LIFTING D-MODULES FROM POSITIVE TO ZERO CHARACTERISTIC

by

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Résumé. — Nous étudions des relèvements des D-modules (D est l'anneau des opérateurs différentiels de EGA IV) de la caractéristique positive en caractéristique nulle en utilisant des idées de Matzat et la théorie de descente par Frobenius (pour les D-modules arithmétiques) de Berthelot. Nous prêtons une attention particulière à la croissance du groupe de Galois différentiel du relèvement. Nous appliquons aussi la théorie locale des déformations (d'après Schlessinger et Mazur) pour analyser l'espace local de modules des relèvements. À la fin, nous comparons la théorie des déformations (locales) d'un D-module avec la théorie des déformations (locales) d'une représentation d'un schéma en groupes naturellement associé.

Abstract. — We study liftings or deformations of D-modules (D is the ring of differential operators from EGA IV) from positive characteristic to characteristic zero using ideas of Matzat and Berthelot's theory of arithmetic D-modules. We pay special attention to the growth of the differential Galois group of the liftings. We also apply formal deformation theory (following Schlessinger and Mazur) to analyze the space of all liftings of a given D-module in positive characteristic. At the end we compare the problems of deforming a D-module with the problem of deforming a representation of a naturally associated group scheme.

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1. Introduction

The present work focuses on deformations of D-modules (stratified modules) from positive characteristic to zero characteristic. Abandoning generality, this can be grasped by the following situation. Let k be an algebraically closed field of characteristic $p > 0$, W be its ring of Witt vectors, $\mathscr{O}$ the ring of convergent power series $\sum_i a_i x^i$ with coefficients in W, so that $\mathscr{O}/p\mathscr{O} = k[x]$. We consider a "linear differential system" (or a $D_{k[x]/k}$-module structure on $k[x]^{\oplus\mu}$)

$$\partial_q y_i = \sum_{j=1}^{\mu} \overline{a}(i,j,q) y_j \tag{1}$$

where $\overline{a}(i,j,q) \in \mathscr{O}/p\mathscr{O}$, ∂_q is the differential operator of order q analogous to $\frac{1}{q!}\frac{\partial^q}{\partial x^q}$ and the matrices $(\overline{a}(i,j,q))_{i,j}$ are required to satisfy certain compatibilities arising from the relations between the various ∂_q. Then we can ask if there is a lifting of these matrices to $\mathscr{O}$ giving rise to a linear differential system. Furthermore, it is reasonable to require that the differential Galois group (DGG) of the lifted system bears resemblance to the differential Galois group of (1). It is the latter question that the present work sets out to analyze (in greater generality). The analysis runs in two distinct directions corresponding to a natural division of the text into two main parts. The first one, comprising sections 3 to 5 deals with the problem of finding such a lifting (with the property concerning the DGG mentioned before). The second part, which occupies sections 6 to 9, deals with the quantitative nature of these liftings or, more precisely, studies the associated deformation problem as understood and proposed as a theory by Schlessinger [**32**] (and named "a scientific approach" by Kontsevich). Of course, the idea to threat the problem like this comes from Mazur [**25**]. We now briefly summarize the contents of each section.

In section 2 we review some standard material concerning monoidal categories and torsors. The categories shaping this article are monoidal categories of D-modules and the algebraic Geometry (commutative Algebra, rather) in a monoidal category plays an important conceptual rôle: we talk about algebras, groups, torsors, comodules etc. These and some minor folkloric results will be discussed in section 2 in order to be applied further on.

The existence of section 3 is justified by its expository nature – we fix relevant notations concerning the Frobenius morphism – and by the explicit constructions made in 3.2.2. The main result, Theorem 11, is not proved or commented on and the work is left to [**6**]; section 3.2.2 will give an operational view of the theory. The principal cognitive gain the reader should look for in section 3 is the understanding that, like D-modules in positive characteristic (after [**13**, Thm. 1.3]), D-modules in mixed characteristic can be controlled by certain "Frobenius divisions" (see Definition 10). This important observation, in the present context, is due to Matzat and van der Put [**23**], [**24**]; in a more general context it is an application of Berthelot's robust theory of Frobenius descent and arithmetic D-modules [**3**], [**4**] and [**5**].

In section 4 we use the concepts from commutative algebra in a monoidal category (§2.3) to develop a "torsor" version of the ideas in section 3. Since the work of Nori [**28**], it is clear that to a theory of objects in a monoidal category, there should exist a theory of torsors (see also [**11**]). The application of this principle to the case of D-modules (see Definition 14) is important to the handling of the problem concerning liftings with controlled DGGs as we propose first to find *liftings of the torsors* (see Theorem 16) and then, by lifting the representation, obtain the desired D-modules in characteristic zero by twisting (§2.4.3). The existence of liftings of the torsors (see Proposition 13, Corollary 15 and Theorem 16) comes fairly mechanically from the amount of *elbow room* left by Grothendieck once we incorporate the ideas of Section 3.

Section 5 collects the fruits of the previous ones. Its main result is the existence of liftings of D-modules from positive to zero characteristic whose differential Galois group is "close" to the one in positive characteristic, see Theorem 17. Here the term "close" should be interpreted in the following way. A lifting $\mathscr{V}$ of a D-module in positive characteristic $\mathscr{V}_0$ has an integral differential Galois group (see [**12**] and its references) which contains the DGG of $\mathscr{V}_0$ on its closed fibre. Then "close" should mean that the DGG of $\mathscr{V}_0$ *is* the closed fibre. To repeat what was said above, the construction of the liftings of the torsor allows us to pass from the problem of finding a lifting of a D-module to the problem of deforming a representation. The principal result, Theorem 17, is not free of restrictive assumptions on the type of group and representations that we allow; the analysis of the hypothesis (§5.3) proves to be an interesting exercise in group theory. It raises pertinent questions – whose answers may be well known – concerning the nature of groups over discrete valuation rings. (For example: are there affine and flat group schemes over a DVR of mixed characteristic whose special fibre is trivial?)

In section 6 we study the natural formal deformation problem associated to liftings (or deformations) of D-modules; the mould is that of [**32**] and [**21**], while the main inspiration is the seminal work of Mazur [**25**] on deformations of Galois representations. The main product, Theorem 26, shows that the corresponding deformation functor or moduli problem (Definition 19) is homogeneous (Definition 20). Due to a celebrated theorem of Schlessinger [**32**, Thm. 2.11], homogeneity in the presence of finite dimensionality of tangent spaces means representability. But, as the tangent space of the deformation functor proposed at this point has no reason whatsoever to be of finite dimension, we will need to be more selective in the deformations allowed; this is the reason for section 7.

In section 7 we propose to study certain classes of deformations – called periodic (Definition 30) – of a D-module in positive characteristic. Using methods similar to those of section 6, we obtain the homogeneity (Definition 20) of the functor associated to this problem (Theorem 31). The introduction of these classes is motivated by the fact that the corresponding deformation functor has a finite dimensional tangent space in many cases, as will be seen in section 8. The idea behind the notion of periodic deformations is that the "solutions" of the deformed D-module will appear in a *fixed* extension of the base ring. Perhaps the mental image, suggested by the analytic picture, is that of a family of representations whose image lies in a fixed subgroup of some general linear group. (We are inclined to use the word isomonodromy, but this may not be helpful.)

In section 8 we carry the calculations of the tangent spaces to the functors introduced (see Definition 21) in sections 6 and 7. (The calculation of t_{Def} is completely standard and is included in the text for lack of a mild reference.) The principal result is Proposition 33, which hints that for a large class of differential Galois groups, the tangent space is finite dimensional; we also state Corollary 35 in order to encapsulate our findings.

The last section, section 9, is devoted to showing that the formal deformation theory proposed in section 7 is *isomorphic* to the formal deformation theory of the representation attached to the DGG of the D-module in positive characteristic (see Theorem 38). This is reminiscent of the well known fact from differential Geometry: deforming a flat connection is equivalent to deforming a representation of the topological fundamental group [**15**]. Clearly, there is a considerable difference here, which is the infancy of a theory for the fundamental group scheme, specially when it varies over a DVR. As an application, we obtain a result (Corollary 41) which allows us to say when two liftings to characteristic zero are isomorphic as D-modules.

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1.2. Conventions and notations. —

1.2.1. The base ring.— We fix Λ a complete discrete valuation ring. Its field of fractions, which will be denoted by K, is of characteristic zero; the residue field, which will be denoted by k, is algebraically closed and of characteristic $p > 0$; the uniformizer will be denoted by ϖ.

1.2.2. From deformation theory.— By $\mathscr{C}_\Lambda$ we shall denote the following subcategory of the category of Λ-algebras. Objects are pairs (A, f) where A is a local Artinian ring and $f : \Lambda \longrightarrow A$ is a *local* homomorphism which induces an *isomorphism between residue fields.* (This is the classical notation of [**32**].) Arrows are local homomorphisms of Λ-algebras.

1.2.3. The ambient space.— Let $\mathscr{O}$ be a noetherian Λ-algebra which is a domain. We assume that $\mathscr{O}$ is complete for the ϖ-adic topology. Furthermore, we suppose that there exists a morphism of Λ-algebras

$$\psi : \Lambda\langle x_1, \ldots, x_d\rangle \longrightarrow \mathscr{O}$$

which is an etale morphism in the following sense: the truncations $\psi \otimes \Lambda/\varpi^l$ are all etale morphisms. (In particular $\mathscr{O} \otimes_\Lambda k$ is of finite type over Λ and hence topologically of finite type over Λ [EGA $\mathbf{0}_I$, Prp. 7.5.5, p. 71].) The spectrum of $\mathscr{O}$ will be denoted by X (note that X and $X \otimes k$ are connected).

1.2.4. A base point.— We let $\xi : \mathscr{O} \longrightarrow \Lambda$ be a section of the structural morphism, i.e. a Λ-rational point of X.

1.2.5. Differential operators and stratified sheaves.— Under the smoothness assumption on X made above, the ring of differential operators (continuous ones that is, see below) is

$$D = D_\Lambda = \bigoplus_{q\in\mathbb{N}^d} \mathscr{O} \cdot \partial_q$$

where $\partial_q : \mathscr{O} \longrightarrow \mathscr{O}$ is the extension to $\mathscr{O}$ of the operator $x^m \mapsto \binom{m}{q} x^{m-q}$ on $\Lambda\langle x_1, \ldots, x_d\rangle$ [EGA IV-4, 16.11.2] or [**2**, 2.6]. For any given $A \in \mathscr{C}_\Lambda$, the ring of A-linear differential operators on $X \otimes_\Lambda A$ will be $D \otimes_\Lambda A$ [EGA IV-4, 16.4.5 or 16.2.3(ii)] (Notation: D_A). The left ideal of D_A consisting of those operators which annihilate 1 will be denoted by D_A^+.

Let $Y \longrightarrow S$ be a smooth morphism of schemes. The category of S-stratified $\mathscr{O}_Y$-modules [**2**, 2.10] which are also *coherent* as $\mathscr{O}_Y$-modules will be denoted by $\mathbf{str}(Y/S)$. The category of stratified $\mathscr{O}_Y$-modules which are only quasi-coherent will be denoted by $\mathbf{Str}(Y/S)$. Since stratified sheaves are simply D-modules [**2**, 2.11], we will sometimes call stratified sheaves by this more suggestive term. *Moreover*, in this work *we will not essentially use the concept of a stratification but that of a D-module.* The only place where there may be a cognitive loss in forgetting the theory of stratifications is in understanding Berthelot's proof of Theorem 11 (which is the subject of [**6**]). The reader who wishes to substitute the notation "$\mathbf{str}(Y/S)$" by something like "$D_{Y/S}$-mod" is invited to do so.

Those objects of $\mathbf{str}(Y/S)$ which are furthermore flat as $\mathscr{O}_Y$-modules form a full subcategory which will be denoted by $\mathbf{str}^{\#}(Y/S)$. (As is customary in algebraic Geometry, we shall abuse notations when the schemes in question are affine, e.g. if $Y = \operatorname{Spec} B$, then $\mathbf{str}(Y/S) = \mathbf{str}(B/S)$ etc.) If $Y \longrightarrow S$ is in fact an adic and formally smooth morphism of locally noetherian formal schemes, then we work with continuous differential operators and continuous stratifications: this means that one has to substitute in the definition of $\mathscr{P}^n_{Y/S}$ [**2**, 2.1] the usual fibered product by the fibered product in the category of formal schemes.

1.2.6. Representation theory. — (i) Let S be a ring and G an affine flat group scheme over S (flatness assumption comes from what is developed in [**17**, Ch. 2, §2.9]). The category of *representations* of G, denoted $\mathrm{Rep}_S(G)$, consists of $\mathscr{O}(G)$-comodules which are of *finite presentation* when regarded as S-modules; except for the finiteness assumption we follow the definitions of Jantzen [**17**, 2.7–2.9, pp. 30-33]. The full subcategory of $\mathrm{Rep}_S(G)$ whose objects are the representations which, when regarded as S-modules, enjoy the property of being projective will be denoted by $\mathrm{Rep}_S^{\#}(G)$. When the need for $\mathscr{O}(G)$-comodules which are not finitely presented over S arises, we shall use the term G-modules to bring them to mind: the category of G-modules will be denoted by $\mathrm{Rep}'_\Lambda(G)$. (These terminologies are not standard.) Using [**8**, 4.2.2] and [**17**, Part I, 2.13] the reader can conclude that

$$\mathrm{Rep}'_S(G) = \mathrm{Ind}\,\mathrm{Rep}_S(G) \tag{2}$$

if S is noetherian. An object of $\mathrm{Rep}'_S(G)$ will tipically be denoted by (V, ρ), where V is an S-module and ρ has an ambiguous meaning: it will either be the arrow of S-modules defining the $\mathscr{O}(G)$-comodule structure or the homomorphism of group schemes $G \longrightarrow \mathrm{GL}(V)$.

(ii) If (V, ρ) and (W, σ) are $\mathscr{O}(G)$-comodules, then $V \otimes W$ carries a natural $\mathscr{O}(G)$-comodule structure, which will be denoted by $\rho \boxtimes \sigma$.

(iii) If Y is an S-scheme on which G acts on the right, then its ring of functions $\mathscr{O}(Y)$ is naturally a G-module. If we let G act on itself on the right by $x \cdot g = g^{-1}x$, then the corresponding G-module will be called the "left regular representation" and will be denoted by $(\mathscr{O}(G), \rho_l)$ or $\mathscr{O}(G)_{\mathrm{left}}$. Analogously we can define $\mathscr{O}(G)_{\mathrm{right}}$ and observe that the coaction in this case is given by the comultiplication map.

1.3. Monoidal categories. — In this work all monoidal categories [**19**, VII, §1] will be assumed *symmetric* [**19**, XI, §1]. If S is a commutative ring, an S-abelian-monoidal category $(C, \otimes, \mathbb{1})$ is a monoidal category such that: (1) C is S-linear and abelian and (2) $\otimes$ is S-bilinear and right exact on each variable. By an abelian-monoidal category we will understand a $\mathbb{Z}$-abelian-monoidal category.

2. Generalities on torsors and monoidal categories

2.1. Intentions. — We use this section to gather some facts and notations from the theory of torsors (§2.2) and the theory of monoidal categories (§2.3). Many results concerning monoidal categories can be found, at least in essence, in [**20**, Ch. III] or in [**7**]. The results on torsors stem from the theory of sheaves [SGA 4 II], [**14**].

2.2. Torsors. —

2.2.1. Definitions. —

Definition 1. — *Let Y be a scheme and let G be a group scheme over Y.*

(a) A G-torsor over Y is a couple (P, α), where P is a faithfully flat and quasi-compact Y-scheme and

$$\alpha : P \times_Y G \longrightarrow P$$

is a morphism of Y-schemes defining a right action *of G on P such that*

$$(\mathrm{pr}_1, \alpha) : P \times_Y G \longrightarrow P \times_Y P$$

is an isomorphism. A morphisms of torsors is just a morphism of Y-schemes which respects the actions. The phrase "P is an Y-torsor under G" will also be employed to specify that P is a G-torsor over Y.

(b) The category of all Y-torsors under G–which forms a fibered category over (Sch/Y)*–will be denoted by* $\mathrm{T{\small ORS}}(G)_Y$*.*

(c) If Y is a scheme over a base scheme S and G_0 is an S-group scheme, a G_0-torsor over Y will be simply a $G_0 \times_S Y$-torsor. The category of G_0-torsor over Y will be denoted by $\mathrm{T{\small ORS}}(G_0)_Y$*. The collection of all* $\mathrm{T{\small ORS}}(G_0)_Y$ *for Y variable forms a fibered category over* (Sch/S)*.*

Remark: If $E_{/Y}$ denotes the site obtained by endowing (Sch/Y) with the fpqc topology, then what we call $\mathrm{T{\small ORS}}(G)_Y$, Giraud calls the category of *representable* objects in $\mathrm{T{\small ORS}}(E_{/Y}; G)$. By descent, if G is affine, then every torsor (in the sense of Giraud) is representable. (See [**14**, 1.1.5, p 107] for the definition of a pseudo-torsor in a category; [**14**, 1.4.1, p. 117] for the definition of torsor over a topos; [**14**, 1.7.1, p. 126] for the definition of a torsor over a site.) The fibered category $\mathrm{T{\small ORS}}(G) \longrightarrow (\mathrm{Sch}/Y)$ is in fact a stack for the fpqc topology [**14**, III, 1.4.5, p. 119].

2.2.2. Contracted products [**17**, I, 5.14], [**9**, III §4 no. 3], [**14**, III, 1.3, p. 114]. — Let Y be a scheme and assume that (Sch/Y) has been endowed with a Grothendieck topology. Let $(\mathrm{Sch}/Y)^{\sim}$ denote the category of sheaves (of sets) on (Sch/Y) (i.e. a topos). The category $(\mathrm{Sch}/Y)^{\sim}$ admits small direct (inductive) limits [SGA 4 II, 4.1, p. 235] and, furthermore, these are *universal* [SGA 4 II, 4.3, p. 237]. (Below we will recall a consequence of universality.) The inductive limits which interest us most in the present work are certain quotients by group actions, called the contracted products: Let G be a group scheme over Y, Z an Y-scheme endowed with a left action of G, and P a G-torsor over Y. The contracted product of P by Z is the quotient (thus a co-kernel, thus a direct limit) of

$$P \times_Y Z$$

by the "diagonal" action of G – $(p, q) \cdot g = (p \cdot g, g^{-1} \cdot q)$. It will be denoted by $P \times^G Z$ (Giraud writes $P \wedge^G Z$). Note that, a priori, $P \times^G Z$ is simply a *sheaf*.

