ON THE MONODROMY OF HOLOMORPHIC DIFFERENTIAL SYSTEMS

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ABSTRACT. First we survey and explain the strategy of some recent results that construct holomorphic sl(2, $\mathbb{C}$)–differential systems over some Riemann surfaces Σ_g of genus $g \geq 2$, satisfying the condition that the image of the associated monodromy homomorphism is (real) Fuchsian [10] or some cocompact Kleinian subgroup

$$\Gamma \subset \mathrm{SL}(2, \mathbb{C})$$

as in [11]. As a consequence, there exist holomorphic maps from Σ_g to the quotient space $\mathrm{SL}(2, \mathbb{C})/\Gamma$, where $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ is a cocompact lattice, that do not factor through any elliptic curve [11]. This answers positively a question of Ghys in [19]; the question was also raised by Huckleberry and Winkelmann in [24].

Then we prove that when M is a Riemann surface, a Torelli type theorem holds for the affine group scheme over $\mathbb{C}$ obtained from the category of holomorphic connections on *étale trivial* holomorphic bundles.

After that, we explain how to compute in a simple way the holonomy of a holomorphic connection on a free vector bundle.

Finally, for a compact Kähler manifold M, we investigate the neutral Tannakian category given by the holomorphic connections on étale trivial holomorphic bundles over M. If ϖ (respectively, Θ) stands for the affine group scheme over $\mathbb{C}$ obtained from the category of connections (respectively, connections on free (trivial) vector bundles), then the natural inclusion produces a morphism $v : \mathcal{O}(\Theta) \longrightarrow \mathcal{O}(\varpi)$ of Hopf algebras. We present a description of the transpose of v in terms of the iterated integrals.

1. INTRODUCTION

In recent times, there has been a resurgent interest in the theory of holomorphic *differential systems*, or connections on trivial bundles, in the special case of a *compact* ambient space. This interest has been mainly triggered by a strategy of Ghys, presented in [16], to answer a question raised first by Huckleberry and Winkelmann in [24] and later by Ghys in [19]. This question in [23, 19] inquires about the existence of holomorphic maps from compact hyperbolic Riemann surfaces to compact complex manifolds of the form $\mathrm{SL}(2, \mathbb{C})/\Gamma$, with Γ a lattice in $\mathrm{SL}(2, \mathbb{C})$ (which does not factor to any elliptic curve). The suggested direction in [16] towards an answer is the study of representations of surface groups into $\mathrm{SL}(2, \mathbb{C})$ giving rise to rank two *free* (trivial) holomorphic vector bundles over some Riemann surface of genus $g \geq 2$ (see Question 2.2).

Although the theory of differential systems (holomorphic connections on free vector bundles) has always been a much studied subject, the restrictions imposed by a compact

ambient space are rather pleasing and produce fruitful geometric situations. The present paper is a contribution to this theme, containing an extended survey presenting different backgrounds and recent advances.

To motivate the reader and display some of the beautiful geometry we are set to explain, we begin by surveying some recent results on the theory of holomorphic $SL(2,\mathbb{C})$–differential systems over compact Riemann surfaces and their monodromy representations. This is done in Section 2, where after reviewing Ghys' questions and its relation to the theory of differential systems (see Question 2.1 and Question 2.2), we move on to explain some local results about the Riemann-Hilbert mapping, proved in [16, 8]. (It is perhaps worth noting that the main results of these works were recently extended to the logarithmic case in [1].)

Despite the local results obtained in [16], for curves of genus $g = 2$, and then in [8, 1], Ghys' question was still open until very recently. In particular, it was not known whether one can realize given real, discrete or Zariski dense subgroups in $SL(2,\mathbb{C})$ as monodromy of $SL(2,\mathbb{C})$–local systems over some Riemann surface Σ_g of genus $g \geq 2$.

An important step toward the understanding of the image of the monodromy homomorphism was very recently realized in [10] where holomorphic connections with (real) Fuchsian monodromy were constructed. This completely answers a question asked in [16].

Eventually Ghys strategy has been successfully implemented in the very recent work [11]. It is first shown in [11] that every irreducible $SL(2,\mathbb{R})$–representation with sufficient many symmetries can be realized as the monodromy of a holomorphic irreducible $SL(2,\mathbb{C})$–connection on the rank two trivial holomorphic bundle over some (very symmetric) Riemann surface Σ such that genus$(\Sigma) \geq 2$. Then an example of such a symmetric representation contained in a cocompact lattice Γ in $SL(2,\mathbb{C})$ is given and a holomorphic map

$$\Sigma \longrightarrow SL(2,\mathbb{C})/\Gamma$$

is constructed (which does not descend on any elliptic curve), answering the open question of Ghys [19] and Huckleberry-Winkelmann [24].

Once light was thrown on the pertinence of the theory of differential systems on compact spaces, it became clear, see for example [6], that a reasonable way to study holomorphic connections is to impose conditions on the underlying vector bundle. It is with this idea in mind that we go over to Section 3. There, we give ourselves a compact complex manifold M and study integrable connections where the underlying vector bundle enjoys the following property: it becomes holomorphically trivial after an étale covering or, it is *étale trivial.* (We explain at the start of Section 3 that the latter property can be recovered from the existence of a connection with finite monodromy.) We start by reviewing recent work [5, 8] which connects étale trivial bundles with *finite* [30, p. 35] and *"properly" trivial* ones [12, Property (T), p. 225]. Then, we follow the path opened by Tannakian categories and define an affine group scheme whose representations amount to the aforementioned connections.

Another of the pleasant properties of differential systems (in compact ambient) is the simplicity in which differential Galois groups can be computed, as was the theme in [4]. (This paper takes the matter from the purely algebro-geometric viewpoint.) Section 4 deals with an analogue of [4] in a complex analytic setting by showing that the smallest

reduction of structure group preserving a connection is given by the algebraic hull of the "connection matrices".

In Section 5, we study the relation between the category of integrable connections and the category of integrable connections on trivial vector bundles by the Tannakian point of view, as in [6]. Moreover, this study is done by putting together Hochschild-Mostow's [22, p.1140] (as understood by [39]) and Saavedra-Grothendieck's [35] approach to Tannaka duality. Recall that the first one is achieved by means of "representative functions" while the second is, from the start, categorical. In explicit terms: Let M be a compact complex manifold, x_0 a point in M, $\mathcal{C}_{\mathrm{dR}}(M)$ and $\mathcal{T}(M)$ respectively the categories of integrable connections and integrable connections on trivial vector bundles. As is well-known, $\mathcal{C}_{\mathrm{dR}}(M)$ is equivalent to the category of $\mathbb{C}\pi_1(M, x_0)$–modules while, as argued in [4], $\mathcal{T}(M)$ is equivalent to the category of representations of a certain co-commutative Hopf algebra

$$\mathfrak{A}_M = \frac{\text{Tensor algebra on } H^0(M, \Omega^1_M)^*}{\text{a certain ideal}}.$$

Following [39, Chapter VI], these categories are then, respectively, equivalent to the categories of comodules over the Hopf-algebras $\mathbb{C}\pi_1(M, x_0)^\circ$ and $\mathfrak{A}_M^\circ$. Hence, the morphism $\mathfrak{A}_M^\circ \longrightarrow \mathbb{C}\pi_1(M, x_0)^\circ$ coming from the inclusion $\mathcal{T}(M) \longrightarrow \mathcal{C}_{\mathrm{dR}}(M)$ through [35] gives rise to an arrow between certain completions:

$$u : \mathbb{C}\pi_1(M, x_0)\widehat{} \longrightarrow \widehat{\mathfrak{A}}_M;$$

it is rendering explicit the composition $\mathbb{C}\pi_1(M, x_0) \longrightarrow \mathbb{C}\pi_1(M, x_0)\widehat{} \longrightarrow \widehat{\mathfrak{A}}_M$ in terms of iterated integrals that occupies Section 5. In doing so, we recover expressions appearing in the works of Chen, Parshin and Hain but, in our context, the source and target of the map u are bigger completions and carry more information. To wit, these are the pro-finite *dimensional* (as opposed to pro-*finite*) completions.

This work is dedicated to Oscar García-Prada on the occasion of his sixtieth birthday.

2. Curves in compact quotients of $\mathrm{SL}(2,\mathbb{C})$

A natural class of compact complex manifolds generalizing complex tori are complex manifolds whose holomorphic tangent bundle is holomorphically trivial. By a classical result of Wang [40], compact complex manifolds with holomorphically trivial tangent bundle are biholomorphic to the complex manifolds of the form G/Γ, where G is a complex Lie group and $\Gamma \subset G$ a discrete cocompact subgroup. Compact complex manifolds in this class are called *(complex) parallelizable manifolds*. They are in general not Kähler. More precisely, a complex parallelizable manifold G/Γ is Kähler if and only if the complex Lie group G is abelian, in which case G/Γ is biholomorphic to a compact compact complex torus [40].

In particular, when G is a complex semi-simple Lie group, and $\Gamma \subset G$ is a cocompact lattice, the complex parallelizable manifold G/Γ is non-Kähler. Moreover, a theorem due to Huckleberry and Margulis [23] says that G/Γ does not admit any complex analytic hypersurface. This implies, in particular, that G/Γ does not admit any nonconstant meromorphic function and, consequently, the algebraic dimension of the complex parallelizable manifold G/Γ is zero.

When G is a complex semi-simple Lie group with no local factor isomorphic to $\mathrm{SL}(2, \mathbb{C})$ (for example, $G = \mathrm{SL}(n, \mathbb{C})$, with $n \geq 3$), a vanishing result of Raghunathan, [33], says that the complex structure of any compact quotient G/Γ, where Γ a discrete cocompact subgroup in G, is *rigid*: any complex structure on the underlying real manifold G/Γ close to the standard one (endowed by the complex structure of G) actually coincides with the standard one.

The remaining question about the rigidity (or flexibility) of the standard complex structure on the compact quotients G/Γ, with $G = \mathrm{SL}(2, \mathbb{C})$, was studied by Ghys in [19] (see also [34]). In this situation Ghys computes in [19] the Kuranishi space of $\mathrm{SL}(2, \mathbb{C})/\Gamma$ and proves that this deformation space is nontrivial for any discrete cocompact subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ with positive first Betti number. This implies that the standard complex structure of these compact quotients of $\mathrm{SL}(2, \mathbb{C})$ is flexible and admit nontrivial deformations described by Ghys.

It may be mentioned that the discrete cocompact subgroups $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ with positive first Betti number are quite abundant (see [27]); their existence is closely related to the compact quotients of the 3–hyperbolic space $\mathbb{H}^3$ (i.e. the compact hyperbolic 3–manifolds). More precisely, the complex Lie group $\mathrm{PSL}(2, \mathbb{C})$ being the orientation preserving isometry group of the 3-hyperbolic space $\mathbb{H}^3$, it is identified with the oriented orthonormal frame bundle of $\mathbb{H}^3$. Consequently, the orthonormal frame bundle of any compact hyperbolic 3–manifold M (diffeomorphic to a quotient $\mathbb{H}^3/\Gamma$, with Γ a lattice in $\mathrm{SL}(2, \mathbb{C})$) admits the structure of a compact complex parallelizable manifold which is biholomorphic to a quotient of $\mathrm{SL}(2, \mathbb{C})$ by its cocompact lattice Γ (isomorphic to the fundamental group of M). One of the reasons for the complex geometry of the compact quotients of $\mathrm{SL}(2, \mathbb{C})$ to be of interest is this fact.

