Fundamental group schemes for stratified sheaves

João Pedro Pinto dos Santos[1]

Abstract

We study stratified sheaves in positive characteristic algebraic geometry using the technique of Tannakian categories. We show that the associated Tannakian group scheme Π is a perfect scheme. We also prove that in the proper case the largest unipotent quotient of Π is pro-etale. Special attention is paid to abelian varieties, where a description of Π is obtained. We finish with a discussion of the case where the ground field is valued. *Keywords*: Differential equations in positive characteristic, fundamental groups, abelian varieties. [2]

1 Introduction

This work studies the philosophy of the Riemann-Hilbert (RH) correspondence in positive characteristic algebraic geometry. The RH correspondence relates differential equations and topology with the topological side being controlled by representation theory of some group. In the complex analytic case the group is the topological fundamental group and the differential equations are the integrable connections on vector-bundles. In finite characteristic, integrable connections on a smooth X/k are not very well behaved (local freeness fails) and we have to consider coherent sheaves with an action of all k-linear differential operators (called stratified sheaves or D-modules.) By Tannakian reconstruction, the category of these sheaves is equivalent to the category of representations of an affine group scheme Π, which is *defined over k*. We show here that Π is a perfect scheme and that provided X/k is proper, the largest unipotent part of Π is in fact pro-etale (Theorems 11 and 15.) We also give a definite description of Π in the case X is an abelian variety (Theorem 21.) This description is later

[1] João Pedro Pinto dos Santos
Current address: IRMAR, Université de Rennes 1, Campus de Beaulieu 35042 Rennes (France).
e-mail: joao-pedro.dos-santos@univ-rennes1.fr

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used to shed some light in the comparison between Π and the rigid fundamental group (case of a complete non-Archimedean ground field, section 4.)

The approach through Tannakian categories is important. By thinking of representations one is led to ask the right questions (Frobenius twists, nilpotent objects, characters and etc) and consequently get more satisfactory answers.

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Some notations and standard definitions (**Base field**) k will always be an algebraically closed field of characteristic $p > 0$. The category of finite dimensional vector spaces over k will be denoted by $(k - \text{vect})$.

(**Algebraic Geometry**) (a) The absolute Frobenius morphism of a k-scheme is denoted by F. For a k-scheme Z, the Frobenius pull-back is denoted $Z^{(1)}$ (see also **group theory** or [15, 9.2, p. 146].) The geometric Frobenius is a k-morphism $F_{\text{geom}} : Z \longrightarrow Z^{(1)}$. We will overlook these technicalities and work only with the absolute Frobenius, leaving the reader with the task of checking k-linearity if needed.

(b) For a k-scheme Y, the ring of k-linear differential operators [2, ch. 1] of order $< m$ is denoted by $\mathscr{D}_Y^{<m}$. The differential operators which kill $1 \in \mathscr{O}(Y)$ are denoted $\mathscr{D}_Y^{+}$ and $\mathscr{D}_Y^{<m,+} := \mathscr{D}_Y^{+} \cap \mathscr{D}_Y^{<m}$.

(c) Torsors will always be affine over the base scheme. Our notations will follow [15, Ch. 5] closely. In particular, the "associated sheaf" functor corresponding to the G-torsor $P \longrightarrow Y$ will be denoted by $\mathscr{L}$, $\mathscr{L}_P$ or $\mathscr{L}_{P/G}$.

(**Group theory**) (a) By a group scheme we will mean an affine group scheme (Hopf algebra) over k.

(b) The category of *finite dimensional* k-representations of a group scheme G will be denoted by $\mathrm{Rep}_k(G)$. The category of representations of any dimension is denoted by $\mathrm{Rep}'_k(G)$ – according to [26, Thm. 3.3] $\mathrm{Rep}'_k(G)$ is the category of direct limits in $\mathrm{Rep}_k(G)$.

(c) The co-module map for the left (resp. right) regular representation of a group scheme G is denoted by ρ_l (resp. ρ_r).

(d) The diagonal group associated to the abstract abelian group $\mathbb{X}$ will be denoted by $\mathrm{Diag}(\mathbb{X})$, [26, 2.2].

(e) Given a k-space V, we will denote its Frobenius pull-back by $V^{(1)}$ (analogous for $V^{(i)}$). The underlying additive group $(V^{(1)}, +)$ is $(V, +)$ but scalar multiplication is twisted: $\lambda \bullet v = \lambda^{p^{-1}} v$. This construction is extended to representations [15, 2.16]: Given $V \in \mathrm{Rep}_k(G)$ let $\rho : V \longrightarrow V \otimes_k \mathscr{O}(G)$ define the co-module structure. The *Frobenius twist* $V^{(1)}$ is the representation of G on $V^{(1)}$ with co-module map

$$V^{(1)} \xrightarrow{\rho^{(1)}} (V \otimes_k \mathscr{O}(G))^{(1)} \xrightarrow{\text{canonical} \cong} V^{(1)} \otimes_k \mathscr{O}(G)^{(1)} \xrightarrow{\mathrm{id}\otimes F} V^{(1)} \otimes \mathscr{O}(G).$$

In down-to-earth terms, $V^{(1)}$ is the representation whose matricial coefficients are p-powers of the matricial coefficients of V.

(f) If Γ is an abstract group, we will also denote by Γ the group scheme (if it exists) representing the functor $(\mathrm{Sch}/k)^{\mathrm{opp}} \longrightarrow \mathrm{Grps}$:

$$\widetilde{\Gamma}(S) = \mathrm{Maps}(\pi_0(S), \Gamma).$$

In this context, a representation will always mean a representation of the group scheme Γ and not of the abstract group (unless otherwise mentioned – see section 4.) In particular, if $\Gamma = \varprojlim \Gamma_i$ is pro-finite $\widetilde{\Gamma}$ is representable and any representation of Γ factors through some Γ_i.

2 Definitions and foundational results

Throughout section 2, we will fix a connected locally noetherian regular k-scheme X with a k-rational point x_0.

2.1 Generalities on Tannakian categories

We review some known facts on Tannakian categories which we will use in this work. For references the reader should consult [7], [9], [8] or [25].

2.1.1 Introducing Tannakian categories

Tannakian categories were invented by Saavedra and Grothendieck to generalize a phenomenon discovered by Tannaka: Given a compact Lie group G, the subring of continuous functions $G \longrightarrow \mathbb{C}$ generated by the matrix coefficients of representations on finite dimensional vector-spaces is the ring of regular functions of an algebraic group $\Gamma/\mathbb{C}$ and the Lie group G can be reconstructed from Γ. This gives rise to a genuine mathematical problem: To what extent is a group determined by its representations? As a neighboring problem (inversion), one asks the question of whether a given abelian category is the category of representations of some group. These last two questions summarize the purpose of Tannakian categories.

To understand the question of inversion, we have to make evident particular properties of $\mathrm{Rep}_k(G)$. Two structures immediately spring to mind: the tensor product and the existence of kernels, cokernels etc. So, given a k-linear category $\mathfrak{A}$ we endow it with a tensor product functor $\otimes : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{A}$ (we let the reader imagine the associativity and commutativity constraints) and an identity object $\mathbb{1}$ ($U \longrightarrow U \otimes \mathbb{1}$ is an equivalence), see §1 of [7]. Furthermore, assume that $\mathfrak{A}$ is abelian and $\otimes$ is k-bilinear:

$$(A_1 \oplus A_2) \otimes (B_1 \oplus B_2) = (A_1 \otimes B_1) \oplus (A_1 \otimes B_2) \oplus (A_2 \otimes B_1) \oplus (A_2 \otimes B_2).$$

These are not yet all the properties sufficient to characterize $\mathrm{Rep}_k(G)$. For a commutative ring R, the category $R-\mathrm{Mod}$ of R-modules is not the of the form $\mathrm{Rep}_k(G)$ because the functor $? \otimes M$ is not exact. This is fixed by introducing the notion of rigidity ([7], Def. 1.7, p. 112 and Prp. 1.16, p. 119) which roughly says that duals (denoted usually by $A^\vee$) exist and satisfy the convenient working properties we are used to:

$$\mathrm{Hom}(A \otimes X, Y) \cong \mathrm{Hom}(X, A^\vee \otimes Y), \quad A^{\vee\vee} \cong A \quad \text{[7], pp. 109–112.}$$

Finally, in order to be equivalent to $\mathrm{Rep}_k(G)$, there must exist an exact faithful functor $\omega : \mathfrak{A} \longrightarrow (k - \mathrm{vect})$ which preserves the tensor product (a tensor functor, [7] Def. 1.8, p. 113.) Such a functor is called *fibre functor*.

The confluence of all these properties deserves a special name:

Definition 1. *Let $(\mathfrak{A}, \otimes)$ be an abelian k-linear rigid tensor category with $\otimes$ k-bilinear. If there exists a fibre functor $\omega : \mathfrak{A} \longrightarrow (k - \mathrm{vect})$ then $\mathfrak{A}$ is called a* neutral Tannakian category*.*

The main result of the theory is:

Theorem 2 (Saavedra,[7], 2.11)**.** *Let $\mathfrak{A}$ be neutral Tannakian with fibre functor $\omega : \mathfrak{A} \longrightarrow (k - \mathrm{vect})$. Then there exists a group scheme G such that ω induces an equivalence between $\mathfrak{A}$ and the subcategory $\mathrm{Rep}_k(G)$ of $(k - \mathrm{vect})$. The group scheme G is called the Tannakian fundamental group scheme associated to $\mathfrak{A}$ via ω.* □

In many applications the existence of ω is the hardest property to verify ([9] is dedicated to generalizing this) and often one can only get away with $\omega : \mathfrak{A} \longrightarrow (K - \mathrm{vect})$ where K/k is an extension.

Example: ([8], 10.24) Let Γ be an abstract group. The category of abstract representations $\mathrm{Rep}_k(\Gamma)$ is certainly neutral Tannakian and thus is equivalent to the category of representations of an affine group scheme Γ^{alg} called the algebraic hull of Γ (even though Γ^{alg} itself is usually not algebraic!) A more constructive description of Γ^{alg} is based on the following. Given a homomorphism $\rho : \Gamma \longrightarrow \mathrm{GL}(V) = \mathbb{GL}(V)(k)$, we obtain a closed reduced subgroup scheme $G \subseteq \mathbb{GL}(V)$ by taking the Zariski closure of $\mathrm{im}(\rho)$ in $\mathbb{GL}(V)$. These groups form a projective system of group schemes and Γ^{alg} is the limit.