We now recall what universality of direct limits entails (for complete definitions, see [SGA 4 I, 2.5, p. 13]); this property together with descent theory will give representability of the contracted products in certain cases. Let I be a small category, $\mathscr{F} : I \longrightarrow (\mathrm{Sch}/Y)^{\sim}$ a functor, $f : Z' \longrightarrow Z$ an Y-morphism, and $\nu : \mathscr{F} \Rightarrow Z$ a natural transformation from $\mathscr{F}$ to the constant functor Z. To the data $(\mathscr{F}, \nu, f : Z' \longrightarrow Z)$ we can associate the functor

$$\mathscr{F}' = \mathscr{F} \times_Z Z' : I \longrightarrow (\mathrm{Sch}/Y)^{\sim}$$

together with a canonical arrow

$$\eta : \varinjlim_I \mathscr{F}' \longrightarrow \left(\varinjlim_I \mathscr{F} \right) \times_Z Z';$$

universality guarantees that η is an isomorphism. This implies that quotients commute with base extension, which in turn has the following application. Let G be a group scheme over Y, Z an Y-scheme endowed with a left action of G, and P a G-torsor over Y. Then, for any $U \longrightarrow Y$, we have a cartesian diagram in $(\mathrm{Sch}/Y)^{\sim}$

$$\begin{array}{ccc} P_U \times^{G_U} Z_U & \longrightarrow & P \times^G Z \\ \downarrow & \square & \downarrow \\ U & \longrightarrow & Y. \end{array}$$

By assuming that $\mathrm{Hom}_Y(U, P) \neq \emptyset$, so that $P_U \cong G_U$, we have

$$\begin{array}{ccc} Z_U & \longrightarrow & P \times^G Z \\ \downarrow & \square & \downarrow \\ U & \longrightarrow & Y. \end{array}$$

By descent, $P \times^G Z$ is a scheme if $Z \longrightarrow Y$ is affine and U is a covering of Y. (Of course here we should work with some specified Grothendieck topology like fppf or fpqc.)

Warning: The reader here is advised to use the above theory with caution. Usually, the word “quotient” is overburdened and the formal constructions made in SGA 4 and partially reproduced above are not equivalent with more intuitive notions as that of “invariant functions.”

2.3. Standard constructions in a monoidal category. — Let $(C, \otimes, \mathbb{1})$ be a monoidal category.

2.3.1. Algebras and modules [**19**, VII, §3, §4],[**20**, III]. — An algebra of C is a triple (a,μ,ε) consisting of an object a endowed with a *multiplication* $\mu : a\otimes a \longrightarrow a$ and an *identity* $\varepsilon : \mathbb{1} \longrightarrow a$; these are subjected to some compatibility constrains which include *commutativity* (therefore differing from [**19**]). Sometimes we will only specify the object a and refer to it as an algebra. A module over a is a pair (x,α) consisting of an object x and an arrow $\alpha : a\otimes x \longrightarrow x$ satisfying the expected axioms. There is an obvious notion of morphism of a-modules; these data produce the category of a-modules which will be denoted by a-Mod. (In [**19**, VII], MacLane adopts the notation $_a\mathbf{Lactn}$.)

2.3.2. Hopf algebras and comodules. — A Hopf algebra of C is a quintuple $(h,\mu,\varepsilon,\Delta,\eta)$ where (h,μ,ε) is an algebra, $\Delta : h \longrightarrow h\otimes h$ is a *co-multiplication*, and $\eta : h \longrightarrow \mathbb{1}$ is a *co-unit*. These are required to satisfy the usual compatibilities [**20**, III, pp. 80-81].

If h is a Hopf algebra of C and $m\in C$ is any object, a *co-action* is an arrow $\kappa : m \longrightarrow m\otimes h$ which satisfies $\mathrm{id}_m \cong (\mathrm{id}_m\otimes\varepsilon)\circ\kappa$ and $(\mathrm{id}_m\otimes\Delta)\circ\kappa = (\kappa\otimes\mathrm{id}_h)\circ\kappa$. The pair (m,κ) will be called an *h-comodule.* The category of h-comodules will be denoted by (h-Comod). Provided that h is flat (Definition 3), (h-Comod) is abelian-monoidal.

2.3.3. Relative tensor products [**20**, III, p. 82]. — Assume that C is abelian-monoidal (§1.3). Let a be an algebra of C and let $(x,\alpha),(y,\beta)$ be a-modules. The tensor product of x and y over a is defined as the cokernel of the obvious pair of arrows

$$x\otimes a\otimes y \xrightarrow{\cong} a\otimes x\otimes y \xrightarrow{\alpha\otimes y} x\otimes y \quad\text{and}\quad x\otimes a\otimes y \xrightarrow{x\otimes\beta} x\otimes y.$$

The relative tensor product will be denoted by $x\otimes_a y$; the category $(a\text{-Mod},\otimes_a,a)$ is a abelian-monoidal [**20**, III, 18.3].

The following Lemma will be useful latter on and the proof can be extracted from [**20**, 18.2, p. 82].

Lemma 2. — *Let $\varphi : a \longrightarrow b$ be a morphism of algebras of C and let $\varphi_* :$ b-Mod $\longrightarrow a$-Mod be the obvious functor. Then*

$$\varphi^* : a\text{-}Mod \longrightarrow b\text{-}Mod; \qquad x \mapsto b \underset{a}{\otimes} x$$

is a left adjoint to φ_*. For an $x \in a$-Mod, the unit $\psi_x : x \longrightarrow b \otimes_a x$ is defined by means of $\varepsilon_b : \mathbb{1} \longrightarrow b$. □

Notation: If the object y carries another structure of a-module, say $\gamma : a \otimes y \longrightarrow y$, then we will use the notations $x \otimes_{\alpha,a} y$ and $x \otimes_{\gamma,a} y$ to distinguish these structures.

2.3.4. Torsors. — Assume that C is abelian-monoidal (§1.3).

Definition 3. — [**7**, §7.7] *An object $x \in C$ such that the functor $\tau_x : y \mapsto x \otimes y$ is exact (resp. exact and faithful) will be called flat (resp. faithfully flat).*

If h is a Hopf algebra and b is an algebra which is *faithfully flat*, we say that b is a torsor under h if there exists a co-action $\kappa : b \longrightarrow b \otimes h$ which furthermore satisfies:
(1) κ is an arrow of *algebras*.
(2) The composition

$$\zeta : b \otimes b \xrightarrow{\mathrm{id}_b \otimes \kappa} b \otimes b \otimes h \xrightarrow{\mu_b \otimes \mathrm{id}_h} b \otimes h$$

is an isomorphism.

(Note that faithful flatness of b implies flatness of h.) Finally, we shall employ another useful

Terminology: Let $(C', \otimes', \mathbb{1}')$ be another monoidal category and $T : C' \longrightarrow C$ a monoidal functor. If h' is a Hopf algebra in C' and b an algebra in C, we will say that b is an h'-torsor if it is a Th'-torsor. If, in particular, C' is the category of modules over a commutative ring S – so that a Hopf algebra "is" an S-group scheme – then we will, as usual, confuse Hopf algebras with their spectra and talk about torsors of C under affine group schemes.

Key example: [**28**, §2], [**11**, §2.3.2] Let S be a noetherian ring and let G be an affine flat group scheme over S. Recall that $\mathscr{O}(G)$ affords two G-module structures: the left regular and the right regular (§1.2.6). It is clear that both $\mathscr{O}(G)_{\text{left}}$ and $\mathscr{O}(G)_{\text{right}}$ are algebras in $\text{Rep}'_S(G)$. Let $\mathscr{O}(G)_{\text{triv}}$ denote the G-module obtained by letting G act trivially on $\mathscr{O}(G)$: it is a Hopf algebra in $\text{Rep}'_S(G)$. Let Δ denote the comultiplication on $\mathscr{O}(G)$:

$$\Delta : \mathscr{O}(G)_{\text{left}} \longrightarrow \mathscr{O}(G)_{\text{left}} \underset{S}{\otimes} \mathscr{O}(G)_{\text{triv}}.$$

Then Δ endows $\mathscr{O}(G)_{\text{left}}$ with a structure of a torsor (of $\text{Rep}'_S(G)$) under $\mathscr{O}(G)_{\text{triv}}$.

2.4. Twisting by representations: applications of (abstract) faithfully flat descent. — The process of twisting a torsor by a representation to get a sheaf on the quotient space is central in this work. Giraud [**14**, III, §1.6] threats this process from the point of view of sheaves on a site, which is very convenient if one wants schemes (sheaves) as the output. Since we want sheaves of modules *with additional structures*, we will take the more algebraic point of view (which is closer to [**17**, Part 1, Ch. 5]).

2.4.1. The case of sheaves of modules [**17**, Part I, 5.8–5.15]. — The contents of this paragraph are presented in a more general form further below (§2.4.2, §2.4.3). Let S be a base affine scheme, X an S-scheme, G an affine group scheme flat over S. Let $\psi : P \longrightarrow X$ be a torsor under G (in particular ψ is fpqc) with action morphism denoted by μ. Using the isomorphism $P \times_S G \cong P \times_X P$, a descent data $\varphi : \text{pr}_2^*\mathscr{E} \longrightarrow \text{pr}_1^*\mathscr{E}$ for $\mathscr{E}$ relative to P [TDTE I, 1.6] gives rise to a morphism $\lambda : \mu^*\mathscr{E} \longrightarrow \text{pr}_1^*\mathscr{E}$ together with a certain "cocycle" condition. (See also [**27**, pp. 110–111].) Quasi-coherent sheaves endowed with such an isomorphism are the objects of a category (arrows defined in the obvious way), denoted by $\text{Qcoh}^G(P)$; they are called G-equivariant sheaves or simply G-sheaves. From the main theorem of descent theory [TDTE I, Thm. 1], we have an equivalence of categories

$$\psi^* : \text{Qcoh}(X) \longrightarrow \text{Qcoh}^G(P).$$

The construction of an equivalence inverse to ψ^* proceeds by taking G-invariants as ruled by the construction (see for example [**26**, 2.21, p. 18]).

Given a representation M of G, we get a canonical G-sheaf structure on the coherent P-module $\mathscr{O}_P \otimes_S M$. By "descending" $\mathscr{O}_P \otimes_S M$ to X, we obtain the *sheaf associated to the representation* M or *twisted product* of M by P. In what follows this sheaf will be denoted by $P \times^G M$. (In [**17**, Part 1, Ch. 5] this sheaf is denoted by $\mathscr{L}_{P/G}(M)$.) We remark that for an open and affine $\operatorname{Spec} A \subseteq X$ with pre-image $\psi^{-1}\operatorname{Spec} A = \operatorname{Spec} B$, we have

$$\Gamma(\operatorname{Spec} A, P \times^G M)) = (M \otimes_S B)^G.$$

This construction is extended in §2.4.3.

2.4.2. Descending equivariant objects. — Let $(C, \otimes, \mathbb{1})$ be abelian-monoidal (§1.3). Let $(h, \mu_h, \varepsilon_h, \Delta, \eta)$ be a Hopf algebra of C and b a torsor under it (§2.3.4); as usual the coaction $b \longrightarrow b \otimes h$ is denoted by κ. The canonical homomorphism of algebras $b \longrightarrow b \otimes h$ will be denoted by d_1.

Definition 4. — *An h-equivariant b-module is a pair (x, λ) consisting of a b-module x together with a morphism of $b \otimes h$-modules*

$$\lambda : \kappa^*(x) = (b \otimes h) \underset{\kappa, b}{\otimes} x \longrightarrow d_1^*(x) = x \underset{b, d_1}{\otimes} (b \otimes h) = x \otimes h$$

which is required to satisfy the following two axioms:

(1) By abuse of notation, let $\eta : b \otimes h \longrightarrow b$ denote the arrow induced by the coidentity of h. Identify $\eta^*(\kappa^* x)$ and $\eta^*(d_1^* x)$ with x. Then the arrow of b-modules

$$\eta^*(\lambda) : x \longrightarrow x$$

is the identity.
(2) The following diagram

$$\begin{array}{ccc} s^*\kappa^*(x) = t^*\kappa^*(x) & \xrightarrow{t^*(\lambda)} & t^*d_1^*(x) = r^*\kappa^*(x) \\ & \searrow^{s^*(\lambda)} & \downarrow{\scriptstyle r^*(\lambda)} \\ & & r^*d_1^*(x) = s^*d_1^*(x) \end{array}$$

is commutative, where:

$$b \otimes h \begin{array}{c} \xrightarrow{r = \mathrm{id}_b \otimes \mathrm{id}_h \otimes \varepsilon_h} \\ \xrightarrow{s = \mathrm{id}_b \otimes \Delta} \\ \xrightarrow{t = \kappa \otimes \mathrm{id}_h} \end{array} b \otimes h \otimes h.$$

Remark: It follows from the axioms that the descent morphism λ of the above definition is actually an *isomorphism.* This observation is usually omitted from the definition of descent data; it was Deligne in [**7**, 4.1] who brought this simple fact to our attention. (And this is of course in Grothendieck's *Fondements.*)

Using the isomorphisms

$$\zeta : b \otimes b \cong b \otimes h$$

and

$$(\zeta \otimes \mathrm{id}_h) \circ (\mathrm{id}_b \otimes \zeta) : b \otimes b \otimes b \cong b \otimes h \otimes h,$$

the definition of an h-equivariant module is simply a reproduction of the definition of a b-module with descent relative to b. (We leave it for the reader to precise this. For the standard case of modules and rings, see [**26**, 2.21], but beware that we have inverted the sense of the descent isomorphism λ.) Therefore, either by copying the proof of the fundamental result of descent theory – that of [**26**, 2.21] can be conveniently reproduced – or by applying the Theorem of Barr-Beck [**19**, VI, §7], [**7**, §4.1], we obtain the following result.

Theorem 5. — *Let (x, λ) be an h-equivariant b-module and let $\rho_\lambda : x \longrightarrow x \otimes h$ be the composition*

$$x \longrightarrow \kappa_* \kappa^*(x) \xrightarrow{\kappa_*(\lambda)} \kappa_* d_1^*(x) = x \otimes h.$$

(Note that ρ_λ is the right adjunct of λ, see p. 81 of [**19**] *for vocabulary and* [**19**, IV §1, Thm. 1] *for a justification.) Define the subobject* $\mathrm{Des}(x, \lambda)$ *of x by*

$$\mathrm{Des}(x, \lambda) = \ker \left(x \underset{\mathrm{id}_x \otimes \varepsilon_h}{\overset{\rho_\lambda}{\rightrightarrows}} x \otimes h \right). \tag{3}$$

Then the canonical arrow

$$b \otimes \mathrm{Des}(x, \lambda) \longrightarrow x$$

is an isomorphism of b-modules. Moreover, this defines an equivalence inverse for the functor

$$C \longrightarrow \{h\text{-equivariant } b\text{-modules}\}, \qquad x_0 \mapsto b \otimes x_0.$$

□

Remark: Alternatively, one can add the assumption that C is a monoidal subcategory of the category of quasi-coherent sheaves on a scheme and apply directly the main result of descent theory. Of course, it would be necessary to make proper arrangements: we do not know a priori that b would correspond to a faithfully flat algebra over the base scheme. In all the applications which the above formalism finds in the present work, this will be the case.

Corollary 6. — *Let C' be another abelian-monoidal category, $T : C \longrightarrow C'$ a monoidal, additive and right-exact functor, x, b, h and λ as in the theorem. Suppose that $T(b)$ is faithfully flat.*

(1) Let $\varphi : a_1 \longrightarrow a_2$ be an arrow of algebras in C, $\varphi' : a'_1 \longrightarrow a'_2$ the arrow of algebras in C' obtained from T. Then the obvious functors $T_i : a_i$-mod $\longrightarrow$ a'_i-mod define a map of adjunctions between (φ^*, φ_*) and (φ'^*, φ'_*) [**19**, IV §7, p. 99].

(2) There exists a canonical isomorphism

$$\delta : T\mathrm{Des}(x, \lambda) \xrightarrow{\cong} \mathrm{Des}(Tx, T\lambda).$$

Proof. — (1) We always have $T_1\varphi_* = \varphi'_* T_2$ and the right exactness of T gives $T_2\varphi^* = \varphi'^* T_1$. Using [**19**, IV §7, Prp. 1] and the explicit expression of the unit $\mathrm{Id} \Rightarrow \varphi_*\varphi^*$ (Lemma 2), we can conclude easily.

(2) By (1), we know that the the h-equivariant b-module structure λ on x is taken to a Th-equivariant Tb-module structure on Tx. Since ρ_λ (notations from Theorem 5) is the right adjunct of λ, $T(\rho_\lambda)$ is the right adjunct of $T(\lambda)$, so that $\rho_{T(\lambda)} = T(\rho_\lambda)$. The construction of δ stems from the commutative diagram (ι is the equalizer)

$$\begin{array}{ccccc} T\mathrm{Des}(x,\lambda) & \longrightarrow & Tx & \overset{T(\rho_\lambda)}{\rightrightarrows} & T(x\otimes h) \\ \delta \downarrow & & \downarrow & & \downarrow \\ \mathrm{Des}(Tx,T\lambda) & \xrightarrow[\iota]{} & Tx & \overset{\rho_{T(\lambda)}}{\rightrightarrows} & T(x)\otimes T(h). \end{array}$$

That δ is an isomorphism follows from the fact – which uses the isomorphisms $T(b) \otimes T\mathrm{Des}(x, \lambda) \cong Tx$ and $T(b) \otimes \mathrm{Des}(Tx, T\lambda) \cong Tx$ – that $T(b) \otimes \delta$ is likewise. □

2.4.3. Twisting by representations. — We keep all notations and assumptions of §2.4.2. An important source of h-equivariant b-modules is the category of h-comodules: Let $\rho : v \longrightarrow v \otimes h$ define an h-comodule. The h-comodule structure on $b \otimes v$, call it $\kappa \boxtimes \rho$, can be regarded as an arrow in b-Mod from $b \otimes v$ to $\kappa_*(b \otimes v \otimes h)$. Lemma 2 gives us the left adjunct of $\kappa \boxtimes \rho$:

$$\lambda : \kappa^*(b \otimes v) \longrightarrow b \otimes v \otimes h = d_1^*(b \otimes v) \in \mathrm{Arr}\,(b \otimes h\text{-Mod})\,.$$

The pair $(b \otimes v, \lambda)$ is then an h-equivariant b-module and, due to (3):

$$\mathrm{Des}(b \otimes v, \lambda) := \ker \left(b \otimes v \underset{b\otimes v\otimes \varepsilon_h}{\overset{\kappa\boxtimes\rho}{\rightrightarrows}} b \otimes v \otimes h \right) = (b \otimes v)^h, \tag{4}$$

where the upper script "h" means that we are taking "h-invariants" of the h-comodule $b \otimes v$.

Definition 7. — *Let (v, ρ) be an h-comodule. The twisted product of v by b is the object $(b \otimes v)^h$ of eq.* (4) *and is denoted by* $\mathrm{Tw}_b(v)$ *or* $b \times^h v$.

Corollary 8. — *Keep the above notations.*

(1) The natural arrow $b \otimes b \times^h v \longrightarrow b \otimes v$ is an isomorphism of b-modules.

(2) The association

$$(h\text{-}Comod) \longrightarrow C; \qquad v \mapsto b \times^h v$$

is a monoidal and exact functor.

(3) Let C' be another abelian-monoidal category and let $T : C \longrightarrow C'$ be a monoidal, additive and right-exact functor. Assume that $T(b)$ is faithfully flat. Then there is a canonical isomorphism

$$\delta : T(b \times^h v) \longrightarrow T(b) \times^{T(h)} T(v).$$

□

3. Φ-divided and stratified sheaves

(Applications of Berthelot's theory of Frobenius descent)

The main difficulty of working with stratified sheaves in positive characteristic is that it is quite hard to fabricate such objects *ex nihilo*. This difficulty is surpassed using a theorem of Katz [**13**, Thm. 1.3] (which in turn is an iteration of Cartier's celebrated theorem on the p-curvature [**18**, Thm. 5.1]) which states that a stratified sheaf is simply a family of (coherent) sheaves together with isomorphisms relating the various pull-backs by Frobenius (see Definition 10). This result is extremely useful due to the fact that it allows us to translate the problem of dealing with an infinite number of differential operators into a question of commutative algebra (this method is crowned by the elegant solution of the "connected" inverse problem in [**23**]).