As mentioned before, the flexibility of the complex structure of $\mathrm{SL}(2, \mathbb{C})/\Gamma$ was discovered by Ghys in [19] (see also [34]) where he showed that the corresponding Kuranishi space has positive dimension for all Γ with positive first Betti number. Notice that compact hyperbolic 3–manifolds with prescribed rational cohomology ring (in particular, with arbitrarily large first Betti number) can be constructed using Thurston's hyperbolisation Theorem (see [19, Lemme 6.2]). It should be mentioned that the complex geometry of the orthonormal frame bundle of the compact hyperbolic 3–manifold $\mathbb{H}^3/\Gamma$ endowed with the complex structure $\mathrm{SL}(2, \mathbb{C})/\Gamma$ is not well understood yet.

When studying the geometry of compact quotients of $\mathrm{SL}(2, \mathbb{C})$, Ghys asked in [19] the following question, also raised by Huckleberry and Winkelmann in [24]:

Question 2.1. *Does there exist a compact quotient* $\mathrm{SL}(2, \mathbb{C})/\Gamma$ *admitting a compact holomorphic curve of genus* $g \geq 2$?

A negative answer to this question would generalize the Huckleberry-Margulis Theorem [23] (for $G = \mathrm{SL}(2, \mathbb{C})$) to compact holomorphic curves of genus $g \geq 2$. Notice that, in the case $g = 1$, elliptic curves covered by one-parameter subgroups in $\mathrm{SL}(2, \mathbb{C})$ are known to exist in certain quotients $\mathrm{SL}(2, \mathbb{C})/\Gamma$.

In an attempt to answer positively Question 2.1, Ghys developed the following strategy to characterize compact holomorphic curves of genus $g \geq 2$ in quotients $\mathrm{SL}(2, \mathbb{C})/\Gamma$ (see [16] where this strategy is explained in detail).

Find a Riemann surface Σ of genus $g \geq 2$ and an irreducible holomorphic $\mathrm{SL}(2, \mathbb{C})$–connection on the rank two trivial holomorphic bundle $\Sigma \times \mathbb{C}^2$ over Σ such that the image of the corresponding monodromy homomorphism lies in a cocompact lattice Γ in $\mathrm{SL}(2, \mathbb{C})$.

Recall that a holomorphic connection on a holomorphic bundle over a Riemann surface is automatically flat.

As Ghys observed, in this context, the parallel frame of the above (flat) holomorphic connection gives rise to a holomorphic map from Σ into the quotient $\mathrm{SL}(2, \mathbb{C})/\Gamma$ (which, due to the irreducibility of the connection, does not factor through any elliptic curve) (see [16] for more details). More generally, Ghys asked:

Question 2.2. *Given a compact orientable surface X of genus $g \geq 2$, describe the space of all irreducible group homomorphisms $\rho : \pi_1(X) \longrightarrow \mathrm{SL}(2, \mathbb{C})$ for which there exists a complex structure Σ on X such that the holomorphic vector bundle over Σ defined by ρ is holomorphically trivial.*

If some irreducible homomorphism $\rho : \pi_1(X) \longrightarrow \mathrm{SL}(2, \mathbb{C})$ satisfying the above condition has the property that $\rho(\pi_1(X))$ is contained in a cocompact lattice $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$, then the local system on the Riemann surface Σ given by ρ produces a nontrivial holomorphic map from Σ to $\mathrm{SL}(2, \mathbb{C})/\Gamma$ (more details about this method can be found in [16]).

A first local result in the direction of Question 2.2 was obtained in [16] for the genus $g = 2$. Other local results generalizing the one in [16] were obtained in [8], and then in [1] for logarithmic differential systems. We briefly recall below those local results obtained in [16, 8].

The aim of both [16, 8] is to study the Riemann-Hilbert correspondence for differential systems over compact Riemann surfaces of genus $g \geq 2$. Denote by X a compact connected oriented topological surface of genus $g \geq 2$. We also denote by G a connected reductive affine algebraic group defined over the field of complex numbers.

Let us now fix a complex structure Σ on X (equivalently, it is the choice of an element in the Teichmüller space of X). We consider also a holomorphic connection ∇ on the trivial holomorphic principal G–bundle $\Sigma \times G$ over Σ. Notice that ∇ is determined by an element $\delta \in \mathfrak{g} \otimes H^0(\Sigma, K_\Sigma)$, where $\mathfrak{g}$ is the complex Lie algebra of G and K_Σ is the canonical bundle of Σ. Fix a base point $x_0 \in X$ and consider the corresponding universal cover $\pi : \widetilde{\Sigma} \longrightarrow \Sigma$ of Σ (endowed with the induced complex structure). On the trivial principal G–bundle $\widetilde{\Sigma} \times G$ over $\widetilde{\Sigma}$ we have the pulled back holomorphic (flat) connection $\pi^*\nabla$.

For any locally defined ∇–parallel section ϕ of $\Sigma \times G$, the pulled back local section $\pi^*\phi$ of $\widetilde{\Sigma} \times G$ extends to a $\pi^*\nabla$–parallel section on entire $\widetilde{\Sigma}$. The above extension of $\pi^*\phi$ identifies with a holomorphic map $\widetilde{\Sigma} \longrightarrow G$ which is $\pi_1(X, x_0)$–equivariant with respect to the natural action of $\pi_1(X, x_0)$ on $\widetilde{\Sigma}$ through deck transformations and the action of $\pi_1(X, x_0)$ on G through a group homomorphism $\pi_1(X, x_0) \longrightarrow G$, which is known as the *monodromy* (homomorphism) of the flat connection ∇.

The above monodromy homomorphism depends on the choice of the holomorphic trivialization of the principal G–bundle over $\widetilde{\Sigma}$. Nevertheless, the corresponding element of

the character variety of G–representations

$$\Xi := \mathrm{Hom}(\pi_1(X),\,G)/\!\!/G$$

does not depend either on the choice of the trivialization of the principal G–bundle over $\widetilde{\Sigma}$, or on the choice of the base point x_0 (or on the choice of the local section ϕ).

The character variety Ξ is known to be a singular complex analytic space of complex dimension $2((g-1)\cdot\dim[G,\,G]+g\cdot(\dim G-\dim[G,\,G]))$; see, for example, [37, Proposition 49]. The dimension of Ξ equals $2(g-1)d + 2gc$, with d being the complex dimension of the commutator group $[G,\,G]$ for G and c being equal to $\dim G - \dim[G,\,G]$.

As in [16, 8], denote by Syst the space of all differential systems, meaning the space of the pairs $(\Sigma,\,\nabla)$, where Σ is a complex structure on X (an element of the Teichmüller space for X) and ∇ is a holomorphic connection on the trivial holomorphic principal G–bundle $\Sigma \times G$ over Σ. This space of differential systems on X is a singular complex analytic space of dimension $(g-1)(d+3) + gc$. Associating to any pair $(\Sigma,\,\nabla) \in$ Syst the element in the character variety Ξ corresponding to the monodromy morphism of ∇, we obtain a holomorphic map from Syst to the character variety Ξ. This map is the restriction to Syst of the Riemann–Hilbert correspondence.

We consider now the nontrivial Zariski open subset $\mathrm{Syst}^{\mathrm{irred}}$ in Syst defined by all pairs $(\Sigma,\,\nabla)$ such that the holomorphic connection ∇ is *irreducible* (i.e., the monodromy homomorphism associated to ∇ does not factor through any proper parabolic subgroup of G). The Zariski open set $\mathrm{Syst}^{\mathrm{irred}}$ is a smooth complex manifold (see, for example, Corollary 50 in [37]). The image of $\mathrm{Syst}^{\mathrm{irred}}$ under the Riemann–Hilbert correspondence described above lies in the smooth Zariski orbifold open subset

$$\Xi^{\mathrm{irred}} \subset \Xi$$

defined by the irreducible homomorphisms $\pi_1(X) \longrightarrow G$ (i.e., homomorphisms that do not factor through some proper parabolic subgroup of G).

Denote by Mon (as monodromy) this restriction of the Riemann–Hilbert correspondence to $\mathrm{Syst}^{\mathrm{irred}}$:

$$\mathrm{Mon}\,:\,\mathrm{Syst}^{\mathrm{irred}} \longrightarrow \Xi^{\mathrm{irred}} \tag{2.1}$$

Then Mon is a holomorphic map between the complex manifolds $\mathrm{Syst}^{\mathrm{irred}}$ and Ξ^{irred}.

The main result proved in [8] is the following:

Theorem 2.3 ([8]). *If the complex dimension of G is at least three, the map* Mon *in* (2.1) *is an immersion at the generic point.*

If $G = \mathrm{SL}(2,\mathbb{C})$, the dimensions of $\mathrm{Syst}^{\mathrm{irred}}$ and Ξ^{irred} are both $6g-6$. Therefore, in that case Theorem 2.3 implies that Mon is a local biholomorphism at the generic point. It should be mentioned that examples constructed in [16] show that for $G = \mathrm{SL}(2,\mathbb{C})$ and Σ of genus $g \geq 3$, the monodromy map Mon is not always a local biholomorphism (over the entire $\mathrm{Syst}^{\mathrm{irred}}$).

When $G = \mathrm{SL}(2,\mathbb{C})$ and $g = 2$, the main result of [16] says that the map Mon in (2.1) is a local biholomorphism over entire $\mathrm{Syst}^{\mathrm{irred}}$. An alternative proof of this result of [16] was also provided in [8, Corollary 5.6].

The main motivation for the authors of [16] and [8] was the case $G = \mathrm{SL}(2,\mathbb{C})$ because Ghys strategy relates the monodromy of $\mathfrak{sl}(2,\mathbb{C})$–differential systems to the question of existence of holomorphic curves of genus $g > 1$ lying in quotients of $\mathrm{SL}(2,\mathbb{C})$ by cocompact lattices.

It was asked in [16] if there exist holomorphic $\mathfrak{sl}(2,\mathbb{C})$–differential systems on Riemann surfaces of genus $g > 1$ having real or discrete monodromy. Indeed, despite those local results obtained in [16], for curves of genus $g = 2$, and then in [8, 1]. Ghys' question was still open until very recently. In particular, it was not known whether it is possible to realize any discrete or Zariski dense subgroup in $\mathrm{SL}(2,\mathbb{C})$ as the monodromy of a $\mathrm{SL}(2,\mathbb{C})$–local system for some complex structure Σ on X.

A first answer to this question was given in [9] where the main result is a construction of an irreducible holomorphic connection with $\mathrm{SL}(2,\mathbb{R})$–monodromy on the rank two trivial holomorphic vector bundle over a compact Riemann surface of genus $g > 1$.

Another important step towards realizing Ghys' strategy was very recently made in [10] where holomorphic connections with (real) Fuchsian monodromy were constructed. The result is the following:

Theorem 2.4 ([10]). *For every integer $g \geq 2$, there exists a (hyperelliptic) Riemann surface Σ_g of genus g, such that the rank two trivial holomorphic vector bundle $\Sigma_g \times \mathbb{C}^2$ over Σ_g admits infinitely many holomorphic $\mathrm{SL}(2,\mathbb{C})$–connections with Fuchsian monodromy representation.*

The above result completely answers the question asked in [16] by Calsamiglia, Deroin, Heu and Loray.