Our preferred example of this situation is the category of complex local systems $\mathbf{LS}(M)$ on a complex manifold M. It is isomorphic to the category of representations of the usual fundamental group $\pi_1(M, x_0)$ and hence the Tannakian fundamental group of $\mathbf{LS}(M)$ associated to the fibre functor x_0^* is $\pi_1(M)^{\mathrm{alg}}$. We note that even if $\pi_1(M)$ is very simple, e.g. $\mathbb{Z}$, $\pi_1(M)^{\mathrm{alg}}$ can be bigger than a first impression might suggest (look at the inclusion $\mathbb{Z} \longrightarrow \mathbb{G}_a(\mathbb{C})$, for example.)

Example: Let X/k be a noetherian connected scheme. The category of coherent sheaves $\mathrm{coh}(X)$ is abelian, k-linear and has a tensor product but fails to have good duals. Nevertheless, $\mathrm{coh}(X)$ might contain certain abelian categories $\mathfrak{A}$ of *locally free* sheaves which are stable under the tensor product and duals. Choosing an $x_0 \in X(k)$ gives a fibre functor $x_0^* : \mathfrak{A} \longrightarrow (k-\mathrm{vect})$. This is the point of view in [22].

2.1.2 Nilpotent objects [8], [25]

Definition 3. *Given an abelian category $\mathfrak{A}$ and a preferred object A, we define the category $\mathfrak{N}_A\mathfrak{A}$ as the full subcategory of $\mathfrak{A}$ whose objects N have a filtration*

$$0 = F^r N \subseteq F^{r-1}N \subseteq \ldots \subseteq F^0 N = N,$$

with $\mathrm{gr}_F^i \cong A$.

If $\mathfrak{A}$ is also a tensor category, by convention, the preferred object is $\mathbb{1}$ and $\mathfrak{N}_{\mathbb{1}}\mathfrak{A} = \mathfrak{N}\mathfrak{A}$ is called the nilpotent category associated to $\mathfrak{A}$. *A tensor category $\mathfrak{A}$ is called* nilpotent *if $\mathfrak{N}\mathfrak{A} = \mathfrak{A}$.*

Example: Let G be a group scheme. The category $\mathfrak{N}\mathrm{Rep}_k(G)$ is certainly neutral Tannakian (forgetful functor as fibre) and hence is equivalent to $\mathrm{Rep}_k(G^{\mathrm{uni}})$ for some group scheme G^{uni}. This group is the largest unipotent [26, Ch. 8] quotient of G and is referred to as *the unipotent part* of G. In particular, G is unipotent if and only if $\mathrm{Rep}_k(G)$ is nilpotent. Shiho [25, 1.2.1, p. 521] gives a general criterion for $\mathfrak{N}\mathfrak{A}$ to be abelian.

2.2 Stratified and F-divided sheaves

We will define the Tannakian categories in which we are interested (Dfn. 4) and which serve as analogues of differential equations. They are related by Katz's Theorem (Thm. 8.)

2.2.1 Definitions and fundamental groups

Definition 4. *The category of stratified sheaves* $\mathbf{str}(X)$ *is the category whose:*

Objects are $(\mathscr{E}, \nabla)$ *with* $\mathscr{E}$ *a coherent* $\mathscr{O}_X$*-module and*

$$\nabla : \mathscr{D}_X \longrightarrow \mathscr{E}nd_k(\mathscr{E})$$

a homomorphism of $\mathscr{O}_X$*-algebras.*

Arrows are *homomorphism of* $\mathscr{D}_X$*-modules.*

The category of F*-divided sheaves* $\mathbf{Fdiv}(X)$ *is the category whose:*

Objects are *sequences of coherent* $\mathscr{O}_X$*-modules* $\{\mathscr{E}_i\}_{i\in\mathbb{N}}$ *and isomorphisms of* $\mathscr{O}_X$*-modules*

$$\sigma_i : F^*\mathscr{E}_{i+1} \longrightarrow \mathscr{E}_i.$$

Arrows are *projective systems;* $\{\alpha_i\} : \{\mathscr{E}_i, \sigma_i\} \longrightarrow \{\mathscr{F}_i, \tau_i\}$ *with* α_i $\mathscr{O}_X$*-linear and* $\tau_i \circ F^*(\alpha_{i+1}) = \alpha_i \circ \sigma_i$.

Assume for the moment that X is smooth over k. Then the sheaf underlying an object in $\mathbf{str}(X)$ is locally free [2, Prp. 2.16]. Also, one can reinterpret the definition in such a way that the tensor structure becomes clear [2, Ch. 2] as well as the existence of duals (rigidity constraint.) As $\mathbf{str}(X)$ is certainly k-linear abelian, the following definition makes sense:

Definition 5. *The group scheme associated to the Tannakian category* $\mathbf{str}(X)$ *via the fibre functor* x_0^* *is denoted* $\Pi^{\mathbf{str}}(X, x_0)$ *or* $\Pi_X^{\mathbf{str}}$ *if no confusion is possible.*

As X is regular, $\mathbf{Fdiv}(X)$ is abelian (F is faithfully flat) and we give it the termwise tensor product. $\mathbf{Fdiv}(X)$ is endowed with a k-linear structure by correcting the p-linearity of F in the following way:

$$\mathrm{Hom}(\{\mathscr{E}, \sigma_i\}, \{\mathscr{F}_i, \tau_i\}) = \varprojlim_i \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}_i, \mathscr{F}_i)^{(i)}.$$

That the category $\mathbf{Fdiv}(X)$ is neutral Tannakian (given $x_0 \in X(k)$) is a consequence of the following Lemma whose proof was communicated to us by Nick Shepherd-Barron.

Lemma 6. *If* $\{\mathscr{E}_i\}$ *is an object of* $\mathbf{Fdiv}(X)$*, then* $\mathscr{E}_0$ *is locally free.*

Proof: This is a local problem so we take A a regular local ring and M_i finite A-modules such that $M_{i+1} \otimes_{A,F} A \cong M_i$, where $F : A \longrightarrow A$ is the Frobenius. We will use Fitting ideals. Let

$$A^n \xrightarrow{\alpha} A^m \longrightarrow M_0 \longrightarrow 0$$

be a finite free presentation of M_0 with m minimal. The r'th Fitting ideals $\Phi_r(\alpha)$ are the ideals generated by the $r \times r$ minors of α ($1 \le r \le \min(m, n)$.) Also, by convention $\Phi_r = (1)$ for $r \le 0$ and $\Phi_r = 0$ for $r > \min(m, n)$. The module M_0 is free of rank $m-r$ if and only if (see [5], Prop. 1.4.10, p. 22) $\Phi_{r+1}(\alpha)=0$ and $\Phi_r(\alpha)=(1)$ holds. The Fitting ideals $\Phi_r(\alpha)$ depend only on the isomorphism class of M_0, that is, for another finite free presentation of M_0

$$A^\nu \xrightarrow{\beta} A^\mu \longrightarrow M_0 \longrightarrow 0$$

we have $\Phi_{\mu-r}(\beta)=\Phi_{m-r}(\alpha)$ for all $r\geq 0$.

Also, they base change nicely (loc.cit., p. 21); thus, given a finite free presentation

$$A^\nu \xrightarrow{\beta} A^\mu \longrightarrow M_i \longrightarrow 0$$

$\Phi_{m-r}(\alpha)=\Phi_{\mu-r}((F^i)^*\beta)=F^i(\Phi_{\mu-r}(\beta))A$. If $\Phi_{m-r}(\alpha)\neq(1)$, then it is contained in $\cap_i \mathrm{rad}(A)^{p^i}=0$ (by Krull's Intersection Theorem.) This shows that M_0 is free. □

Definition 7. *The group scheme associated via Tannakian duality to the category* $\mathbf{Fdiv}(X)$ *with the fibre functor* x_0^* *is denoted* $\Pi^{\mathbf{Fdiv}}(X,x_0)$ *or* $\Pi_X^{\mathbf{Fdiv}}$ *is no confusion is possible.*

Remark: The category of stratified sheaves was defined by Grothendieck in [14]. Now-a-days it is commonly known as the category of D-modules. The category of F-divided sheaves was defined in [11]. They where then called *flat* sheaves.

2.2.2 A theorem of Katz

Theorem 8. *Let X/k be smooth. The categories* $\mathbf{Fdiv}(X)$ *and* $\mathbf{str}(X)$ *are equivalent tensor categories.*

This theorem was proved by Katz and appeared in [11], Thm. 1.3. It is an explicit equivalence which iterates Cartier's Theorem on the p-curvature. We recall the construction of the equivalences.

Let $(\mathscr{E},\nabla)$ be a stratified sheaf and let $\mathscr{E}_i$ be the sheaf

$$U\mapsto\{e\in\mathscr{E}(U);\ \nabla(D)(e_x)=0,\forall D\in\mathscr{D}_{X,x}^{+,<p^i},\forall x\in U\}.$$

Each $\mathscr{E}_i$ can be seen as a coherent sheaf of $\mathscr{O}_X$-modules via multiplication

$$f\bullet e=f^{p^i}e.$$

Natural inclusion $\mathscr{E}_{i+1}\subset\mathscr{E}_i$ gives rise to an $\mathscr{O}_X$-linear homomorphism

$$\mathscr{E}_{i+1}\otimes_{\mathscr{O}_X,F}\mathscr{O}_X \longrightarrow \mathscr{E}_i. \tag{1}$$

Katz's idea is to give an integrable connection on $\mathscr{E}_i$ in such a way that $\mathscr{E}_{i+1}$ is the sheaf of horizontal sections and then conclude, by Cartier's Theorem

[16] on the p-curvature, that arrow (1) is an isomorphism. Using the stratification it is not hard to find the right connection. This gives a functor $\mathbf{str}(X) \longrightarrow \mathbf{Fdiv}(X)$.