After Berthelot's seminal work on arithmetic D-modules and "Frobenius descent" [**3**], [**4**], [**5**], it becomes clear that the generalization of Katz's theorem should also hold. Nevertheless, the precise statement of the desired generalization, which is Theorem 11, is not directly available from [**3**], [**4**], [**5**], but is the subject of [**6**]. Finally, it should be pointed out that Matzat had already discovered and successfully used such a fundamental principle in [**24**]; in fact, the inspiration to pursue such a direction comes directly from loc.cit. (Unfortunately the setting in [**24**] is not very convenient if one wants to apply all the machinery of algebraic Geometry and commutative Algebra.)

The proof of the main theorem of this section, Theorem 16, will be the subject of [**6**]. We also present, in section 3.2.2, a technique which explicitly associates to a Φ-division (Definition 10) an action of the ring of differential operators. That aside, we merely gather notations and introduce terminology pertinent to the rest of the work.

3.1. The Frobenius morphisms. — We let $\sigma : \Lambda \longrightarrow \Lambda$ be an isomorphism lifting the Frobenius of k. For any given Λ-algebra A and any given integer m, we let $A^{(m)}$ be the Λ-algebra obtained by endowing the ring A with the twisted multiplication by Λ: $\lambda \cdot a = \sigma^{-m}(\lambda)a$. This is simply the Λ-algebra $A \otimes_{\Lambda,\sigma^m} \Lambda$. Analogous notations are in force for schemes over Λ.

Let us fix once and for all a σ-linear ring homomorphism

$$\Phi : \mathscr{O} \longrightarrow \mathscr{O}$$

which satisfies

$$\Phi(f) \equiv f^p \mod \varpi\mathscr{O};$$

this is a *lifting of the Frobenius*

$$F : \mathscr{O}_k \longrightarrow \mathscr{O}_k, \qquad f \mapsto f^p,$$

and it will be called simply a Frobenius morphism. (Here we will reserve the letter F for the Frobenius in positive characteristic.) To establish the existence of Φ, one needs to use the formal smoothness of $\mathscr{O}$ over Λ [EGA IV-1, Ch. 0, 19.0.1] to construct liftings $\Phi_n : \mathscr{O}/\varpi^{n+1}\mathscr{O} \longrightarrow \mathscr{O}/\varpi^{n+1}\mathscr{O}$, step by step, of the "true" Frobenius F. We remark that since the rings $\mathscr{O}^{(m)}$ are all the same (only their Λ-algebra structures vary), Φ, which is σ-linear, induces homomorphisms of Λ-algebras $\mathscr{O}^{(m+1)} \longrightarrow \mathscr{O}^{(m)}$ for any $m \in \mathbb{N}$; we will abuse notation and let Φ denote not only these morphisms but also the corresponding one between the spectra.

The following result is useful and can be proved combining [**4**, 2.3.1], [**22**, Thm. 8.4, p. 58] and the local flatness criterion [**22**, Thm. 2.23, p. 74].

Lemma 9. — *Φ is finite and flat (hence faithfully flat).* □

3.2. F and Φ divided modules. —

3.2.1. Statement of Berthelot's Theorem. —

Definition 10. — *The category of Φ-divided modules (or sheaves) over $X =$ Spec $\mathscr{O}$, denoted $\Phi\mathbf{div}(\mathscr{O}/\Lambda)$, has as*

objects: *families $\{M_i, \alpha_i\}_{i=0,1\dots}$, where M_i a finitely generated $\mathscr{O}^{(i)}$-module and $\alpha_i : \Phi^* M_{i+1} \longrightarrow M_i$ is an isomorphism of $\mathscr{O}^{(i)}$-modules;*

arrows: *families of morphisms of modules $\{\psi_i\}_{i=0,\dots}$ which are compatible with the given isomorphisms.*

The category of F-divided sheaves over $X_k =$ Spec $\mathscr{O}_k$, $F\mathbf{div}(\mathscr{O}_k/k)$, is defined analogously, by replacing Φ by F [**13**, 1.1],[**11**, Def. 4].

We can now state the main result of this section. We warn the reader that the hypothesis are not the weakest possible.

***Theorem 11* ([**6**]).** — *Let*

$$\Delta_k : F\mathbf{div}(\mathscr{O}_k/k) \longrightarrow \mathbf{str}(\mathscr{O}_k/k)$$

be the equivalence defined in [**13**, Thm. 1.3]*. There exists a monoidal,* Λ*-linear and exact equivalence*

$$\Delta_\Lambda : \Phi\mathbf{div}(\mathscr{O}/\Lambda) \longrightarrow \mathbf{str}(\mathscr{O}/\Lambda)$$

which makes the diagram

$$\begin{array}{ccc} \Phi\mathbf{div}(\mathscr{O}/\Lambda) & \xrightarrow{\Delta_\Lambda} & \mathbf{str}(\mathscr{O}/\Lambda) \\ \downarrow & & \downarrow \\ F\mathbf{div}(\mathscr{O}_k/k) & \xrightarrow[\Delta_k]{} & \mathbf{str}(\mathscr{O}_k/k) \end{array} \tag{5}$$

commute (vertical arrows are reduction modulo ϖ*). The* $\mathscr{O}$*-module underlying* $\Delta_\Lambda(\{M_i\})$ *is simply* M_0*.*

3.2.2. Construction of a D-module from a Φ*-division (the functor* Δ*).* — We now present Matzat's technique [**24**]; this construction is *equivalent* to the one presented in [**6**], to which we refer the reader for more details. Let $\{M_i, \alpha_i\}$ be a family of $\mathscr{O}^{(i)}$-modules and isomorphisms

$$\alpha_i : \Phi^* M_{i+1} = \mathscr{O}^{(i)} \otimes_{\mathscr{O}^{(i+1)}} M_{i+1} \longrightarrow M_i$$

of $\mathscr{O}^{(i)}$-modules. We further assume that $M := M_0$ is flat over $\mathscr{O}$, which entails that M_i is flat over $\mathscr{O}^{(i)}$, since Φ is faithfully flat. The isomorphisms of $\mathscr{O}$-modules

$$\mathscr{O} \otimes_{\mathscr{O}^{(i)}} M_i \xrightarrow{\cong} M_0 = M$$

naturally obtained from the $\{\alpha_i\}$ will be denoted by β_i.

Step 1. Let $s \in M$ and $i \in \mathbb{N}$ be an index. Let

$$\sum_j f_j \otimes s_j \qquad \text{and} \qquad \sum_j f'_j \otimes s'_j$$

be elements of $\mathscr{O} \otimes_{\mathscr{O}^{(i)}} M_i$ corresponding to s under β_i (so they are in fact equal).

Lemma 12. — *(i) Let $\partial : \mathscr{O} \longrightarrow \mathscr{O}$ be a Λ-linear derivation. For each $\nu \geq 1$ and each $f \in \mathscr{O}$, $\partial[\Phi^\nu(f)] \in \varpi^\nu \mathscr{O}$.*
(ii) Let $\mathfrak{p}$ denote the prime ideal $\varpi\mathscr{O}$ and let $\mathfrak{p}_i$ be the i-th symbolic power of $\mathfrak{p}$, i.e. $\mathfrak{p}_i = (\mathfrak{p}\mathscr{O}_\mathfrak{p})^i \cap \mathscr{O}$. Then we have

$$\beta_i\left(\sum_j \partial(f_j) \otimes s_j\right) \equiv \beta_i\left(\sum_j \partial(f'_j) \otimes s'_j\right) \mod \mathfrak{p}_i M.$$

Proof. — (i) This follows from a simple induction; the equations

$$\begin{aligned}\partial\left\{\Phi^{\nu+1}(f)\right\} &= \partial\left\{\Phi^\nu(f^p + \varpi g)\right\}\\ &= \partial\left\{[\Phi^\nu(f)]^p + \sigma^\nu(\varpi)\Phi^\nu(g)\right\}\\ &= p \cdot \partial[\Phi^\nu(f)] \cdot [\Phi^\nu(f)]^{p-1} + \sigma^\nu(\varpi) \cdot \partial[\Phi^\nu(g)]\end{aligned}$$

form the heart of a proof.

(ii) Let $\mathfrak{p}'$ be the prime ideal of $\mathscr{O}^{(i)}$ generated by ϖ. Localizing we have an isomorphism

$$\beta_i : \mathscr{O}_\mathfrak{p} \otimes_{\mathscr{O}^{(i)}_{\mathfrak{p}'}} (M_i)_{\mathfrak{p}'} \xrightarrow{\cong} M_\mathfrak{p}.$$

Let $\{v_1, \ldots, v_r\} \subset (M_i)_{\mathfrak{p}'}$ be a basis of this free $\mathscr{O}^{(i)}_{\mathfrak{p}'}$-module and let $a_{\nu j} \in \mathscr{O}^{(i)}_{\mathfrak{p}'}$ (resp. $a'_{\nu j}$) be the coordinates of s_j (resp. s'_j) with respect to this basis. Then

$$\sum_j f_j \cdot \Phi^i(a_{\nu j}) = \sum_j f'_j \cdot \Phi^i(a'_{\nu j}).$$

Applying ∂ to both sides and using (i) we have:

$$\sum_j \partial(f_j) \cdot \Phi^i(a_{\nu j}) \equiv \sum_j \partial(f'_j) \cdot \Phi^i(a'_{\nu j}) \mod \mathfrak{p}^i \mathscr{O}_\mathfrak{p}.$$

This gives

$$\sum_j \partial(f_j) \otimes s_j \equiv \sum_j \partial(f'_j) \otimes s'_j \mod (\mathfrak{p}\mathscr{O}_\mathfrak{p})^i \otimes_{\mathscr{O}^{(i)}_{\mathfrak{p}'}} (M_i)_{\mathfrak{p}'}.$$

Which implies that the difference

$$\beta_i\left(\sum_j \partial(f_j) \otimes s_j\right) - \beta_i\left(\sum_j \partial(f'_j) \otimes s'_j\right)$$

belongs to $M \cap \left(\mathfrak{p}^i M_\mathfrak{p}\right)$. The proof of the lemma is then reduced to the proof of the following:

Claim: $M \cap \left(\mathfrak{p}^i M_{\mathfrak{p}}\right) = \mathfrak{p}_i M$.

Proof: We clearly have that $M \cap \left(\mathfrak{p}^i M_{\mathfrak{p}}\right) \supseteq \mathfrak{p}_i M$. Now let F be a free $\mathscr{O}$-module, on the basis $\{e_1, \ldots, e_t\}$ say, which contains M and whose quotient F/M is also flat over $\mathscr{O}$. (This is possible since M is projective [**34**, Thm. 3.2.7, p. 71] and since any projective module is flat.) Let

$$m = \sum_\nu a_\nu \cdot e_\nu \in M \cap \left(\mathfrak{p}^i M_{\mathfrak{p}}\right).$$

By definition, there exists $s \in A \setminus \mathfrak{p}$ such that

$$s \cdot m = \sum_\nu s \cdot a_\nu \cdot e_\nu$$

belongs to $\mathfrak{p}^i M \subseteq \mathfrak{p}^i F$. This entails that $sa_\nu \in \mathfrak{p}^i$ for each ν, which implies that $a_\nu \in \mathfrak{p}_i$ and consequently that $m \in \mathfrak{p}_i F$. As the quotient F/M is flat, it follows that $m \in \mathfrak{p}_i M$. The claim is proved, and so is Lemma 12. □

Step 2. Let

$$\overline{s}(\partial, i) \in M/\mathfrak{p}_i M$$

be the class of

$$\beta_i \left(\sum_j \partial(f_j) \otimes s_j \right)$$

modulo $\mathfrak{p}_i M$, where $\sum_j f_j \otimes s_j \in \mathscr{O} \otimes_{\mathscr{O}(i)} M_i$ is taken, via β_i, to $s \in M$. Due to Lemma 12:

$$\overline{s}(\partial, i) \equiv \overline{s}(\partial, i+1) \mod \mathfrak{p}_i M.$$

This defines an element

$$\overline{s}(\partial) \in \varprojlim_i M/\mathfrak{p}_i M.$$

By (one of) Zariski's theorem on holomorphic functions, the $\mathfrak{p}$-adic topology is equivalent to the topology defined by the filtration $\mathfrak{p}_\bullet$ [**35**, Lemma 3, p. 104]. Since the $\mathscr{O}$-module M is complete for the $\mathfrak{p}$-adic topology [**22**, 8.7, p. 60], it is complete for the $\{\mathfrak{p}_\bullet\}$-adic topology. We have thus obtained a unique element $\nabla(\partial) \cdot s$ such that

$$\nabla(\partial) \cdot s \equiv \beta_i \left(\sum_j \partial(f_j) \otimes s_j \right) \mod \mathfrak{p}_i M \tag{6}$$

for each $i \in \mathbb{N}$ and for each $\sum_j f_j \otimes s_j \in \mathscr{O} \otimes_{\mathscr{O}^{(i)}} M_i$ which corresponds to s under β_i.

Step 3. From the constructions it is obvious that $\nabla : \mathrm{Der}_\Lambda(\mathscr{O}) \longrightarrow \mathrm{End}_\Lambda(M)$ is an integrable connection.

Step 4. Finally we show that ∇ prolongs to a stratification, that is, to a homomorphisms of $\mathscr{O}$-algebras

$$\tilde{\nabla} : D_\Lambda \longrightarrow \mathrm{End}_\Lambda(M).$$

(Recall that D_Λ is an $\mathscr{O}$-algebra on the left.) As the algebra $D_\Lambda[1/p]$ is generated over $\mathscr{O}[1/p]$ by the derivations, it is clear that, on the generic fibre, we have an extension of ∇ to a homomorphism (of $\mathscr{O}[1/p]$-algebras)

$$\nabla[1/p] : D_\Lambda[1/p] \longrightarrow \mathrm{End}_{\Lambda[1/p]}(M[1/p]).$$

We need to show that

$$\nabla[1/p]\,(D_\Lambda) \cdot M = \sum_{q \in \mathbb{N}^d} \frac{1}{q!} \nabla\left(\frac{\partial^q}{\partial x_q}\right) \cdot M \subseteq M.$$

This is verified once since M admits a set of "small" generators (compare with [**23**, §8]). That is, if for any given $\kappa \in \mathbb{N}$, we can find generators $\{s_\nu\}$ of the $\mathscr{O}$-module M such that

$$\nabla\left(\frac{\partial}{\partial x_j}\right) \cdot s_\nu \in \varpi^\kappa M \qquad (\forall \nu, j).$$

But the image of M_i via β_i, for i sufficiently large, always contains such a "small" set of generators, since by Zariski's theorem [**35**, Lemma 3, p. 104] $\mathfrak{p}_i \subseteq \mathfrak{p}^\kappa = \varpi^\kappa \mathscr{O}$ for i sufficiently large (notations of Lemma 12).

Summary of the construction. Let $\{M_i, \alpha_i\}$ be Φ-divided; assume furthermore that $M := M_0$ is flat. Let $\partial : \mathscr{O} \longrightarrow \mathscr{O}$ be a Λ-linear derivation, s an element of M and, for each $i \in \mathbb{N}$,

$$\{s_{i,j} : j \in S_j\}$$

a set of generators of the $\mathscr{O}^{(i)}$-module M_i. Imagine that M_i is contained in M. If $f_{i,j} \in \mathscr{O}$ are such that

$$\sum_{j \in S_i} f_{i,j} \cdot s_{i,j} = s$$

let ∂ act on s by the rule

$$\nabla(\partial)\cdot s = \lim_{i\to\infty}\sum_{j\in S_i}\partial(f_{i,j})\cdot s_{i,j}.$$

Let $q=(q_1,\dots,q_d)\in\mathbb{N}^d$ be a multi-index. For any $t\in M_i$ and any derivation ∂, we know that

$$\nabla(\partial)\cdot t\in\mathfrak{p}_iM\subseteq\varpi^{\tau(i)}\cdot M,$$

for some $\tau(i)\in\mathbb{N}$. Choosing i wisely, we can assume that $\tau(i)\geq\nu+\mathrm{ord}(q!)$, for any fixed ν. This shows that there exists a unique $t(q)\in\varpi^\nu M$ such that

$$\nabla\left(\frac{\partial^q}{\partial x^q}\right)\cdot t = q!\cdot t(q).$$

Thus

$$\nabla(\partial_q)\cdot t = t(q).$$

In particular $\nabla(\partial_q)\cdot t\in\varpi^\nu M$. Since any element of M can be written as a linear combination of elements of some M_i, we have an action of $D_{\mathscr{O}/\Lambda}$ on M. Explicitly and *informally*:

$$\begin{aligned}\nabla(\partial_\kappa)\cdot s &= \sum_{j\in S_i}\sum_{\lambda+q=\kappa}\partial_\lambda(f_{i,j})\cdot\nabla(\partial_q)\cdot s_{i,j}\\ &=\sum_{j\in S_i}\sum_{\lambda+q=\kappa}\partial_\lambda(f_{i,j})\cdot s_{i,j}(q).\end{aligned}\tag{7}$$

With these explicit definitions, it is not hard to see that the D_k-module structure induced on $M\otimes k$ is the one coming from the F-division $\{M_i\otimes k,\alpha_i\otimes k\}$ through Katz theorem [**13**, Thm. 1.3].

4. Φ-divided torsors

In [**24**] Matzat uses the key principle discussed in section 3 to construct liftings of stratified modules from positive characteristic to characteristic zero (this can be seen already in [**23**]). But, since the differential Galois group is understood by looking at the corresponding torsor (in the appropriate monoidal category), it is interesting to regard liftings of the latter if one wishes to curb the growth of the former. This conducts to the study of Φ-divided torsors (see [**11**] for the introduction of the analogous objects in positive characteristic). One of the ground ideas in this work is to use the fact that the torsors do always lift to characteristic zero in order to translate the problem of lifting of stratified modules into the problem of lifting of representations.

4.1. Lifting of torsors. — In this section we will concentrate on some standard facts about torsors and the lifting of those. All that follows is thus pure algebraic Geometry with no mention to differential structures. Let us now write $S = \operatorname{Spec}\Lambda$. For any given S-scheme Z, the reduction modulo ϖ^{n+1} of Z will be denoted by Z_n. The following proposition is central to the remaining of this work, its proof is a collection of standard facts from the theory of deformations (as in [SGA 1]) and from the theory of torsors.

Proposition 13. — *Let G/Λ be an affine and smooth group scheme, $P_0 \longrightarrow X_0$ a G_0-torsor.*

1. *There exists a G-torsor $P \longrightarrow X$ which induces $P_0 \longrightarrow X_0$ upon reduction modulo ϖ.*
2. *Given any other G-torsor $Q \longrightarrow X$ and an isomorphisms of G-torsors $\sigma_0 : P_0 \longrightarrow Q_0$, then there exists an isomorphism of G-torsors $\sigma : P \longrightarrow Q$ inducing σ_0.*

Proof. — We note that to find a formal lifting of P_0, i.e. a compatible family of liftings $P_n \longrightarrow X_n$, we could simply quote [**14**, VII, 1.3, p. 374]. But as we are *not only* looking for a formal lift, we will need to do some more work.