In this context it is also natural to ask which holomorphic rank two bundles over a given Riemann surface Σ admit holomorphic connections with Fuchsian monodromy representations. This question was indeed raised by Katz in [25, p. 555–556] (where the question is attributed to Bers) in 1978 and is still unsolved. Even when restricting to the trivial rank two holomorphic bundle, it was not known before [10] whether a holomorphic connection ∇ with Fuchsian monodromy representation exists.

Eventually, Ghys strategy was successfully realized in the recent work [11]. In [11] it is first shown that every irreducible $\mathrm{SL}(2,\mathbb{R})$–representation with sufficient many symmetries can be realized as the monodromy of a holomorphic irreducible $\mathrm{SL}(2,\mathbb{C})$–connection on the rank two trivial holomorphic bundle over some (very symmetric) Riemann surface Σ such that genus$(\Sigma) \geq 2$. Then an example of such a symmetric representation with image contained in a cocompact lattice Γ in $\mathrm{SL}(2,\mathbb{C})$ is constructed, with Γ being by the dodecahedron tiling $\{5, 3, 4\}$ of the hyperbolic 3-space $\mathbb{H}^3$. The results in [11] imply the following:

Theorem 2.5 ([11]). *Let Γ be the cocompact lattice in $\mathbb{H}^3$ given by the dodecahedron tiling $\{5, 3, 4\}$ of the hyperbolic 3-space. Then there exists a non constant holomorphic map from a compact curve Σ of genus 4 in $\mathrm{SL}(2,\mathbb{C})/\Gamma$.*

Remark 2.6. The holomorphic map from Σ to $\mathrm{SL}(2,\mathbb{C})/\Gamma$, constructed in Theorem 2.5, has 4 simple branch points and by Riemann-Hurwitz Theorem it cannot factor through a lower genus surface.

This gives an affirmative answer to above question raised by Huckleberry and Winkelmann [24] and by Ghys [19].

2.1. **Strategy of the proofs of Theorem 2.4 and Theorem 2.5.** For both above results, namely Theorem 2.4 and Theorem 2.5, straightforward methods such as computing the monodromy representation explicitly or using geometric existence results, such as the uniformization metric for Fuchsian representations on the Gunning bundle, is not developed or applicable. Therefore, the general philosophy is to start with a given monodromy of a known holomorphic differential system and try to deform it until the deformed monodromy has the desired properties. This leaves of course a lot of freedom for implementation. Though initial setup of both proofs are the same, Theorem 2.5 requires a completely new set of further ideas as well as some further applications of the ideas in Theorem 2.4. Hence, we begin with describing the general setup as well as the strategy for Theorem 2.4.

The first step is to reduce complexity. We first impose various symmetries of the underlying Riemann surface and the considered holomorphic differential systems as follows. For every genus g consider the complex 1-dimensional family of Riemann surfaces which admit a $\mathbb{Z}_{g+1}$–action with 4 fixed points of order g, i.e, the Riemann surface is given by a $(g+1)-$fold covering of the complex projective line totally branched over 4 points. Then the connected components of $\mathbb{Z}_{g+1}$–equivariant connections are complex 2-dimensional subspaces of the de Rham moduli space of flat $\mathrm{SL}(2,\mathbb{C})$–connections. The Fuchsian representations of the symmetric Riemann surfaces automatically give rise to $\mathbb{Z}_{g+1}$–equivariant connections. By [3], all these equivariant connections are determined by logarithmic $\mathrm{SL}(2,\mathbb{C})$–connections on the 4-punctured sphere with prescribed local conjugacy classes depending (only) on both the genus of the surface and the connected component of $\mathbb{Z}_{g+1}$–equivariant connections inside the de Rham moduli space.

For example, the eigenvalues of the residues of a logarithmic connection over the complex projective line have to be $\pm\frac{g}{2(g+1)}$, at each of the 4 singular points, in order to correspond to an equivariant Fuchsian representation, see [10, Section 3]. Moreover, there exist exactly one affine (complex) line of logarithmic connections in each of these connected components which correspond to holomorphic systems on the compact Riemann surface of genus g, given by a specific parabolic structure (see [21, Theorem 3.2 (5)] or [10, Proposition 3.1]).

By imposing compatible reality conditions on the Riemann surface and the flat connections respectively, we can reduce further such that (the smooth components of) the moduli space of real symmetric Riemann surfaces is real 1-dimensional, and the real subspace of the Betti moduli space is real 2-dimensional. In other words, the considered Riemann surfaces possess a real involution τ compatible with the $\mathbb{Z}_{g+1}$–action (which is equivalent to the 4 branch points lying on one circle) and the considered connections ∇ lies in the same gauge class as $\tau^*\overline{\nabla}$, we refer to ∇ then as a $\tau-$real connection. Considering $\tau-$real connections on real Riemann surfaces leads to certain traces of the monodromies being automatically real. In order for the connection to be called real in following, we require all the monodromy traces along every closed curve to be real.

2.1.1. *Theorem 2.4.* Theorem 2.4 asserts the existence of infinitely many holomorphic systems with Fuchsian monodromy on a real symmetric Riemann surface. The first observation is that τ–real connections in the appropriate component (determined by the eigenvalues) give rise to Fuchsian representations if it is real and the traces satisfy some additional inequalities. Since the underlying Riemann surface is a 4-punctured sphere, the character variety, or its Betti moduli space, is determined by 3 traces x, y and z. For given x and y the third one must satisfy the character variety equation, which has two solutions z and $\widetilde{z}$. Due to $\tau-$symmetry z and $\widetilde{z}$ are automatically real and the reality of x implies then y being real as well. When working with the one-puncture torus instead of the 4-punctured sphere we can avoid dealing with further inequalities determining the connected component of Fuchsian representations. In other words, in order to find a Fuchsian representation it suffices to show that just one (additional) trace x is real on the one-punctured torus.

The trivial holomorphic structure when lifting the rank two flat bundle to the compact Riemann surface Σ_g of genus g is determined by a specific parabolic structure (see [21, Theorem 3.2 (5)] or [10, Proposition 3.1]). Since the space of (τ–symmetric) Higgs fields on the other hand is only (real) one-dimensional, i.e., generated by some Ψ, it remains to show that there exist (infinitely many) parameters $t \in \mathbb{R}$ such that $\nabla + t\Psi$ has real x, where ∇ is the unique unitary connection inducing that parabolic structure. Again, computing the monodromy directly is out of reach, but its asymptotic for large t can be studied using WKB analysis. Though the method has been carried out successfully in various situations, somehow surprisingly, the existing mathematical literature did not cover the asymptotic analysis of the trace of the monodromy, at least to the best of our knowledge. It was then shown by Takuro Mochizuki in the appendix of [10] that

$$\lim_{t \to \infty} \operatorname{tr}(P_\gamma(\nabla + t\Psi))e^{t \int_\gamma \sqrt{\det \Psi}} \; = \; c,$$

where $P_\gamma(\nabla + t\Psi)$ is the monodromy of $\nabla + t\Psi$ along the curve γ which has to satisfy

$$\Re(\sqrt{\det \Psi}(\gamma')) \; < \; 0, \tag{2.2}$$

and $c \neq 0$ is a non-zero constant. As the curve γ corresponding to the trace x satisfies (2.2) and $\int_\gamma \sqrt{\det \Psi} \notin \mathbb{R}$ we find infinitely many parameters t_n accumulating at $t = \infty$ for which all monodromies of the connections $\nabla + t_n\Psi$ are real and therefore give Fuchsian representations.

2.1.2. *Theorem 2.5.* While in the space of $\mathrm{SL}(2, \mathbb{R})$–representations, the Fuchsian ones form a whole connected component, cocompact lattices in $\mathrm{SL}(2, \mathbb{C})$ are discrete. Therefore, a new set of ideas is needed to attain a specific representation with holomorphic connections rather than just reaching the component when varying a parameter. In contrast to Theorem 2.4 where we fixed the Riemann surface type and show existence of countably infinite many Fuchsian representations on the trivial holomorphic bundle, we now vary the Riemann surface type and show that the representations obtained from holomorphic system exhaust the whole (real 1-dimensional) space of symmetric and real representations. Theorem 2.5 then follows from the fact that the dodecahedron tilling has a sublattice inducing such a symmetric and real representation.

In other words, let ρ be an arbitrary real and symmetric representation and let ρ^F be a suitable real, τ–real and $\mathbb{Z}_{g+1}$–equivariant representation induced by the monodromy of

a holomorphic system on a real, symmetric Riemann surface. Then the aim is to deform ρ^F via deforming the underlying (real, symmetric) Riemann surface structure and the corresponding holomorphic system until we hit ρ.

The existence of the initial value ρ^F is guaranteed by [10], but the proof of 2.5 in fact gives an independent proof for the existence of the Fuchsian representations of [10]. Local existence and uniqueness of the deformation by varying the underlying real symmetric Riemann surface is based on a technical Lemma [11, Lemma 5.2] which was motivated by the main result of [16] for genus 2 surfaces and whose proof uses the detailed analysis of the isomonodromic deformations of the Lamé equation by Loray in [28]. An important observation is the following. We can squeeze a real, τ–real and equivariant holomorphic system between the (orbifold) uniformization of the Riemann surface and a grafting of the (orbifold) uniformization of another real, symmetric Riemann surface inside the ordered space of real, τ–real and equivariant representations. The ordering of all three connections is preserved along the deformation, and it is shown that the orbifold uniformization and their graftings both cover the whole 1-dimensional space of real, τ–real and equivariant representations. We should emphasize that the induced deformation of the representations is not necessarily continuous introducing some technicalities to show the exhaustion property.

The last point to explain here is why and how we can squeeze the holomorphic system with real monodromy between a (orbifold) uniformization and a grafting of another uniformization. The main ingredient is the new observation that grafting changes the spin structure in a very natural way. Using this we can show the existence of a non-trivial path in the moduli space of bundles which admit a unique lift to real, τ–real and equivariant connections, which starts and ends at two different Gunning bundles (respectively maximally unstable bundles in a given component of equivariant connections) and passes through the trivial holomorphic structure. This path naturally deforms under the deformation of the underlying Riemann surface structure, and as orbifold uniformisations and their graftings exhaust the 1-dimensional space of real, τ–real equivariant connections the existence of holomorphic curves of genus $g > 1$ in certain quotients $\mathrm{SL}(2, \mathbb{C})/\Gamma$ by cocompact lattices follows.

3. ÉTALE TRIVIAL BUNDLES

Let M be a compact connected complex manifold. The tangent bundle and the i–exterior product of the cotangent bundle of M will be denoted by TM and Ω_M^i respectively. A holomorphic connection on a holomorphic vector bundle E on M is a first order holomorphic differential operator $D : E \longrightarrow E \otimes \Omega_M^1$ satisfying the Leibniz identity, which says that $D(fs) = fD(s) + s \otimes df$ for all locally defined holomorphic section s of E and all locally defined holomorphic function f on M (see [2]). Any coherent analytic sheaf on M admitting a holomorphic connection is locally free (see [13, p. 211, Proposition 1.7] and [6, p. 1037, Remark 2.1]). A holomorphic connection D is called integrable if its curvature $D^2 \in H^0(M, \mathrm{End}(E) \otimes \Omega_M^2)$ vanishes identically. For an integrable holomorphic connection D we have the monodromy homomorphism

$$\mathrm{Mon}(D) : \pi_1(M, x_0) \longrightarrow \mathrm{GL}(E|_{x_0}), \qquad (3.1)$$

where $x_0 \in M$ is any base point.

