# Finite torsors (with Hai)

§1 Nori's theory / field (3 approaches)

§2. Gasbarri's & Metha-Subramanian's theory

§3. Modules trivialized proper morphs.

§4. Properties + links:

$\boxed{1}$ $k=k^{p}$. $X/k$ complete $x_0 \in X(k)$.

**Dfn:** $\mathcal{E} \in VB(X)$ Nori-ss $\Leftrightarrow$ $\forall\, C \xrightarrow{\gamma} X$, $\gamma^*\mathcal{E}$ s.s., $\deg = 0$.

$\uparrow$ sm, proper curve

Notation: $NS(X)$.

**Thm.** (Narasimhan-Seshadri-Nori) $NS(X)$ abel.,

$x_0^*: NS \longrightarrow k\text{-mod}$ exact + faithful.

$NS$ = big!

**Dfn:** $\mathcal{E} \in NS(X)$ finite $\Leftrightarrow$ $\mathcal{E}^{\otimes d} \oplus \left( \bigoplus \left( \mathcal{E}^{\otimes i}\right)^{a_i}\right) \cong \bigoplus^{d-1} \left(\mathcal{E}^{\otimes i}\right)^{b_i}$

( $\mathcal{E}$ "integral)

$EF(X) = \{ \mathcal{E} = \mathcal{E}'/\mathcal{E}''\,;\ \overset{NS}{\mathcal{E}''} \subset \overset{NS}{\mathcal{E}'} \subset \overset{\text{finite}}{F} \}$

<u>Thm</u>: $EF(X) \xrightarrow[x_0^*]{\sim} \mathrm{Rep}\,\pi^N(X, x_0)$. ← pro-finite

★: $\mathrm{Hom}\big(\pi^N(X,x_0), \underset{\substack{\nearrow\\ \text{fin.}}}{G}\big) = \left\{ \begin{matrix} G\text{-torsors} \\ P \to X \text{ with pt.}/x_0 \end{matrix} \right\}$

- Filtering Approach:

$$\begin{tikzcd} & P' \circlearrowleft G' \arrow[dr] & \\ Q \arrow[ur, dashed] \arrow[dr, dashed] & & P \circlearrowleft G \\ & \underset{\substack{\circlearrowleft\\ G''}}{P''} \arrow[ur] & \end{tikzcd}$$

$\pi = \underset{\longleftarrow}{\lim}\, G$

(Nori, Antei, Emsalem)

- Trivialising approach → Def: $\underset{\|}{\Pi_X}$

<u>Thm</u> $EF(X) = \{ \mathcal{E} \in VB_X : \exists\, Y \to X$ proper surj. $\mathcal{E}|_Y = \mathcal{O}_Y^r \}$

(Biswas-…, Antei-Mehta, Tonini-Zhang)

$\boxed{2}$ $A = DVR$, $X/A$ flat$^{(\cdots)}$, $x_0 \in X(A)$.

Filtering OK if reduced fibres.

<u>Thm</u>. Exists $\Pi^{fil}/A$ flat + profinite with ★.

(Gasbarri, Antei-Emsalem-Gasbarri)

<u>Q</u>. Link $\mathrm{Rep}_A \Pi^{fil}$ & $coh(X)$?

Thm. (Mehta-Subramanian) $X/W(k)$ smooth proj.

$$\mathrm{Hom}(\Pi^{fil} \otimes \overline{W}, GL_n) = \{ \mathcal{E} \in VB_{X_{\overline{W}}} : \mathcal{E} \text{ is EF on fibres} \}.$$

Not abelian →
Defined / $\overline{W}$

$\boxed{3}$ $X/A$ proper, reduced fibres, $x_0 \in X(A)$

- $Y \xrightarrow{\varphi} X$ proper & surj.

Dfn: $\Pi_Y = \{ \mathcal{E} \in \mathrm{Coh}\, X : \varphi^* \mathcal{E} = \mathcal{O}_Y \otimes_A E \}$.

Thm: If • $Y/A$ coh. flat
• $Y$ has $H^0(\mathcal{O}_Y)$-point over $x_0$ $\Rightarrow \Pi_Y \xrightarrow[x_0^*]{\sim} \mathrm{Rep}\, \Pi^{tr}(X;Y)$

flat+affine

Rmk: coh. flat = "$H^0(\mathcal{O}_Y \otimes_A M) = H^0(\mathcal{O}_Y) \otimes_A M$"
delicate.

Q. Put $\{\Pi_Y\}_Y$ together?

Thm: (Reduced fibre). $A = A^h$. $Y/A$ • flat • $Y_{\overline{K}}$ reduced.