(1) We will first suppose that $G = \mathrm{GL}(r)$. It is well known (see for example [**9**, III, §4, Cor. 2.4, p. 367]) that there is an equivalence of fibered categories over (Sch/S)

$$\textsc{Tors}(G) \xrightarrow{\cong} \textsc{Fib}(r), \tag{8}$$

where $\textsc{Fib}(r)$ is the fibered category whose fibre over $X \in (\mathrm{Sch}/S)$ is the groupoid of all coherent $\mathscr{O}_X$-modules which are locally free of rank r (morphisms are isomorphisms). Therefore, in order to find $P \longrightarrow X$ reducing to $P_0 \longrightarrow X_0$, we need to show that it is possible to lift the locally free sheaf of rank r associated to P_0, call it $\mathscr{E}_0$, to a locally free sheaf of rank r over X. Using [SGA 1 III, Prp. 7.1], it is possible to find a compatible family of liftings $\mathscr{E}_n \in \textsc{Fib}(r)_{X_n}$. Using the completeness of $\mathscr{O}$, the existence of a lifting is now a consequence of Thm. 8.4 and Thm. 22.3 of [**22**].

Let us now pass to the general case. Consider $\rho : G \longrightarrow \mathrm{GL}(r)$ a faithful representation of G; this means that ρ is a closed embedding. Let $E_0 \longrightarrow X_0$ be the induced $\mathrm{GL}(r)_0$-torsor (§2.2):

$$E_0 = P_0 \times^{G_0} \mathrm{GL}(r)_0.$$

(That this is a scheme follows from fpqc descent, see §2.2.2.) From the first part of the proof it follows that there exists a lifting of the $\mathrm{GL}(r)_0$-torsor E_0 to a $\mathrm{GL}(r)$-torsor $E \longrightarrow X$.

Now let Q be the scheme representing the fppf sheaf $\mathrm{GL}(r)/G$ [**1**, Thm. 4.C, p. 53]. We contend that Q is smooth and separated over S. From [**9**, III §3, 2.5, p. 328] the projection $\mathrm{GL}(r) \longrightarrow Q$ is faithfully flat and of finite presentation, in particular Q is of finite presentation over S. So we are allowed to use [**9**, III §3, 2.7, p. 329] to conclude that Q is smooth over S. Now we observe that the following diagram is cartesian (all arrows are the obvious ones)

$$\begin{array}{ccc} \mathrm{GL}(r) \times_Q \mathrm{GL}(r) & \longrightarrow & \mathrm{GL}(r) \times_S \mathrm{GL}(r) \\ \downarrow & & \downarrow \\ Q & \xrightarrow[\text{diag.}]{} & Q \times_S Q. \end{array}$$

As $\mathrm{GL}(r) \times_Q \mathrm{GL}(r) \cong \mathrm{GL}(r) \times_S G$, we obtain another cartesian diagram

$$\begin{array}{ccc} \mathrm{GL}(r) \times_S G & \xrightarrow{(g,h)\mapsto(g,gh)} & \mathrm{GL}(r) \times_S \mathrm{GL}(r) \\ \downarrow & & \downarrow \\ Q & \xrightarrow[\text{diag.}]{} & Q \times_S Q. \end{array}$$

Since the upper horizontal arrow is a closed embedding and the right vertical arrow is fppf, [EGA IV-2, 2.7.1] shows that $Q \longrightarrow Q \times_S Q$ is a closed embedding: Q is separated.

Claim: The sheaf $E \times^{\mathrm{GL}(r)} Q = E/G$ is represented by a smooth and separated X-scheme. In particular, the sheaves E_n/G_n are represented by smooth and separated X_n-schemes $(E/G)_n$.

Proof: There exists a covering of X by affine open subsets $U_1, \dots, U_m$ such that $\mathrm{Hom}_X(U_i, E) \neq \emptyset$. Hence, by §2.2.2, we obtain a cartesian diagram

$$\begin{array}{ccc} U \times_S Q & \longrightarrow & E/G \\ \big\downarrow & & \big\downarrow \\ \sqcup U_i = U & \longrightarrow & X. \end{array}$$

Using Zariski descent [SGA 1 VII, Cor. 7.3] (or simply common sense) it follows that E/G is a scheme. Smoothness and separateness follow from [EGA IV-2, 2.7.1] and [EGA IV-4, 17.7.4]. As the second assertion is a direct consequence of the explications made in §2.2.2, we have proved the claim.

Now let $\mathscr{S} \longrightarrow (\mathrm{Sch}/S)$ be the fibered category whose category of sections over $U \in (\mathrm{Sch}/S)$ is defined as follows:

Objects are pairs (R, σ), with $R \longrightarrow U$ a $\mathrm{GL}(r)$-torsor and $\sigma \in \mathrm{Hom}_U(U, R/G)$ a section.

Arrows from (R, σ) to (R', σ') are morphisms of torsors which are compatible with the sections.

Another standard result from the theory of torsors [**14**, III, §3, 3.2.1, p. 159] is that the functor

$$\textsc{Tors}(G)_U \longrightarrow \mathscr{S}_U,$$

which maps a G-torsor P to the pair $(P \times^G \mathrm{GL}(r), \sigma_{\mathrm{can}})$, where

$$\sigma_{\mathrm{can}} : U = P/G \longrightarrow [P \times^G \mathrm{GL}(r)]/G$$

is the canonical section, defines an equivalence of fibered categories. Therefore, exhibiting a lifting P of $P_0 \longrightarrow X_0$ amounts to finding a section $\sigma : X \longrightarrow E/G$ which induces the canonical section $\sigma_0 : X_0 \longrightarrow E_0/G_0 = (E/G)_0$. Now we first find a sequence of liftings

$$(\sigma_n) \in \varprojlim_n \mathrm{Hom}_{X_n}(X_n, (E/G)_n),$$

then, using Grothendieck's algebraization, find a $\sigma \in \mathrm{Hom}_X(X, E/G)$ lifting (σ_n). The existence of (σ_n) is guaranteed by [SGA 1 III, Cor. 5.5] and that of $\sigma \in \mathrm{Hom}_X(X, E/G)$ by [EGA III-1, 5.4.2, p. 157].

(2) Let Q be another G-torsor over X. Then the functor

$$(\mathrm{Sch}/X) \longrightarrow \mathrm{Set};\ f : Y \longrightarrow X \mapsto \mathrm{Hom}_{\mathrm{T{\scriptsize ORS}}(G)_Y}(f^*P, f^*Q)$$

is represented by the smooth and affine X-scheme $P \times^G \check{Q}$, where $\check{Q}$ is the scheme Q with the obvious action of G on the *left* by $g \cdot q = q \cdot g^{-1}$; see [**14**, III, §1, Thm. 1.6.1, p. 123]. Thus the isomorphism $\sigma_0 : P_0 \longrightarrow Q_0$ can be seen as an element of

$$\mathrm{Hom}_X(X_0, P \times^G \check{Q});$$

by the smoothness of $P \times^G \check{Q} \longrightarrow X$, σ_0 can be lifted to a sequence

$$(\sigma_n) \in \varprojlim \mathrm{Hom}_{X_n}(X_n, P \times^G \check{Q}).$$

It follows easily that σ_0 can be lifted to a

$$\sigma \in \mathrm{Hom}_X(X, P \times^G \check{Q}) = \mathrm{Isom}_G(P, Q).$$

□

4.2. Φ-divided torsors. — We will use the notations introduced in §3.1. Throughout, G will stand for an affine and flat Λ-group scheme.

Definition 14. — *Let A be a Λ-algebra. The category of Φ_A-divided G-torsors over $X \otimes_\Lambda A$, denoted by $\Phi\mathrm{T{\scriptsize ORS}}(X, G)_A$, has as objects $\mathbb{N}$-indexed families of $G \otimes_\Lambda A$-torsors $\{P_i\}$ over $X^{(i)} \otimes_\Lambda A$ together with isomorphisms*

$$\alpha_i : (\Phi \otimes A)^* P_{i+1} \longrightarrow P_i$$

of $G \otimes A$-torsors. A morphisms between Φ_A-divided torsors is simply a family of morphisms of torsors which respects these isomorphisms. The category $\Phi\mathrm{T{\scriptsize ORS}}(X, G)_A$ is the category of sections of a co-fibered category $\Phi\mathrm{T{\scriptsize ORS}}(X, G) \longrightarrow (\Lambda$-alg$)$.

Reconciliation: Using the terminology of §2.3.4, every torsor of the category $\mathrm{Ind}\,\Phi\mathbf{div}(\mathscr{O}/\Lambda)$ under the Hopf algebra $\mathscr{O}(G) \otimes \mathbb{1}$ (Definition 10) gives rise to a Φ-divided G-torsor. In fact, these two categories coincide; to show that one needs to use the fact that any G-module is a direct limit of finitely generated G-modules (representations) [**17**, Part 1, 2.13]. This will be explained further in the proof of Theorem 16.

Warning: The notation here can lead to confusions: while $\textsc{Tors}(G)_X$ denoted the category of G-torsors over X, $\Phi\textsc{Tors}(G)_A$ denote the category of Φ-torsors under G over $X \otimes_\Lambda A$, and not over $\operatorname{Spec} A$.

Note that Proposition 13 immediately gives:

Corollary 15. — *Assume that G is smooth over Λ. Then the restriction functor $\Phi\textsc{Tors}(X, G)_\Lambda \longrightarrow \Phi\textsc{Tors}(X, G)_k$ is essentially surjective.* □

The following important result is a consequence of Theorem 11 and of Corollary 15.

Theorem 16. — *Let G/Λ be a smooth affine group scheme. Let $\mathscr{B}_k$ be a torsor under G_k in $\operatorname{Ind}\mathbf{str}(\mathscr{O}_k/k)$ (§2.3.4). Then there exists a torsor $\mathscr{B}_\Lambda$ of $\operatorname{Ind}\mathbf{str}(\mathscr{O}/\Lambda)$ under G and an isomorphism of torsors of $\operatorname{Ind}\mathbf{str}(\mathscr{O}_k/k)$*

$$\mathscr{B}_\Lambda \otimes_\Lambda k \cong \mathscr{B}_k.$$

Proof. — The *only purpose* of this proof is to verify some *technicalities concerning inductive limits.* Since $\mathbf{str}(\mathscr{O}_k/k)$ and $F\mathbf{div}(\mathscr{O}_k/k)$ are equivalent monoidal categories, $\mathscr{B}_k$ corresponds to a torsor of $\operatorname{Ind}\Phi\mathbf{div}(X/\Lambda)_k$, $\mathscr{B}_{k,\bullet} = \{\mathscr{B}_{k,i}\}_i$ say, under G_k. Using Theorem 11 we only need to find a G-torsor of $\operatorname{Ind}\Phi\mathbf{div}(\mathscr{O}/\Lambda)$ above $\mathscr{B}_{k,\bullet}$. Let

$$P_{k,\bullet} = \{P_{k,i}\}_{i\in\mathbb{N}} \in \Phi\textsc{Tors}(X, G)_k$$

be the F-divided torsor associated to $\mathscr{B}_{k,\bullet}$ by taking the spectra of the quasi-coherent algebras. Let us lift $P_{k,\bullet}$ to an object

$$P_\bullet = \{P_i\}_i \in \Phi\textsc{Tors}(X, G)_\Lambda$$

using Corollary 15. Let $L_i : \operatorname{Rep}_\Lambda(G) \longrightarrow \operatorname{coh}(X^{(i)})$ be the associated sheaf: $L_i(V) = P_i \times^G V$; these functors produce a monoidal and exact functor

$$L = \{L_i\} : \operatorname{Rep}_\Lambda(G) \longrightarrow \Phi\mathbf{div}(\mathscr{O}/\Lambda),$$

which can be extended to a monoidal functor of Ind categories, it will be denoted likewise. Since the left regular G-module $R_{\text{left}} = (\mathscr{O}(G), \rho_l)$ is a torsor of $\operatorname{Ind}\operatorname{Rep}_\Lambda(G)$ under the Hopf algebra $\mathscr{O}(G)_{\text{triv}}$ (§2.3.4) it follows that L takes R_{left} to an algebra of $\operatorname{Ind}\Phi\mathbf{div}(\mathscr{O}/\Lambda)$,

$$\mathscr{B}_\bullet = \{\mathscr{B}_i\} = \{P_i \times^G R_{\text{left}}\}_i = \{[\mathscr{O}_{P_i} \otimes R_{\text{left}}]^G\}_i,$$

which automatically fulfills all axioms defining a torsor under G except faithful flatness. This missing property can be demonstrated with the help of two observations: (1) $\mathscr{B}_0$ is faithfully flat over $\mathscr{O}$ since $\mathscr{B}_0 \otimes_{\mathscr{O}} \mathscr{O}_{P_0} \cong R \otimes_{\Lambda} \mathscr{O}_{P_0}$, and (2) the canonical functor $\mathrm{Ind}\,\Phi\mathbf{div}(\mathscr{O}/\Lambda) \longrightarrow \mathrm{Qcoh}(X)$ is exact (c.f. [SGA4 I], 8.9.9(c) and 8.9.7), faithful [SGA4 I 8.6.4] and monoidal.

Applying Corollary 8 to the category of quasi-coherent sheaves we see that the reduction modulo ϖ of $\mathscr{B}_\bullet$,

$$\{(P_i \times^G R_{\mathrm{left}}) \otimes k\}_i,$$

is isomorphic to the object

$$\{P_{k,i} \times^{G_k} (R_{\mathrm{left}} \otimes k)\}_i$$

of $\mathrm{Ind}\,\Phi\mathbf{div}(\mathscr{O}/\Lambda)_k$. Now the co-identity produces an arrow of G_k-torsors of $\mathrm{Ind}\,\Phi\mathbf{div}(\mathscr{O}/\Lambda)_k$

$$\mathrm{id} \otimes \varepsilon : \{[\mathscr{B}_{k,i} \otimes_k (R_{\mathrm{left}} \otimes k)]^{G_k}\}_i \longrightarrow \{\mathscr{B}_{k,i}\}_i. \tag{9}$$

The homomorphism of quasi-coherent modules

$$[\mathscr{B}_{k,0} \otimes_k (R_{\mathrm{left}} \otimes k)]^{G_k} \longrightarrow \mathscr{B}_{k,0} \tag{10}$$

corresponds to the natural morphism of torsors

$$P_{k,0} \longrightarrow P_{k,0} \times^{G_k} G_{k,\mathrm{left}};$$

as a morphism of G_k-torsors is always an isomorphism, we conclude that (10) is an isomorphism. Due to the fact that the obvious functor $\mathrm{Ind}\,\Phi\mathbf{div}(\mathscr{O}/\Lambda)_k \longrightarrow \mathrm{Qcoh}(X)$ is exact and faithful (references as above), it follows that the arrow in (9) is an isomorphism. Therefore $\mathscr{B}_\bullet$ is the required lifting of $\mathscr{B}_{k,\bullet}$. □

5. Liftings to characteristic zero which preserve the differential Galois group

5.1. Introduction. — Let $\mathscr{V}_k$ be an $\mathscr{O}_k$-module of finite type which possesses a stratification

$$\nabla_k : D_k \longrightarrow \mathrm{End}_k(\mathscr{V}_k)$$

and let $\mathscr{V}_\Lambda$ be a flat $\mathscr{O}$-module lifting $\mathscr{V}_k$ (this is gratuitous [SGA 1 III, Prop. 7.1], [**22**, 22.3, p. 174]). We will use the ideas developed in §4, specially Theorem 16, to guarantee the existence of deformations (or liftings)

$$\nabla_\Lambda : D_\Lambda \longrightarrow \mathrm{End}_\Lambda(\mathscr{V}_\Lambda); \qquad \nabla_\Lambda \otimes k = \nabla_k,$$

whose integral differential Galois group (see [**12**] and the references there) is "close" to the differential Galois group of $(\mathscr{V}_k, \nabla_k)$. The precise definition of "close" in the previous sentence is: the closed embedding ([**24**], [**12**]) of the differential Galois group of $(\mathscr{V}_k, \nabla_k)$ into the special fibre of the integral differential Galois group of $(\mathscr{V}_\Lambda, \nabla_\Lambda)$ is an isomorphism, see Theorem 17.

5.2. The principal result. — Let us keep the notations of the introduction. We will make some natural constructions concerning the differential Galois group of $(\mathscr{V}_k, \nabla_k)$ and its natural inclusion into a general linear group. These constructions, which involve liftings of objects in positive to characteristic zero, are not gratuitous and the cost will be analyzed in §5.3.

Assume (see Hypothesis 1 in §5.3) the existence of a smooth and affine group scheme G/Λ such that

$$G \otimes k = \text{differential Galois group of } (\mathscr{V}_k, \nabla_k) \text{ at } \xi_k.$$

The functor

$$(\xi_k)^* : (\mathscr{O}_k\text{-mod}) \longrightarrow (k\text{-mod})$$

factors through an equivalence of monoidal categories

$$\overline{\xi}_k^* : \langle (\mathscr{V}_k, \nabla_k) \rangle_\otimes \longrightarrow \mathrm{Rep}_k(G \otimes k).$$

Here $\langle \mathscr{V}_k \rangle_\otimes = \langle (\mathscr{V}_k, \nabla_k) \rangle_\otimes$ is the full subcategory of $\mathbf{str}(X_k/k)$ whose objects are subquotients of the generalized tensors

$$\bigoplus_{i=1}^{r} \left[(\mathscr{V}_k, \nabla_k)^{\otimes a_i} \otimes \mathrm{Dual}(\mathscr{V}_k, \nabla_k)^{\otimes b_i} \right] \qquad (a_i, b_i \in \mathbb{N}).$$

The canonical representation $\overline{\xi}_k^*(\mathscr{V}_k)$ will be denoted by

$$\rho_k : G_k \longrightarrow \mathrm{GL}(V_k)$$

in the sequel.

Let $\mathscr{O}(G_k)_{\rm triv}$ denote the object of $\operatorname{Ind}\operatorname{Rep}_k(G_k)$ obtained by letting G_k act trivially on its ring of functions $\mathscr{O}(G_k)$. Then $\mathscr{O}(G_k)_{\rm left}$ (see §1.2 for notation) is a torsor under $\mathscr{O}(G_k)_{\rm triv}$ (§2.3.4). Transporting this construction to $\operatorname{Ind}\mathbf{str}(\mathscr{O}_k/k)$ by using Nori's method (see [**28**, §2, Prop. 2.5] or [**11**, 2.3.2]), we obtain a torsor

$$\mathscr{B}_k$$

of $\operatorname{Ind}\mathbf{str}(\mathscr{O}_k/k)$ under the Hopf algebra of $\operatorname{Ind}\mathbf{str}(\mathscr{O}_k/k)$ corresponding to $\mathscr{O}(G_k)_{\rm triv}$: $\mathscr{O}(G_k)\otimes_k\mathscr{O}_k$ (§2.3.4). This torsor has the property that

$$\mathscr{B}_k\times^{G_k}V_k\cong\mathscr{V}_k \tag{11}$$

in $\mathbf{str}(\mathscr{O}_k/k)$, as $\mathscr{B}_k\times^{G_k}(\bullet)$ induces an equivalence inverse to $\overline{\xi}_k^*$. (Here we are abusing the notation introduced in Definition 7 and replacing the Hopf algebra by "G_k".) Using Theorem 16 we obtain a torsor of $\operatorname{Ind}\mathbf{str}(\mathscr{O}/\Lambda)$ under G – or better, under the Hopf algebra $\mathscr{O}(G)\otimes_\Lambda\mathscr{O}$ constructed by letting D_Λ act "horizontally" on $\mathscr{O}(G)$ – call it

$$\mathscr{B},$$

lifting $\mathscr{B}_k$.

