{E}; \mathfrak{T}_\zeta \rangle_\otimes$ .

(2). We pick  $\zeta$  and  $\mathcal{E}$  as in item (1) and define  $H = \text{Gal}'(\mathcal{E}; \mathfrak{T}_X, x_0)$  so that  $\bullet|_{x_0} : \langle \mathcal{E}; \mathfrak{T}_X \rangle_\otimes \rightarrow \text{Rep}_A(H)$  is an equivalence of tensor categories. Because of Theorem 8.10,  $H$  is a *finite* group scheme over  $A$ . Using the  $A$ -point  $q_0$  of  $Q$  above  $x_0$ , the functor  $(\bullet|_{x_0}) \circ \theta_Q : \text{Rep}_A(G) \rightarrow A\text{-mod}$  is naturally isomorphic to the forgetful functor and hence we derive a morphism of group schemes

$$\rho : H \longrightarrow G$$

such that  $(\bullet|_{x_0}) \circ \theta_Q \simeq \rho^\#$ . Now, similarly to [Nor76] (see §2 and the argument on p.39), there exists a  $H$ -torsor  $R \rightarrow X$  with an  $A$ -point  $r_0$  above  $x_0$  such that  $\theta_R \circ (\bullet|_{x_0}) \simeq \text{id}$  and  $(\bullet|_{x_0}) \circ \theta_R \simeq \text{id}$  as tensor functors. (The quasi-coherent  $\mathcal{O}_X$ -algebra of the torsor  $R$  corresponds, in the category  $\text{Rep}_A(H)$ , to  $A[H]_{\text{left}}$ , see [Nor76, Definition, p. 32].) Now

$$\begin{aligned} \theta_Q &\simeq \theta_R \circ (\bullet|_{x_0}) \circ \theta_Q \\ &\simeq \theta_R \circ \rho^\# \end{aligned}$$

which shows, just as in [Nor76, Proposition 2.9(c), p. 34], that  $Q \simeq R \times^H G$ .

To verify the last statement, we note  $\rho$  can be decomposed into  $H \rightarrow H' \xrightarrow{\sigma} G$ , where  $\sigma$  is a closed immersion and  $H'$  is finite. This being so, we have  $Q \simeq (R \times^H H') \times^{H'} G$ .

(3). Let  $H \in (\mathbf{FGSch}/A)$  be finite,  $Q \rightarrow X$  be an  $H$ -torsor,  $H \rightarrow G$  be a closed immersion, and  $R \times^H G \simeq Q$  be an isomorphism as in (2). Now, employing the arrow

$$R \times G \longrightarrow G, \quad (r, g) \longmapsto g^{-1}$$

we obtain an injection  $\text{Mor}(G, \mathbb{A}^1) \rightarrow \text{Mor}(R \times G, \mathbb{A}^1)$  and hence an injection

$$\{H\text{-equivariant } G \rightarrow \mathbb{A}^1\} \longrightarrow \{H\text{-equivariant } R \times G \rightarrow \mathbb{A}^1\}.$$

Since the right-hand-side above is simply the ring of functions of  $R \times^H G \simeq Q$ , the hypothesis then forces  $A = A[G]^H$ . But  $A[G]$  is a finite and locally free  $A[G]^H$ -module whose rank equals that of  $A[H]$  (see III.2.4 of [DG70]). It is then easy to see that the closed immersion  $H \rightarrow G$  is an isomorphism.  $\square$

We shall now gather some consequences of Theorem 12.1 and in doing so connect it to [Nor82, Chapter II] and [AEG20]. *Notations are as in the statement of Theorem 12.1.*

Let us abbreviate  $\Pi = \Pi(X, x_0)$ . As already explained in Section 10 (see the discussion preceding Corollary 10.4), there exists a  $\Pi$ -torsor

$$\tilde{X} \longrightarrow X$$

with an  $A$ -point  $\tilde{x}_0$  above  $x_0$  such that  $\theta_{\tilde{X}} \circ (\bullet|_{x_0}) \simeq \text{id}$  and  $(\bullet|_{x_0}) \circ \theta_{\tilde{X}} \simeq \text{id}$  as tensor functors. Recall that for each homomorphism  $\rho : \Pi \rightarrow G$ , the fpqc sheaf of the contracted product  $\tilde{X} \times^\rho G$  (see [DG70, III.4.3.2, p.368] or [J87, Part I, 5.14]) has the following

description: it is the quotient of  $\tilde{X} \times G$  by the right action of  $\Pi$  defined, on the level of points, by

$$(9) \quad (\tilde{x}, g) \cdot \gamma = (\tilde{x}\gamma, \rho(\gamma)^{-1}g).$$