A holomorphic vector bundle E on M is called étale trivial if there is a finite étale covering $\psi : \widetilde{M} \longrightarrow M$ such that ψ^*E is holomorphically trivial.

The following lemma is straightforward.

Lemma 3.1. *A holomorphic vector bundle E on M is étale trivial if and only if E admits an integrable holomorphic connection D such the image of the monodromy homomorphism $\mathrm{Mon}(D)$ in (3.1) is a finite subgroup of $\mathrm{GL}(E|_{x_0})$.*

Such a connection is uniquely determined by the following property. If $\psi : \widetilde{M} \longrightarrow M$ is a finite Galois covering for which there exists an isomorphism $h : \psi^*E \longrightarrow \mathcal{O}^r_{\widetilde{M}}$, then ψ^*D corresponds, under h, to the trivial connection on $\mathcal{O}^r_{\widetilde{M}}$.

Proof. If the image of $\mathrm{Mon}(D)$ is finite, then consider the étale Galois covering $\psi : \widetilde{M} \longrightarrow M$ corresponding to $\mathrm{kernel}(\mathrm{Mon}(D)) \subset \pi_1(M, x_0)$. The pulled back holomorphic connection ψ^*D on ψ^*E has trivial monodromy, and hence ψ^*E is holomorphically trivial.

Given any a connected finite étale covering $\psi_1 : \widetilde{M_1} \longrightarrow M$, there is a connected finite étale covering $\psi_2 : \widetilde{M} \longrightarrow \widetilde{M_1}$ such that $\psi := \psi_1 \circ \psi_2$ is Galois. If ψ_1^*E is holomorphically trivial, then ψ^*E is also holomorphically trivial. So if E is étale trivial, then we may assume that the trivializing connected finite étale covering ψ is Galois. If ψ^*E is holomorphically trivial, then the natural evaluation map

$$\widetilde{M} \times H^0(\widetilde{M}, \psi^*E) \longrightarrow \psi^*E$$

is a holomorphic isomorphism. Using this isomorphism, the trivial holomorphic connection on $\widetilde{M} \times H^0(\widetilde{M}, \psi^*E)$ produces an integrable holomorphic connection $\widetilde{D}$ on ψ^*E with trivial monodromy. This holomorphic connection $\widetilde{D}$ is preserved by the natural action of $\mathrm{Gal}(\psi)$ on ψ^*E, and hence it descends to a holomorphic connection on E. This descended holomorphic connection on E is evidently integrable and its monodromy homomorphism has finite image.

To verify the last property, we observe the following. Let $f : E \longrightarrow F$ be an arrow between holomorphic vector bundles and $\psi : \widetilde{M} \longrightarrow M$ an étale covering such that ψ^*E and ψ^*F are trivial vector bundles. Suppose that ∂_E and ∂_F are integrable connections on E and F, respectively, such that

$$\psi^*(f) : (\psi^*E, \psi^*\partial_E) \longrightarrow (\psi^*F, \psi^*\partial_F)$$

is horizontal. It then follows that f is horizontal. $\square$

Take any polynomial $p(x) = \sum_{i=0}^d a_i x^i$, where a_i are nonnegative integers. For any holomorphic vector bundle E on M, define the holomorphic vector bundle

$$p(E) := \bigoplus \left(E^{\otimes i} \right)^{\oplus a_i},$$

where $E^{\otimes 0}$ is the trivial holomorphic line bundle $\mathcal{O}_M$, and $F^{\oplus 0} = 0$ for any vector bundle F. A holomorphic vector bundle E is called finite if there are two polynomials p and q as above, with $p \neq q$, such that the two holomorphic vector bundles $p(E)$ and $q(E)$ are holomorphically isomorphic [30, p. 35], [31].

The following is a criterion for a vector bundle to be finite.

Theorem 3.2 ([5, Theorem 1.1]). *A holomorphic vector bundle E on M is finite if and only if it is étale trivial.*

In [30], [31], Theorem 3.2 was proved in the algebro-geometric set-up. Theorem 3.2 is a key ingredient in the proof of the following result (whose algebro-geometric counterpart appears in [12, Remarks, pp. 232-3].)

Theorem 3.3 ([7, Theorem 1.1]). *Let $f : X \longrightarrow M$ be a surjective holomorphic maps between compact connected complex manifolds and $E \longrightarrow M$ a holomorphic vector bundle, such that f^*E is holomorphically trivial. Then E is étale trivial.*

3.1. **A neutral Tannakian category.** Let M be a compact connected complex Kähler manifold. Fix a base point $x_0 \in M$. In [6], the following was studied: The category $\mathcal{T}(M)$ consisting of trivial vector bundles endowed with an integrable connection. Using the fiber functor $E \longmapsto E|_{x_0}$, it was deduced that $\mathcal{T}(M)$ is equivalent to the category of representations of an affine group scheme

$$\Theta(M, \ x_0).$$

A basic property of $\Theta(M, \ x_0)$ is that it is pro-connected [6, Proposition 3.7].

Let $\mathcal{C}_{\mathrm{dR}}(M)$ denote the category whose objects are pairs of the form $(E, \ D)$, where E is a holomorphic vector bundle on M and D is an integrable holomorphic connection on E. As before, morphisms from $(E, \ D)$ to $(E', \ D')$ are all holomorphic homomorphisms of vector bundles $h : E \longrightarrow E'$ satisfying $D' \circ h = (h \otimes \mathrm{Id}_{\Omega^1_M}) \circ D$ as differential operators from E to $E' \otimes \Omega^1_M$. This category is equipped with the operators of direct sum, tensor product and dualization. More precisely, $\mathcal{C}_{\mathrm{dR}}(M)$ is a rigid abelian tensor category (see [18, p. 118, definition 1.14] for rigid abelian tensor categories). This category $\mathcal{C}_{\mathrm{dR}}(M)$, endowed with the faithful fiber functor that sends any object $(E, \ D)$ to the fiber $E|_{x_0}$ defines a neutral Tannakian category [38, p. 67]. The corresponding pro-algebraic affine group scheme over $\mathbb{C}$ will be denoted by $\varpi(M, x_0)$ (see [38, p. 69], [6, Section 2]). Concerning the relation between $\mathcal{C}_{\mathrm{dR}}(M)$ and $\mathcal{T}(M)$, we can say that the morphism of group schemes

$$\mathbf{q} : \varpi(M, x_0) \longrightarrow \Theta(M, x_0) \tag{3.2}$$

induced by the inclusion $\mathcal{T}(M) \longrightarrow \mathcal{C}_{\mathrm{dR}}(M)$ is *faithfully flat*, or a *quotient* morphism. See Proposition 3.1 in [10].

Let us now modify our point of view and enlarge the category of vector bundles on which we allow a connection. Let $\mathcal{T}^{\mathrm{et}}(M)$ denote the category of all pairs of the form $(E, \ D)$, where E is an étale trivial vector bundle on M and D is a holomorphic connection on E. This $\mathcal{T}^{\mathrm{et}}(M)$ is evidently closed under the operations of direct sum and tensor product, just as it is closed under the operation of taking dual. For any holomorphic homomorphism between two holomorphically trivial vector bundles over a compact connected complex manifold, both kernel and cokernel are holomorphically trivializable because the homomorphism is given by a constant matrix. From this it follows that for any two pairs $(E_1, \ D_1)$ and $(E_2, \ D_2)$ in $\mathcal{T}^{\mathrm{et}}(M)$, any homomorphism $\rho : E_1 \longrightarrow E_2$, satisfying the condition that

$$D_2 \circ \rho = (\rho \otimes \mathrm{Id}_{\Omega^1_M}) \circ D_1,$$

has the property that both kernel(ρ) and cokernel(ρ) are étale trivial bundles, and furthermore, D_1 (respectively, D_2) induces a holomorphic connection on kernel(ρ) (respectively, cokernel(ρ)).

Therefore, $\mathcal{T}^{\mathrm{et}}(M)$ together with the faithful fiber functor on it that sends any (E, D) to the fiber $E\big|_{x_0}$ produce a neutral Tannakian category (see [18, p. 138, Definition 2.19], [35], [31, p. 76] for neutral Tannakian category). For any neutral Tannakian category there is a naturally associated pro-algebraic affine group scheme over $\mathbb{C}$ [18, p. 130, Theorem 2.11] (and the remark following [18, p. 138, Definition 2.19]), [35], [31, p. 77, Theorem 1.1], [38, p. 69]. Consequently, the neutral Tannakian category $\mathcal{T}^{\mathrm{et}}(M)$ produces a pro-algebraic group (= affine group scheme)

$$\mathbb{E}(M, x_0)$$

over $\mathbb{C}$ such that by means of the functor $\bullet|_{x_0}$, the category $\mathcal{T}^{\mathrm{et}}(M)$ becomes equivalent to the category of representations of $\mathbb{E}(M, x_0)$ on finite dimensional complex vector spaces.

The tautological functor $\mathcal{T}^{\mathrm{et}}(M) \longrightarrow \mathcal{C}_{\mathrm{dR}}(M)$ produces a homomorphism

$$\mathbf{r} \; : \; \varpi(M, x_0) \; \longrightarrow \; \mathbb{E}(M, x_0). \tag{3.3}$$

Lemma 3.4. *The homomorphism $\mathbf{r}$ in (3.3) is faithfully flat (in other words, it is surjective).*

Proof. This can be proved by the same method employed to verify Proposition 3.1 in [6]. $\square$

Let $\pi^{\mathrm{et}}(M, x_0)$ be the pro-finite completion of the fundamental group of M at x_0; it carries a natural structure of pro-algebraic group, or affine group scheme, once regarded as a projective limit of finite groups. This is commonly known as the *étale fundamental group scheme*.

Lemma 3.5. *There is a natural faithfully flat morphism (same as surjective morphism)*

$$\gamma \; : \; \mathbb{E}(M, x_0) \; \longrightarrow \; \pi^{\mathrm{et}}(M, x_0)$$

to the étale fundamental group scheme $\pi^{\mathrm{et}}(M, x_0)$.

Proof. The étale fundamental group $\pi^{\mathrm{et}}(M, x_0)$ corresponds to the neutral Tannakian category defined by the étale trivial holomorphic vector bundles on M. In the proof of Lemma 3.1 it was shown that any étale trivial vector bundle has a canonical integrable holomorphic connection whose monodromy is finite. Any homomorphism between two étale trivial holomorphic vector bundles is flat with respect to this canonical connection. Indeed, write d_E for the canonical connection on an étale trivial bundle E. Then, if $\psi : \widetilde{M} \longrightarrow M$ is a finite étale covering such that $\psi^*E \simeq \mathcal{O}_{\widetilde{M}}^r$, then the pulled-back connection $\psi^*\mathrm{d}_E$ corresponds to the trivial connection on $\mathcal{O}_{\widetilde{M}}^r$. (Note that this notion is independent of the isomorphism $\psi^*E \simeq \mathcal{O}_{\widetilde{M}}^r$.) This being so, if $f : E \longrightarrow F$ is an arrow between vector bundles which become trivial on the étale covering $\psi : \widetilde{M} \longrightarrow M$, then the arrow ψ^*f is also horizontal with respect to the connections $\psi^*(\mathrm{d}_E)$ and $\psi^*(\mathrm{d}_F)$. Consequently, surjectivity of ψ assures that h is horizontal.