Let V be a flat Λ-module lifting $V_k=\xi_k^*(\mathscr{V}_k)$ and *assume* that there exists a *faithful representation*

$$\rho:G\longrightarrow\mathrm{GL}(V)$$

lifting $\rho_k:G_k\longrightarrow\mathrm{GL}(V_k)$. (See §5.3 for some comments on this hypothesis.) Hence

$$(\mathscr{V},\nabla):=\mathscr{B}\times^G V$$

(we are again abusing the notation in Definition 7 and replacing "$\mathscr{O}(G)\otimes\mathscr{O}$" by "$G$") is a stratified module over $\mathscr{O}$ which, using Corollary 8, induces $\mathscr{V}_k$ on the special fibre. Note that $\mathscr{V}$ is $\mathscr{O}$-flat, since its obtained by "descending" $\mathscr{B}\otimes V$. Let

$$\Pi=\Pi(\mathscr{V},\xi)$$

be the corresponding integral differential Galois group at the point ξ as proposed in [**12**, 4.3.1], see also (12) below.

Theorem 17. — *Assume that each connected component of G maps surjectively onto* $\operatorname{Spec}\Lambda$ *(see §5.3). Then the group scheme* Π *is isomorphic to* G

and the closed embedding of the differential Galois group $G \otimes k$ *of* $(\mathscr{V}_k, \nabla_k)$ *into* $\Pi \otimes k$ [**12**, Thm. 27] *is an isomorphism.*

Proof. — We will construct, using Tannakian formalism, a closed embedding

$$\varphi : \Pi \hookrightarrow G.$$

This will give us all properties envisaged in the statement via the following general arguments. Let

$$\alpha : G \otimes k \hookrightarrow \Pi \otimes k$$

denote the closed embedding referred to in the statement. As $G \otimes k$ is smooth, the closed embedding $(\varphi \otimes k) \circ \alpha : G \otimes k \hookrightarrow G \otimes k$ is an isomorphism (this is firstly checked on the neutral component using [**9**, II, §5, 5.6, p. 252]). Consequently $\varphi \otimes k$ and α are isomorphisms. Since Π is Λ-flat, this implies that, if

$$\mathfrak{A} := \ker \varphi^* : \mathscr{O}(G) \twoheadrightarrow \mathscr{O}(\Pi),$$

then $\varpi \cdot \mathfrak{A} = \mathfrak{A}$. Due to our assumption on G and the regularity of $\mathscr{O}(G)$, exercise 9.11 on p. 70 of [**22**] shows that

$$\mathscr{O}(G) = \bigoplus R_\nu,$$

with R_ν an integral domain which is faithfully flat over Λ. Applying Krull's Intersection Theorem [**22**, 8.10, p. 60] to each one of the ideals $\mathfrak{A}R_\nu \subseteq R_\nu$, we obtain that $\mathfrak{A} = 0$. This shows that φ is an isomorphism.

Construction of the closed embedding φ*:* Since $\rho : G \longrightarrow \mathrm{GL}(V)$ is faithful, every representation $W \in \mathrm{Rep}^{\#}_{\Lambda}(G)$ is of the form W'/W'', with W' a subrepresentation of some tensor power

$$V^a_b = \bigoplus_{i=1}^{s} V^{\otimes a_i} \otimes \mathrm{Dual}(V)^{\otimes b_i}$$

such that V^a_b/W' is in $\mathrm{Rep}^{\#}_{\Lambda}(G)$ [**12**, §3]. Therefore, each $\mathscr{B} \times^G W$ ($W \in \mathrm{Rep}^{\#}_{\Lambda}(G)$) is an object of the following *full* subcategory of $\mathbf{str}^{\#}(\mathscr{O}/\Lambda)$:

$$\langle \mathscr{V} \rangle^s_{\otimes} := \left\{ \begin{array}{l} \mathscr{V}'/\mathscr{V}'' \text{ where } \mathscr{V}' \text{ is a subobject (in } \mathbf{str}\text{) of some } \mathscr{V}^a_b \\ \text{for which } \mathscr{V}^a_b/\mathscr{V}' \text{ is also in } \mathbf{str}^{\#}(\mathscr{O}/\Lambda) \end{array} \right\}.$$

Due to [**12**, Thm. 24], we know that the functor ξ^* induces an equivalence of monoidal categories

$$\overline{\xi}^* : \langle \mathscr{V} \rangle_{\otimes}^{s} \longrightarrow \mathrm{Rep}_{\Lambda}^{\#}(\Pi) \tag{12}$$

(in fact, this is the definition of Π in [**12**]) and that $\overline{\xi}^*(\mathscr{V})$ is a *faithful* representation of Π. This produces a monoidal functor

$$\nu := \overline{\xi}^* \circ [\mathscr{B} \times^G (?)] : \mathrm{Rep}_{\Lambda}^{\#}(G) \longrightarrow \mathrm{Rep}_{\Lambda}^{\#}(\Pi).$$

Claim: There exists a homomorphism of group schemes $\varphi : \Pi \longrightarrow G$ such that $\mathrm{res}(\varphi) = \nu$.

Granted the claim, the verification that φ is a closed embedding is immediate.

Let us now justify the claim using [**31**]. Let Γ/Λ be any flat and affine group scheme, $\omega_\Gamma : \mathrm{Rep}_{\Lambda}^{\#}(\Gamma) \longrightarrow (\Lambda\text{-mod})$ the forgetful functor. Then Γ, regarded as a functor from $(\Lambda\text{-alg})$ to sets, is naturally isomorphic to the functor

$$\underline{\mathrm{End}}^{\otimes,\mathbb{1}}(\omega_\Gamma) : (\Lambda\text{-alg}) \longrightarrow \mathrm{Set}$$

defined by

$$R \mapsto \{\text{endomorphisms of the monoidal functor } \omega_\Gamma \otimes_\Lambda R\};$$

the isomorphism being given by the obvious natural transformation $\Gamma \longrightarrow \underline{\mathrm{End}}^{\otimes,\mathbb{1}}(\omega_\Gamma)$ [**31**, Ch. II, §3.4, p. 151]. (We observe that by [**31**, Ch. I, 5.2.3, p. 82], $\underline{\mathrm{End}}$ could in fact be replaced by $\underline{\mathrm{Aut}}$.) Thus, the homomorphism φ will be obtained once we establish the existence of a natural isomorphism $\eta : \omega_\Pi \circ \nu \Longrightarrow \omega_G$ of monoidal functors. The monoidal functor $\omega_\Pi \circ \overline{\xi}^*$ is naturally isomorphic (as a monoidal functor) to ξ^*, so we need to show that $\xi^* \circ [\mathscr{B} \times^G (?)]$ is naturally isomorphic to ω_G. Such an isomorphism can be obtained from the existence—which is guaranteed by the triviality of torsors under smooth group schemes over strictly henselian rings [SGA 3 XXIV, Prp. 8.1(i), p. 401]—of a Λ-point on $\mathrm{Spec}\,\mathscr{B}$ above $\xi \in X(\Lambda)$. □

Apart from finite and etale $\mathscr{O}_k$-algebras, the most successful way to construct D-modules is via the F-division. We chose matrices $\alpha_i \in \mathrm{GL}_\mu(\mathscr{O}_k^{(i)})$ to define an F-divided module $M_\bullet := \{\mathscr{O}_k^{(i)\oplus\mu}, \alpha_i\}$. If the α_i are elements of $G(\mathscr{O}_k^{(i)})$, where $G \hookrightarrow \mathrm{GL}_\mu$ is a closed Λ-subgroup scheme satisfying the hypothesis of Theorem 17, then the differential Galois group $\Pi(M_\bullet, \xi_k)$ is naturally a closed subgroup scheme of $G \otimes k$ [**23**, Prp. 5.3]. In the sequel, we *assume that this closed embedding is an isomorphism*; explicit examples of this phenomenon – which are not easily available – are described in [**23**, §7] (see specially the end of p. 30 there). The choice of the $\alpha_i \in G_k(\mathscr{O}_k^{(i)}) = \mathrm{Hom}_k(X_k^{(i)}, G_k)$ is simply the data of an F-divided G_k-torsor: $P_\bullet = \{X_k^{(i)} \times G_k; \alpha_i\}$, so that any family of liftings $\widetilde{\alpha}_i \in G(\mathscr{O}^{(i)})$ will produce a Φ-divided torsor $\widetilde{P}_\bullet = \{X^{(i)} \times G, \widetilde{\alpha}_i\}$ (resp. a Φ-divided module $\widetilde{M}_\bullet := \{\mathscr{O}^{(i)\oplus\mu}, \widetilde{\alpha}_i\}$) lifting $P_\bullet$ (resp. $M_\bullet$). From the proof of Theorem 17, we see that the differential Galois group $\Pi(\widetilde{M}_\bullet, \xi)$ is isomorphic to G, since $M_\bullet$ (resp. $\widetilde{M}_\bullet$) is derived from $P_\bullet$ (resp. $\widetilde{P}_\bullet$) using the twisting construction applied to the representation $G_k \hookrightarrow \mathrm{GL}_{\mu,k}$ (resp. $G \hookrightarrow \mathrm{GL}_\mu$). This is, of course, a slight generalization of the last example in [**24**].

5.3. Analysis of the hypothesis in Theorem 17. — There are four main hypothesis in Theorem 17, which we now enumerate and comment on.

Hypothesis 1. "There exists a lifting of the differential Galois group $\Pi(\mathscr{V}_k, \xi_k)$ of $(\mathscr{V}_k, \nabla_k)$ to a flat (hence smooth) group scheme over Λ." As usual this problem is divided into two other.

Formal: Using [SGA 3 III, 3.7] we obtain cohomological criteria for infinitesimal liftings of $\Pi(\mathscr{V}_k, \xi_k)$. In particular, if

$$\mathrm{H}^3(\Pi(\mathscr{V}_k, \xi_k); \mathrm{Lie}) = 0,$$

then there exists a *formal* group scheme $\hat{\Pi}$ over $\mathrm{Spf}\,\Lambda$ lifting $\Pi(\mathscr{V}_k, \xi_k)$.

Algebraization: Unless $\Pi(\mathscr{V}_k, \xi_k)$ is etale, we are unable to mention something relevant here if the generality is as above; that is to say, we are unable to find algebraization techniques for formal groups (similar to those of Artin and Elkik).

Therefore, in a very general case, we cannot say that the standard methods give something interesting. But, since each reductive group over k comes from a base change of a reductive group over $\mathbb{Z}$ (see Thm. 3.6.6 and Prp. 5.1.6 of [**10**]), we do not need to enter into these algebraization question–even though they are quite interesting *per se*–if we know that $\Pi(\mathscr{V}_k, \xi_k)$ is a reductive group or a product of such by an etale group.

Hypothesis 2: "There exists a lifting of ρ_k to a homomorphism $\rho : G \longrightarrow \mathrm{GL}(V)$." To verify this hypothesis in general, we proceed, as above, in two steps.

Formal: Using deformation theory, notably [SGA 3 III, Cor. 2.6, p. 117], one can obtain cohomological constrains to infinitesimal liftings of a representation. In particular, if $\mathrm{H}^2(G_k, \mathrm{Ad}(\rho_k)) = \mathrm{Ext}^2_{G_k}(\rho_k, \rho_k) = 0$, then we can find a compatible family of liftings

$$\rho^j : G \otimes \Lambda_j \longrightarrow \mathrm{GL}(V) \otimes \Lambda_j, \qquad j = 0, 1 \ldots,$$

where $\Lambda_j = \Lambda/\varpi^{j+1}$ and $\rho_k := \rho^0$.

Algebraization: If G is a split reductive group scheme over Λ [SGA 3 XXII 1.13, p. 162], [**10**, 3.1.1], then the functor

$$(\Lambda\text{-alg}) \longrightarrow \mathrm{Set}; \qquad R \mapsto \mathrm{Hom}_{R\text{-gr}}(G_R, \mathrm{GL}(V)_R)$$

is representable by a scheme locally of finite presentation over Λ [SGA 3 XXIV 7.1.10, p. 389] and hence, the obvious map

$$\mathrm{Hom}_{\Lambda\text{-gr}}(G, \mathrm{GL}(V)) \longrightarrow \varprojlim \mathrm{Hom}_{\Lambda_j\text{-gr}}(G \otimes \Lambda_j, \mathrm{GL}(V) \otimes \Lambda_j)$$

is a bijection. (For a counter example in the case of arbitrary generality, the reader is directed to [SGA 3 IX, 7.4, p. 73].) Consequently, in this case, if we know that $\mathrm{H}^2(G_k, \mathrm{Ad}(\rho_k)) = 0$, then there exists $\rho : G \longrightarrow \mathrm{GL}(V)$ reducing to ρ_k modulo ϖ.

Nevertheless, it is possible to show that certain irreducible representations of $\mathrm{GL}(2)_k$ cannot be lifted to characteristic zero: simply pick a dominant weight for which the highest weight module in positive characteristic has a smaller dimension than the corresponding one in characteristic zero (see [**17**, Part I, 2.16]).

Hypothesis 3: "The lift guaranteed in Hypothesis 2 is a closed embedding into $\mathrm{GL}(V)$." This leads to a problem which we are unable to deal with: Let G/Λ be a smooth group scheme over Λ and let Z be a flat Λ scheme on which G acts (on the right). Is it true that a faithful action on the special fibre entails a faithful action? If we replace our DVR of characteristic $(0, p)$ by a DVR of characteristic p, then the following is a counter example (thanks are due to M. Florence for pointing this out to us): Define G to be the additive group and let G act on itself by means of the homomorphism

$$\varpi \cdot \mathrm{Frobenius} - \mathrm{Id} : G \longrightarrow G.$$

Hypothesis 4: "Each connected component of G dominates $\operatorname{Spec}\Lambda$." This will be satisfied once $\mathscr{O}(G)$ is a *projective* Λ-module or when the generic fibre $G \otimes K$ is connected (straightforward proofs).

6. Unrestricted deformations

We now enter the second part of the article, where we will study the deformation theory (as envisaged by [**32**]) of a D-module in positive characteristic. We fix $\mathscr{V}$ a flat and coherent $\mathscr{O}$-module of rank μ and assume that $\mathscr{V} \otimes k = \mathscr{V}_k$ is endowed with a stratification

$$\nabla_k : D_k \longrightarrow \mathrm{End}_k(\mathscr{V}_k).$$

This notation will be in force for the remaining of the text.

The deformations of ∇_k to stratifications on $\mathscr{V}_A$, for $A \in \mathscr{C}_\Lambda$, are our main subject of interest.

6.1.

Definition 18. — *Let* $A \in \mathscr{C}_\Lambda$. *Define the set* $\mathrm{Def}^+(A)$ *by*

$$\{\nabla_A : D_A \longrightarrow \mathrm{End}_A(\mathscr{V}_A) \text{ stratifications which induce the stratification } \nabla_k\}.$$

The functor from $\mathscr{C}_\Lambda$ to Set, $A \mapsto \mathrm{Def}^+(A)$, is not really the object we wish to study in this section; its introduction is convenient due to the simple nature of the deformation theory obtained. What we really want is to understand deformations modulo isomorphisms. Define the automorphisms functor

$$\mathbf{G} : \mathscr{C}_\Lambda \longrightarrow \mathrm{Grp}$$

by

$$\mathbf{G}(A) := \ker\left(\mathrm{Aut}_{\mathscr{O}_A}(\mathscr{V}_A) \longrightarrow \mathrm{Aut}_{\mathscr{O}_k}(\mathscr{V}_k)\right) = \{\varphi \in \mathrm{End}_{\mathscr{O}_A}(\mathscr{V}_A) : \ \varphi \otimes_A k = \mathrm{Id}\}.$$

(The second equality being a consequence of [**32**, Lemma 3.3].) The functor $\mathbf{G}$ acts naturally (i.e. functorially) on the right of Def^{+}:

$$(\nabla_A * g)(\partial) \cdot v = g^{-1}\nabla_A(\partial) \cdot gv, \qquad (g \in \mathbf{G}(A),\ \partial \in D_A,\ v \in \mathscr{V}_A).$$

Definition 19. — *The functor* $\mathrm{Def} : \mathscr{C}_\Lambda \longrightarrow \mathrm{Set}$ *takes* $A \in \mathscr{C}_\Lambda$ *to the set of equivalence classes of* $\mathrm{Def}^{+}(A)$ *modulo* $\mathbf{G}(A)$*; symbolically:*

$$\mathrm{Def}(A) = \mathrm{Def}^{+}(A)/\mathbf{G}(A).$$

Remark: Here is another way to describe the functor Def. Let $A \in \mathscr{C}_\Lambda$. Consider the set E_A of all pairs $(\mathscr{M}_A, \alpha)$, where $\mathscr{M}_A$ is a D_A-module which is flat as an $\mathscr{O}_A$-module and $\alpha : \mathscr{M}_A \otimes_\Lambda k \longrightarrow \mathscr{V}_k$ is an isomorphism of D_k-modules. Define an isomorphism $f : (\mathscr{M}_A, \alpha) \longrightarrow (\mathscr{N}_A, \beta)$ as an isomorphism of D_A-modules which satisfies $\beta \circ f \otimes A = \alpha$. Since any two flat $\mathscr{O}_A$-modules deforming $\mathscr{V}_k$ are (non-canonically) isomorphic, we have

$$\mathrm{Def}(A) = E_A/\text{isomorphisms}.$$

Abstract deformation theory is the study of certain classes of functors $\mathscr{C}_\Lambda \longrightarrow$ Set. Let us recall from [**21**, §2] what some of these classes are.

Definition 20. —

1. *A functor* $\Phi : \mathscr{C}_\Lambda \longrightarrow (\mathrm{Set})$ *is a functor of Artin rings if* $\Phi(k) = \{\bullet\}$.
2. *A functor of Artin rings is homogeneous if, given a diagram in* $\mathscr{C}_\Lambda$*:*

$$\begin{array}{ccc} & & C \\ & & \downarrow \\ B & \longrightarrow & A, \end{array}$$

the natural map

$$\eta_\Phi : \Phi(B \times_A C) \longrightarrow \Phi(B) \times_{\Phi(A)} \Phi(C) \tag{13}$$

is bijective whenever $B \longrightarrow A$ *is surjective (as usual* $B \times_A C$ *is the fibered product of these rings).*

3. *A functor of Artin rings* Φ *is a deformation functor if* η_Φ *is (i) surjective provided that* $B \longrightarrow A$ *is surjective and (ii) bijective whenever* $A = k$.
4. *A functor of Artin rings is smooth if for every surjection* $B \longrightarrow A$ *in* $\mathscr{C}_\Lambda$*, the map* $\Phi(B) \longrightarrow \Phi(A)$ *is surjective.*

Of course, every homogeneous functor is a deformation functor. Smoothness is not generically related to homogeneity. Following Schlessinger's celebrated theorem [**32**, 2.11], homogeneity gives "pro-representability" once the tangent spaces are finite dimensional. For the convenience of the reader we recall the definition of tangent space.

