Group schemes from ODEs defined over a discrete valuation ring

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Definition

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- "Splitting field" or the Picard-Vessiot extension: $\mathbb{C}(x, \mathcal{E}) := \mathrm{Frac}(R/(\text{some ideal}))$.

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Example

Take $(\mathcal{E})$ to be $y' = y$. Then $\mathrm{Gal} = \mathbb{G}_m$ but $\mathrm{Mon} = 1$.

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Theorem (Schlesinger)

If $(\mathcal{E})$ *only has* **regular-singular** *points, then*

$$\mathrm{Gal} = \textit{Zariski closure of } \mathrm{Mon}$$

□

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$$\langle E \rangle_{\otimes} = \left\{ E'/E'' \ : \ E'' \subset E' \subset \bigoplus_i E^{\otimes a_i} \otimes \check{E}^{\otimes b_i} \right\}.$$

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If $\varrho_E : \Pi(\mathfrak{T}) \to \mathrm{GL}(\omega E)$ associated to $E \rightsquigarrow \Pi_E = \mathrm{Im}(\varrho_E)$. In particular Π_E is linear algebraic group.

Some examples: Abstract groups

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Corollary (Schlesinger)

$$\Pi_{\mathrm{DE}}(\mathcal{E}) \xrightarrow{\quad\sim\quad} \Pi_{\mathrm{DE}_{\mathrm{rs}}}(\mathcal{E}) \xrightarrow{\quad\sim\quad} \overline{\mathrm{Im}(Mon_{\mathcal{E}})}$$

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$k = \overline{k} \rightsquigarrow$ there exists $\omega : \mathrm{DE}^{\mathrm{formal}} \hookrightarrow k$-vect (Deligne). In case of $\mathrm{DE}^{\mathrm{formal}}_{\mathrm{rs}}$ can just "classify".

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$\Rightarrow$ *Exists* $\Pi(\mathcal{T})$ **flat** *group scheme such that* $\mathcal{T} \xrightarrow{\sim} \mathrm{Rep}_R(\Pi)$. □

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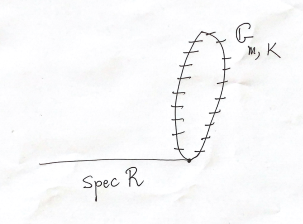

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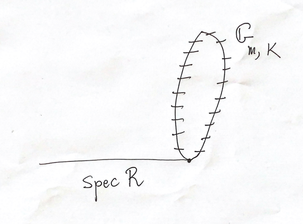

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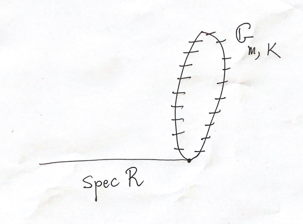

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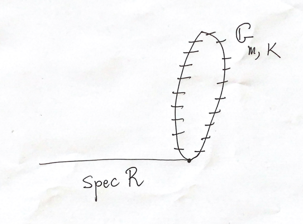

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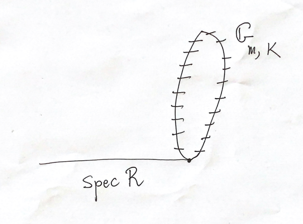

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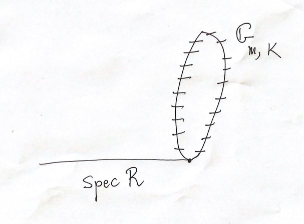

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   (b) $\mathcal{G} \simeq \mathcal{N}^{\infty}_{\mathfrak{H}'}(G')$. *(Hai-dS 2020)*

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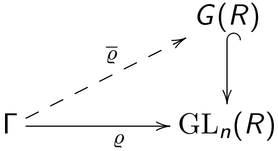

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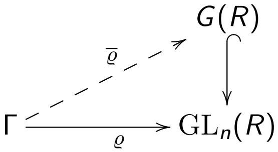

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- To prove our result: Use theory of **exponents** to bound order of poles in the logarithmic models.

Texts

1. N. D. Duong, P. H. Hai. *Tannakian duality over Dedekind rings and applications*. Math. Zeit. 2018.
2. N. D. Duong, P. H. Hai and J. P dos Santos. *On the structure of affine flat group schemes over discrete valuation rings, I*.Ann. SNS Pisa 2018
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5. P. H. Hai and J. P dos Santos P. T. Tam. *Algebraic theory of formal regular-singular connections with parameters.*