The lemma is a straightforward consequence of these. $\square$

The group of multiplicative characters $G \longrightarrow \mathbb{G}_m = \mathbb{C}^*$ of an affine group scheme G will be denoted by $\mathbb{X}(G)$.

Let

$$\gamma^* : \mathbb{X}(\pi^{\mathrm{et}}(M, x_0)) \longrightarrow \mathbb{X}(\mathbb{E}(M, x_0)) \tag{3.4}$$

be the homomorphism induced by γ in Lemma 3.5.

Proposition 3.6. *The image of the homomorphism γ in (3.4) is the subgroup consisting of all characters of finite order.*

Proof. Any character in $\mathbb{X}(\pi^{\mathrm{et}}(M, x_0))$ is of finite order, and hence the image of γ^* is contained in the subgroup of $\mathbb{X}(\mathbb{E}(M, x_0))$ consisting of all characters of finite order.

Let L be an étale trivial line bundle on M and D a holomorphic connection on L. If the curvature of D is $R \in H^0(M, \Omega_M^2)$, then the curvature of the connection on $L^{\otimes n}$ induced by D is $n \cdot R$. Therefore, if (L, D) corresponds to a multiplicative character of $\mathbb{E}(M, x_0)$ of finite order, then D is integrable.

To prove the proposition we need to show the following: Let D be an integrable holomorphic connection on an étale trivial line bundle L on M. If the monodromy homomorphism for D has finite image, then D is the canonical connection on L constructed in Lemma 3.1.

Since the monodromy homomorphism for D has finite image, there is a connected étale Galois covering

$$\psi : \widetilde{M} \longrightarrow M$$

such that $\psi^* L$ is the trivial holomorphic line bundle and $\psi^* D$ is the trivial connection on $\psi^* L$. From the construction of the connection in the proof of Lemma 3.1 it follows immediately that D coincides with the canonical connection on L in Lemma 3.1. This completes the proof. $\qquad \square$

Let us end this section by rendering explicit the group of *additive characters* of $\mathbb{E}(M, x_0)$; this is a key observation in [6]. The natural functor $\mathcal{T}(M) \longrightarrow \mathcal{T}^{\mathrm{et}}(M)$ defines a morphism of group schemes

$$\mathbf{p} : \mathbb{E}(M, x_0) \longrightarrow \Theta(M, x_0)$$

which must be automatically surjective in view of the fact that $\mathbf{q} : \varpi(M, x_0) \longrightarrow \Theta(M, x_0)$ is. For an affine group scheme, let $\mathbb{X}_a(G)$ stand for the group of homomorphisms $G \longrightarrow \mathbb{G}_a$. We then obtain an injection

$$\mathbb{X}_a(\Theta(M, x_0)) \longrightarrow \mathbb{X}_a(\mathbb{E}(M, x_0)) \tag{3.5}$$

Lemma 3.7. *The arrow in eq. (3.5) is a bijection. In particular, the group of additive characters of $\mathbb{E}(M, x_0)$ is isomorphic to $H^0(M, \Omega_M^1)$.*

Proof. Let G be an affine group scheme. We first note that $\mathbb{X}_a(G)$ can be canonically identified with the group $\mathrm{Ext}_G^1(\mathbb{C}, \mathbb{C})$ of extensions of the trivial representation by itself. On the side of vector bundles, we are then required to show that given an extension

$$0 \longrightarrow (\mathcal{O}_M, \mathrm{d}) \longrightarrow (E, D) \longrightarrow (\mathcal{O}_M, \mathrm{d}) \longrightarrow 0,$$

where $E \in \mathcal{T}^{\text{et}}(M)$, we must immediately have $E \simeq \mathcal{O}_M^2$. Let $\psi : \widetilde{M} \longrightarrow M$ be the an étale Galois covering with group G trivializing E and let $y_0 \in \psi^{-1}(x_0)$. Giving E the canonical connection (see Lemma 3.1), call it d_E, the sequence

$$1 \longrightarrow (\mathcal{O}_M, \mathrm{d}) \longrightarrow (E, \mathrm{d}_E) \longrightarrow (\mathcal{O}_M, \mathrm{d}) \longrightarrow 0$$

is also exact. Since $\psi^* E$ is trivial, it is the case that

$$\pi_1(\widetilde{M}, y_0) \longrightarrow \mathrm{GL}(E|_{x_0}) = \mathrm{GL}_2(\mathbb{C})$$

is trivial and hence $\pi_1(M, x_0) \longrightarrow \mathrm{GL}_2(\mathbb{C})$ factors through the finite quotient G. Now $\mathrm{Mon}(\mathrm{d}_E)$ has image in the subgroup of strict upper triangular matrices, which is isomorphic to $\mathbb{C}$. Since $\mathbb{C}$ does not have any finite subgroup other than the trivial, we conclude that $\mathrm{Mon}(\mathrm{d}_E)$ is trivial. Hence, E is holomorphically trivial.

The last claim follows from [6, Lemma 3.5]. □

3.2. **The special case of Riemann surfaces.** In this subsection assume that $\dim_{\mathbb{C}} M = 1$. Since $\Omega_M^2 = 0$, any holomorphic connection on M is integrable,

Let $\mathcal{C}_B(M)$ denote the category whose objects are all finite dimensional complex representations of the topological fundamental group $\pi_1(M, x_0)$. Sending a holomorphic connection (E, D) to its monodromy homomorphism $\pi_1(M, x_0) \longrightarrow \mathrm{GL}(E|_{x_0})$ we obtain an equivalence between the two categories $\mathcal{C}_{\mathrm{dR}}(M)$ and $\mathcal{C}_B(M)$. Using the tautological fiber functor $\mathcal{C}_B(M)$ defines a neutral Tannakian category. In view of the above equivalence of categories, the pro-algebraic affine group scheme over $\mathbb{C}$ corresponding to $\mathcal{C}_B(X)$ is the group scheme $\varpi(X, x_0)$ [38, p. 69, Lemma 6.1]. In other words, $\varpi(M, x_0)$ is the pro-algebraic completion of $\pi_1(M, x_0)$.

Let N be a compact connected Riemann surface and

$$\phi : M \longrightarrow N \tag{3.6}$$

an orientation preserving map. The corresponding homomorphism of fundamental groups

$$\phi_* : \pi_1(M, x_0) \longrightarrow \pi_1(N, \phi(x_0))$$

produces a homomorphism

$$\phi_* : \varpi(M, x_0) \longrightarrow \varpi(N, \phi(x_0)) \tag{3.7}$$
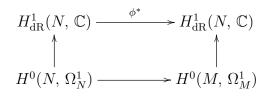
between the pro-algebraic completions.

Proposition 3.8. *Take any ϕ as in (3.6) which is a homeomorphism. If the homomorphism ϕ_* is (3.7) descends to a homomorphism*

$$\mathbb{E}(M, x_0) \longrightarrow \mathbb{E}(N, \phi(x_0)) \, .$$

of quotients (see Lemma 3.4), then the two Riemann surfaces M and N are isomorphic.

Proof. Its proof is very similar to the proof of Theorem 4.1 of [6, p. 1050]; the key point is the existence of an arrow $\mathbb{E}(M, x_0) \longrightarrow \mathbb{E}(N, \phi(x_0))$ which then assures, through

Lemma 3.7, the existence of an arrow $H^0(N, \Omega_N^1) \longrightarrow H^0(M, \Omega_M^1)$ rendering

$$
\begin{array}{ccc}
H_{\mathrm{dR}}^1(N, \mathbb{C}) & \xrightarrow{\ \phi^*\ } & H_{\mathrm{dR}}^1(N, \mathbb{C}) \\
\uparrow & & \uparrow \\
H^0(N, \Omega_N^1) & \longrightarrow & H^0(M, \Omega_M^1)
\end{array}
$$

commutative. □

4. HOLONOMY OF CONNECTION ON TRIVIAL BUNDLE

Let M be a compact connected complex manifold and G a complex connected Lie group. The Lie algebra of G will be denoted by $\mathfrak{g}$. Consider the trivial holomorphic principal G–bundle

$$
E := M \times G \longrightarrow M
$$

on M. Take a holomorphic connection D on E. Denoting the trivial connection on E by D_0, we have

$$
D - D_0 \ \in \ H^0(M, \Omega_M^1 \otimes_{\mathbb{C}} \mathfrak{g}) \ = \ H^0(M, \Omega_M^1) \otimes_{\mathbb{C}} \mathfrak{g} .
$$

Let

$$
\mathbb{D} \ : \ H^0(M, \Omega_M^1)^* \ \longrightarrow \ \mathfrak{g}
$$

be the homomorphism given by $D - D_0$. The complex Lie subalgebra of $\mathfrak{g}$ generated by the image of the homomorphism $\mathbb{D}$ will be denoted by $\mathfrak{g}_D$. Let G_D denote the unique smallest connected closed complex Lie subgroup of G such that the Lie algebra of G_D contains $\mathfrak{g}_D$.

Consider the holomorphic reduction of structure group of E

$$
E_D := M \times G_D \subset M \times G = E \tag{4.1}
$$

to the subgroup G_D.

Proposition 4.1. *The connection D preserves E_D in (4.1) (equivalently, D is induced by a connection on E_D).*

If $E_H \subset E$ is a holomorphic reduction of the structure group of E to a complex Lie subgroup $H \subset G$ such that

- *the connection D preserves E_H, and*
- *there is a point $x_0 \in M$ such that $(x_0, e) \in E_H$, where $e \in G$ is the identity element,*

then $G_D \subset H$ and $E_D \subset E_H$.

Proof. Since $\mathbb{D}(H^0(M, \Omega_M^1)^*)) \subset \mathfrak{g}_D \subset \mathrm{Lie}(G_D)$, it follows that the connection D preserves the reduction E_D in (4.1). The second statement is equally straightforward. □

5. Expressing $\varpi \longrightarrow \Theta$ with the aid of iterated integrals

Let M be a complex compact Kähler manifold and x_0 a point of it. Recall that $\varpi = \varpi(M, x_0)$ (respectively, $\Theta = \Theta(M, x_0)$) is the Tannakian group associated to the category of integrable holomorphic connections $\mathcal{C}_{\mathrm{dR}}(M)$ (respectively, integrable holomorphic connections on trivial vector bundles $\mathcal{T}(M)$) over M. And our objective in this section is to render the arrow $\mathbf{q}$ of eq. (3.2) more explicit by using *iterated integrals*, therefore linking it to other works like [32] and [20].