Definition 21. — [**32**, 2.10], [**21**, Prp. 2.6, p. 30]. *Let* $\Phi : \mathscr{C}_\Lambda \longrightarrow \mathrm{Set}$ *be a deformation functor. Let* $k[\varepsilon] = k[T]/T^2$ *be regarded as a* Λ*-algebra through the trivial homomorphism* $\Lambda \longrightarrow k \longrightarrow k[\varepsilon]$. *Then* $t_\Phi := \Phi(k[\varepsilon])$ *has a natural structure of* k*-vector space and is called the tangent space of* Φ.

Lemma 22. — *The functor* Def^+ *is homogeneous.*

Proof. — Let $\alpha : B \longrightarrow A$ and $\beta : C \longrightarrow A$ be arrows in $\mathscr{C}_\Lambda$ with α surjective; let $S := B \times_A C$. Note that we have natural identifications $\mathscr{V}_S \otimes_S B = \mathscr{V}_B$, $\mathscr{O}_S \otimes_S B = \mathscr{O}_B$, $D_S \otimes_S B = D_B$ etc. Applying Corollary 3.6 of [**32**], we see that the natural maps $\mathscr{O}_S \longrightarrow \mathscr{O}_B \times_{\mathscr{O}_A} \mathscr{O}_C$, $\mathscr{V}_S \longrightarrow \mathscr{V}_B \times_{\mathscr{V}_A} \mathscr{V}_C$ and $D_S \longrightarrow D_B \times_{D_A} D_C$ are isomorphisms. Furthermore, the natural maps $\mathscr{V}_S \longrightarrow \mathscr{V}_S \otimes_S B$, $D_S \longrightarrow D_S \otimes_S B$, etc. are identified to the projections $\mathscr{V}_B \times_{\mathscr{V}_A} \mathscr{V}_C \longrightarrow \mathscr{V}_B$, $D_B \times_{D_A} D_C \longrightarrow D_B$, etc.

If we are given $\nabla_B \in \mathrm{Def}^+(B)$ and $\nabla_C \in \mathrm{Def}^+(C)$ which induce $\nabla_A \in \mathrm{Def}^+(A)$ we can then form the homomorphism $\nabla_B \times_{\nabla_A} \nabla_C$ by

$$\nabla_B \times_{\nabla_A} \nabla_C(\partial_B, \partial_C) \cdot (v_B, v_C) = (\nabla_B(\partial_B) \cdot v_B, \nabla_C(\partial_C) \cdot v_C). \tag{14}$$

In other words η_{Def^+} (notation from (13)) is surjective.

To prove injectivity of η_{Def^+}, we let $\nabla_S : D_S \longrightarrow \mathrm{End}_S(\mathscr{V}_S)$ be a stratification which induces $\nabla_?$ on $\mathscr{V}_?$ for $? = A, B, C$. Let us spell-out what this last assertion means using the identification $D_S = D_B \times_{D_A} D_C$. Take $(\partial_B, \partial_C) \in D_B \times_{D_A} D_C$. Then for any $(v_B, v_C) \in \mathscr{V}_B \times_{\mathscr{V}_A} \mathscr{V}_C$ we have (by definition) that $\nabla_B(\partial_B) \cdot v_B$ is the image of $\nabla_S(\partial_B, \partial_C) \cdot (v_B, v_C) \in \mathscr{V}_S$ in $\mathscr{V}_B$. This is just the component corresponding to $\mathscr{V}_B$ in the identification $\mathscr{V}_S = \mathscr{V}_B \times_{\mathscr{V}_A} \mathscr{V}_C$. The same holds if we replace B by C. We have thus obtained that

$$\nabla_S(\partial_B, \partial_C) \cdot (v_B, v_C) = (\nabla_B(\partial_B) \cdot v_B, \nabla_C(\partial_C) \cdot v_C) \in \mathscr{V}_B \times_{\mathscr{V}_A} \mathscr{V}_C;$$

∇_S is uniquely determined by the elements it induces in $\mathrm{Def}^+(B) \times \mathrm{Def}^+(C)$. We have just proved that Def^+ is homogeneous. □

Lemma 23. — *The functor* Def *is a deformation functor.*

Proof. — Since we already know that Def^+ is homogeneous, [**21**, Lemma 2.20] will take care of the proof provided that we show that $\mathbf{G}$ is a smooth deformation functor. Smoothness is verified using deformation theory as developed in [SGA 1 I]; in [**16**, 5.3(a)] Illusie gives a result which demonstrates smoothness promptly. Let us now analyze homogeneity. Consider $A, B, C, S := B \times_A C$ as in Definition 20 and assume that $B \longrightarrow A$ is surjective. Applying [**32**, Cor. 3.6] we know that the obvious maps $\mathscr{V}_S \longrightarrow \mathscr{V}_B \times_{\mathscr{V}_A} \mathscr{V}_C$ and $\mathscr{O}_S \longrightarrow \mathscr{O}_B \times_{\mathscr{O}_A} \mathscr{O}_C$ are bijective; the homogeneity of $\mathbf{G}$ follows without difficulty. □

6.2. Lifting of automorphisms and homogeneity. — In this paragraph we review some results from abstract deformation theory which we learned from appendix A of [**30**]. (Unfortunately, the criterion Pridham presents there–Corollary A.1.15–does not fit exactly our purposes.) They are well known to deformation theorists; the main result, Proposition 24, is silently used in [**25**, Prp. 1]. The considerations made here will be later applied to the functor Def (see Theorem 26).

We introduce some notations. Let

$$F : \mathscr{C}_\Lambda \longrightarrow \mathrm{Set}$$

and

$$G : \mathscr{C}_\Lambda \longrightarrow (\mathrm{Group})$$

be deformation functors (Definition 20) with G acting on the right of F.

Proposition 24. — *Assume furthermore that G is smooth and F is homogeneous. The functor $Q = F/G : \mathscr{C}_\Lambda \longrightarrow$* Set *is homogeneous if the following condition holds:*

Lifting of automorphisms: *For every surjection $A' \longrightarrow A$ and every $a' \in F(A')$, the natural map*

$$\{g \in G(A');\ a' \cdot g = a'\} \longrightarrow \{g \in G(A);\ (a'|A) \cdot g = (a'|A)\}$$

is surjective.

Proof. — We need to show that for every cartesian diagram

$$\begin{array}{ccc} A' \times_A A'' & \longrightarrow & A' \\ \downarrow & & \downarrow \\ A'' & \longrightarrow & A \end{array}$$

in $\mathscr{C}_\Lambda$ with $A' \longrightarrow A$ surjective the natural map

$$\eta : Q(A' \times_A A'') \longrightarrow Q(A') \times_{Q(A)} Q(A'')$$

is bijective. As surjectivity of η is already documented knowledge [**21**, Lemma 2.20], we concentrate on the injectivity. Take $a, b \in F(A' \times_A A'')$ such that the induced elements in $Q(A' \times_A A'')$ have the same image under η. This means that there are $g' \in G(A')$ and $g'' \in G(A'')$ such that $a' \cdot g' = b'$ and $a'' \cdot g'' = b''$ (here $a' := a|A'$, $a'' := a|A''$, etc.). As G is smooth and $A' \times_A A'' \longrightarrow A''$ is surjective, we can assume that $g'' = \mathrm{id}$. Therefore,

$$(a'|A) \cdot (g'|A) = b'|A = b|A = b''|A = a''|A = a|A = a'|A;$$

from the lifting of automorphism assumption, there exists $\gamma' \in G(A')$ above $g'|A$ such that $a' \cdot \gamma' = a'$. Hence we obtain an element $((\gamma')^{-1} \cdot g', \mathrm{id}) \in G(A') \times_{G(A)} G(A'')$; as G is a deformation functor there exists $\gamma \in G(A' \times_A A'')$ such that $\gamma|A' = (\gamma')^{-1} \cdot g'$ and $\gamma|A'' = \mathrm{id}$. Hence

$$a \cdot \gamma|A' = a' \cdot (\gamma')^{-1} \cdot g' = a' \cdot g' = b' = b|A'$$

and $a \cdot \gamma|A'' = b'' = b|A''$. The homogeneity of F proves that $a \cdot G(A' \times_A A'') = b \cdot G(A' \times_A A'')$, so that the images of a and b in $Q(A' \times_A A'')$ coincide. □

In order to study the lifting of "automorphisms" property of Proposition 24, we can still use abstract deformation theory a little more. Let us keep the assumptions *preceding* Proposition 24. Let $R \in \mathscr{C}_\Lambda$ and take $\xi \in F(R)$. Define the following functor of Artin rings:

$$(15) \qquad I_\xi : \mathscr{C}_R \longrightarrow (\mathrm{Grp}); \qquad I_\xi(A) = \{g \in G(A) : \ (\xi|A) \cdot g = \xi|A\}.$$

Proposition 25. — *Assume that F is homogeneous (and that G is a deformation functor). Then I_ξ is a deformation functor with tangent space*

$$t_{I_\xi} = \{g \in t_G :\ 0 \cdot g = 0\}.$$

If G is homogeneous, then I_ξ is homogeneous.

Proof. — With the exception of the last statement–which is immediate–this is proved in Proposition 2.21 of [**21**]. □

6.3. Applications to Def. — After this interlude on abstract deformation theory, we come back to our study of Def.

Theorem 26. — *Assume that* $\mathrm{End}_{D_k}(\mathscr{V}_k) = k$. *Then* $\mathrm{Def} : \mathscr{C}_\Lambda \longrightarrow \mathrm{Set}$ *is homogeneous.*

Proof. — We want to apply Proposition 24. Let $R \longrightarrow A$ be a surjection in $\mathscr{C}_\Lambda$ and let $\nabla_R \in \mathrm{Def}^+(R)$ induce $\nabla_A \in \mathrm{Def}^+(A)$. We want to show that

$$\{g \in \mathbf{G}(R) :\ \nabla_R * g = \nabla_R\} = I_{\nabla_R}(R) \longrightarrow I_{\nabla_R}(A) = \{g \in \mathbf{G}(A) :\ \nabla_A * g = \nabla_A\} \tag{16}$$

is surjective. This will be accomplished using a direct description of I_{∇_R}, as in [**25**, Prp. 1]. Define

$$\hat{\mathbb{G}}_{m,R} : \mathscr{C}_R \longrightarrow (\mathrm{Grp});\quad \hat{\mathbb{G}}_{m,R}(B) = \ker\left(B^\times \longrightarrow k^\times\right) = 1 + \mathrm{rad}(B).$$

It is immediate to see that $\hat{\mathbb{G}}_{m,R} \subseteq I_{\nabla_R}$ and we wish to show that this is an equality.

Claim: $\hat{\mathbb{G}}_{m,R}(k[\varepsilon]) = I_{\nabla_R}(k[\varepsilon])$.

Proof: Let $\nabla_\varepsilon : D_{k[\varepsilon]} \longrightarrow \mathrm{End}_{k[\varepsilon]}(\mathscr{V}_{k[\varepsilon]})$ be the stratification induced from ∇_R. It is the same as the stratification induced from ∇_k via the inclusion $k \longrightarrow k[\varepsilon]$. By definition, $I_{\nabla_R}(k[\varepsilon])$ is just the group of automorphisms of the $D_{k[\varepsilon]}$-module $(\mathscr{V}_{k[\varepsilon]}, \nabla_\varepsilon)$ which induce the identity when reduced modulo ε. But it is not hard to see that any such element is of the form $\mathrm{Id}_{\mathscr{V}_{k[\varepsilon]}} + \varepsilon \cdot g$ with $g \in \mathrm{End}_{D_k}(\mathscr{V}_k)$. This proves the claim.

Using Schlessinger's main theorem [**32**, Thm. 2.11], we know that

$$I_{\nabla_R} \cong \mathrm{Hom}_R(S, \bullet),$$

where S is a complete local R-algebra of residue field k. Furthermore, the inclusion of functors $\hat{\mathbb{G}}_{m,R} \subseteq I_{\nabla_R}$ comes from a continuous homomorphism of R-algebras $f : S \longrightarrow R[[t]]$, since $R[[t]]$ pro-represents $\hat{\mathbb{G}}_{m,R}$. The claim guarantees that the induced k-linear map $df : t^*_{R[[t]]/R} \longrightarrow t^*_{S/R}$ on Zariski cotangent spaces is an isomorphism. It is easy to prove that such a homomorphism is an isomorphism ([**32**, Lemma 1.1] can be used here). In summary, we showed that $\hat{\mathbb{G}}_{m,R} = I_{\nabla_R}$, so that the map in eq. (16) has to be surjective. This allows us to apply Proposition 24 to conclude that $\mathrm{Def} = \mathrm{Def}^+/\mathbf{G}$ is homogeneous. □

Example 27. — By Schur's Lemma, once the stratified sheaf $\mathscr{V}_k$ is simple, we have $\mathrm{End}_{D_k}(\mathscr{V}_k) = k$. Another interesting case happens when the differential Galois group G of $\mathscr{V}_k$ at the point ξ_k is reductive and the canonical representation attached to it is of the form $\mathrm{ind}_B^G(\lambda)$, where $B \subset G$ is a Borel subgroup and λ is a dominant character in a maximal torus of B [**17**, Part II, Prp. 2.8, p. 202].

7. Periodic deformations

We maintain the notations introduced in the beginning of §6. In section 6 we saw that Def is a homogeneous functor. Unfortunately, its tangent space may be much too large, so that the local moduli space is too big to be of any interest (see Lemma 32 and the remark after it). With this in mind, we consider certain restrictions on the kind of deformations allowed. In section 8 we show how to describe the tangent spaces in cohomological terms (see Lemma 32 and Proposition 33).

Let $(\mathscr{B}, \lambda) \in \mathrm{Ind}\ \mathbf{str}(\mathscr{O}/\Lambda)$ be an algebra (§2.3) which satisfies the following hypothesis:

B1 For any $A \in \mathscr{C}_\Lambda$, the $\mathscr{O}_A$-algebra $\mathscr{B}_A$ is faithfully flat and $\mathscr{B}_k$ is a domain.

B2 The only elements of $\mathscr{B}_k$ which are annihilated by D_k^+ are of the form $a \cdot 1_{\mathscr{B}}$, with $a \in k$. (The only horizontal sections are the constants.)

B3 There exists a map

$$\mathscr{B}_k^{\oplus \mu} \longrightarrow \mathscr{B}_k \otimes_{\mathscr{O}_k} \mathscr{V}_k$$

which is simultaneously an isomorphism of $\mathscr{B}_k$-modules and of D_k-modules.

Before we give the main definition of this section, let us pause to gather two results which will be useful further on.

Lemma 28. — *Let $A \in \mathscr{C}_\Lambda$. Then the only elements of $\mathscr{B}_A$ which are annihilated by the ideal D_A^+ are of the form $a \cdot 1_{\mathscr{B}}$ with $a \in A$. (The only horizontal sections are the constant ones.)*

Proof. — We proceed by induction on the length of the Artin ring A; the case of length one is guaranteed by axiom. So let

$$0 \longrightarrow I \longrightarrow A' \longrightarrow A \longrightarrow 0$$

be a small extension in $\mathscr{C}_\Lambda$: this means that the kernel I is generated by one element t which is annihilated by the radical $\mathfrak{r}(A')$–consequently I is a one dimensional k-space. Let $\alpha \in \mathscr{B} \otimes_\Lambda A'$ be annihilated by $D_{A'}^+$. By the inductive hypothesis, we can assume that there exists $a \in A'$ such that $\alpha - a \cdot 1_{\mathscr{B}'_A}$ belongs to the kernel $\mathscr{B} \otimes_\Lambda I$. Since $\mathscr{B} \otimes_\Lambda I$ is isomorphic both as an $\mathscr{O}$-module and as a D_Λ-module to $\mathscr{B}_k$, it follows that $\alpha - a \cdot 1_{\mathscr{B}_{A'}}$ belongs to the image of A', which finishes the proof. □

Lemma 29. — *Let $\mathscr{W} \in \mathbf{str}(\mathscr{O}_k/k)$ and let G denote its differential Galois group at the point ξ_k. As usual, we denote the monoidal equivalence from $\langle \mathscr{W} \rangle_\otimes$ to $\mathrm{Rep}_k(G)$ by $\overline{\xi}_k^*$ (§5.2). Regard the left regular representation $\mathscr{O}(G)_{\mathrm{left}}$ as a torsor under G (§2.3.4) and let $\mathscr{B}_0 \in \mathrm{Ind}\,\mathbf{str}(\mathscr{O}_k/k)$ be the G-torsor which corresponds under $\overline{\xi}_k^*$ to $\mathscr{O}(G)_{\mathrm{left}}$. If $\mathscr{B}$ is an algebra of $\mathrm{Ind}\,\mathbf{str}(\mathscr{O}/\Lambda)$ which is flat over $\mathscr{O}$ and reduces modulo ϖ to $\mathscr{B}_0$, then conditions **B1** and **B2** are satisfied.*

Proof. — This is well known and standard: one can say that it provides the link between classical differential Galois theory and its Tannakian counterpart, since $\mathscr{B}_0$ is then a PV extension. Condition **B2** is immediately verified as we have an isomorphism of k-spaces

$$\mathrm{Hom}_{D_k}(\mathbb{1}, \mathscr{B}_0) \cong \mathrm{Hom}_G(\mathbb{1}, \mathscr{O}(G)_{\mathrm{left}}) \cong k \cdot 1.$$

It is also easy to verify that $\mathscr{B}_0$ is a faithfully flat $\mathscr{O}_k$-algebra and thus, by transport of structure,

$$P_0 = \mathrm{Spec}\, \mathscr{B}_0 \longrightarrow X_k$$

is a G-torsor. (Note that $\mathscr{B}_A$ is consequently faithfully flat over $\mathscr{O}_A$.) Since G is a smooth group scheme over k [**11**, Cor. 12(iii)], P_0 is a smooth k-scheme [EGA IV-2, 2.7.1] and [EGA IV-4, 17.7.4]. Now assume that $\mathscr{B}_0$ fails to be a domain. As a regular ring with more than one minimal prime has to admit non-trivial idempotents [**22**, ex. 9.11, p. 70] and idempotents are always annihilated by D_k^+, we would obtain $\dim_k \mathrm{Hom}_{D_k}(\mathbb{1}, \mathscr{B}_0) > 1$, which is impossible. Therefore $\mathscr{B}_0$ is a domain. □

Definition 30. — *Let $\mathscr{B}$ be as before. We say that a deformation $(\mathscr{V}_A, \nabla_A) \in$ $\mathrm{Def}^+(A)$ is $\mathscr{B}$-periodic if there exists a map*

$$\mathscr{B}_A^{\oplus\mu} \longrightarrow \mathscr{B}_A \otimes_{\mathscr{O}_A} \mathscr{V}_A$$

which is an isomorphism of D_A-modules and of $\mathscr{B}_A$-modules. In an equivalent way: There exists a basis $\{v_1, \dots, v_\mu\}$ of the $\mathscr{B}_A$-module $\mathscr{B}_A \otimes_{\mathscr{O}_A} \mathscr{V}_A$ which is annihilated by D_A^+. Such a basis will be called a periodic basis.

For each $A \in \mathscr{C}_\Lambda$, we let $\mathrm{Def}^+_{\mathscr{B}}(A)$ denote the set of all $\mathscr{B}$-periodic deformations in $\mathrm{Def}^+(A)$. It is clear that $\mathrm{Def}^+_{\mathscr{B}}$ defines a functor from $\mathscr{C}_\Lambda$ to Set.

Remark: From Lemma 28, it follows that any two periodic basis of $\mathscr{B}_A \otimes_{\mathscr{O}_A} \mathscr{V}_A$ are related by a matrix in $\mathrm{GL}_\mu(A)$.

