ON THE VECTOR BUNDLES OVER RATIONALLY CONNECTED VARIETIES

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Abstract. Let X be a rationally connected smooth projective variety defined over $\mathbb{C}$ and $E \longrightarrow X$ a vector bundle such that for every morphism $\gamma : \mathbb{CP}^1 \longrightarrow X$, the pullback γ^*E is trivial. We prove that E is trivial. Using this we show that if γ^*E is isomorphic to $L(\gamma)^{\oplus r}$ for all γ of the above type, where $L(\gamma) \longrightarrow \mathbb{CP}^1$ is some line bundle, then there is a line bundle ζ over X such that $E = \zeta^{\oplus r}$.

Résumé. Soit X une variété rationnellement connexe sur $\mathbb{C}$ et soit $E \longrightarrow X$ un fibré vectoriel tel que, pour tout morphisme $\gamma : \mathbb{CP}^1 \longrightarrow X$, le fibré γ^*E est trivial. Nous montrons que E est trivial. Nous en déduisons que si, pour tout γ comme avant, γ^*E est isomorphe à $L(\gamma)^{\oplus r}$, où $L(\gamma) \longrightarrow \mathbb{CP}^1$ est un fibré en droites, alors il existe un fibré en droites ζ sur X et un isomorphisme $E \cong \zeta^{\oplus r}$

1. Introduction

Let E be a holomorphic vector bundle over a connected complex projective manifold X. If for every pair of the form (C, γ), where C is a compact connected Riemann surface, and $\gamma : C \longrightarrow X$ is a holomorphic map, the pullback γ^*E is semistable, then it is known that E is semistable, and $c_i(End(E)) = 0$ for all $i \geq 1$ [BB, pp. 3–4, Theorem 1.2]. Our aim here is to show that if X is rationally connected, then the above conclusion remains valid even if we insert in the condition that C is a rational curve. We recall that a complex projective variety X is said to be *rationally connected* if any two points of X can be joined by an irreducible rational curve on X; see [KMM, Theorem 2.1] for equivalent conditions. We prove the following theorem:

Theorem 1.1. *Let E be a vector bundle of rank r over a rationally connected smooth projective variety X defined over $\mathbb{C}$ such that for every morphism*

$$\gamma : \mathbb{CP}^1 \longrightarrow X ,$$

the pullback γ^*E is isomorphic to $L(\gamma)^{\oplus r}$ for some line bundle $L(\gamma) \longrightarrow \mathbb{CP}^1$. Then there is a line bundle ζ over X such that $E = \zeta^{\oplus r}$.

In [AW] this was proved under the extra assumption that $\mathrm{Pic}(X) = \mathbb{Z}$ (see [AW, p. 211, Proposition 1.2]).

Theorem 1.1 is deduced from the following proposition (see Proposition 2.1):

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Proposition 1.2. *Let X be as in Theorem 1.1. Let $E \longrightarrow X$ be a vector bundle such that for every morphism $\gamma : \mathbb{CP}^1 \longrightarrow X$, the pullback $\gamma^* E$ is trivial. Then E itself is trivial.*

The condition in Theorem 1.1 that $\gamma^* E$ is of the form $L(\gamma)^{\oplus r}$ can be replaced by an equivalent condition which says that $\gamma^* E$ is semistable (see Corollary 2.3).

2. Criterion for triviality

Let X be a rationally connected smooth projective variety defined over $\mathbb{C}$. Let $E \longrightarrow X$ be a vector bundle.

Proposition 2.1. *Assume that for every morphism*

$$\gamma : \mathbb{CP}^1 \longrightarrow X$$

the vector bundle $\gamma^* E \longrightarrow \mathbb{CP}^1$ is trivial. Then E itself is trivial.

Proof. Let $x \in X$ be a closed point. There is a smooth family of rational curves on X

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & X \\ {\scriptstyle \sigma}\uparrow\downarrow{\scriptstyle f} & & \\ T & & \end{array} \tag{2.1}$$

where

(1) T is open in $\mathrm{Mor}(\mathbb{CP}^1, X; (0:1) \longmapsto x)$ (hence T is quasiprojective),
(2) $f \circ \sigma = \mathrm{Id}_T$,
(3) φ is dominant, and
(4) $\varphi(\sigma(t)) = x$ for all $t \in T$.

(See [Ca, Section 3], [Ko2, Theorem 3].)