We construct $\mathbf{Fdiv}(X) \longrightarrow \mathbf{str}(X)$. Let $\{\mathscr{E}_i\}$ be F-divided. For each $D \in \mathscr{D}_X^{<p^i}(U)$, $e_1, \ldots, e_r \in \mathscr{E}_i(U)$ and $f_1, \ldots, f_r \in \mathscr{O}_X(U)$, define

$$\nabla(D)(f_1e_1 + \ldots + f_re_r) = D(f_1)e_1 + \ldots + D(f_r)e_r;$$

in other words, $\mathscr{E}_i \subseteq \mathscr{E}_0$ is the sheaf of sections killed by differential operators of order $< p^i$. This is well defined since by Lemma 6 $\mathscr{E}_i$ is locally free.

Remarks: (a) In the light of this equivalence, we may confuse the notions of F-divided and stratified sheaves. The concept of an F-divided sheaf is much more useful to us and we will only work with F-divided sheaves from now on. Nevertheless, the notations $\Pi_X^{\mathbf{Fdiv}}$ and $\Pi_X^{\mathbf{str}}$ are cumbersome and hence we have:

> **Notation:** From now on $\Pi_X := \Pi^{\mathbf{Fdiv}}(X, x_0)$ unless we are also dealing with other Tannakian categories; in this case we write $\Pi_X^{\mathbf{str}}$ or $\Pi^{\mathbf{str}}$.

(b) Another name for Theorem 8 is "Cartier-Katz Theorem". The name of Cartier is present to remind us of a proof.

2.3 Torsors and the Frobenius on Π_X

In [22, §2] tensor functors $\mathscr{L} : \mathrm{Rep}_k(G) \longrightarrow \mathrm{coh}(X)$ are studied. Nori shows that under certain conditions $\mathscr{L}$ is isomorphic to $\mathscr{L}_{P/G}$, where $P \longrightarrow X$ is a G-torsor uniquely determined by $\mathscr{L}$ (see also [7, 3.2, 3.8].) The construction is quite explicit and in fact works in more generality (details are in 2.3.2 below.) Its importance relies on showing how to invert the main construction of Tannakian duality (Thm. 2.)

In this section we apply Nori's method to understand the torsors controlling F-divided sheaves (Prp. 10) and obtain that the Frobenius twist is an equivalence (Thm. 11.) Proposition 10 will also be used to study the relation between Π_X and the etale fundamental group (2.4.)

2.3.1 Main result: Π_X is a perfect scheme

Let G be any group scheme.

Definition 9. *An F-division of a G-torsor P is a family of G-torsors $\psi_i : P_i \longrightarrow X$, with $P_0 = P$, and isomorphisms of G-torsors between P_i and the Frobenius pull-back of P_{i+1}. If an F-division $\{P_i\}$ of P is fixed, we call it F-divided. Morphisms of F-divided torsors are given in the natural fashion.*

In (2.3.2) we discuss a more categorical version of this definition. Given an F-divided torsor $P = \{P_i\}$ we obtain a tensor functor

$$\mathscr{L}_P : \mathrm{Rep}_k(G) \longrightarrow \mathbf{Fdiv}(X); \qquad \mathscr{L}_P(V) = \{\mathscr{L}_{P_i}(V)\}. \tag{2}$$

as the transition isomorphisms $F^*P_i \longrightarrow P_{i-1}$ will induce transition isomorphisms

$$\sigma_i : F^*\mathscr{L}_{P_i}(V) \longrightarrow \mathscr{L}_{P_{i-1}}(V).$$

Proposition 10. *Let $\mathscr{L} : \mathrm{Rep}_k(G) \longrightarrow \mathbf{Fdiv}(X)$ be a faithful and exact tensor functor. Then there exists an F-divided G-torsor $P = \{P_i\}$ such that the functor $\mathscr{L}_P$ of eq. (2) is naturally isomorphic to $\mathscr{L}$.*

Proof: The proof is an application of the material in (2.3.2) below and is given there.

Theorem 11. *The Frobenius homomorphism $F : \Pi_X \longrightarrow \Pi_X^{(1)}$ is an isomorphism.*

Proof: We will show that taking the Frobenius twist Ft of a representation induces an equivalence on $\mathrm{Rep}_k(\Pi_X)$. Since Ft sits in a diagram

$$\begin{tikzcd} \mathrm{Rep}_k(\Pi_X) \arrow[rr, "\mathrm{Ft}"] \arrow[dr, "\text{equivalence}"'] & & \mathrm{Rep}_k(\Pi_X) \\ & \mathrm{Rep}_k(\Pi_X^{(1)}) \arrow[ur, "\mathrm{Res}(F)"'] & \end{tikzcd} \tag{3}$$

by general theory of Tannakian categories we will be done ($\mathrm{Res}(F)$ is the tensor functor induced by the k-linear Frobenius $F : \Pi_X \longrightarrow \Pi_X^{(1)}$).

Let $\mathscr{L} : \mathrm{Rep}_k(\Pi_X) \longrightarrow \mathbf{Fdiv}(X)$ be a tensor equivalence. It exists by a general result of Saavedra. According to Prop. 10, there exists an F-divided torsor $P = \{P_i\}$ such that $\mathscr{L} = \mathscr{L}_P$.

Consider the shifting functor $\Phi : \{\mathscr{E}_i; \sigma_i\} \longrightarrow \{F^*(\mathscr{E}_i); F^*(\sigma_i)\}$. It is obviously an equivalence and because of our explicit description of $\mathscr{L}$, we can easily show that the diagram

$$\begin{tikzcd} \mathbf{Fdiv}(X) \arrow[r, "\Phi"] & \mathbf{Fdiv}(X) \\ \mathrm{Rep}_k(\Pi_X) \arrow[u, "\mathscr{L}"] \arrow[r, "\mathrm{Ft}"'] & \mathrm{Rep}_k(\Pi_X) \arrow[u, "\mathscr{L}"'] \end{tikzcd}$$

is commutative up to natural isomorphism, thus finishing the proof. This is done as follows.

For a G-torsor $\varphi : Q \longrightarrow X$, there is a natural isomorphism of $\mathscr{O}_X$-modules

$$\lambda_V : F^*\mathscr{L}_Q(V) \longrightarrow \mathscr{L}_Q(V^{(1)}), \quad V \in \mathrm{Rep}_k(G),$$

functorial in V. Explicitly, if $U = \operatorname{Spec} A$ is an affine open of X and $\varphi^{-1}(U) = \operatorname{Spec} B$, then $\lambda_V(U)$ is

$$\begin{array}{ccc} A \otimes_{F,A} (B \otimes V)^G & \longrightarrow & (B \otimes V^{(1)})^G \\ & & \\ a \otimes \sum_j b_j \otimes v_j & \longmapsto & \sum_j ab_j^p \otimes v_j. \end{array}$$

Hence, for any representation V of Π_X, there are natural isomorphisms

$$\lambda_i : F^*\mathscr{L}_{P_i}(V) \longrightarrow \mathscr{L}_{P_i}(V^{(1)}).$$

All we have to do is check that $\{\lambda_i\} : \{F^*\mathscr{L}_{P_i}(V)\} \longrightarrow \{\mathscr{L}_{P_i}(V^{(1)})\}$ is an arrow in $\mathbf{Fdiv}(X)$ i.e., that the diagrams

$$\begin{tikzcd} F^*(F^*\mathscr{L}_{P_{i+1}}(V)) \arrow[r, "F^*(\lambda_{i+1})"] \arrow[d, "F^*(\sigma_{i+1})"'] & F^*(\mathscr{L}_{P_{i+1}}(V^{(1)})) \arrow[d, "\sigma_{i+1}"] \\ F^*\mathscr{L}_{P_i}(V) \arrow[r, "\lambda_i"'] & \mathscr{L}_{P_i}(V^{(1)}). \end{tikzcd} \tag{4}$$

are commutative. The question is then local on X and we can use the local expression of λ_i. It is now a tedious but straightforward algebraic manipulation to check commutativity of diagram (4). □

Corollary 12. *i)* $\mathrm{Lie}(\Pi_X) = 0$.

ii) If Π_X is finite or smooth it is etale.

iii) Any pro-finite quotient of Π_X is pro-etale.

Proof: *i*). Because the relative Frobenius is a closed embedding, the differential at the identity $dF_e = 0$ is injective [26, Cor. p. 94].

ii). An affine algebraic group scheme is smooth if and only if its dimension coincides with the dimension of its Lie algebra [26, Cor. p. 94]. If Π_X is finite, part *i*) shows that it is smooth, hence etale. Again, part *i*) shows that if it is smooth then it has dimension zero.

iii). By definition, if G is a quotient of Π_X then $\mathscr{O}(G)$ is a sub-Hopf-algebra of the Hopf-algebra of Π_X. It follows that $\mathscr{O}(G)$ is reduced and if it is finite dimensional it is etale over k [26, Thm. 6.2, p. 46].

□

Remark: Take X/k smooth. If $\mathrm{DE}(X/k)$ denotes the category of integrable connections, then $\mathbf{dR}(X) = \mathfrak{N}\mathrm{DE}(X/k)$ is neutral Tannakian with fibre functor x_0^* [25]. It can be shown that the fundamental group scheme associated to $\mathbf{dR}(X)$ is not reduced [24, Ch. 2]. This shows that the category $\mathbf{Fdiv}(X)$ is a reasonable category of differential equations.

2.3.2 Reviewing the method of Nori

Let $(\mathfrak{A}, \otimes)$ be an abelian tensor category and $G = \operatorname{Spec} R$ an affine group scheme (all over k.) We will show how to prove Prp. 10 in a natural way.