It is showed in [4] that Θ is the Tannakian envelope of a co-commutative Hopf algebra $\mathfrak{A}$ constructed via the free algebra on the vector space $H^0(M, \Omega_M^1)^*$. Constructing $\mathfrak{A}$ is quite simple: letting $\{\beta_i\}$ be a basis of $H^0(M, \Omega_M^2)$, define

$$B_i \in H^0(M, \Omega_M^1)^* \otimes H^0(M, \Omega_M^1)^*, \quad \omega \otimes \omega' \longmapsto \beta_i(\omega \wedge \omega').$$

Then $\mathfrak{A} = \mathbb{T}(H^0(M, \Omega_M^1)^*)/(B_i)$. See [4, Section 2] for details about $\mathfrak{A}$. Concerning the relation between Θ and $\mathfrak{A}$, we can say, in a nutshell, that the restricted dual of $\mathfrak{A}$ [39, VI],

$$\mathfrak{A}^\circ = \varinjlim(\mathfrak{A}/I)^*, \quad I \text{ an ideal of finite codimension,}$$

is the $\mathbb{C}$–algebra of an affine group scheme isomorphic to Θ. In fact, we have natural tensor equivalences

$$\mathbf{comod\text{-}}\mathfrak{A}^\circ \xrightarrow{\sim} \mathfrak{A}\text{-}\mathbf{mod} \xrightarrow{\mathcal{V}} \mathcal{T}(M),$$

where $\mathcal{V}$ is constructed as follows (see [4, Section 3] for more). Given

$$f : \mathfrak{A} \longrightarrow \mathrm{End}(E)$$

an object of $\mathfrak{A}$-**mod**, we associate to it the connection d_f on $\mathcal{O}_M \otimes E$ defined by the $\mathrm{End}(E)$–valued form derived from the composition

$$H^0(M, \Omega_M^1)^* \longrightarrow \mathfrak{A} \longrightarrow \mathrm{End}(E). \tag{5.1}$$

Under this equivalence, the forgetful functor is naturally isomorphic to $\bullet|_{x_0}$.

Now, $\mathcal{C}_{\mathrm{dR}}(M)$ is also equivalent to the category of representations of the topological fundamental group $\Gamma = \pi_1(M, x_0)$ by means of the monodromy, and the latter category can be shown to be isomorphic to the category of "continuous" modules over a certain topological $\mathbb{C}$–algebra, $\widehat{\mathbb{C}\Gamma}$, associated to $\mathbb{C}\Gamma$. Thus, in order to render $\mathbf{q}$ explicit, we shall elaborate on the expression of the monodromy representation for objects in $\mathcal{T}(M)$.

5.1. **Parallel transport and monodromy.** Let Γ be the fundamental group of M based at x_0. Given $(\mathcal{E}, \nabla) \in \mathcal{C}_{\mathrm{dR}}(M)$, the monodromy representation

$$\mathrm{Mon}_\nabla^{x_0} : \Gamma \longrightarrow \mathrm{GL}(\mathcal{E}|_{x_0})$$

is obtained via parallel transport. Write E for the (geometric) holomorphic vector bundle whose sheaf of sections is $\mathcal{E}$. Given a smooth loop $\gamma : [0, 1] \longrightarrow M$ based at x_0, let

$$P_\gamma : \mathcal{E}|_{\gamma(0)} \xrightarrow{\sim} \mathcal{E}|_{\gamma(1)}$$

be the isomorphism of complex vector spaces obtained from parallel transport

$$P_\gamma : E|_{\gamma(0)} \longrightarrow E|_{\gamma(1)}$$

via the canonical identification $\mathcal{E}|_x \xrightarrow{\sim} E|_x$. In other words, we extend ∇ to a C^∞ connection, find a C^∞ lift $\widetilde{\gamma} : [0, 1] \longrightarrow E$ of γ such that $\widetilde{\gamma}(0) = e$ and $\nabla_{\gamma'}\widetilde{\gamma} = 0$, and then define

$$P_\gamma(e) = \widetilde{\gamma}(1).$$

As the composition $\gamma_1 * \gamma_2$ in Γ is the path γ_1 followed by γ_2, we must write

$$\mathrm{Mon}_\nabla^{x_0}(\gamma) = P_{\gamma^{-1}}.$$

5.2. **Parallel transport, monodromy in $\mathcal{T}(M)$ and iterated integrals.** In this section we explain how to write the monodromy representation in terms of iterated integrals. This is explained in [20, § 2], and attributed to Chen, but Hain adopts different conventions (like letting parallel transport "act on the right" and working with connections defined by the negative of a form, see (2.1) and line -11 on p. 614 of [20]), so that it is worth explaining it in full here. Let E be a vector space of finite dimension and

$$\omega \in H^0(M, \Omega_M^1) \otimes \mathrm{End}(E)$$

a differential form with values on $\mathrm{End}(E)$. Write d_ω for the connection on the trivial vector bundle $M \times E \longrightarrow M$ whose associated 1–form is ω. By definition, if $s = (\mathrm{id}, e)$ is a section on some open set then, for any C^∞ vector field v,

$$(\mathrm{d}_\omega)_v(s) = (\mathrm{id}, v(e) + \omega(v) \cdot e).$$

Given a smooth loop $\gamma : [0, 1] \longrightarrow M$ about x_0, a lift $t \longmapsto (\gamma(t), e(t))$ is parallel if and only if

$$e'(t) + A(t)e(t) = 0,$$

where $A(t) = \omega(\gamma'(t))$. It then follows that the parallel transport of $(x_0, e_0) \in (M \times E)|_{x_0}$ along γ, call it

$$P_\gamma(x_0, e_0)$$

is the value $y(1)$ of a solution to

$$\begin{aligned} y'(t) &= -A(t)y(t) \\ y(0) &= e_0. \end{aligned}$$

Solutions to this sort of equation are described concisely using Picard iteration.

Recall the following fundamental result. (A proof can be adapted from the proofs of the Picard-Lindelöf Theorem, which is to be found in almost all texts on ordinary differential equations, e.g. [17], Chapter 6, Theorem 4 and Theorem 6.)

Theorem 5.1 (Picard-Lindelöf). *Let R be a finite dimensional associative $\mathbb{C}$–algebra and $F : [0, 1] \longrightarrow R$ be a continuous map. Given $f_0 \in R$, define inductively continuous functions $[0, 1] \longrightarrow R$ by*

$$f_{k+1}(t) = f_0 + \int_0^t F(s) \cdot f_k(s)\,\mathrm{d}s.$$

(1) The limit $f(t) = \lim_k f_k(t)$ exists for each $t \in [0, 1]$ and $f' = F \cdot f$.
(2) Let $\delta_k = f_k - f_{k-1}$, for each $k \geq 1$. Then, $\delta_1(t) = \int_0^t F(s) \cdot f_0\,\mathrm{d}s$,

$$\delta_{k+1}(t) = \int_0^t F(s)\delta_k(s)\,\mathrm{d}s,$$

for $k \geq 1$ *and*

$$f(t) \;=\; f_0 + \sum_{k=1}^{\infty} \delta_k(t).$$

The functions $\{\delta_k\}$ appearing in the above statement have a well-known expression in terms of iterated integrals. Let R be as in Theorem 5.1 and consider R–valued C^{∞} differential forms $\rho_1, \cdots, \rho_m$. Given a smooth path $c : [0, 1] \longrightarrow M$, let $F_i = \rho_i(c')$ and define

$$\underbrace{G_1(t) \;=\; \int_0^t F_1(s)\,\mathrm{d}s}_{\in R}, \quad \underbrace{G_{i+1}(t) = \int_0^t G_i(s) \cdot F_{i+1}(s)\,\mathrm{d}s}_{\in R}$$

and

$$\underbrace{\int_c \rho_1 \cdots \rho_m \;=\; G_m(1)}_{\in R}$$

(See [15, p. 359] and [20, § 1], among others.) For example,

$$\int_c \rho_1 \rho_2 \;=\; \int_0^1 \left(\int_0^{s_2} F_1(s_1)\,\mathrm{d}s_1 \right) \cdot F_2(s_2)\,\mathrm{d}s_2.$$

Alternatively, it is sometimes convenient to think of the iterated integral as an integral over a simplex:

$$\int_c \rho_1 \cdots \rho_m \;=\; \int_{0 \leq s_1 \leq \cdots \leq s_n \leq 1} F_1(s_1) \cdots F_m(s_m)\,\mathrm{d}s_1 \cdots \mathrm{d}s_m.$$

Returning to the problem of expressing the parallel transport of

$$(x_0,\, e_0) \;\in\; (M \times E)|_{x_0}$$

along the path γ with respect to d_ω, the above terminology and Theorem 5.1 allow us to write

$$P_\gamma(x_0, e_0) \;=\; \left(x_0\,,\; e_0 - \int_0^1 A(s_1)\,\mathrm{d}s_1 \cdot e_0 + \int_{0 \leq s_1 \leq s_2 \leq 1} A(s_1)A(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 \cdot e_0 - \cdots \right).$$

A simple change of variables shows that

$$\int_{\gamma^{-1}} \omega_1 \cdots \omega_m \;=\; (-1)^m \int_\gamma \omega_m \cdots \omega_1$$

and hence

$$\mathrm{Mon}_{\mathrm{d}_\omega}^{x_0}(\gamma) : (x_0, e_0) \longmapsto \left(x_0\,,\; e_0 + \int_\gamma \omega \cdot e_0 + \int_\gamma \omega^2 \cdot e_0 + \int_\gamma \omega^3 \cdot e_0 + \cdots \right). \quad (5.2)$$

(Here, obviously, ω^n stands for $\underbrace{\omega \cdots \omega}_{n}$.)

5.3. **Algebraic interlude.**
To sustain the arguments concerning the relation between $\mathfrak{A}$ and $\mathbb{C}\Gamma$ necessary in explaining the morphism $\mathbf{q}$, we shall require some material on completions of rings and Hopf duals. We fix a field K.

Completions. Let A be an associative and unital K–algebra. Write

$$S_A = \left\{ \begin{array}{c} \text{two sided ideals of} \\ \text{finite codimension in } A \end{array} \right\};$$

this set it partially ordered by decreeing $I \leq J$ if $I \supset J$. Consider the subset of 2^A defined by

$$\{a + I\}_{a \in A, \, I \in S_A}$$

and let $\mathfrak{O}$ be the unique topology on A having this set as a sub-base [26, p. 47–48]. Of course, S_A is local base at 0 [26, p. 50] and all ideals of S_A are open, hence closed.

Put

$$\widehat{A} = \varprojlim_{I \in S_A} A/I;$$

endowing each A/I with the *discrete topology* and $\widehat{A}$ with the projective limit topology, $\widehat{A}$ becomes a topological ring. (Rings as $\widehat{A}$ are called pseudo-compact [36, VIIB, § 0].) Let $\iota : A \longrightarrow \widehat{A}$ be the evident map; according to [14, III.7.3, Cor. 1], ι is a completion of A for $\mathfrak{O}$.

Given $\mathfrak{a} \in S_A$, let $p_{\mathfrak{a}} : \widehat{A} \longrightarrow A/\mathfrak{a}$ be the canonical map and denote by $\widehat{\mathfrak{a}}$ the open and closed ideal $\mathrm{Ker}(p_{\mathfrak{a}})$. Note that $A/\mathfrak{a} \xrightarrow{\sim} \widehat{A}/\widehat{\mathfrak{a}}$.