Let us write

$$\chi_\rho : \tilde{X} \times G \longrightarrow \tilde{X} \times^\rho G$$

for the canonical quotient morphism. Then, the arrow

$$(\text{pr}_{\tilde{X}}, \chi_\rho) : \tilde{X} \times G \longrightarrow \tilde{X} \times_X (\tilde{X} \times^\rho G)$$

is an isomorphism of  $G$ -torsors over  $\tilde{X}$  (see [DG70, III.4.3.1] or [J87, Part 1, 5.14(3)]). In addition, if we let  $\Pi$  act on (the right of)  $\tilde{X} \times G$  as implied by (9) and on  $\tilde{X} \times_X (\tilde{X} \times^\rho G)$  by the action solely on  $\tilde{X}$ , then  $(\text{pr}_{\tilde{X}}, \chi_\rho)$  is  $\Pi$ -equivariant, as a simple verification shows.

Now we note that  $A = H^0(\mathcal{O}_{\tilde{X}})$  because  $A[\Pi]_{\text{left}}$ , which corresponds to the quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{O}_{\tilde{X}}$ , has only constant invariants and because of [Har77, Exercise II.1.11, p.67]. Since  $G$  is affine, any morphism of schemes  $\tilde{X} \rightarrow G$  must factor through the structural morphism  $\tilde{X} \rightarrow \text{Spec } A$  [EGA I, 2.2.4, p.99]; we conclude that any arrow  $\beta : \tilde{X} \times G \rightarrow \tilde{X} \times G$  between  $G$ -torsors must be of the form  $(\tilde{x}, g) \mapsto (\tilde{x}, cg)$ , where  $c \in G(A)$ . This being so, if  $\beta$  in addition fixes the  $A$ -point  $(\tilde{x}_0, e)$ , we see that  $\beta = \text{id}$ . This has the following pleasing consequence (implicit in [Nor76, Proposition 3.11]):

**Lemma 12.2.** *Let  $G \in \mathbf{FGSch}/A$  and let  $\rho : \Pi \rightarrow G$  and  $\sigma : \Pi \rightarrow G$  be arrows of group schemes over  $A$ . Let  $\alpha : \tilde{X} \times^\rho G \rightarrow \tilde{X} \times^\sigma G$  be a morphism of  $G$ -torsors sending  $\chi_\rho(\tilde{x}_0, e)$  to  $\chi_\sigma(\tilde{x}_0, e)$ . Then  $\rho = \sigma$  and  $\alpha = \text{id}$ .*

*Proof.* We consider the following commutative diagram

$$\begin{array}{ccc} \tilde{X} \times G & \xrightarrow{\tilde{\alpha}} & \tilde{X} \times G \\ (\text{pr}, \chi_\rho) \downarrow \sim & & \sim \downarrow (\text{pr}, \chi_\sigma) \\ \tilde{X} \times_X (\tilde{X} \times^\rho G) & \xrightarrow{\text{id} \times \alpha} & \tilde{X} \times_X (\tilde{X} \times^\sigma G). \end{array}$$

Since  $\alpha$  takes  $\chi_\rho(\tilde{x}_0, e)$  to  $\chi_\sigma(\tilde{x}_0, e)$ , we conclude that  $\tilde{\alpha}(\tilde{x}_0, e) = (\tilde{x}_0, e)$  and hence, by the above discussion,  $\tilde{\alpha}$  is the identity. Because  $(\text{pr}, \chi_\rho)$  is  $\Pi$ -equivariant (for the actions explained above), we conclude that  $\tilde{\alpha} = \text{id}$  is  $\Pi$ -equivariant, and this is only possible when  $\rho = \sigma$ . Since  $\text{id} \times \alpha = \text{id}$ , fpqc descent [EGA IV<sub>2</sub>, 2.7.1, p.29] assures that  $\alpha = \text{id}$ .  $\square$

Let  $G \in (\mathbf{FGSch}/A)$  be *quasi-finite* and consider the category  $\mathbf{Tors}_*(G)$  whose **objects** are couples  $(Q, q_0)$  consisting of a  $G$ -torsor over  $X$  and an  $A$ -point  $q_0$  of  $Q$  above  $x_0$ , and **arrows** are isomorphisms of  $G$ -torsors which preserve the  $A$ -rational point.