Let

$$\beta := [\varphi(f^{-1}(t))] \in H_2(X, \mathbb{Z})$$

be the homology class, where $t \in T(\mathbb{C})$. Let $\overline{\mathcal{M}}_{0,1}(X, \beta)$ be the moduli stack classifying families of stable maps from 1-pointed genus zero curves to X which represent the class β. (We are following the terminology of [FP].) We know that $\overline{\mathcal{M}}_{0,1}(X, \beta)$ is a proper Deligne–Mumford stack [BM, p. 27, Theorem 3.14].

Let

$$\rho : T \longrightarrow \overline{\mathcal{M}}_{0,1}(X, \beta) \tag{2.2}$$

be the morphism associated to the family in (2.1).

By "Chow's Lemma" [LMB, p. 154, Corollaire 16.6.1], there exists a projective $\mathbb{C}$–scheme Y together with a proper surjective morphism $\psi : Y \longrightarrow \overline{\mathcal{M}}_{0,1}(X, \beta)$. There exists a Cartesian diagram

$$\begin{array}{ccc} T_1 & \xrightarrow{\rho_1} & Y \\ {\scriptstyle \psi_1}\downarrow & \square & \downarrow{\scriptstyle \psi} \\ T & \xrightarrow[\rho]{} & \overline{\mathcal{M}}_{0,1}(X,\beta) \end{array}$$

where T_1 is a scheme and ψ_1 is proper and surjective. This last assertion is justified by the fact that the diagonal of a Deligne–Mumford stack is *schematic* [LMB, p. 26, Lemme 4.2] and [LMB, p. 21, Corollaire 3.13]. As T is separated (it is open in $\mathrm{Mor}(\mathbb{CP}^1, X)$), we can apply Nagata's Theorem [Lü, p. 106, Theorem 3.2] to find a proper $\mathbb{C}$–scheme $\overline{T}_1$ and a schematically dense open immersion

$$i \,:\, T_1 \hookrightarrow \overline{T}_1 \,.$$

Eliminating the "indeterminacy locus" (see e.g. [Lü, pp. 99–100]), we can find a blow–up

$$\xi \,:\, \overline{T} \longrightarrow \overline{T}_1$$

whose center is disjoint from T_1 and a morphism

$$\overline{\rho} \,:\, \overline{T} \longrightarrow Y$$

which extends $\rho_1 : T_1 \longrightarrow Y$. The composition $\psi \circ \overline{\rho} \,:\, \overline{T} \longrightarrow \overline{\mathcal{M}}_{0,1}(X,\beta)$ represents a family of 1–pointed genus zero stable maps

$$\begin{array}{ccc} \overline{Z} & \xrightarrow{\overline{\varphi}} & X \\ {\scriptstyle \overline{\sigma}}\uparrow\downarrow{\scriptstyle \overline{f}} & & \\ \overline{T} & & \end{array} \tag{2.3}$$

whose pull-back via $i \,:\, T_1 \hookrightarrow \overline{T}$ is the pull–back of the family in (2.1) via ψ_1. Clearly $\overline{\varphi}$ is dominant (hence surjective) and $\overline{\varphi} \circ \overline{\sigma}$ is a constant morphism. Note that, without loss of generality, we can assume $\overline{T}$ to be *reduced*.

We recall that the pullback of E by any map from $\mathbb{CP}^1$ is trivial. Consequently, for any point $t \in \overline{T}(\mathbb{C})$, the restriction of

$$\overline{E} \,:=\, \overline{\varphi}^* E$$

to the curve $\overline{f}^{-1}(t)$ — which is a tree of $\mathbb{CP}^1$ — is trivial. Therefore, $\overline{E}$ descends to $\overline{T}$. More precisely, the direct image $\overline{f}_* \overline{E}$ is a vector bundle on $\overline{T}$, and the canonical arrow

$$\overline{f}^* \overline{f}_* \overline{E} \longrightarrow \overline{E} \tag{2.4}$$

is an isomorphism [Mu, §5]. The homomorphism in (2.4) is injective because any section of a trivial vector bundle, over a connected projective scheme, that vanishes at one point actually vanishes identically; the homomorphism is surjective also because $\overline{E}|_{\overline{f}^{-1}(t)}$ is trivial for all t. We also note that the image of (2.4) by $\overline{\sigma}^*$ defines an isomorphism between $\overline{\sigma}^* \overline{E}$ and $\overline{f}_* \overline{E}$. Therefore, using (2.4),

$$\overline{f}^* \overline{\sigma}^* \overline{E} \,=\, \overline{E} \,. \tag{2.5}$$

Now from the condition that $\overline{\varphi} \circ \overline{\sigma}$ is a constant map it follows immediately that $\overline{\sigma}^*\overline{\varphi}^*E = \overline{\sigma}^*\overline{E}$ is a trivial vector bundle. Consequently, using (2.5) we conclude that the vector bundle $\overline{\varphi}^*E$ is trivial.