Lemma 5.2. *The family $\{\widehat{\mathfrak{a}}\}_{\mathfrak{a} \in S_A}$ is a local base at 0. The ideal $\widehat{\mathfrak{a}}$ is the closure of $\iota\mathfrak{a}$ in $\widehat{A}$.*

Proof. We only prove the second statement. Clearly $\iota(\mathfrak{a}) \subset \widehat{\mathfrak{a}}$. Let

$$x = (x_I) \in \widehat{\mathfrak{a}}.$$

Let U be an open neighbourhood of x and let $I_1, \cdots, I_r \in S_A$ be such that

$$V = \{(y_I) \in \widehat{A} : y_{I_j} = x_{I_j}, \, \forall j\}$$

is contained in U. Let $J = I_1 \cap \cdots \cap I_r \cap \mathfrak{a}$; it clearly belongs to S_A. Then, if $\xi \in A$ is such that $\xi + J = x_J$, we conclude that $\iota\xi \in V \subset U$ and $\xi \in \mathfrak{a}$. $\qquad\square$

Example 5.3. Suppose that K is algebraically closed and A is a commutative K–algebra of finite type. Then $\widehat{A}$ is $\prod_{\mathfrak{m}} \widehat{A}_{\mathfrak{m}}$, where $\mathfrak{m}$ ranges over the maximal ideals.

Let A-**mod** be the category of A–modules which have finite dimension as K–spaces. Analogously, let $\widehat{A}$-**modc** be the category of *continuous* $\widehat{A}$–modules, i.e., those (left) $\widehat{A}$–modules of finite dimension over K which are annihilated by some $\widehat{\mathfrak{a}}$ with $\mathfrak{a} \in S_A$. (Note that, although any $\widehat{A}$–module of finite dimension is annihilated by an ideal $\mathfrak{I} \in S_{\widehat{A}}$, we cannot assure a priori that such an ideal contains a certain $\widehat{I}$.) Using the natural morphism $A \longrightarrow \widehat{A}$, we obtain a functor

$$\widehat{A}\text{-}\mathbf{modc} \longrightarrow A\text{-}\mathbf{mod}$$

which is easily seen to be an *isomorphism of categories*. Indeed, for each $E \in A$-**mod**, the ideal $\mathfrak{a} = \mathrm{Ann}(E)$ belongs to S_A, and hence we can define a structure of $\widehat{A}$–module by

$$(x_I) \cdot e = x_{\mathfrak{a}} \cdot e, \quad (x_I) \in \widehat{A}, \, e \in E.$$

The restricted dual. Let A be an associative and unital K–algebra and define, as in [39, Ch. VI], the restricted dual

$$A^\circ = \varinjlim_{I \in S_A} (A/I)^*$$
$$= \{\alpha \in A^* : \alpha \text{ vanishes on some } I \in S_A\}.$$

This is a co-algebra [39, 6.0.2, p. 113] and

$$\widehat{A} = (A^\circ)^*$$

since dualization takes direct limits into projective limits. Letting **comod**-A° be the category of (right) comodules over A° having finite dimension as K–spaces, we have functors

$$\sigma_A : \textbf{comod-}A^\circ \longrightarrow A\textbf{-mod} \quad \text{and} \quad \tau_A : A\textbf{-mod} \longrightarrow \textbf{comod-}A^\circ \qquad (5.3)$$

which together define an *isomorphism of categories*, cf. [39, Theorem 2.1.3] [29, 1.6.4, p. 11]. For future application, note that if $E \in \textbf{comod-}A^\circ$ has basis $\{e_i\}_{i=1}^m$ and co-multiplication defined by $(a_{ij}^\circ) \in \mathrm{Mat}_m((A/I)^*)$, then

$$ae_j = \sum_i a_{ij}^\circ(a)e_i.$$

Conversely, if $F \in A\textbf{-mod}$ has basis $\{f_i\}_{i=1}^n$, is annihilated by $\mathfrak{a}$, and the action is given by

$$af_j = \sum_i \rho_{ij}(a)f_i,$$

then the co-module structure of F is defined by the matrix $(\rho_{ij}) \in \mathrm{Mat}_n((A/\mathfrak{a})^*)$.

Note that $(A^\circ)^*$ is now an algebra [39, Proposition 1.1.1, p. 9] which is easily seen to be the completion $\widehat{A}$ studied above.

Let now B be another algebra and

$$v : A^\circ \longrightarrow B^\circ$$

an arrow of co-algebras. Taking duals, we obtain an arrow of K–algebras $u : \widehat{B} \longrightarrow \widehat{A}$. Alternatively, u can be described as follows. For each $I \in S_A$, there exists $J \in S_B$ such that $v((A/I)^*) \subset (B/J)^*$. In addition, let $I_v \in S_B$ be the largest ideal with this property. We arrive at an order preserving map $(-)_v : S_A \longrightarrow S_B$ and to an arrow of K–spaces

$$u_I : B/I_v \longrightarrow A/I$$

whose transpose is $v|_{(A/I)^*}$. Note that u_I *is a morphism of algebras* since $v|_{(A/I)^*}$ is a morphism of co-algebras. A moment's thought shows that for $J \subset I$ in S_A, the diagram

$$
\begin{array}{ccc}
B/I_v & \xrightarrow{u_I} & A/I \\
\text{canonic}\big\uparrow & & \big\uparrow\text{canonic} \\
B/J_v & \xrightarrow[u_J]{} & A/J
\end{array}
$$

commutes and we obtain an arrow of K–algebras

$$u : \varprojlim_{J \in (S_A)_v} B/J \longrightarrow \varprojlim_{I \in S_A} A/I. \qquad (5.4)$$

For each $I \in S_A$ (respectively, $J \in S_B$), endow A/I, (respectively, B/J) with the *discrete* topology and give the above projective limits the product topology. This being so, general topology assures that u is continuous.

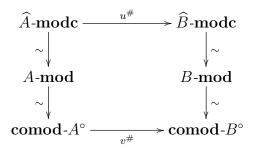We then obtain from eq. (5.4) a morphism of topological K–algebras

$$u : \widehat{B} \longrightarrow \widehat{A}$$

and consequently a functor

$$u^{\#} : \widehat{A}\text{-}\mathbf{modc} \longrightarrow \widehat{B}\text{-}\mathbf{modc}.$$

For the sake of communication, the morphism u shall be called the *continuous transpose of v*.

Lemma 5.4. *The following diagram of categories*

$$
\begin{array}{ccc}
\widehat{A}\text{-}\mathbf{modc} & \xrightarrow{\ u^{\#}\ } & \widehat{B}\text{-}\mathbf{modc} \\
{\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\
A\text{-}\mathbf{mod} & & B\text{-}\mathbf{mod} \\
{\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\
\mathbf{comod}\text{-}A^{\circ} & \xrightarrow[\ v^{\#}\ ]{} & \mathbf{comod}\text{-}B^{\circ}
\end{array}
$$

is strictly commutative. In particular, given $E \in \mathbf{comod}\text{-}A^{\circ}$, $e \in E$ and $b \in \widehat{B}$, we have

$$b \cdot e = u(b) \cdot e, \tag{5.5}$$

where on the left-hand-side of this equation, $\widehat{B}$ acts by means of $\widehat{B}\text{-}\mathbf{modc} \xrightarrow{\sim} \mathbf{comod}\text{-}B^{\circ}$, and on the right-hand-side, $u(b)$ it acts by means of $\widehat{A}\text{-}\mathbf{modc} \xrightarrow{\sim} \mathbf{comod}\text{-}A^{\circ}$.

Proof. We require the constructions made in the beginning of the section. Let $E \in \widehat{A}\text{-}\mathbf{modc}$ and pick a basis $\{e_i\}$. Let $\mathfrak{a} \in S_A$ be such that $\widehat{\mathfrak{a}}$ annihilates E. Now, the coefficients $\{\varphi_{ij}\}$ determining the A°–comodule structure of E with respect to the basis $\{e_i\}$ are determined by

$$ae_j = \sum_i \varphi_{ij}(a) \cdot e_i. \tag{5.6}$$

(Note that $\varphi_{ij} \in (A/\mathfrak{a})^* \subset A^{\circ}$.) Consequently, the coefficients of the B°–comodule E coming from the composition via $v^{\#}$ are $v(\varphi_{ij})$. By construction, $v(\varphi_{ij}) \in (B/\mathfrak{a}_v)^*$.

We now examine the structure of B°–comodule on E obtained through the composition via $u^{\#}$. Clearly, $\widehat{\mathfrak{a}}_v \subset \widehat{B}$ annihilates E. For each $\alpha \in (A/\mathfrak{a})^*$ and $b \in B/\mathfrak{a}_v$, we have, as $u_{\mathfrak{a}} = v^*$, that

$$\alpha \circ u_{\mathfrak{a}}(b) = v(\alpha)(b).$$

This shows that $v(\varphi_{ij}) = \varphi_{ij} \circ u_{\mathfrak{a}}$. By definition of $u^{\#}$ and (5.6) we have

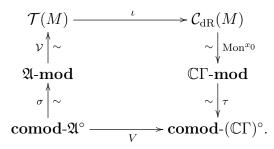$$be_j = u_{\mathfrak{a}}(b)e_j = \sum_i \varphi_{ij}(u_{\mathfrak{a}}(b))e_i$$

and the coefficients of the B°–module E are $\varphi_{ij} \circ u_{\mathfrak{a}}$. This shows commutativity of the diagram on the level of object. The case of arrows being trivially verified, we conclude our proof. $\square$

5.4. **Fiber functor and the Tannakian picture.** Using the inclusion $\mathcal{T}(M) \longrightarrow \mathcal{C}_{\mathrm{dR}}(M)$ and Tannakian theory, we shall construct an arrow of Hopf algebras $\mathfrak{A}^\circ \longrightarrow (\mathbb{C}\Gamma)^\circ$ and from it define, using Section 5.3, an arrow of $\mathbb{C}$–algebras $u : \mathbb{C}\Gamma^{\widehat{\ }} \longrightarrow \widehat{\mathfrak{A}}$. In what follows, let us denote by an upper hashtag the functor between categories of modules or comodules obtained from a morphism of algebras or co-algebras.