(a) We remind the reader that the notions of algebras, co-algebras, Hopf algebras, co-modules and etc. can be easily translated into the more abstract setting of tensor categories. For example, an algebra $\mathscr{A}$ of $\mathfrak{A}$ is an object of $\operatorname{Ind}(\mathfrak{A})$ endowed with a unity $\mathbb{1} \longrightarrow \mathscr{A}$, a multiplication $\cdot : \mathscr{A} \otimes \mathscr{A} \longrightarrow \mathscr{A}$ which should be associative and commutative (note that there is no addition.) In [8, §5] this point of view is observed, but we will avoid, for the sake of concreteness, the use of spectra (the dual category of algebras) in $\mathfrak{A}$. This has one caveat: there is no algebraic name for the ring of functions of a torsor; we abuse terminology and call such an object an *$\mathscr{R}$-torsor-algebra* or *torsor-algebra under $\mathscr{R}$* ($\mathscr{R}$ is a Hopf-algebra.) If $\mathscr{B} \in \operatorname{Ind}(\mathfrak{A})$ is an algebra with co-action $\mu : \mathscr{B} \longrightarrow \mathscr{B} \otimes \mathscr{R}$, then $\mathscr{B}$ is an $\mathscr{R}$-torsor-algebra provided the natural map

$$\mathscr{B} \otimes \mathscr{B} \longrightarrow \mathscr{B} \otimes \mathscr{R}$$

is an isomorphism.

(b) Assume that we have an exact and k-linear tensor functor

$$\mathscr{L} : \operatorname{Rep}_k(G) \longrightarrow \mathfrak{A}$$

and let $\mathscr{L}'$ denote the natural extension

$$\mathscr{L}' : \operatorname{Rep}'_k(G) \longrightarrow \operatorname{Ind}(\mathfrak{A});$$

it is a k-linear exact tensor functor between abelian tensor categories.

Let $\mathfrak{G}$ be the category of affine G-schemes with a right G-action. Taking global sections induces a functor $\Gamma : \mathfrak{G} \longrightarrow \operatorname{Rep}'_k(G)^{\mathrm{op}}$ which maps the direct product $S \times T$ to the tensor product $\Gamma(S) \otimes \Gamma(T)$. Let $(R, \rho_l) = \Gamma(G_{\mathrm{left}})$ be the left regular representation — here G_{left} is the affine scheme G with the right G-action $x \cdot g = g^{-1}x$. Consider the two morphisms in $\operatorname{Rep}'_k(G)$:

$$(R, \rho_l) \otimes (R, \rho_l) \longrightarrow (R, \rho_l), \qquad (R, \rho_l) \longrightarrow (R, \rho_l) \otimes (R, \mathrm{id}_R \otimes 1), \tag{5}$$

where the first is just multiplication on the ring R and the second is the image of the group multiplication $G_{\mathrm{left}} \times G_{\mathrm{triv}} \longrightarrow G_{\mathrm{left}}$ under Γ (G_{triv} is G with the trivial action.) The first map in (5) makes (R, ρ_l) into an algebra in $\operatorname{Rep}'_k(G)$ and hence

$$\mathscr{B} := \mathscr{L}'((R, \rho_l))$$

is an algebra of $\mathfrak{A}$. The second map in (5) will give $\mathscr{B}$ the co-action of the Hopf algebra $\mathscr{R}_{\mathfrak{A}} := \mathscr{L}'((R, \mathrm{id}_R \otimes 1))$. Note that there is an isomorphism

$$\mathscr{B} \otimes \mathscr{B} \longrightarrow \mathscr{B} \otimes \mathscr{R}_{\mathfrak{A}} \tag{6}$$

coming from the isomorphism in $\mathfrak{G}$, $G_{\text{left}} \times G_{\text{triv}} \longrightarrow G_{\text{left}} \times G_{\text{left}}$, $(g,h) \mapsto (g, gh)$: $\mathscr{B}$ is a $\mathscr{R}_{\mathfrak{A}}$-torsor-algebra. If on $\mathscr{B} \otimes \mathscr{R}_{\mathfrak{A}}$ the $\mathscr{R}_{\mathfrak{A}}$-co-action is the one provided by the co-action on $\mathscr{R}_{\mathfrak{A}}$ and on $\mathscr{B} \otimes \mathscr{B}$ it is the one provided by the co-action on the second term of the tensor product, then the isomorphism in (6) is also equivariant.

(c) If $\iota : \mathfrak{A} \longrightarrow \mathrm{coh}(X)$ is an exact tensor functor with values in the locally free sheaves of X, by applying the functor ι to the above constructions, we obtain an action (in the category of X-schemes) of the X-group scheme $\mathrm{Spec}\,\iota(\mathscr{R}_{\mathfrak{A}}) = G \times X$ on the *flat* X-scheme $P := \mathrm{Spec}\,\iota(\mathscr{B})$. It is not hard to see (from the exactness of ι) that P is actually faithfully flat. Isomorphism (6) shows that P is a G-torsor over X.

(d) We show how the torsor-algebra $\mathscr{B}$ helps us to rebuild $\mathscr{L}$. Consider two functors

$$\nu_1, \nu_2 : \mathrm{Rep}_k(G) \longrightarrow \mathrm{Rep}'_k(G)$$

given by

$$\nu_1 : (V, \rho) \mapsto (V, \rho) \otimes (R, \rho_l), \qquad \nu_2 : (V, \rho) \mapsto (V, \mathrm{id}_V \otimes 1) \otimes (R, \rho_l).$$

These functors are naturally isomorphic via

$$(\mathrm{id}_V \otimes \text{mult.}) \circ (\mathrm{id}_V \otimes \sigma \otimes \mathrm{id}_R) \circ (\rho \otimes \mathrm{id}_R) : \nu_1(V) \longrightarrow \nu_2(V),$$

where $\sigma : R \longrightarrow R$ represents $g \mapsto g^{-1}$. This natural isomorphism is the algebraic analogue of the much more intuitive map: Identify $\nu_1(V)$ and $\nu_2(V)$ with the vector space of morphisms $G \longrightarrow V_a$ and let G act on $\nu_1(V)$ by $gf : x \mapsto \rho(g)f(g^{-1}x)$ and on $\nu_2(V)$ by $gf : x \mapsto f(g^{-1}x)$. Then the composition giving the isomorphism between the R-co-modules is $f \mapsto (g \mapsto \rho(g)^{-1}f(g))$.

Nori discovered that ν_i are actually related to a category finer than $\mathrm{Rep}'_k(G)$: the category of co-modules for the Hopf-algebra $(R, \mathrm{id}_R \otimes 1) \in \mathrm{Rep}'_k(G)$. We denote it by $\mathrm{Rep}''_k(G)$. As objects $\mathrm{Rep}''_k(G)$ has pairs (W, τ_W), where $W \in \mathrm{Rep}'_k(G)$ and τ_W is an arrow in $\mathrm{Rep}'_k(G)$

$$\tau_W : W \longrightarrow W \otimes (R, \mathrm{id}_R \otimes 1) \quad \text{(plus the usual commutative diagrams.)}$$

Both ν_i factor through functors $\overline{\nu}_i : \mathrm{Rep}_k(G) \longrightarrow \mathrm{Rep}''_k(G)$ and the natural isomorphism above is in fact a natural isomorphism $\overline{\nu}_1 \Rightarrow \overline{\nu}_2$.

To define $\overline{\nu}_i$, we have to declare what the arrows $\tau_i : \nu_i(V) \longrightarrow \nu_i(V) \otimes (R, \mathrm{id}_R \otimes 1)$ are.

- τ_1 is the R-co-module map $V \otimes R \longrightarrow V \otimes R \otimes R$ associated to the representation $(V, \mathrm{id}_V \otimes 1) \otimes (R, \rho_r)$ (V has the trivial action and R the right regular action!)
- τ_2 is the R-co-module map $V \otimes R \longrightarrow V \otimes R \otimes R$ associated to the representation $(V, \rho) \otimes (R, \rho_r)$.

It is immediate to verify that the above natural isomorphism $\nu_1 \Rightarrow \nu_2$ actually comes from a natural isomorphism $\overline{\nu}_1 \Rightarrow \overline{\nu}_2$.

Proof of Proposition 10: We will use notations from the previous paragraphs. Let $\mathscr{R}$ be the Hopf-algebra of $\mathbf{Fdiv}(X)$ defined by $\{R \otimes \mathscr{O}_X\}_i = \mathscr{L}'(R, \mathrm{id}_R \otimes 1)$. Torsor-algebras under $\mathscr{R}$ on $\mathbf{Fdiv}(X)$ correspond to F-divided G-torsors. If $\mathscr{B}$ is an $\mathscr{R}$-torsor-algebra corresponding to P, then the natural functor $\mathscr{L}_P : \mathrm{Rep}_k(G) \longrightarrow \mathbf{Fdiv}(X)$ in eq. (2) takes $(V, \rho) \in \mathrm{Rep}_k(G)$ to the F-divided sheaf

$$\ker\ V \otimes_k \mathscr{B} \xrightarrow{\theta_V} (V \otimes \mathscr{B}) \otimes \mathscr{R}\ ,$$

where θ_V is the co-module map for the $\mathscr{R}$-co-module $\{V \otimes \mathscr{O}_X\}_i \otimes \mathscr{B}$ (co-module structure on $\{V \otimes \mathscr{O}_X\}_i$ is induced from ρ and on $\mathscr{B}$ is the right-regular action.) In particular, taking $\mathscr{B} := \mathscr{L}'((R, \rho_l))$ we have

$$\mathscr{L}_P(V) = \ker\ \mathscr{L}' \circ \overline{\nu}_2(V) \xrightarrow{\mathscr{L}'(\tau_2)} \mathscr{L}' \circ \overline{\nu}_2(V) \otimes \mathscr{R} \tag{7}$$

On the other hand $\mathscr{L}' \circ \overline{\nu}_1(V) = \mathscr{L}(V) \otimes \mathscr{R}$ and

$$\mathscr{L}(V) = \ker\ \mathscr{L}' \circ \overline{\nu}_1(V) \xrightarrow{\mathscr{L}'(\tau_1)} \mathscr{L}' \circ \overline{\nu}_1(V) \otimes \mathscr{R}\ .$$

The proposition is a consequence of the natural isomorphism $\mathscr{L}' \circ \overline{\nu}_1 \Rightarrow \mathscr{L}' \circ \overline{\nu}_2$. $\square$

Remark: Deligne has constructed universal torsors associated to fibre functors over Tannakian categories ([9, §8] and [8, §5].) Unfortunately, we were not able to find in Deligne's writtings a precise general statement implying the constructions of (a)–(d) above, even though this sort of result–in the context of integrable connections–is used (without proof) in [8, 10.10–10.29] and (in an ad hoc way) [9, 9.2]. For the reader intimately familiar with [9] and [8], the proof of Prp. 10 should be known.