**Theorem 12.3.** *We maintain the above notations.*

(1) Write  $|\mathbf{Tors}_*(G)|$  for the set of isomorphism classes in  $\mathbf{Tors}_*(G)$  and  $\tilde{X}[\rho]$  for the class of the couple  $(\tilde{X} \times^\rho G, \chi_\rho(\tilde{x}_0, e))$ . Then the map

$$\text{Hom}(\Pi, G) \longrightarrow |\mathbf{Tors}_*(G)|, \quad \rho \longmapsto \tilde{X}[\rho]$$

is bijective.

(1') The category  $\mathbf{Tors}_*(G)$  is discrete [Mac98, p.11].

(2) For each  $(Q, q_0)$  in  $\mathbf{Tors}_*(G)$ , there exists a unique generalized morphism of torsors  $\chi : \tilde{X} \rightarrow Q$  taking  $\tilde{x}_0$  to  $q_0$ .

*Proof.* (1) We first establish surjectivity: the argument is identical to the one in [Nor76, Proposition 3.11]. From Theorem 12.1-(1), the functor  $\theta_Q$  takes values in  $\mathfrak{T}_X^{\text{tan}}$ . Using the  $A$ -point  $q_0$  of  $Q$  above  $x_0$ , we see that  $(\bullet|_{x_0}) \circ \theta_Q : \text{Rep}_A(G) \rightarrow A\text{-}\mathbf{mod}$  is naturally isomorphic to the forgetful functor and, since  $\bullet|_{x_0} : \mathfrak{T}_X^{\text{tan}} \xrightarrow{\sim} \text{Rep}_A(\Pi)$ , we derive a morphism of group schemes

$$\rho : \Pi \longrightarrow G$$

such that  $(\bullet|_{x_0}) \circ \theta_Q \simeq \rho^\#$ . Now

$$\begin{aligned} \theta_Q &\simeq \theta_{\tilde{X}} \circ (\bullet|_{x_0}) \circ \theta_Q \\ &\simeq \theta_{\tilde{X}} \circ \rho^\# \end{aligned}$$

which shows, just as in [Nor76, Proposition 2.9(c), p. 34], that  $Q \simeq \tilde{X} \times^\rho G$ . Injectivity is a direct consequence of Lemma 12.2.

(1') This is a direct consequence of (1) and Lemma 12.2.

(2) This is standard, but we run the argument for the convenience of the reader. Let  $\tau : \tilde{X} \rightarrow Q$  and  $\tau' : \tilde{X} \rightarrow Q$  be morphisms as in the statement covering morphisms  $\rho : \Pi \rightarrow G$  and  $\rho' : \Pi \rightarrow G$  respectively. We therefore deduce arrows  $\bar{\tau} : \tilde{X} \times^\rho G \rightarrow Q$  and  $\bar{\tau}' : \tilde{X} \times^{\rho'} G \rightarrow Q$  in  $\mathbf{Tors}_*(G)$  such that

$$\tilde{X} \xrightarrow{(\text{id}, e)} \tilde{X} \times G \xrightarrow{\chi_\rho} \tilde{X} \times^\rho G \xrightarrow{\bar{\tau}} Q$$

and

$$\tilde{X} \xrightarrow{(\text{id}, e)} \tilde{X} \times G \xrightarrow{\chi_{\rho'}} \tilde{X} \times^{\rho'} G \xrightarrow{\bar{\tau}'} Q$$

are respectively  $\tau$  and  $\tau'$ . This produces an arrow  $\alpha : \tilde{X} \times^\rho G \rightarrow \tilde{X} \times^{\rho'} G$  in  $\mathbf{Tors}_*(G)$ . According to Lemma 12.2,  $\rho = \rho'$  and  $\alpha = \text{id}$  so that  $\tau = \tau'$ .  $\square$