Theorem 31. — *Assume that $\mathrm{End}_{D_k}(\mathscr{V}_k) = k$.*

(i) The subset $\mathrm{Def}^+_{\mathscr{B}}(A)$ of $\mathrm{Def}^+(A)$ is invariant under $\mathbf{G}(A)$.

(ii) The functor $\mathrm{Def}_{\mathscr{B}} := \mathrm{Def}^+_{\mathscr{B}}/\mathbf{G}$ is homogeneous.

Proof. — Assertion (i) is easy and we concentrate on (ii). Let $\alpha : A' \longrightarrow A$ and $A'' \longrightarrow A$ be morphisms in $\mathscr{C}_\Lambda$ with α surjective. We need to show that

$$\eta : \mathrm{Def}_{\mathscr{B}}(A' \times_A A'') \longrightarrow \mathrm{Def}_{\mathscr{B}}(A') \times_{\mathrm{Def}_{\mathscr{B}}(A)} \mathrm{Def}_{\mathscr{B}}(A'')$$

is bijective. Since $\mathrm{Def}_{\mathscr{B}}$ is a subfunctor of Def, the injectivity of η follows from the homogeneity of Def (see Theorem 26). We are left with the proof of surjectivity. Since $\mathbf{G}$ is smooth, we only need to show that

$$\eta^+ : \mathrm{Def}^+_{\mathscr{B}}(A' \times_A A'') \longrightarrow \mathrm{Def}^+_{\mathscr{B}}(A') \times_{\mathrm{Def}^+_{\mathscr{B}}(A)} \mathrm{Def}^+_{\mathscr{B}}(A'')$$

is surjective. So let $\nabla' \in \mathrm{Def}^+_{\mathscr{B}}(A')$ and $\nabla'' \in \mathrm{Def}^+_{\mathscr{B}}(C)$ induce $\nabla \in \mathrm{Def}^+_{\mathscr{B}}(A)$. Let $S = A' \times_A A''$. From Lemma 22 (or rather, its proof), we only need to show that the diagonal stratification defined in eq. (14)

$$\nabla' \times_\nabla \nabla'' : D_S = D_{A'} \times_{D_A} D_{A''} \longrightarrow \mathrm{End}_S(\mathscr{V}_S)$$

is $\mathscr{B}$-periodic. This will be done if we can find periodic basis $\{v'_1, \ldots, v'_\mu\}$ of $\mathscr{B}_{A'} \otimes_{\mathscr{O}_{A'}} \mathscr{V}_{A'}$ and $\{v''_1, \ldots, v''_\mu\}$ of $\mathscr{B}_{A''} \otimes_{\mathscr{O}_{A''}} \mathscr{V}_{A''}$ which induce *the same* periodic basis of $\mathscr{B}_A \otimes_{\mathscr{O}_A} \mathscr{V}_A$. (Here the reader should recall that the proof of Lemma 22 shows that the only stratification on $\mathscr{B}_S = \mathscr{B}_{A'} \times_{\mathscr{B}_A} \mathscr{B}_{A''}$, resp. $\mathscr{B}_S \otimes_{\mathscr{O}_S} \mathscr{V}_S = \mathscr{B}_{A'} \otimes \mathscr{V}_{A'} \times_{\mathscr{B}_A \otimes \mathscr{V}_A} \mathscr{B}_{A''} \otimes \mathscr{V}_{A''}$, compatible with those of $\mathscr{B}_{A'}$ and of $\mathscr{B}_{A''}$, resp. $\mathscr{B}_{A'} \otimes \mathscr{V}_{A'}$ and $\mathscr{B}_{A''} \otimes \mathscr{V}_{A''}$, is the one given "diagonally".) But this is possible since we know that any two periodic basis of $\mathscr{B}_A \otimes_{\mathscr{O}_A} \mathscr{V}_A$ are related by a matrix in $\mathrm{GL}_\mu(A)$ (see Lemma 28 and the remark after it). □

The importance of the concept of $\mathscr{B}$-periodic deformations comes from the calculation of its tangent space made in Section 8 (see Proposition 33 and Corollary 34).

8. The tangent spaces and representability

8.1. General remarks on deformations of structures from k to $k[\varepsilon]$. — We shall work with deformations of structures from k to $k[\varepsilon] = k[T]/T^2$. Let E be a k-algebra (k is central); we note that an $E \otimes_k k[\varepsilon] = E[\varepsilon]$-module M is the same as a pair $(e(M), \iota)$ of an an E-module $e(M)$ and an element $\iota \in \mathrm{End}_k(e(M))$ of square zero which commutes with the elements in the image of E in $\mathrm{End}_k(e(M))$. In particular, the E-module obtained from M via the inclusion $E \subseteq E[\varepsilon]$ is just $e(M)$ (the E-module underlying M). In this paragraph, a deformation of an E-module M will stand for a couple (N, α) of an $E[\varepsilon]$-module and an isomorphism of E-modules $\alpha : N/\varepsilon N \longrightarrow M$.

The following Lemma is standard.

Lemma 32. — *Keep the above notations and assumptions. Let M be an E-module (on the left). Let N be an $E[\varepsilon]$-module which is* flat *over $k[\varepsilon]$ and which deforms the E-module M. Then the E-module underlying N, $e(N)$, is an extension of M by itself and this association defines a bijection between the isomorphism classes of deformations of M which are flat as $k[\varepsilon]$-modules and* $\mathrm{Ext}^1_E(M, M)$.

Proof. — Let N be an $E[\varepsilon]$-module deforming the E-module M. Let $\{m_i\}$ be a basis of M over k and let $n_i \in N$ be above m_i. From [**32**, Lemma 3.3] it follows that $\{n_i\}$ is a $k[\varepsilon]$-basis of N. The k-linear map which takes m_i to εn_i defines an injection of k-spaces $M \longrightarrow e(N)$; in order to check E-linearity, the reader can write the action of E in terms of matrices. Thus we have associated to each deformation N an extension of M by itself. Let us now associate to an extension

$$0 \longrightarrow M \xrightarrow{\beta} N_0 \xrightarrow{\alpha} M \longrightarrow 0$$

of E-modules an $E[\varepsilon]$-module structure on N_0, call it N, by letting ι (see above) be $\beta \circ \alpha$. Then N is flat over $k[\varepsilon]$ since the natural map

$$M \otimes_k k[\varepsilon] \longrightarrow N; \qquad m \otimes (a + \varepsilon b) \mapsto a\alpha'(m) + b\beta(m),$$

where $\alpha' : M \longrightarrow N_0$ is a section of α, is an isomorphism of $k[\varepsilon]$-modules. Furthermore, $N/\varepsilon N \cong M$ (through α). Finally, it is easy to see that the two constructions define the desired bijection. □

Remark: From the above lemma, we see that the tangent space of Def is isomorphic to $\mathrm{Ext}^1_{D_k}(\mathscr{V}_k, \mathscr{V}_k)$. Taking the point of view of Tannakian categories, the latter extension group can be seen as $\mathrm{Ext}^1_{\Pi}(V, V)$, where Π is the (affine) fundamental group scheme of the category $\mathbf{str}(\mathscr{O}_k/k)$ at $\xi_k \in X_k(k)$ and V is the representation of Π obtained from $\mathscr{V}_k$. This Ext group can easily be infinite dimensional: Assume that $\mathbb{G}_a$ appears as a quotient of Π. Then we have an inclusion of the infinite dimensional vector space $\mathrm{Ext}^1_{\mathbb{G}_a}(\mathbb{1},\mathbb{1}) \cong \mathrm{Hom}_{k\text{-gr}}(\mathbb{G}_a,\mathbb{G}_a) = \oplus_i k\cdot \mathrm{Frob}^i$ in $\mathrm{Ext}^1_{\Pi}(\mathbb{1},\mathbb{1})$.

8.2. The tangent space to the functor $\mathrm{Def}_{\mathscr{B}}$. —

Proposition 33. — *Let $e : \mathrm{Def}(k[\varepsilon]) \longrightarrow \mathrm{Ext}^1_{D_k}(\mathscr{V}_k,\mathscr{V}_k)$ be the bijection constructed in Lemma 32. Then the $\mathscr{B}$-periodic deformations correspond to the $\mathscr{B}$-periodic extensions (that is, extensions $\mathscr{W}$ of $\mathscr{V}_k$ by itself which satisfy $\mathscr{W}\otimes\mathscr{B}_k \cong \mathscr{B}_k^{\oplus 2\mu}$ as $\mathscr{B}_k$- and D_k-modules).*

Proof. — We recall that $\mathscr{B}_{k[\varepsilon]}$ is endowed with the obvious action of $D_{k[\varepsilon]}$ arising from the action of D_k on $\mathscr{B}_k$. Let $(\mathscr{V}_\varepsilon,\nabla_\varepsilon) = (\mathscr{V}_{k[\varepsilon]},\nabla_\varepsilon)$ be a deformation of $(\mathscr{V}_k,\nabla_k)$ to the ring of dual numbers $k[\varepsilon]$, i.e. an element of $\mathrm{Def}^+(k[\varepsilon])$. From Lemma 32, we know that the D_k-module underlying $\mathscr{V}_\varepsilon$, denoted by $e(\mathscr{V}_\varepsilon)$, determines an extension of $\mathscr{V}_k$ by itself:

$$0 \longrightarrow \mathscr{V}_k(=\varepsilon\mathscr{V}_k) \xrightarrow{\ \beta\ } e(\mathscr{V}_\varepsilon) \xrightarrow{\ \alpha\ } \mathscr{V}_k \longrightarrow 0; \tag{17}$$

where α is reduction modulo ε. (In terms of $\mathscr{O}_k$-modules seq. (17) is the obvious one.) We now assume that $e(\mathscr{V}_\varepsilon)$ is $\mathscr{B}_k$-periodic – this means that $\mathscr{B}_k\otimes_{\mathscr{O}_k} e(\mathscr{V}_\varepsilon)$ is isomorphic, both as a D_k-module and as a $\mathscr{B}_k$-module, to $\mathscr{B}_k^{\oplus 2\mu}$ – and we want to establish that

$$\mathscr{B}_{k[\varepsilon]}\otimes_{\mathscr{O}_{k[\varepsilon]}}\mathscr{V}_\varepsilon$$

has a $\mathscr{B}_{k[\varepsilon]}$-basis which is annihilated by $D^+_{k[\varepsilon]}$. As $\mathscr{B}_k$ is flat over $\mathscr{O}_k$, the exact sequence (17) becomes the exact sequence of $\mathscr{B}_k$- and D_k-modules

$$0 \longrightarrow \mathscr{B}_k^{\oplus\mu} \xrightarrow{\ \overline{\beta}\ } \mathscr{B}_k^{\oplus 2\mu} \xrightarrow{\ \overline{\alpha}\ } \mathscr{B}_k^{\oplus\mu} \longrightarrow 0.$$

Let (a_{ij}) be the matrix of $\overline{\alpha}$. Due to **B2**, it follows that each a_{ij} belongs to $k\subset\mathscr{B}_k$. Because $\mathscr{B}_k$ is a domain, it follows that some $\mu\times\mu$-minor of (a_{ij}), which is in k, is invertible in $\mathscr{B}_k$. This means that μ-vectors among the

$$\overline{\alpha}(\overrightarrow{e_j}) = \begin{pmatrix} a_{1j}\\ \vdots \\ a_{\mu j}\end{pmatrix} \qquad j=1,\dots,2\mu,$$

say the first μ, form a $\mathscr{B}_k$-basis of $\mathscr{B}_k^{\oplus\mu}$. Let us now make the following identification

$$\mathscr{B}_k \underset{\mathscr{O}_k}{\otimes} e(\mathscr{V}_\varepsilon) \xrightarrow{=} \mathscr{B}_{k[\varepsilon]} \underset{\mathscr{O}_{k[\varepsilon]}}{\otimes} \mathscr{V}_\varepsilon, \tag{18}$$

under which $\overline{\alpha} : \mathscr{B}_{k[\varepsilon]} \otimes \mathscr{V}_\varepsilon \longrightarrow \mathscr{B}_k \otimes \mathscr{V}_k$ corresponds to reduction modulo ε. Hence, $\{\overrightarrow{e}_1, \ldots, \overrightarrow{e}_\mu\} \subset \mathscr{B}_{k[\varepsilon]} \otimes_{\mathscr{O}_{k[\varepsilon]}} \mathscr{V}_\varepsilon$ is a $\mathscr{B}_{k[\varepsilon]}$-basis (again by [**32**, Lemma 3.3]). By the construction of $e(\mathscr{V}_k)$, it follows that the above identification also respects the D_k-module structures. (Here we use that the stratification on $\mathscr{B}_{k[\varepsilon]}$ is the obvious one obtained from that on $\mathscr{B}_k$.) Furthermore, $\overrightarrow{e}_i$ is annihilated by $D_{k[\varepsilon]}^+$, as it is by D_k^+. In conclusion, $\{\overrightarrow{e}_1, \ldots, \overrightarrow{e}_\mu\}$ is a periodic basis.

Now assume that $(\mathscr{V}_{k[\varepsilon]}, \nabla_\varepsilon)$ is $\mathscr{B}_{k[\varepsilon]}$-periodic. This means that the $\mathscr{B}_{k[\varepsilon]}$-module $\mathscr{B}_{k[\varepsilon]} \otimes_{\mathscr{O}_{k[\varepsilon]}} \mathscr{V}_{k[\varepsilon]}$ has a basis $\{w_1, \ldots, w_\mu\}$ which is annihilated by $D_{k[\varepsilon]}^+$. Using that (a) the natural identification in (18) also preserves the action of D_k and (b) $\{w_i, \varepsilon w_i\}$ is a basis of the $\mathscr{B}_k$-module underlying $\mathscr{B}_{k[\varepsilon]} \otimes_{\mathscr{O}_{k[\varepsilon]}} \mathscr{V}_{k[\varepsilon]}$, and so of $\mathscr{B}_k \otimes_{\mathscr{O}_k} e(\mathscr{V}_{k[\varepsilon]})$, the $\mathscr{B}_k$-periodicity of $e(\mathscr{V}_{k[\varepsilon]})$ follows immediately. □

Linguistic remark: In the above statement we have introduced the term "$\mathscr{B}_k$-periodic extension", which is not directly defined by the concept of $\mathscr{B}$-periodic deformation.

Corollary 34. — *Let* $(\mathscr{W}_k, \Delta_k) \in \mathbf{str}(\mathscr{O}_k/k)$*,* G_0 *its differential Galois group at the point* ξ_k *and* $\overline{\xi}_k^* : \langle \mathscr{W}_k \rangle_\otimes \longrightarrow \mathrm{Rep}_k(G_0)$ *the defining equivalence (see §5.2 for notations). Let* $\mathscr{B} \in \mathrm{Ind}\,\mathbf{str}(\mathscr{O}/\Lambda)$ *be an algebra. Assume that* $\mathscr{B}_k$ *is a torsor of* $\mathrm{Ind}\,\mathbf{str}(\mathscr{O}_k/k)$ *under* G_0 *which corresponds to* $\mathscr{O}(G_0)_{\mathrm{left}}$ *(see the example in §2.3.4) via* $\overline{\xi}_k^*$*. Suppose that* $(\mathscr{V}_k, \nabla_k) \in \langle \mathscr{W}_k \rangle_\otimes$*. Then* $\mathscr{B}$ *satisfies* ***B1****,* ***B2*** *and* ***B3*** *and the* k*-vector-space* $\mathrm{Def}_{\mathscr{B}}(k[\varepsilon])$ *is isomorphic to* $\mathrm{Ext}^1_{G_0}(\overline{\xi}_k^* \mathscr{V}_k, \overline{\xi}_k^* \mathscr{V}_k)$*.*

Proof. — We will show that the objects of $\langle \mathscr{W}_k \rangle_\otimes$ are precisely the $\mathscr{B}_k$-periodic objects of $\mathbf{str}(\mathscr{O}_k/k)$. This will establish that $\mathscr{B}_k$ satisfies **B3**. That $\mathscr{B}$ fulfills **B1** and **B2** was already treated by Lemma 29. Applying Proposition 33 we conclude that

$$\mathrm{Def}_{\mathscr{B}}(k[\varepsilon]) \cong \mathrm{Ext}^1_{\langle \mathscr{W} \rangle_\otimes}(\mathscr{V}_k, \mathscr{V}_k) \cong \mathrm{Ext}^1_{G_0}(\overline{\xi}_k^* \mathscr{V}_k, \overline{\xi}_k^* \mathscr{V}_k).$$

Let $(\mathscr{V}',\nabla')$ be $\mathscr{B}_k$-periodic, so that $\mathscr{V}'\otimes\mathscr{B}_k\cong\mathscr{B}_k^{\oplus\delta}$ as D_k-modules. We obtain an inclusion of D_k-modules $\mathscr{V}'\subseteq\mathscr{B}_k^{\oplus\delta}$. As $\mathscr{W}_k$ is finitely generated as an $\mathscr{O}_k$-module and $\mathscr{B}_k$ is, as a D_k-module, a direct limit of objects in $\langle\mathscr{W}_k\rangle_\otimes$, it follows that $\mathscr{V}'$ is a sub-object of an element in $\langle\mathscr{W}_k\rangle_\otimes$ and consequently is also an object of $\langle\mathscr{W}_k\rangle_\otimes$.

Let us suppose now that $\mathscr{V}'$ is in the category $\langle\mathscr{W}_k\rangle_\otimes$; we wish to show that $\mathscr{V}'$ is $\mathscr{B}_k$-periodic. The equivalence $\overline{\xi}_k^*$ of tensor categories can be extended to an equivalence between $\mathrm{Ind}(\langle\mathscr{W}_k\rangle_\otimes)$ and the category of all G_0-modules (see §1.2.6 for conventions). Thus, we are left with the task of showing that there is an isomorphism of G_0-modules

$$\mathscr{O}(G_0)_{\mathrm{left}}^{\oplus\delta}\cong\overline{\xi}_k^*(\mathscr{V}')\otimes_k\mathscr{O}(G_0)_{\mathrm{left}} \tag{19}$$

which is also an isomorphism of $\mathscr{O}(G_0)$-modules. This is accomplished by noting that for any $(W,\omega)\in\mathrm{Rep}_k(G_0)$:

$$\mathrm{Mor}\left(G_0,\mathbb{A}^{\dim W}\right)\longrightarrow\mathrm{Mor}(G_0,W_a),\qquad f\mapsto(x\mapsto\omega(x)\cdot f(x)) \tag{20}$$

induces the sought isomorphism

$$[\mathscr{O}(G_0)_{\mathrm{left}}]^{\oplus\dim W}\xrightarrow{\cong}\mathscr{O}(G_0)_{\mathrm{left}}\otimes(W,\omega). \tag{21}$$

□

As an application of [**32**, Thm. 2.11], we have:

Corollary 35. — *Notations as in corollary 34. Assume that* $\mathrm{Ext}^1_{G_0}(\overline{\xi}_k^*\mathscr{V}_k,\overline{\xi}_k^*\mathscr{V}_k)$ *is finite dimensional. Then the functor* $\mathrm{Def}_{\mathscr{B}}:\mathscr{C}_\Lambda\longrightarrow\mathrm{Set}$ *is pro-represented by a complete noetherian* Λ*-algebra* $R(\nabla_k,\mathscr{B})$. □

We now give some examples of affine group schemes G/k satisfying $\dim H^1(G,V)<\infty$ for all $V\in\mathrm{Rep}_k(G)$ (and hence the hypothesis of Corollary 35). As one can use the Hochschild complex to compute group cohomology [**17**, Ch. 1, 4.16, p. 63], finite group schemes are an obvious class of examples. Another important class are reductive groups. Let B be a Borel subgroup of the reductive group G (all over k) and let M and N be representations of B and G respectively. Using the spectral sequence in [**17**, 4.5, p. 58], we see that $H^1(G,\mathrm{ind}_B^G M)\hookrightarrow H^1(B,M)$. Since $H^1(B,M)$ is finite dimensional [**17**, 4.10(a), p. 235], $H^1(G, \mathrm{ind}_B^G M)$ is likewise. Let L be a simple G-module; L is then a submodule of some induced character $\mathrm{ind}_B^G(\lambda)$ [**17**, 2.4, p. 200] – which is finite dimensional. It follows that $\dim H^1(G, L) < \infty$. Consequently, $\dim H^1(G; V) < \infty$ for any representation.