2.4 Connection with the etale fundamental group

The goal of this section is to relate the F-divided fundamental group with the etale fundamental group. It generalizes and brings to the right context [11, Prp. 1.9]. The best way to encapsulate the result below is: "etale coverings are differential equations with finite monodromy".

Let us remind the reader that we regard the etale fundamental group as a constant group scheme.

Proposition 13. *There is a natural quotient homomorphism of group schemes* $\nu : \Pi_X \longrightarrow \pi_1^{\mathrm{et}}(X, x_0)$ *which identifies* $\pi_0(\Pi_X)$ *with* π_1^{et}*.*

Proof: We will abuse notation and write π to denote π_1^{et} as well as its group of k-points. For any pro-etale group scheme G, the Frobenius twist of representations is an equivalence and hence, to every representation V of G, there exist unique representations V_i such that $V_{i+1}^{(1)} = V_i$ and $V_0 = V$. Also, if $P \longrightarrow X$ is a G-torsor, we obtain a tensor functor

$$\varphi_P : \mathrm{Rep}_k(G) \longrightarrow \mathbf{Fdiv}(X), \qquad \varphi_P(V) := \{\mathscr{L}_{P/G}(V_i)\}. \tag{8}$$

If P has a k-rational point above x_0, then $x_0^*\mathscr{L}_{P/G}$ is isomorphic to the forgetful functor. By Tannakian duality, we obtain a homomorphism $\Pi_X \longrightarrow G$. If $\pi = \varprojlim_i \pi_i$ with π_i a quotient, there are connected pointed π_i-torsors $E_i \longrightarrow X$, such that $E_i \times^{\pi_i} \pi_{i-1} = E_{i-1}$ as *pointed* torsors. It follows easily (e.g. from [22, 2.9 (c)]) that these torsors give rise to a homomorphism $\nu : \Pi_X \longrightarrow \pi$; such a homomorphism factors through $\overline{\nu} : \pi_0(\Pi_X) \longrightarrow \pi$. We will prove that $\overline{\nu}$ is an isomorphism by showing that $\mathrm{Res}(\overline{\nu})$ is an equivalence of categories.

$\mathrm{Res}(\overline{\nu})$ **is full and faithful** This amounts to showing that given a connected G-torsor P, G-etale, φ_P is full and faithful (as the subcategory of $\mathbf{Fdiv}(X)$ corresponding to $\pi_0(\Pi_X)$ is full.) Faithfulness is obvious and we concentrate on the fullness. By linear algebra, it is enough to show that any arrow $\theta : \mathbb{1} \longrightarrow \varphi_P(V)$ in $\mathbf{Fdiv}(X)$ comes from a $v \in V^G$. But such a θ is given by a sequence of morphisms $s_i : P \longrightarrow V_i$ such that the natural composition

$$\mathscr{L}_{P/G}(V_i)(X) \longrightarrow \mathscr{L}_{P/G}(V_{i-1})(X)$$

takes s_i to s_{i-1}. The above composition is just (after giving V a basis)

$$(f_1, \ldots, f_d) \mapsto (f_1^p, \ldots, f_d^p)$$

and hence, by Krull's intersection theorem, $s_0 \in k^d = V$, as the scheme P is regular, connected, locally noetherian and has a k-rational point.

$\mathrm{Res}(\overline{\nu})$ **is essentially surjective** Let $\{\mathscr{E}_i\}$ be an object of $\mathbf{Fdiv}(X)$ which is in the subcategory corresponding to $\pi_0(\Pi_X)$. There exists an etale group scheme G and an F-divided pointed G-torsor $\{P = P_0, P_1, \ldots\}$ such that $\{\mathscr{E}_i\} = \{\mathscr{L}_{P_i/G}(V)\}$ as objects of $\mathbf{Fdiv}(X)$.

From the claim below, it follows that $\{\mathscr{L}_{P_i/G}(?)\}_i$ and $\varphi_P(?)$ are naturally isomorphic tensor functors. Now, $P = E_j \times^{\pi_j} G$ for some j and hence $\{\mathscr{E}_i\} \cong \varphi_{E_j}(\mathrm{Res}^G_{\pi_j}(V))$ thereby proving that $\mathrm{Res}(\overline{\nu})$ is essentially surjective.

Claim: The functors $\{\mathscr{L}_{P_i/G}(?)\}_i$ and $\varphi_P(?)$ are naturally isomorphic.

Proof: From [21, Prp. 7.2.2, p. 146] it follows that the Frobenius pull-back F^* induces an equivalence on the category of G-torsors over X. From Prp. 10 it follows that the functor φ_P is associated to some F-divided G-torsor $\{Q_i\}$ such that $Q_0 = P = P_0$. But the F-division $\{Q_i\}$ will be isomorphic to the F-division $\{P_i\}$ and we conclude that $\varphi_P = \{\mathscr{L}_{Q_i/G}\}$ and $\{\mathscr{L}_{P_i/G}\}$ are naturally isomorphic.

□

Remark: The above proposition cannot be true if k is not algebraically closed. The reason is quite obvious, as, for example, the categories $\mathbf{Fdiv}(\operatorname{Spec} k)$ and $\mathbf{Fdiv}(\mathbb{P}^r_k)$ are trivial (for $\mathbb{P}^r$ see [11].)

2.5 The category of nilpotent objects of $\mathbf{Fdiv}(X)$

To ease notation, we abbreviate $\mathfrak{N}\mathbf{Fdiv}(X)$ by $\mathbf{nstr}$. The Tannakian group scheme associated to $\mathbf{nstr}$ is Π_X^{uni}. We want to show that if X is proper, then Π_X^{uni} is pro-etale. This is Corollary 16 below. The proof will be based on a general result (Lemma 17) used *en passant* by Nori [23]. Below we give a clear and group-theoretical proof of Lemma 17 (which should be well-known to group theorists.)

2.5.1 Main result

We start with the useful

Lemma 14 ([11], Prp. 1.7)**.** *If $\{\mathscr{E}_i\}_{i\in\mathbb{N}}$ and $\{\mathscr{F}_i\}_{i\in\mathbb{N}}$ are two objects of $\mathbf{Fdiv}(X)$ such that each $\mathscr{E}_i$ is isomorphic as an $\mathscr{O}_X$-module to $\mathscr{F}_i$ and $\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{E}_i,\mathscr{E}_i)$ is finite dimensional, then $\{\mathscr{E}_i\}_{i\in\mathbb{N}} \cong \{\mathscr{F}_i\}_{i\in\mathbb{N}}$ in $\mathbf{Fdiv}(X)$.* □

Theorem 15. *Assume that $\mathrm{H}^1(X,\mathscr{O}_X)$ and $\mathrm{H}^0(X,\mathscr{O}_X)$ are finite dimensional. Then Π_X^{uni} is pro-finite.*

Proof: According to Lemma 17, we need to show that

$$\mathrm{Ext}^1_{\mathbf{Fdiv}}(\mathbb{1},\mathbb{1}) = \mathrm{Hom}_{\mathbf{Fdiv}}(\Pi^{\mathbf{nstr}},\mathbb{G}_a)$$

is a finite dimensional k-space. An extension of $\mathbb{1}$ by $\mathbb{1}$ in $\mathbf{Fdiv}(X)$ is given by an F-divided sheaf $\{\mathscr{E}_i\}_{i\in\mathbb{N}}$ such that each $\mathscr{E}_i$ is an extension of $\mathscr{O}_X$ by itself and the diagram

$$\begin{tikzcd}
0 \arrow[r] & F^*\mathscr{O}_X \arrow[r] \arrow[d] & F^*\mathscr{E}_{i+1} \arrow[r] \arrow[d, "\cong"'] & F^*\mathscr{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}_X \arrow[r] & \mathscr{E}_i \arrow[r] & \mathscr{O}_X \arrow[r] & 0
\end{tikzcd}$$

commutes. Hence we obtain a homomorphism from $\mathrm{Ext}^1_{\mathbf{Fdiv}}(\mathbb{1},\mathbb{1})$ to

$$E := \varprojlim \mathrm{Ext}^1_{\mathscr{O}}(\mathscr{O},\mathscr{O}) = \varprojlim \mathrm{H}^1(X,\mathscr{O}_X), \tag{9}$$

with F^* being used to form the projective limit. We claim that this homomorphism is actually bijective. Surjectivity is obvious and we prove injectivity. Let $\{\mathscr{E}_i\}_{i\in\mathbb{N}}$ be an extension such that $\mathscr{E}_i \cong \mathscr{O}_X^{\oplus 2}$ for each i. Since $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X^{\oplus 2},\mathscr{O}_X^{\oplus 2})$ is finite dimensional, we can apply Lemma 14 to prove that $\{\mathscr{E}_i\}_{i\in\mathbb{N}} \cong \mathbb{1}\oplus\mathbb{1}$ in $\mathbf{Fdiv}(X)$. It is easy to verify that, in this case, the extension $\{\mathscr{E}_i\}$ is equivalent to the trivial extension.

By using [20, Cor. §14, p.143], we decompose $V := \mathrm{H}^1(X,\mathscr{O}_X)$ as $V_{\mathrm{s}} \oplus V_{\mathrm{n}}$; each summand is stable under F^* and $F^{*r}|V_{\mathrm{n}} = 0$ (for $r \gg 0$) while $F^*|V_{\mathrm{s}}$ is bijective. It follows immediately that the first projection induces an isomorphism between E and V_{s}. □

Corollary 16. *If X is as above, then Π_X^{uni} is pro-etale. In fact, it is the largest unipotent quotient of the etale fundamental group.*

Proof: The first assertions follows from *iii*) of Cor. 12. The second assertion follows from Prp. 13.

□

Remark: The pro-etaleness of Π_X^{uni} is a particular phenomenon of positive characteristic. In fact, for an elliptic curve X over $\mathbb{C}$, the algebraic hull of $\pi_1^{\mathrm{top}}(X)$ (the analogue of Π_X) has $\mathbb{G}_a$ as a quotient as one sees by considering the inclusion $\mathbb{Z} \subset \mathbb{G}_a(\mathbb{C})$.