This allows us to compare  $\Pi$  to the fundamental group scheme introduced in [AEG20]. Recall that these authors show the existence of a pro-quasi-finite (see Definition 8.7) flat group scheme  $\Pi^*$ , a  $\Pi^*$ -torsor  $X^* \rightarrow X$  and an  $A$ -point  $x_0^*$  above  $x_0$  enjoying the following universal property. If  $G$  is flat and quasi-finite,  $Q \rightarrow X$  is a  $G$ -torsor with a point  $q_0$  above  $x_0$ , then there exists a *unique* generalized morphism of torsors  $X^* \rightarrow Q$  taking  $x_0^*$  to  $q_0$ . (See the paragraph after the proof of Theorem 5.2 in [AEG20].) From Theorem 12.3 we have:

**Corollary 12.4.** *Under the assumptions of Theorem 12.1, there exists a unique generalized isomorphism of torsors  $\tilde{X} \rightarrow X^*$  taking  $\tilde{x}_0$  to  $x_0^*$ . In particular,  $\Pi \simeq \Pi^*$ .  $\square$*

In particular, Theorem 8.10 says that  $\Pi^*$  is in fact pro-finite.

## REFERENCES

- [AEG20] M. Antei, M. Emsalem and C. Gasbarri, *Sur l'existence du schéma en groupes fondamental*. Épijournal Géom. Algébrique 4 (2020), Art. 5, 15 pp.
- [AM11] M. Antei and V. Mehta, *Vector bundles over normal varieties trivialized by finite morphisms*. Arch. Math. 97 (2011), 523–527.
- [A69] M. Artin, *Algebraic approximation of structures over complete local rings*. Inst. Hautes Études Sci. Publ. Math. 36 (1969) 23–58.
- [BdS11] I. Biswas and J. P. dos Santos, *Vector bundles trivialized by proper morphisms and the fundamental group scheme*. J. Inst. Math. Jussieu 10(2), (2011), 225–234.
- [BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 21. Springer-Verlag, Berlin, 1990.
- [BLR95] S. Bosch, W. Lütkebohmert and M. Raynaud, *Formal and rigid geometry IV. The reduced fibre theorem*. Invent. math 119 (1995), 361–398.
- [BAC] N. Bourbaki, *Algèbre commutative*. Masson, Paris, 1985.