$\Rightarrow B \supset A$ finite of DVRs &

$$\begin{tikzcd} Y \arrow[d] & Y_B \arrow[l] \arrow[d] & Z \arrow[l, "\substack{\cdot \text{finite} \\ \cdot \text{gen. iso.}}"'] \\ \mathrm{Spec}\, A & \mathrm{Spec}\, B \arrow[l] & \end{tikzcd}$$

with $Z/B$ geom. red. fibres.

**Thm:** $A =$ • henselian.
• Japanese

$$\Pi_X = \bigcup_{Y \xrightarrow{\varphi} X} \Pi_Y \xrightarrow[x_0^*]{\sim} \mathrm{Rep}_A \Pi^{tr}(X, x_0)$$

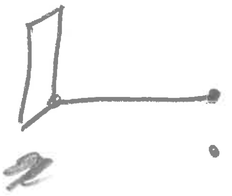
**Q.** $\Pi^{tr}/A$ pro-finite?

$\boxed{4}$

**Dfn:** $\overset{\text{loc. free}}{\mathcal{E}} \in \Pi_X$ ; $\langle \mathcal{E} \rangle_{\otimes} = \{ \mathcal{F}'/\mathcal{F}'' : \mathcal{F}'' \subset \mathcal{F}' \subset \bigoplus \mathcal{E}^{\otimes a_i} \otimes \mathcal{E}^{\vee \otimes b_i} \}$

$\langle \mathcal{E} \rangle_{\otimes} \cong \mathrm{Rep}_A \mathrm{Gal}(\mathcal{E}) \leftarrow$ • flat
• generically alg.

**Ex:** • $W_e^{\infty}(\mathbb{G}_a) = \mathrm{Spec}\{P \in K[x] : P(0) \in A\} =$



• $\mu_2' =$ Néron blowup of $\mu_{2, \mathbb{C}[[t]]}$ at $e$ $=$

**Thm 1:** $X_K$ $\overset{\text{geo. red fibres}}{\text{normal}} \Rightarrow \mathrm{Gal}(\mathcal{E})$ finite.

Key points: 1) $\mathcal{E}_K$ ess. finite and $\mathrm{Gal}(\mathcal{E})_K = \mathrm{Gal}(\mathcal{E}_K)$.
quasi-finite

2) Suffices $A = \hat{A}$.

3) Prudence

**Dfn:** $G/A$ prudent if $\forall R^n \circlearrowleft G, \forall L \subset R^n$ line,

"$L/\pi^i \subset (R^n/\pi^i)$ invariant $\Longrightarrow$ $L$ invariant."

$\forall i$ $\bar{A}$

**Thm:** $G/A$ " prudent. $G_k$ finite type. $\Rightarrow$ $\mathcal{O}(G)$ free

**Proof Thm 1:** $\mathcal{L}/\pi^i$ trivial $\forall i \Rightarrow \mathcal{L}$ trivial.

This is Grothendieck's algebraisation.

**Cor** $\Pi^{fil} = \Pi^{tr}$. ("Cassel's Theorem)

**Cor:** $A$ of char $(0,p)$, $p = \pi^e$.

$\mathcal{L} \in \mathrm{Pic}X$ of order $p^r$ & $\mathcal{L}_k \cong \mathcal{O}_{X_k}$.

$\Longrightarrow p^{r-1}(p-1) \leq e$.

(Idea: Blowup $\mu_p$ at $e \subset \mu_p \otimes k$ finite $\Leftrightarrow p-1 \leq e$)

**Thm:** $k = k^r$, $X/A$ normal fibres.

$\mathrm{Rep}^{VB}(\Pi^{tr}) = \{ \mathcal{E} \in VB_X : \text{fibres are } EF \}$.

"$\supset$": Need:

<u>Thm.</u> $\mathcal{E}_k \in VB_{X_k}$ & $\mathrm{Frob}^{*e}\mathcal{E}_k = \mathcal{O}_{X_k}^{r}$.

Then $\mathcal{E}_k$ trivial on $Y_k$ with

$Y \longrightarrow X$ proper & $Y/A$ flat.

Deninger - Werner:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\square"] & X \arrow[d, phantom, "\cap"] \\ \mathbb{P}^n_A \arrow[r, "x \mapsto x^p"'] & \mathbb{P}^n_A \end{tikzcd} \Rightarrow (Y_k)_{red} \longrightarrow X_k = \mathrm{Frob}.$$

<u>Cor</u>: Each quasi-finite $Q \to X$ has reduction to finite. So $\mu_p$ doesn't show if $p-1 > e$