2.5.2 A Lemma of Nori

We prove

Lemma 17. *Let $G = \operatorname{Spec} R$ be an unipotent affine group scheme over k. If $\mathrm{Hom}(G,\mathbb{G}_a)$ is finite dimensional, then G is pro-finite.*

For the sake of elegance, we shall use a well known elementary result from the theory of additive polynomials (proof can be found in [13].) Recall that an additive polynomial $P \in k[x]-\{0\}$ is a polynomial which satisfies the following identity in $k[x_1,x_2]$:

$$P(x_1+x_2) = P(x_1)+P(x_2).$$

If P and Q are additive, then so are $P+Q$ and $P\circ Q$. The ring of consisting of additive polynomials together with the zero polynomial (multiplication = composition) is $\mathrm{Hom}_{k-\mathrm{group}}(\mathbb{G}_a,\mathbb{G}_a)$. By [26, Thm. 8.4] this last ring is the twisted polynomial ring $k\{F\}$ with $\lambda^p F = F\lambda$ and F is the Frobenius endomorphism of $\mathbb{G}_a$.

Lemma 18 (existence of lcm)**.** *Given additive polynomials P and Q, there exists an additive polynomial L which is right divisible by P and Q.* $\square$

Proof of Lemma 17: Because G is a projective limit of unipotent algebraic quotients ([26], 3.3, Corollary, p. 24 and exercise 3. of Chapter 8, p. 66), we can assume that G itself is algebraic and we shall prove, under this assumption, that G is finite.

By [26], 8.3, Theorem, p. 64 we can assume that G is a closed subgroup of $\mathbb{U}_n$. Let x_{ij} denote the restriction to G of the coordinate functions on $\mathbb{A}^{n^2}$. If $\Delta : R \longrightarrow R \otimes R$ is the co-multiplication we have, for $i < j$,

$$\Delta(x_{ij}) = \begin{cases} x_{ij} \otimes 1 + 1 \otimes x_{ij}, & \text{if } j = i+1, \\ x_{ij} \otimes 1 + 1 \otimes x_{ij} + \sum_{i<l<j} x_{il} \otimes x_{lj}, & \text{if } j > i+1. \end{cases} \tag{10}$$

Thus, $k[x] \longrightarrow R, x \mapsto x_{i,i+1}$ determines a homomorphism $G \longrightarrow \mathbb{G}_a$. By assumption, the vector space $V_{i,i+1} := \operatorname{span}\{x_{i,i+1}^{p^m};\ m \in \mathbb{N}\}$ is finite dimensional. In consequence, there exist additive polynomials $\widetilde{P}_i$ such that $\widetilde{P}_i(x_{i,i+1}) = 0$. By the existence of lcm, there exists an additive polynomial P_1 with $P_1(x_{i,i+1}) = 0$ for all $i = 1, \dots, n-1$.

Assume that we have proved, for $r > 1$, that

> There are additive polynomials $P_1, \dots, P_{r-1}$ such that $P_j(x_{i,i+j}) = 0$. (†)

We want to prove that (†) holds for r and this will be sufficient to prove the lemma. Let $V_{ij} := \operatorname{span}\{x_{ij}^{p^m};\ m \in \mathbb{N}\}$. From the expressions in (10),

$$\Delta(x_{i,i+r}^{p^m}) - x_{i,i+r}^{p^m} \otimes 1 - 1 \otimes x_{i,i+r}^{p^m} \in \sum_{l=i+1}^{i+r-1} V_{il} \otimes V_{l,i+r}, \qquad (m \in \mathbb{N}).$$

Since

$$\sum_{l=i+1}^{i+r-1} V_{il} \otimes V_{l,i+r}$$

is finite dimensional, there is a non-trivial relation

$$\sum_m \lambda_{im} \Delta(x_{i,i+r}^{p^m}) - \lambda_{im}(x_{i,i+r}^{p^m} \otimes 1) - \lambda_{im}(1 \otimes x_{i,i+r}^{p^m}) = 0, \qquad i = 1, \dots, n-r.$$

Hence there exist additive polynomials $\widetilde{Q}_i$ such that $\widetilde{Q}_i(x_{i,i+r}) : G \longrightarrow \mathbb{G}_a$ defines a homomorphism and from the hypothesis of the lemma there exist additive polynomials Q_i such that $Q_i(x_{i,i+r}) = 0$. The existence of the lcm now completes the induction in (†). $\square$

3 $\Pi^{\mathbf{str}}$ of an Abelian variety

Let X be a g-dimensional abelian variety over k. When dealing with the stratified fundamental group scheme we will take the identity as base point.

We want to use the results of the previous sections to find a definite expression for Π_X. Recall that in the complex analytic case, the fundamental group is abelian and hence its algebraic hull will decompose as a diagonal and an unipotent part. Using results of Gieseker and Miyanishi-Mukai we are able to prove that the same holds for Π_X.

To state our theorem, we need the following lemma and the

Definition 19. *Let G be an abelian abstract group and let $[p] : G \longrightarrow G$ be multiplication by p. We shall denote by $G\langle p\rangle$ the group*

$$\varprojlim \left(\cdots \xrightarrow{[p]} G \xrightarrow{[p]} G \xrightarrow{[p]} \cdots \right).$$

Note that $G \mapsto G\langle p\rangle$ is an additive left-exact functor from the category of abelian groups to itself.

Lemma 20. *Let Y be a proper and smooth scheme over k. Assume that the Picard scheme $\mathrm{Pic}_{Y/k}$ is reduced. Then the character group $\mathbb{X} = \mathbb{X}(\Pi_Y)$ is isomorphic to*

$$\mathrm{NS}(Y)' \oplus \mathrm{Pic}^0(Y)\langle p\rangle,$$

where $\mathrm{NS}(Y)'$ is the group of elements of finite prime-to-p order. In particular,

$$\mathbb{X} \cong \mathrm{Pic}^0(Y)\langle p\rangle$$

for Y an abelian variety or a curve.

Proof: Since we are assuming that the Picard scheme $\mathrm{Pic}_{Y/k}$ is reduced, the connected component of the identity $\mathrm{Pic}^0_{X/k}$ is proper and smooth (properness follows from the smoothness of Y/k.) Hence $\mathrm{Pic}^0_{Y/k}(k)$ is divisible.

We will denote $\mathrm{Pic}_{Y/k}(k) = \mathrm{Pic}(Y)$ by P and $\mathrm{Pic}^0_{Y/k}(k) = \mathrm{Pic}^0(Y)$ by P_0. Recall that for α a class in P, the class corresponding to $F^*\alpha$ is $p \cdot \alpha$ (think of cocycles.) There is a natural homomorphism $\mathbb{X} \longrightarrow P\langle p\rangle$ which from Lemma 14 is an isomorphism. The rest of the proof now follows from abstract nonsense and the deep fact that $\mathrm{NS}(Y)$ is finitely generated (SGA 6, Exp. 13, Thm. 5.1.) Applying the functor $\langle p\rangle$ to the exact sequence

$$0 \longrightarrow P_0 \longrightarrow P \longrightarrow \mathrm{NS}(Y) \longrightarrow 0,$$

we obtain the exact sequence

$$0 \longrightarrow P_0\langle p\rangle \longrightarrow \mathbb{X} \longrightarrow \mathrm{NS}(Y) \longrightarrow R^1\langle p\rangle(P_0).$$

Since the projective system used to form $P_0\langle p\rangle$ satisfies the Mittag-Leffler condition ($[p] : P_0 \longrightarrow P_0$ is surjective), [27, Prp. 3.5.7, p. 83] shows that $R^1\langle p\rangle(P_0) = 0$. Using that $P_0\langle p\rangle$ is divisible (this follows easily from the divisibility of P_0) the exact sequence

$$0 \longrightarrow P_0\langle p\rangle \longrightarrow \mathbb{X} \longrightarrow \mathrm{NS}(Y)\langle p\rangle \longrightarrow 0$$

splits and we are left to show that $\mathrm{NS}(Y)\langle p\rangle \cong \mathrm{NS}(Y)'$. But this is easy since $\mathrm{NS}(Y) \cong \mathrm{NS}(Y)' \oplus \mathrm{NS}'' \oplus \mathbb{Z}^t$ with NS'' finite and p-torsion. That the Néron-Severi group of an abelian variety is in [20, Cor. 2, p. 178]. $\square$

Theorem 21. *There is a natural isomorphism*

$$\Pi_X \xrightarrow{\cong} T_p(X) \times \mathrm{Diag}(\mathrm{Pic}^0(X)\langle p\rangle),$$

where $T_p(X)$ is the p-adic Tate module (of k-rational points.)

Proof: The work has essentially been done. We just need to put it together. Using Corollary 16, the unipotent part Π_X^{uni} is the unipotent part of the etale fundamental group of X:

$$\pi_1^{\mathbf{et}} \cong \prod_{l \text{ prime}} T_l(X), \quad \text{canonically by [20, §18, p. 171].}$$

For $l \neq p$, the group schemes $T_l(X)$ are non-canonically isomorphic to the diagonal group $\mathbb{Z}_l^{2g}$ and $T_p(X)$ is non-canonically isomorphic to the unipotent [26, Thm. 8.5, p. 66] group scheme $\mathbb{Z}_p^r$. So $\Pi_X^{\mathrm{uni}} \cong T_p(X)$.

It now remains to show that Π_X decomposes as a direct product of its diagonal and unipotent parts. In terms of tensor products of Tannakian categories [9, 5.18, 6.21], this is equivalent to the statement that every object $\mathscr{E} \in \mathbf{Fdiv}(X)$ can be written as a direct sum

$$\bigoplus_i \mathscr{N}_i \otimes \mathscr{L}_i,$$

where $\mathscr{L}_i$ is of rank one and $\mathscr{N}_i$ is in **nstr**. Alternatively, we want to show that every indecomposable object in $\mathbf{Fdiv}(X)$ is the tensor product of a nilpotent and a rank one object. So let $\mathscr{E} = \{\mathscr{E}_i\}$ be indecomposable.