- [DM82] P. Deligne and J. Milne, *Tannakian categories*. Lecture Notes in Mathematics 900, pp. 101–228, Springer-Verlag, Berlin-New York, 1982.
- [DG70] M. Demazure and P. Gabriel, *Groupes algébriques*. Tome I Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970.
- [DW05] C. Deninger and A. Werner, *Vector bundles on  $p$ -adic curves and parallel transport*. Ann. Sci. École Norm. Sup. (4) 38 (2005), no. 4, 553–597.
- [DuHa18] N. D. Duong and P. H. Hai, *Tannakian duality over Dedekind rings and applications*. Math. Z. (2018) 288:1103–1142.
- [DHdS18] N. D. Duong, P. H. Hai and J. P. dos Santos, *On the structure of affine flat group schemes over discrete valuation rings, I*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) Vol. XVIII (2018), 977–1032.
- [Ga03] C. Gasbarri, *Height of vector bundles and the fundamental group scheme of a curve*. Duke Math. Jour. Vol. 117, No. 2, 2003, 287–311.
- [Gr13] D. Grant, *On an analogue of the Lutz–Nagell Theorem for hyperelliptic curves*. Journal of Number Theory 133 (2013) 963–969.
- [HdS21] P. H. Hai and J. P. dos Santos, *On the structure of affine flat group schemes over discrete valuation rings, II*. Int. Math. Res. Not. IMRN 2021, no. 12, 9375–9424.
- [Har77] R. Hartshorne, *Algebraic geometry*. Graduate texts in mathematics 52, Springer, 1977.
- [Har66] R. Hartshorne, *Residues and duality*. Lecture notes in mathematics 20, Springer, 1966.
- [Ill06] L. Illusie, *Grothendieck’s existence theorem in formal geometry*. With a letter (in French) of Jean-Pierre Serre. Math. Surveys Monogr., 123, Fundamental algebraic geometry, 179–233, Amer. Math. Soc., Providence, RI, 2005
- [J87] J. C. Jantzen, *Representations of algebraic groups*. Pure and Applied Mathematics, 131. Academic Press, Inc., Boston, MA, 1987.
- [Ka52] I. Kaplansky, *Modules over Dedekind rings and valuation rings*. Trans. Amer. Math. Soc. 72, (1952). 327–340.
- [Liu02] Q. Liu, *Algebraic geometry and arithmetic curves*. Translated from the French by Reinie Ern e. Oxford Graduate Texts in Mathematics, 6. Oxford Science Publications. Oxford University Press, Oxford, 2002.
- [Mac98] S. Mac Lane, *Categories for the working mathematician*, GTM 5, Second edition, Springer-Verlag, New York, 1998.
- [Mat89] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1989.
- [Mat80] H. Matsumura. *Commutative Algebra*. Second edition. Mathematics Lecture Note Series, 56. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980
- [MS13] V. Mehta and S. Subramanian, *The fundamental group scheme of a smooth projective variety over a ring of Witt vectors*. J. Ramanujan Math. Soc. 28A (Special Issue-2013) 341–351.
- [MS02] V. Mehta and S. Subramanian, *On the fundamental group scheme*. Invent. math. 148, 143–150 (2002).
- [Mu70] D. Mumford, *Abelian varieties*. Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Bombay, Oxford University Press, London 1970.
- [Ni05] N. Nitsure, *Construction of Hilbert and Quot schemes, in* Fundamental algebraic geometry. Mathematical Surveys and Monographs, 123. American Mathematical Society, Providence, RI, 2005.
- [Nor76] M. Nori, *On the representations of the fundamental group*. Compositio Math. 33 (1976), no. 1, 29–41.
- [Nor82] M. V. Nori, *The fundamental group scheme*, Proc. Indian Acad. Sci. Math. Sci. 91 (1982), no. 2, 73–122.
- [Ray70] M. Raynaud, *Anneaux Locaux Henséliens*. Lecture Notes in Mathematics 169, Springer, 1970.
- [Saa72] N. Saavedra Rivano, *Catégories Tannakiennes*. Lecture Notes in Mathematics 265, Springer-Verlag, Berlin (1972).
- [dS09] J. P. dos Santos, *The behaviour of the differential Galois group on the generic and special fibres: A Tannakian approach*. J. Reine Angew. Math. 637 (2009), 63–98.
- [Se68] J. P. Serre, *Groupe de Grothendieck des schémas en groupes réductifs déployés*, Publ. Math. IHES 34 (1968), pp. 37–52.
- [Sil86] J. H. Silverman, *The arithmetic of elliptic curves*. Second edition. Graduate Texts in Mathematics, 106. Springer, Dordrecht, 2009.
- [SP] The Stacks Project Authors, *Stacks Project*. <http://stacks.math.columbia.edu>.
- [Tem10] M. Temkin, *Stable modifications of relative curves*. J. Algebraic Geometry 19 (2010) 603–677.
- [TZ20] F. Tonini and L. Zhang, *Essentially Finite Vector Bundles on Normal Pseudo-proper Algebraic Stacks*. Documenta Mathematica, 25, 159–169, 2020.

- [Wa79] W. C. Waterhouse, *Introduction to affine group schemes*, Graduate Texts in Mathematics, 66. Springer-Verlag, New York-Berlin, 1979.
- [WW80] W. C. Waterhouse and B. Weisfeiler, *One dimensional affine group schemes*. J. Algebra 66 (1980), no. 2, 550–568.

INSTITUTE OF MATHEMATICS, VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY, HANOI, VIETNAM  
*Email address:* `phung@math.ac.vn`

INSTITUT DE MATHÉMATIQUES DE JUSSIEU – PARIS RIVE GAUCHE, 4 PLACE JUSSIEU, CASE 247,  
75252 PARIS CEDEX 5, FRANCE  
*Email address:* `joao_pedro.dos_santos@yahoo.com`