Since $\overline{\varphi}$ is a surjective and proper morphism, and $\overline{\varphi}^*E$ is trivial, we conclude that the Chern class $c_i(E)$ is numerically equivalent to zero for all $i \geq 1$.

Next we will show that the vector bundle E is semistable.

Let $C \hookrightarrow X$ be a smooth irreducible (proper) curve on X, and let $C' \hookrightarrow \overline{Z}$ be an irreducible curve such that $\overline{\varphi}(C') = C$. (The curve C' can be constructed as the closure of a closed point of the generic fiber of $\overline{\varphi}^{-1}(C) \longrightarrow C$.) Since the pull-back of $E|_C$ to C' is trivial, so is the pull-back of $E|_C$ to the normalization of C'. Consequently, the vector bundle $E|_C$ is semistable. This allows us to conclude that E is semistable with respect to any chosen polarization on X.

Since E is semistable, and both $c_1(E)$ and $c_2(E)$ are numerically equivalent to zero, a theorem of Simpson says that E admits a flat connection (see [Si, p. 40, Corollary 3.10]). On the other hand, X is simply connected because it is rationally connected [Ca, p. 545, Theorem 3.5], [Ko1, p. 362, Proposition 2.3]. Therefore, any flat vector bundle on X is trivial. In particular, the vector bundle E is trivial. □

As before, let E be a vector bundle over the rationally connected variety X. Let r be the rank of E.

Theorem 2.2. *Assume that for every morphism*

$$\gamma : \mathbb{CP}^1 \longrightarrow X ,$$

there is a line bundle $L(\gamma) \longrightarrow \mathbb{CP}^1$ *such that* $\gamma^*E = L(\gamma)^{\oplus r}$. *Then there is a line bundle* $\zeta \longrightarrow X$ *such that* $E = \zeta^{\oplus r}$.

Proof. The above condition on γ^*E and Proposition 2.1 ensure that the vector bundle $End(E)$ is trivial. This implies that, for any $x_0 \in X(\mathbb{C})$, the evaluation map

$$H^0(X,\, End(E)) \longrightarrow \mathrm{End}_{\mathbb{C}}(E(x_0)) \tag{2.6}$$

is as isomorphism; let $A : E \longrightarrow E$ be an isomorphism such that all the eigenvalues $\lambda_1, \ldots, \lambda_r$ of $A(x_0)$ are distinct. As the eigenvalues of $A(x)$ are independent of $x \in X$, it follows that E is isomorphic to the direct sum of the line subbundles

$$\mathcal{L}_i := \mathrm{kernel}(\lambda_i - A) \subseteq E ,$$

$1 \leq i \leq r$.

Since the evaluation map in (2.6) is an isomorphism, we have

$$\dim H^0(X,\, \mathcal{L}_i \otimes \mathcal{L}_j^*) \leq 1$$

for all $i, j \in [1, r]$. Note that if $H^0(X,\, \mathcal{L}_i \otimes \mathcal{L}_j^*) = 0$ for some i, j, then

$$\dim H^0(X,\, End(E)) < r^2 ,$$

which contradicts the fact that $End(E)$ is trivial. For $s_{ij} \in H^0(X, \mathcal{L}_i \otimes \mathcal{L}_j^*) \setminus \{0\}$, $i, j \in [1, r]$, the composition $s_{ij} \circ s_{ji}$ is an automorphism of $\mathcal{L}_i$, hence each s_{ij} is an isomorphism. This completes the proof of the theorem. □

A theorem due to Grothendieck says that any vector bundle over $\mathbb{CP}^1$ decomposes into a direct sum of line bundles [Gr, p. 126, Théorème 2.1]. Therefore, Theorem 2.2 has the following corollary:

Corollary 2.3. *If for every morphism $\gamma : \mathbb{CP}^1 \longrightarrow X$, the vector bundle γ^*E is semistable, then there is a line bundle $\zeta \longrightarrow X$ such that $E = \zeta^{\oplus r}$.*

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