Define a tensor functor

$$V : \mathbf{comod}\text{-}\mathfrak{A}^\circ \longrightarrow \mathbf{comod}\text{-}(\mathbb{C}\Gamma)^\circ$$

by means of a composition as indicated in

$$
\begin{array}{ccc}
\mathcal{T}(M) & \xrightarrow{\quad \iota \quad} & \mathcal{C}_{\mathrm{dR}}(M) \\
\mathcal{V} \uparrow \sim & & \sim \downarrow \mathrm{Mon}^{x_0} \\
\mathfrak{A}\text{-}\mathbf{mod} & & \mathbb{C}\Gamma\text{-}\mathbf{mod} \\
\sigma \uparrow \sim & & \sim \downarrow \tau \\
\mathbf{comod}\text{-}\mathfrak{A}^\circ & \xrightarrow[\quad V \quad]{} & \mathbf{comod}\text{-}(\mathbb{C}\Gamma)^\circ.
\end{array}
$$

Here, τ and σ are the equivalences mentioned in (5.3). Letting $\omega_\star$ denote "the" forgetful functor, we have a natural tensor isomorphism

$$\eta : \omega_{\mathfrak{A}^\circ} \xrightarrow{\ \sim\ } \omega_{(\mathbb{C}\Gamma)^\circ} \circ V$$

constructed from the canonical isomorphisms

$$\omega_{\mathfrak{A}} \xrightarrow{\ \sim\ } \bullet|_{x_0} \circ \mathcal{V} \quad \text{and} \quad \omega_{\mathbb{C}\Gamma} \circ \mathrm{Mon}^{x_0} \xrightarrow{\ \sim\ } \bullet|_{x_0}.$$

(It should be noted that $\omega_{\mathfrak{A}} \circ \sigma = \omega_{\mathfrak{A}^\circ_M}$ and $\omega_{(\mathbb{C}\Gamma)^\circ} \circ \tau = \omega_{\mathbb{C}\Gamma}$.) In simpler terms, given $E \in \mathbf{comod}\text{-}\mathfrak{A}^\circ$, the vector space $V(E)$ is $\mathcal{V}(E)|_{x_0} = (\mathcal{O}_M \otimes E)|_{x_0}$ and η_E is the natural isomorphism $E \xrightarrow{\ \sim\ } \mathcal{V}(E)|_{x_0}$.

We obtain an arrow of *Hopf algebras*

$$v : \mathfrak{A}^\circ \longrightarrow (\mathbb{C}\Gamma)^\circ$$

jointly with an isomorphism of tensor functors

$$\overline{\eta} : v^\# \longrightarrow V$$

determined by η [35, II.2.1.2.1-3, p.116]. That is, η equals the composition

$$\omega_{\mathfrak{A}^\circ} = \omega_{(\mathbb{C}\Gamma)^\circ} \circ v^\# \xrightarrow{\ \omega_{(\mathbb{C}\Gamma)^\circ} \star \overline{\eta}\ } \omega_{(\mathbb{C}\Gamma)^\circ} \circ V.$$

Again, in simpler terms, given $E \in \mathbf{comod}\text{-}\mathfrak{A}^\circ$, the canonical isomorphism of vector spaces $\eta_E : E \xrightarrow{\ \sim\ } \mathcal{V}(E)|_{x_0}$ is an isomorphism of $(\mathbb{C}\Gamma)^\circ$–comodules, where $\mathcal{V}(E)|_{x_0}$ carries the co-action coming from the monodromy and E the co-action coming from v.

Applying the results of Section 5.3 to v, we derive a continuous morphism of topological $\mathbb{C}$–algebras

$$u : \mathbb{C}\Gamma^{\widehat{\ }} \longrightarrow \widehat{\mathfrak{A}}$$

which makes the analogue diagram in Lemma 5.4 commute strictly. We can then say the following. Let $E \in \mathbf{comod}\text{-}\mathfrak{A}^\circ$ be given. Endow E, respectively $\mathcal{V}(E)|_{x_0}$, with its $\mathbb{C}\Gamma$–module structure coming from u, respectively from the monodromy. Then $\eta : E \longrightarrow \mathcal{V}(E)|_{x_0}$ is an arrow of $\mathbb{C}\Gamma$–modules.

5.5. **Description of** $u : \mathbb{C}\Gamma\widehat{\ } \to \widehat{\mathfrak{A}}$ **in terms of iterated integrals.** We can now explicitly describe the action of Γ on $\mathfrak{A}$–modules and hence the morphism of $\mathbb{C}$–algebras $u : \mathbb{C}\Gamma\widehat{\ } \longrightarrow \widehat{\mathfrak{A}}$.

Let R be a finite dimensional associative $\mathbb{C}$–algebra and $f : \mathfrak{A} \longrightarrow R$ a surjective morphism. Regarding R as a left $\mathfrak{A}$–module, let d_f be the connection on $\mathcal{O}_M \otimes R$ defining $\mathcal{V}(R)$ and

$$\varphi_f = \mathrm{d}_f(1 \otimes 1) \in H^0(M, \Omega^1_M) \otimes R.$$

Proposition 5.5. *For any smooth loop γ based at x_0, the series*

$$1 + \int_\gamma \varphi_f + \int_\gamma \varphi_f^2 + \dots$$

in R converges, with respect to the Euclidean *topology, to $f(u(\gamma))$.*

Proof. Let $\lambda : R \longrightarrow \mathrm{End}(R)$ be the morphism of left multiplication. With this notation,

$$\Phi_f := (\mathrm{id} \otimes \lambda)(\varphi_f) \in H^0(M, \Omega^1_M) \otimes \mathrm{End}(R)$$

is the 1–form associated to the connection d_f (see (5.1)). As seen in (5.2), the series

$$1 + \sum_{n \geq 1} \int_\gamma \Phi_f^n$$

converges to $\mathrm{Mon}_\gamma \in \mathrm{End}(R)$. (We write Mon_γ instead of $\mathrm{Mon}^{x_0}_{\mathrm{d}_{\Phi_f}}(\gamma)$.) Now, for each $n \in \mathbb{N}$, we have in $\mathrm{End}(R)$ that

$$\lambda \int_\gamma \varphi_f^n = \int_\gamma \Phi_f^n,$$

so that $1 + \int_\gamma \varphi_f + \int_\gamma \varphi_f^2 + \dots$ converges to an element m_γ which satisfies $\lambda(m_\gamma) = \mathrm{Mon}_\gamma$. In particular, $\mathrm{Mon}_\gamma(1_R) = m_\gamma$. But, by construction of $u : \mathbb{C}\Gamma\widehat{\ } \longrightarrow \widehat{\mathfrak{A}}$ (see (5.5) and Section 5.4), we know that

$$\mathrm{Mon}_\gamma(1_R) = f(u(\gamma)) \cdot 1_R,$$

so that $m_\gamma = f(u(\gamma))$. $\qquad\square$

As a consequence, we have the following. Let γ be a smooth loop based at x_0 and define, for every $n \in \mathbb{N} \setminus \{0\}$, the linear functional

$$J_{\gamma,n} : H^0(M, \Omega^1_N)^{\otimes n} \longrightarrow \mathbb{C}$$

by

$$\varphi_1 \otimes \dots \otimes \varphi_n \longmapsto \int_\gamma \varphi_1 \cdots \varphi_n.$$

This gives rise to an element of $[H^0(M, \Omega^1_M)^*]^{\otimes n}$, then to an element of $\mathfrak{A}$ and hence to an element

$$J_{\gamma,n} \in \widehat{\mathfrak{A}}.$$

For convenience, we decree that $J_{\gamma,0} = 1$. By fixing a basis $\{\theta_i\}_{i=1}^g$ of $H^0(M, \Omega^1_M)$, a dual basis $\{t_i\}_{i=1}^g$ and writing $t_{i_1,\dots,i_n}$ for $t_{i_1} \otimes \dots \otimes t_{i_n}$, we have

$$J_{\gamma,n} = \sum_{1 \leq i_1,\dots,i_n \leq g} \left(\int_\gamma \theta_{i_1} \cdots \theta_{i_n} \right) \cdot t_{i_1,\dots,i_n} \qquad \begin{array}{c} \text{modulo integrability} \\ \text{relations.} \end{array}$$

These elements were studied firstly by Chen and Parshin [32] and hence we call the arrow

$$\mathrm{cp}_N \: : \: \Gamma \: \longrightarrow \: \widehat{\mathfrak{A}}, \quad \gamma \longmapsto \sum_{n=0}^{N} J_{\gamma,n}$$

the Chen-Parshin maps of order N (as of now, these are maps of sets) and (cp_N) the Chen-Parshin sequence.

Lemma 5.6. *Let R be a finite dimensional associative $\mathbb{C}$–algebra and $f : \mathfrak{A} \longrightarrow R$ a surjection. Let d_f stand for the connection on $\mathcal{O}_M \otimes R$ associated to the left $\mathfrak{A}$–module structure on R, and let $\varphi_f \in R \otimes H^0(M, \Omega_M^1)$ be the 1–form $\mathrm{d}_f(1 \otimes 1)$. Then, for each $n \in \mathbb{N}$, we have*

$$f(J_{\gamma,n}) \: = \: \int_\gamma \varphi_f^n.$$

Sketch proof. Let $\varphi_f \: = \: \sum_{i=1}^{g} e_i \otimes \theta_i$ and denote the function $\langle \theta_i, \gamma' \rangle$ by a_i. It follows that $\langle \varphi_f, \gamma' \rangle = \sum_i e_i a_i \in C^\infty([0,1], R)$. Consequently,

$$
\begin{aligned}
\int_\gamma \varphi_f^n &= \int_{0 \leq s_1 \leq \cdots \leq s_n \leq 1} \left(\sum_{1 \leq i_1 \leq g} e_{i_1} a_{i_1}(s_1) \right) \cdots \left(\sum_{1 \leq i_n \leq g} e_{i_1} a_{i_n}(s_n) \right) \, ds_1 \cdots ds_n \\
&= \sum_{1 \leq i_1, \ldots, i_n \leq g} \int_{0 \leq s_1 \leq \cdots \leq s_n \leq 1} e_{i_1} \cdots e_{i_n} a_{i_1}(s_1) \cdots a_{i_n}(s_n) \, ds_1 \ldots ds_n \\
&= \sum_{1 \leq i_1, \ldots, i_n \leq g} e_{i_1, \ldots, i_n} \int_\gamma \theta_{i_1} \cdots \theta_{i_n}.
\end{aligned}
$$

$\square$

For each $I \in S_\mathfrak{A}$, endow the complex vector space $\mathfrak{A}/I$ with the *canonical Euclidean topology* and give $\widehat{\mathfrak{A}}$ the projective limit (product) topology. This topology shall be called the PE-topology (projective-Euclidean). Hence, a sequence (α_n) in $\widehat{\mathfrak{A}}$ converges to $\alpha \in \widehat{\mathfrak{A}}$ if and only if, for each $I \in S_\mathfrak{A}$, the induced sequence $(\alpha_{n,I})$ in $\mathfrak{A}/I$ converges α_I [26, 4 Theorem, p .91].

Corollary 5.7. *In $\widehat{\mathfrak{A}}$, we have the equality*

$$\sum_{n=0}^{\infty} J_{\gamma,n} \: = \: u(\gamma),$$

where convergence is taken in the PE-topology. In other words, $u(\gamma)$ is the limit of the Chen-Parshin sequence $(\mathrm{cp}_N(\gamma))$ in the PE topology.

Proof. Let $I \in S_\mathfrak{A}$ and denote by $f : \mathfrak{A} \longrightarrow R$ the quotient morphism. As in Lemma 5.6, we write $\varphi_f \: = \: \mathrm{d}_f(1 \otimes 1)$. Then,

$$f\left(\sum_{n=0}^{N} J_{\gamma,n} \right) \: = \: \sum_{n=0}^{N} \int_\gamma \varphi_f^n$$

according to Lemma 5.6. Since the sequence $N \longmapsto \sum_{n=0}^{N} \int_{\gamma} \varphi_f^n$ converges (Proposition 5.5), the definition of the product topology proves that $\sum_{n \geq 0} J_{\gamma,n}$ converges in $\widehat{\mathfrak{A}}$. The limit is $u(\gamma)$. $\qquad\square$

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