1. In [11, thm. 2.6] it is proved that every *simple* object of $\mathbf{Fdiv}(X)$ is one dimensional and hence we obtain a filtration

$$\mathscr{E} = \mathscr{E}^{(0)} \supset \mathscr{E}^{(1)} \supset \cdots \supset \mathscr{E}^{(r)} = 0, \tag{11}$$

with $\mathscr{E}^{(\nu)}/\mathscr{E}^{(\nu+1)} =: \mathscr{L}^{(\nu)}$ an F-divided sheaf of rank one.

2. Using 1. one shows that each $\mathscr{E}_i$ is a translation invariant (homogeneous) sheaf; this is well-known and a proof can be found in [18, Ex. 3.2, p. 158].
3. Using the Hilbert-Schmidt property of vector bundles, the indecomposability of $\mathscr{E}$ and Lemma 14 imply that $\mathscr{E}_i$ is indecomposable (as a coherent sheaf) for $i \gg 0$.
4. It follows that for $i \gg 0$, $\mathscr{L}_i^{(\nu)} \cong \mathscr{L}_i^{(\nu+1)}$ (see [19, Thm. 2.3] or [18, Thm. p. 159].)
5. Another application of Lemma 14 shows that $\mathscr{L}^{(\nu)} \cong \mathscr{L}^{(\nu+1)}$ as F-divided sheaves. This allows us to write $\mathscr{E} = \mathscr{N} \otimes \mathscr{L}^{(0)}$ with $\mathscr{N} \in \mathbf{nstr}$.

□

Let C be a smooth and projective curve over k with a k-rational point P. Let J be its Jacobian and let $f : C \longrightarrow J$ be the natural morphism which sends P to identity.

Given any affine algebraic group scheme G, we can form the largest abelian quotient $G^{\mathrm{ab}} := G/[G,G]$ which has the expected universal property: any homomorphism $G \longrightarrow A$ with A abelian group scheme factors through G^{ab} ([26, ex. 1, p. 125].) For a general group scheme $G = \varprojlim_\alpha G_\alpha$ with $\mathscr{O}(G_\alpha) \subseteq \mathscr{O}(G)$ of finite type, we can form G^{ab} as the limit $\varprojlim_\alpha G_\alpha^{\mathrm{ab}}$; obviously we have the same universal property as before. The next corollary gives the expected version in positive characteristic of the known complex analytic analogue.

Corollary 22. *The natural homomorphism* $\Pi^{\mathbf{str}}(C,P) \longrightarrow \Pi^{\mathbf{str}}(J,0)$ *induces an isomorphism* $\Pi^{\mathbf{str}}(C,P)^{\mathrm{ab}} \longrightarrow \Pi^{\mathbf{str}}(J,0)$.

Proof: Because $\Pi^{\mathbf{str}}(C,P)^{\mathrm{ab}}$ is abelian it can be decomposed into a diagonal and an unipotent part. The diagonal part is controlled by the character group $\mathrm{Pic}^0(C)\langle p\rangle$ while the unipotent part is controlled by the largest etale quotient of $\Pi^{\mathbf{nstr}}(C,P)^{\mathrm{ab}}$. Now the corollary follows from the fact that $f^* : \mathrm{Pic}^0(J) \longrightarrow \mathrm{Pic}^0(C)$ and $f^* : \pi_1^{\mathbf{et}}(C,P)^{\mathrm{ab}} \longrightarrow \pi_1^{\mathbf{et}}(J,0)$ are isomorphisms ([17], 9.3, p. 196 and 9.1, p. 195.)

□

Corollary 23. $\Pi^{\mathbf{str}}$ *will not base change in a functorial way.*

Proof: First note that given a diagonal group G over k, the natural map $\mathbb{X}(G) \longrightarrow \mathbb{X}(G_K)$ is bijective for any extension field $K \supseteq k$. Let K be an algebraically closed field containing the function field of $X^\vee$ (the dual). Then $X^\vee(k) \longrightarrow (X_K)^\vee(K) = X^\vee(K)$ will *not* be bijective (the point corresponding to the inclusion in K of the function field will not be in the image of $X^\vee(k)$.) Consequently, $\mathbb{X}(\Pi_X) \longrightarrow \mathbb{X}(\Pi_{X_K})$ will not be bijective.

□

This is a different picture from the etale case because characters of $\pi_1^{\mathbf{et}}$ lie in $X^\vee(k)_{\text{tors}} \cong X^\vee(K)_{\text{tors}}$.

Example: In certain cases it is possible to say something more about the group of characters $\mathbb{X}(\Pi_X)$. If k is the algebraic closure of $\mathbb{F}_p$ (or if every $\xi \in X^\vee(k)$ is of finite order) then

$$\mathbb{X} \cong V_p(X^\vee) \oplus \left(\bigoplus_{l \neq p} (\mathbb{Q}_l/\mathbb{Z}_l)^{\oplus 2g} \right).$$

Another computation of $\mathbb{X}$ will be given in (4.2.)

Remarks: (a) The above proof of the decomposition of Π_X avoids full use of Fourier-Mukai and was suggested by N. Shepherd-Barron and A. Scholl. We have presented it here because it is more elementary. A more direct proof is as follows. Let Φ be the Fourier-Mukai transform. The category $\mathbf{Fdiv}(X)$ is equivalent, via Φ, to the category $\mathbf{FSky}(X)$ whose objects are sequences $\{E_i, \tau_i\}_{i \in \mathbb{N}}$ of skyscraper sheaves E_i on $X^\vee$ and isomorphisms $\tau_i : (F^\vee)_*(E_{i+1}) \longrightarrow E_i$. Under Φ a homogeneous line bundle $\mathscr{L}$ is taken to a sheaf supported at $[\mathscr{L}^{-1}] \in X^\vee(k)$. As skyscraper sheaves supported at more than one point can be written as a direct sum, we obtain another proof of item 4. in the proof of Thm. 21.

(b) Theorem 21 is proved by Gieseker [11] in the case of an elliptic curve.

(c) As said in the introduction, thinking of representations is very fructuous. Even though the computation of the character group $\mathbb{X}$ is a straightforward application of Lemma 14, this important structural result is overlooked in [11]. Also, the methods to determine a decomposition of Π_X into a unipotent and a diagonal part have been available for quite some time but only group theory gives the necessary insight to obtain a definite result as Thm. 21.

4 A link with rigid geometry

It is well known (opening pages of [6]) that over a complex analytic manifold, stratified sheaves are controlled by the topological fundamental group π_1^{top}.

In this section we want to examine a little closer the analogous situation in rigid analytic geometry. The topological fundamental group is substituted by the rigid analytic fundamental group [10, 5.7]. The first result we obtain (Proposition 24) is much weaker than its complex analytic analogue and the proof is the usual one. Later we use rigid analytic abelian varieties to give some measure of this weakness (see Prp. 25 and the remark at the end of 4.2.)

Throughout we will take k complete with respect to a non-Archimedean absolute value $|\cdot| : k^* \longrightarrow \mathbb{R}_{>0}$. We let X be a connected and smooth rigid analytic variety [4] – the G-topology $\mathscr{T}_X$ is the strong topology. Let $x_0 \in X$.

We break the convention and let, for an abstract group Λ, $\mathrm{Rep}_k(\Lambda)$ denote the category of abstract representations of Λ (homomorphisms $\rho : \Lambda \longrightarrow \mathrm{GL}(m)$.)

4.1 The algebraic hull of the rigid fundamental group and Π_X

4.1.1 Setting

We start by clarifying what we mean by stratified sheaves on X. As a reasonable theory of differential operators over smooth rigid analytic varieties is still to be carefully written (but see [24, Ch. 5 Appendix]) we take the point of view of F-divided sheaves; to do that we comment on the Frobenius morphism.

The Frobenius morphism $F : X \longrightarrow X$ is a morphism of locally ringed G-topological spaces which is the identity on $\mathscr{T}_X$ and has the absolute Frobenius $a \mapsto a^p$ as defining homomorphism $\mathscr{O}_X \longrightarrow \mathscr{O}_X = F_*\mathscr{O}_X$. As the local rings in X are all regular the $F^{-1}\mathscr{O}_X$-algebra $\mathscr{O}_X$ is coherent and faithfully flat. *We define* $\mathbf{str}(X)$ in the same way as we defined the category of F-divided sheaves over a regular k-scheme (Def. 4.) In particular, Lemma 6 still applies and x_0^* gives a fibre functor.

The Tannakian fundamental group (Thm. 2) associated to $\mathbf{str}(X)$ via x_0^* is denoted by Π_X. If X_0/k is proper connected with $X_0^{\mathrm{an}} = X$, then rigid GAGA [10, 4.10.5, p. 113] gives

$$\Pi_X \cong \Pi_{X_0}.$$

4.1.2 Construction of a functor [12]

Let $\pi : \Omega \longrightarrow X$ be an analytic covering [10, 5.7] of X by a simply connected rigid analytic variety Ω (the universal covering.) Let $\Lambda := \mathrm{Aut}_X(\Omega)$ be the *rigid fundamental group* of X; note that it acts freely and transitively on $\pi^{-1}(x_0)$. To each k-representation of the abstract group Λ we can associate a stratified sheaf over X in a functorial way

$$\mathfrak{L} : \mathrm{Rep}_k(\Lambda) \longrightarrow \mathbf{str}(X).$$

This clever construction respects the tensor product and avoids the use of differential operators.

Let $\rho : \Lambda \longrightarrow \mathrm{GL}(V)$ be a finite dimensional representation of Λ and consider the associated sheaf on X, $\mathfrak{L}(V)$. For the convenience of the reader we recall the definition of $\mathfrak{L}$. Given an open $U \subseteq X$, the open $\pi^{-1}(U)$ is Λ-invariant. Let $\lambda \in \Lambda$ act on $V \otimes_k \mathscr{O}_\Omega(\pi^{-1}(U))$ by

$$\lambda \cdot (\sum_i v_i \otimes f_i) = \sum_i \rho(\lambda) v_i \otimes f \circ \lambda^{-1}.$$

Then $\mathfrak{L}(V)(U)$ is the $\mathscr{O}_\Omega(\pi^{-1}(U))^\Lambda = \mathscr{O}_X(U)$-module of all invariant elements of $V \otimes \mathscr{O}_\Omega(\pi^{-1}(U))$. It is easy to see that $\mathfrak{L}(V)$ is always a coherent analytic sheaf on X. We also note that the natural map $\pi^*\mathfrak{L}(V) \longrightarrow V \otimes_k \mathscr{O}_\Omega$ is an isomorphism.