9. Deformations of representations and deformations of D-modules

We maintain the notations introduced in the beginning of §6.

9.1. Organization and assumptions. — We assume that $\mathrm{End}_{D_k}(\mathscr{V}_k) = k$. Let G_k be the differential Galois group of $\mathscr{V}_k$ at the point ξ_k: this means that we have a commutative diagram

$$\begin{array}{ccc} \langle(\mathscr{V}_k, \nabla_k)\rangle_\otimes & \xrightarrow{\xi_k^*} & (k\text{-mod}) \\ & \searrow^{\overline{\xi}_k^*} & \uparrow \text{forget} \\ & & \mathrm{Rep}_k(G_k), \end{array}$$

where $\overline{\xi}_k^*$ is a monoidal equivalence. Assume that G_k lifts to a smooth and affine group scheme over Λ: G. We shall fix a free Λ-module V_Λ such that $V_k = V_\Lambda \otimes k$ affords the representation

$$\overline{\xi}_k^*(\mathscr{V}_k) = \rho_k : G_k \longrightarrow \mathrm{GL}(V_k).$$

These hypothesis produce a G_k-torsor

$$\mathscr{B}_k \in \mathrm{Ind}\,\mathbf{str}(\mathscr{O}_k/k)$$

(terminology from §2.3) by using the left regular representation (as in §2.3.4); see the discussion adjacent to formula (11), which becomes valid in the present situation. We let

$$\mathscr{B} \in \mathrm{Ind}\,\mathbf{str}(\mathscr{O}/\Lambda)$$

be a G-torsor reducing to $\mathscr{B}_k$ modulo ϖ; the existence is, of course, guaranteed by Theorem 16. Lemma 29 assures that the assumptions **B1** and **B2** are satified. The fact that assumption **B3** holds in this case was discussed in the proof of Corollary 34, see eqs. (20) and (21)).

Notation: If we let $\mathscr{H}$ denote the object of $\mathrm{Ind}\,\mathbf{str}(\mathscr{O}/\Lambda)$ obtained by endowing $\mathscr{O}\otimes_\Lambda\mathscr{O}(G)$ with the trivial stratification, then the co-action morphism

$$\mathscr{B}\longrightarrow\mathscr{B}\otimes_{\mathscr{O}}\mathscr{H}$$

(which is an arrow of algebras in $\mathrm{Ind}\,\mathbf{str}(\mathscr{O}/\Lambda)$) will be denoted by κ.

9.2. The functor of deformations of group representations. —

Definition 36. — *For each $A\in\mathscr{C}_\Lambda$, let $\mathrm{Def}^+_{\rho_k}(A)$ denote the set of all homomorphisms of group schemes*

$$\rho_A : G_A\longrightarrow \mathrm{GL}(V_A)$$

such that $\rho_A\otimes_A k=\rho_k$. Define $\Gamma(A)$ as the group

$$\ker\left(\mathrm{Aut}_A(V_A)\longrightarrow\mathrm{Aut}_k(V_k)\right).$$

Let Γ act on the right of $\mathrm{Def}^+_{\rho_k}(A)$ by conjugation. The functor

$$A\mapsto \mathrm{Def}^+_{\rho_k}(A)/\Gamma(A)$$

will be denoted by Def_{ρ_k}.

Proposition 37. — *(i) The functor Def_{ρ_k} is homogeneous. (ii) Its tangent space $\mathrm{Def}_{\rho_k}(k[\varepsilon])$ is isomorphic to $\mathrm{Ext}^1_{G_k}(V_k,V_k)$.*

Proof. — The proof of the homogeneity of Def_{ρ_k} – in the presence of $\mathrm{End}_{G_k}(V_k)=k$ – is similar to the proof of homogeneity of Def. (That is, one uses the criterion given in Proposition 24 and the same method $I_\rho\cong\widehat{\mathbb{G}}_m$ as in Theorem 26.) A calculation of the tangent space can be extracted from [SGA 3 III], Theorem 2.1 and its corollaries. A more methodic way of proceeding uses Pridham's approach [**29**] of simplicial deformation complexes and the co-monadic adjunction

$$\mathscr{O}(G_A)\text{-Comod}^{\#}\underset{U_*}{\overset{U^*}{\rightleftarrows}} A\text{-Mod}^{\#}\ ,\quad U^*\dashv U_*, \tag{22}$$

where, for $A\in\mathscr{C}_\Lambda$, $\mathscr{O}(G_A)$-Comod$^{\#}$ (resp. A-Mod$^{\#}$) denotes the category of $\mathscr{O}(G_A)$-comodules which are flat as A-modules (resp. A-flat modules), U^* the forgetful functor and $U_*:M\mapsto\mathscr{O}(G_A)_{\mathrm{right}}\otimes_A M$. ◻

9.3. Main result: A natural isomorphism $\mathrm{Def}_{\rho_k} \longrightarrow \mathrm{Def}_{\mathscr{B}}$**.** — Fix $A \in \mathscr{C}_\Lambda$. Since $\mathscr{B}$ is a torsor under G, the twisting construction (§2.4.3) defines a (monoidal) functor

$$\mathscr{B}_A \times^{G_A} (?) : \mathrm{Rep}_A(G_A) \longrightarrow \mathbf{str}(\mathscr{O}_A/A).$$

(We are again abusing notation as we did in §5.2, p. 32: to perform the twisting construction we are using the Hopf algebra $\mathscr{H}_A = \mathscr{O}(G_A) \otimes_A \mathscr{O}_A$.) If $\rho_A : G_A \longrightarrow \mathrm{GL}(V_A)$ is a lifting of ρ_k, Corollary 8 assures that $\mathscr{B}_A \times^{G_A} (\rho_A)$ is a deformation of $(\mathscr{V}_k, \nabla_k)$, so that we obtain a *natural transformation* of deformation functors

$$\mathrm{Def}_{\rho_k} \longrightarrow \mathrm{Def}.$$

In fact, this construction gives a natural transformation

$$\tau_{\mathscr{B}} : \mathrm{Def}_{\rho_k} \longrightarrow \mathrm{Def}_{\mathscr{B}}$$

due to the following:
Observation (1): the functor $\mathscr{B}_A \times^{G_A} (?)$ transports the torsor $\mathscr{O}(G_A)_{\mathrm{left}}$ of the example in §2.3.4 to the torsor $\mathscr{B}_A$ and
Observation (2): for each $W_A \in \mathrm{Rep}_A^{\#}(G_A)$ of rank r, the representation $W_A \otimes \mathscr{O}(G_A)_{\mathrm{left}}$ is isomorphic to $\mathscr{O}(G_A)_{\mathrm{left}}^{\oplus r}$ as indicate eqs. (20) and (21).

We can now state the main result.

Theorem 38. — *The natural transformation $\tau_{\mathscr{B}}$ defined above is an isomorphism of functors.*

The proof relies mainly on the procedure that associates to every $\mathscr{B}$-periodic deformation a representation of the group. It is the topic of the following paragraph.

The holonomy. — We recall in a more algebraic setting some usual constructions from differential Galois theory (see for example [**33**, Obs. 1.26(1), p. 19] and [**33**, 2.33]).

Let $A \in \mathscr{C}_\Lambda$ and assume that $(\mathscr{V}_A, \nabla_A)$ is a $\mathscr{B}$-periodic deformation. By definition, there exists an isomorphism of D_A-modules

$$\Theta : \mathscr{B}_A^{\oplus \mu} \longrightarrow \mathscr{B}_A \underset{\mathscr{O}_A}{\otimes} \mathscr{V}_A \qquad (23)$$

which is also an isomorphism of $\mathscr{B}_A$-modules. Bearing in mind the notations introduced on §9.1, we obtain an $\mathscr{H}_A = \mathscr{O}(G_A) \otimes_A \mathscr{O}_A$-comodule structure on $\mathscr{B}_A \otimes_{\mathscr{O}_A} \mathscr{V}_A$:

$$\kappa \underset{\mathscr{O}_A}{\otimes} \mathscr{V}_A : \mathscr{B}_A \underset{\mathscr{O}_A}{\otimes} \mathscr{V}_A \longrightarrow \left(\mathscr{B}_A \underset{\mathscr{O}_A}{\otimes} \mathscr{V}_A \right) \underset{\mathscr{O}_A}{\otimes} \mathscr{H}_A.$$

Using the isomorphism Θ, we obtain an $\mathscr{H}_A$-comodule structure

$$\chi : \mathscr{B}_A^{\oplus\mu} \longrightarrow \mathscr{B}_A^{\oplus\mu} \underset{\mathscr{O}_A}{\otimes} \mathscr{H}_A$$

on $\mathscr{B}_A^{\oplus\mu}$ (in the category $\mathbf{Str}(\mathscr{O}_A/A)$). This produces an $\mathscr{O}(G_A)$-comodule structure

$$\chi : \mathscr{B}_A^{\oplus\mu} \longrightarrow \mathscr{B}_A^{\oplus\mu} \underset{A}{\otimes} \mathscr{O}(G_A)$$

on the A-module $\mathscr{B}_A$ (with the additional property that χ is an arrow of D_A-modules). If we let

$$\mathrm{Hor} = \mathrm{Hom}(\mathbb{1}, ?) : \mathbf{Str}(\mathscr{O}_A/A) \longrightarrow (A\text{-Mod})$$

denote the functor of horizontal sections, Lemma 28 allows us to deduce a $\mathscr{O}(G_A)$-comodule structure on $\mathrm{Hor}(\mathscr{B}_A^{\oplus\mu}) = A^{\oplus\mu}$, i.e. a representation

$$\mathrm{hol}(\nabla_A; \Theta) : G_A \longrightarrow \mathrm{GL}_{\mu,A}. \tag{24}$$

Of course, $\mathrm{hol}(\nabla_A, \Theta)$ does not depend too much from the particular choice of Θ, due to the fact that any other such isomorphism is of the form $\Theta \circ (a_{ij})$ with $(a_{ij}) \in \mathrm{GL}_\mu(A)$. The representation in (24) will be called the *holonomy*.

Lemma 39. — *Let (V_A, η) be a representation of G_A whose underlying A-module is free of rank μ. Let $\mathscr{V}_A := \mathscr{B}_A \times^{G_A} V_A$ be the stratified module obtained from V_A. Then the holonomy of $\mathscr{V}_A$ is isomorphic to (V_A, η). In particular, the functor $\tau_{\mathscr{B}}$ is a monomorphism (injective on "A-valued points").*

Proof. — To ease notation we will omit the subscripts alluding to A. Also, we will write R_{left} (resp. R_{right}) instead of $\mathscr{O}(G_A)_{\mathrm{left}}$ (resp. $\mathscr{O}(G_A)_{\mathrm{right}}$). Let $\overline{R}$ and $\overline{V}$ denote the G-modules obtained by letting G act trivially; using eq. (20) we have an isomorphism of G-modules

$$\theta : R_{\mathrm{left}} \otimes \overline{V} \longrightarrow R_{\mathrm{left}} \otimes (V, \eta).$$

The isomorphism

$$\Theta := \mathscr{B} \times^G (\theta) : \mathscr{B}^{\oplus\mu} \longrightarrow \mathscr{B} \underset{\mathscr{O}}{\otimes} \mathscr{V}$$

defines the $\mathscr{B}$-periodicity of $\mathscr{V}$. The holonomy representation is constructed, by transport of structure using Θ, from the $\mathscr{H}$-comodule structure on $\mathscr{B} \otimes \mathscr{V}$ arising from the $\mathscr{H}$-comodule structure of $\mathscr{B}$. The latter comodule structure already "comes" from the category $\mathrm{Rep}'_A(G)$; the defining arrow is the image of

$$\Delta \otimes V : R_{\mathrm{left}} \otimes V \longrightarrow R_{\mathrm{left}} \otimes V \otimes \overline{R} \in \mathrm{Arrow}\left(\mathrm{Rep}'_A(G_A)\right)$$

by $\mathscr{B} \times^G (?)$ (see Obs. (1) on page 53). Thus, for a better understanding of $\mathrm{hol}(\Theta; \mathscr{V})$, we need to consider the arrow ζ in the commutative diagram (in $\mathrm{Rep}'_A(G)$)

(25)
$$\begin{array}{ccc} R_{\mathrm{left}} \otimes \overline{V} & \xrightarrow[\cong]{\theta} & R_{\mathrm{left}} \otimes V \\ \zeta \downarrow & & \downarrow \Delta\otimes V \\ [R_{\mathrm{left}} \otimes \overline{V}] \otimes \overline{R} & \xrightarrow[\theta\otimes\overline{R}]{\cong} & [R_{\mathrm{left}} \otimes V] \otimes \overline{R}. \end{array}$$

It is not hard to verify that ζ is the arrow $\Delta \boxtimes \eta$ (see §1.2.6 for notation) defining the G-module structure on $R_{\mathrm{right}} \otimes (V, \eta)$. (It is sufficient to show that the the diagram is still commutative if one replaces ζ by $\Delta \boxtimes \eta$, and one way to verify this is to give V a basis and do the calculations.) Therefore, the $\mathscr{H}$-comodule structure on $\mathscr{B}^{\oplus\mu}$, named χ in the above construction of $\mathrm{hol}(\Theta; \nabla_A)$, is obtained as the tensor product of the two $\mathscr{H}$-comodule structures $(\mathscr{B}, \kappa)$ and $(V \otimes_A \mathscr{O}, \eta \otimes \mathscr{O})$. Since $\mathrm{Hor}(\mathscr{B}) = A \cdot 1$, it follows that the representation $\mathrm{hol}(\Theta, \mathscr{V})$ is η. □

Proof of Theorem 38.— We make use of some basic terminology from the theory of functor of Artin rings. Let $\Phi, \Psi : \mathscr{C}_\Lambda \longrightarrow \mathrm{Set}$ be functors of Artin rings and let $h : \Phi \longrightarrow \Psi$ be a morphism between them. We say that h is smooth if for any given surjection $B \longrightarrow A$ in $\mathscr{C}_\Lambda$, the natural map

$$\Phi(B) \longrightarrow \Phi(A) \times_{\Psi(A)} \Psi(B)$$

is also surjective. (In particular, a functor Φ is smooth if and only if $\Phi \longrightarrow$ (trivial functor) is smooth.) We say that h is etale if it is smooth and $dh : t_\Phi \longrightarrow t_\Psi$ is an isomorphism.

As $\mathrm{Def}_{\mathscr{B}}$ is homogeneous (Theorem 31), it is sufficient to show that $\tau_{\mathscr{B}}$ is *etale* [**21**, Cor. 2.11]. It is possible to show that $d\tau_{\mathscr{B}}$ is an isomorphism by tracking the natural identifications giving the tangent spaces in Propositions 37 and 33, but, since the case of real interest is $\dim_k t_{\mathrm{Def}_{\mathscr{B}}} < \infty$, we only need to apply Lemma 39. Hence we are left with the verification of smoothness. This will follow from the *proof* of the standard smoothness criterion [**21**, Prp. 2.17] once we verify the following claim. (The hypothesis in [**21**, Prp. 2.17] concerning injectivity on obstruction spaces is essentially the content of the claim.)

Claim: Let $A' \longrightarrow A$ be a small extension in $\mathscr{C}_\Lambda$ and let $(V_A, \eta_A) \in \mathrm{Def}_{\rho_k}(A)$ be such that $\mathscr{B}_A \times^{G_A} \eta_A =: (\mathscr{V}_A, \nabla) \in \mathrm{Def}_{\mathscr{B}}(A)$ admits a lifting $(\mathscr{V}_{A'}, \nabla') \in \mathrm{Def}_{\mathscr{B}}(A')$. Then there exists a lift of (V_A, η_A) to $\mathrm{Def}_{\rho_k}(A')$.

Proof: The holonomy representation $\mathrm{hol}(\nabla')$ is a lifting of $\mathrm{hol}(\nabla)$, the latter is isomorphic, as we have seen in Lemma 39, to (V_A, η_A).

We have proved Theorem 38. □

Corollary 40. — *Let us assume that G_k is a reductive and that V_k is a simple G_k-module. Assume furthermore that, for each $l \in \mathbb{N}$, there exists a lifting $\rho_{\Lambda/\varpi^l} : G \otimes \Lambda/\varpi^l \longrightarrow \mathrm{GL}(V_\Lambda \otimes \Lambda/\varpi^l)$ of ρ_k. Then Def_{ρ_k} and $\mathrm{Def}_{\mathscr{B}}$ are pro-represented by Λ.*

Proof. — The tangent space of Def_{ρ_k} is zero dimensional since $\mathrm{Ext}^1_{G_k}(V_k, V_k) = 0$ [**17**, Part II, Ch 2, 2.1.2]. Hence, the ring R pro-representing Def_{ρ_k} (and $\mathrm{Def}_{\mathscr{B}}$) is isomorphic to either Λ or Λ/ϖ^j for some $j \in \mathbb{N}$. The existence of the lifting to Λ/ϖ^{j+1} means that the possibility $R = \Lambda/\varpi^j$ is to be discarded. Thus $R = \Lambda$. □

Corollary 41. — *Keep the assumptions and notations of Corollary 40. Let $(\mathscr{V}, \nabla)$ and $(\mathscr{V}, \nabla')$ be two objects of $\mathbf{str}(\mathscr{O}/\Lambda)$ such that (1) $\nabla \otimes k = \nabla' \otimes k = \nabla_k$ and (2) the reductions of $(\mathscr{V}, \nabla)$ and $(\mathscr{V}, \nabla')$ modulo ϖ^n are $\mathscr{B}$-periodic for each given $n \in \mathbb{N}$. Then $(\mathscr{V}, \nabla) \cong (\mathscr{V}, \nabla')$ as objects in $\mathbf{str}(\mathscr{O}/\Lambda)$.*

Proof. — Denote the reductions modulo ϖ^{n+1} by a subscript n. From Corollary 40, we know that, for each $n \in \mathbb{N}$, there exists $g_n \in \mathbf{G}(\Lambda/\varpi^n)$ such that

$\nabla_n * g_n = \nabla'_n$. We want to conclude that there exists an element

$$\gamma = (\gamma_n) \in \varprojlim_n \mathbf{G}(\Lambda_n) = \mathrm{Aut}_{\mathscr{O}}(\mathscr{V})$$

which verifies $\nabla_n * \gamma_n = \nabla'_n$ for each n. This is proved by induction using that the homomorphism in eq. (16) is surjective. □

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