Because k is perfect, the Frobenius twist $\mathrm{Ft} : \mathrm{Rep}_k(\Lambda) \longrightarrow \mathrm{Rep}_k(\Lambda)$ is an equivalence: fixing a basis of $V_0 \in \mathrm{Rep}_k(\Lambda)$ with respect to which $\rho : \Lambda \longrightarrow \mathrm{GL}(V_0)$ is given by the matrices $(a_{ij}(\lambda))$, the representation V_1, defined by $(a_{ij}(\lambda)^{p^{-1}})$, is such that $\mathrm{Ft}(V_1) = V$.

Inductively, we let V_i be the representation obtained in the same way from V_{i-1}. Note that there is a natural p-linear homomorphism

$$\mathfrak{L}(V_{i+1})(U) \longrightarrow \mathfrak{L}(V_i)(U), \quad (f_1, \dots, f_d) \mapsto (f_1^p, \dots, f_d^p),$$

which induces an isomorphism $F^*\mathfrak{L}(V_{i+1}) \cong \mathfrak{L}(V_i)$. Hence, $\mathfrak{L}$ (abusing notation) is naturally an exact tensor functor from the category of representations of Λ to the category of F-divided sheaves on X.

4.1.3 The algebraic hull Λ^{alg} is a quotient of Π_X

Proposition 24. *The functor $\mathfrak{L}$ identifies $\mathrm{Rep}_k(\Lambda)$ with a tensor subcategory of $\mathbf{str}(X)$. That is, (a) $\mathfrak{L}$ is full and faithful and (b) any sub-object $M \subseteq \mathfrak{L}(V)$ is the image of a sub-object $W \subseteq V$. In particular, by choosing $\omega_0 \in \pi^{-1}(x_0)$ the natural homomorphism $\Pi_X \longrightarrow \Lambda^{\mathrm{alg}}$ is a quotient map.*

Proof: Fully faithfulness follows along the lines of the analogous statement in Prp. 13 by using that a global analytic function $f \neq 0$ is never zero in $\mathscr{O}_{\Omega,\omega_0}$ (this is analytic continuation, a proof can be found in [3, 0.1.13].) More precisely: By linear algebra, it is enough to show that any global section of $\mathfrak{L}(V)$ is induced by $v \in V^\Lambda$; such a global section corresponds to a d-uple ($d = \dim_k V$) of global analytic functions $(f_1, \dots, f_d)$ with the property that for each $r \in \mathbb{N}$ there exits $g_j \in \mathscr{O}(\Omega)$ with $g_j^{p^r} = f_j$. By analytic continuation such a function is in k.

The proof of (b) runs through the usual argument and we merely sketch it. The reader familiar with Berkovich geometry might find [1, III-3.4.4] useful, but the best way to fill the proof is to copy the proceedure from algebraic topology (observing that the "local system" associated to $\{M_n\} \in \mathbf{str}(X)$ is $\cap_n M_n$.) If $M \subseteq \mathfrak{L}(V)$ is a sub-object in $\mathbf{str}(X)$, then there is an admissible cover of $X = \cup_\alpha U_\alpha$ such that the restriction $M|U_\alpha$ is trivial as F-divided sheaf on U_α – this uses the Tannakian property of $\mathbf{str}(U_\alpha)$. Hence, there is a sub-representation $W \subseteq V$ such that the inclusion $M \subseteq \mathfrak{L}(V)$ is $\mathfrak{L}(W) \subseteq \mathfrak{L}(V)$. □

Remark: It is a known phenomenon in rigid analytic geometry that the rigid fundamental group cannot control all differential equations. The reason is simple: there are coverings by admissibles which are not admissible coverings. In positive characteristic another impediment shows up: it is not possible to "integrate" the equations in a neighbourhood of an arbitrary point [24, Ch. 5].

4.2 Uniformizable abelian varieties

Let X_0/k be an abelian variety such that $X_0^{\mathrm{an}} = X = \mathbb{G}_m^g/\Lambda$, $\Lambda = \mathbb{Z}^g$ (the existence of X_0 is not strictly necessary since we can rewrite all relevant results in the previous sections in terms of rigid analytic geometry.) We want to prove the following.

Proposition 25. *The kernel of the quotient map*

$$\Pi_X \longrightarrow \Lambda^{\mathrm{alg}}$$

is $\mathrm{Diag}(\mathbb{Z}_p/\mathbb{Z})^g$*. In fact, there is a non-canonical decomposition*

$$\Pi_X \cong \mathrm{Diag}(\mathbb{Z}_p/\mathbb{Z})^g \times \Lambda^{\mathrm{alg}}.$$

Proof: The proof follows from the two lemmas below and the decomposition of both Π_X and Λ^{alg} into a diagonal and an unipotent part (Theorem 21 and the Jordan decomposition.)

Lemma 26. *Let* $\mathbb{X}$ *be the character group of* Π_X*. There is an exact sequence*

$$0 \longrightarrow (k^*)^g \longrightarrow \mathbb{X} \longrightarrow (\mathbb{Z}_p/\mathbb{Z})^g \longrightarrow 0$$

which has a non-canonical splitting.

Proof: The analytification of the dual abelian variety $X_0^\vee$ is also a torus so that $\mathrm{Pic}^0(X) \cong (k^*)^g/\mathbb{Z}^g$. To ease notation, we assume $X = \mathbb{G}_m/q^{\mathbb{Z}}$ (Tate curve.) From Lemma 20, $\mathbb{X} \cong (k^*/q^{\mathbb{Z}})\langle p\rangle$ and this is what we are going to compute.

Applying the functor $\langle p\rangle$ to the exact sequence

$$1 \longrightarrow q^{\mathbb{Z}} \longrightarrow k^* \longrightarrow k^*/q^{\mathbb{Z}} \longrightarrow 1$$

we get

$$1 \longrightarrow k^* \longrightarrow (k^*/q^{\mathbb{Z}})\langle p\rangle \longrightarrow R^1\langle p\rangle(\mathbb{Z}) \longrightarrow R^1\langle p\rangle(k^*).$$

By [27, Prp. 3.5.7, p. 83] $R^1\langle p\rangle(k^*) = 0$ and hence $(k^*/q^{\mathbb{Z}})\langle p\rangle$ is an extension of $R^1\langle p\rangle(\mathbb{Z})$ by k^*. Now k^* is divisible and the extension above splits: $(k^*/q^{\mathbb{Z}})\langle p\rangle \cong k^* \oplus R^1\langle p\rangle(\mathbb{Z})$. We are left with the computation of $R^1\langle p\rangle(\mathbb{Z})$. As the projective system used to form $\mathbb{Z}\langle p\rangle$ is isomorphic to the projective system

$$\cdots \subset p^{n+1}\mathbb{Z} \subset p^n\mathbb{Z} \subset \cdots,$$

the lemma follows from the computation made in loc.cit., example 3.5.5, p. 82. $\square$

Lemma 27. *The quotient map* $\Pi_X \longrightarrow \Lambda^{\mathrm{alg}}$ *induces an isomorphism between the unipotent parts.*

Proof: Both unipotent parts Π_X^{uni} and $(\Lambda^{\mathrm{alg}})^{\mathrm{uni}}$ are pro-etale isomorphic to $\mathbb{Z}_p^g$. For Π_X this is Cor. 16 plus the fact that the p-rank of X_0 is g. For Λ^{alg}, this follows from the recipe to construct the algebraic hull (2.1.1) together with the observation that any homomorphism $\rho : \mathbb{Z}^g \longrightarrow \mathbb{U}_n(k)$ will factor through $(\mathbb{Z}/p^n\mathbb{Z})^g$.

Hence, any $\mathscr{E} \in \mathbf{nstr}(X_0)$ is the sheaf associated to a representation of $(\mathbb{Z}/p^n\mathbb{Z})^g$ (for some n) via the obvious $(\mathbb{Z}/p^n\mathbb{Z})^g$-torsor

$$\pi : X_0/(\ker[p^n])^\circ \longrightarrow X_0$$

that is, $\mathscr{E} = \varphi(V)$ with $V \in \mathrm{Rep}_k((\mathbb{Z}/p^n\mathbb{Z})^g)$ (the notation is as in Prp. 13, see eq. (8).) So, in order to show that the induced functor

$$\mathrm{Rep}_k((\Lambda^{\mathrm{alg}})^{\mathrm{uni}}) \longrightarrow \mathrm{Rep}_k(\Pi_X^{\mathrm{uni}})$$

is essentially surjective, we have to prove that π is the analytification of some rigid analytic covering $Y \longrightarrow X$. It is an exercise to show that the natural analytic covering $\nu : \mathbb{G}_m^g/p^n\Lambda \longrightarrow \mathbb{G}_m^g/\Lambda$ is what we search.

□

Remarks: (a) To ease notation, assume $g = 1$. We can topologize $\mathbb{X}$ in such a way that it becomes a Hausdorff group. This is done by noting that the Hausdorff group

$$\frac{k^* \oplus \mathbb{Z}_p}{(q,1)^{\mathbb{Z}}}$$

contains k^* as $k^* \oplus \{0\}$ and the quotient is $\mathbb{Z}_p/\mathbb{Z}$. Since there is only one extension of $\mathbb{Z}_p/\mathbb{Z}$ by k^* up to isomorphism, follows that $\mathbb{X} \cong (k^* \oplus \mathbb{Z}_p)/(q,1)^{\mathbb{Z}}$. I thank A. Scholl for pointing this out to us.

(b) The most striking property of the stratified fundamental group (Thm. 11) is also shared by the algebraic hull of any abstract group. It is natural to ask if there is an abstract group Γ whose algebraic hull is "close" to the stratified fundamental group of a projective smooth rigid analytic variety X. Both Prp. 24 and Prp. 25 were designed to shed some light on this problem. Remark (a) sustains the hope that Pontryagin duality could help us find Γ; but this seems too optimistic.

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