SINGULAR VARIETIES AND INFINITESIMAL NON-COMMUTATIVE WITT VECTORS

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ABSTRACT. Given a projective variety X over an algebraically closed field k, M. V. Nori introduced in [No76] a group scheme $\pi(X)$ which accounts for principal bundles $P \to X$ with finite structure, obtaining in this way an amplification the etale fundamental group. One drawback of this theory is that it is quite difficult to arrive at an explicit description of $\pi(X)$, whenever it does not vanish altogether. To wit, there are no known non-trivial examples in the literature where $\pi(X)$ is local, or local of some given height, etc. In this paper we obtain a description of $\pi(X)$ through amalgamated products of certain non-commutative local group schemes — we called them infinitesimal non-commutative Witt group schemes — in the case where X is a non-normal variety obtained by pinching a simply connected one.

1. INTRODUCTION

In this work we are concerned with finite group schemes which appear as structure groups of principal bundles over projective varieties in positive characteristic, that is to say, with the theory of the fundamental group scheme initiated by M. V. Nori [No82, No76]. To begin our discussion, given a projective variety X over an algebraically closed field k of characteristic $p > 0$, let us write $\pi(X, x_0)$ for the essentially finite fundamental group scheme of X at $x_0 \in X(k)$ [No76, Section 3]. One of the drawbacks of this object — and for that matter also of its predecessor, the etale fundamental group — is that *precise* calculation turns out to be quite difficult. Indeed, although there are many instances where one can assure that $\pi(X, x_0)$ is *trivial* [Bi09, BH07, Me11, BKP24], or at least *small* [AB16, BHdS21], it is far from being an easy task to find actual descriptions outside of these cases. For example, the present work shall give the *first* known example for which $\pi(X, x_0)$ is *certainly* local and non-trivial, for which $\pi(X, x_0)$ is of height precisely 1, etc. (If we abandon the requirement of projectivity imposed on X, it is worth noticing that describing $\pi(X, x_0)$ precisely seems somewhat pointless, and putting the question as an existence theorem "à la Abhyankar" is more reasonable [Ot18, OTZ22].)

One of the difficulties behind the determination of $\pi(X, x_0)$ is: what we ultimately want is to identify when $\pi(X, x_0)$ *already appears somewhere else*. Now, prime examples of pro-finite group schemes are (those associated to) pro-finite groups and Frobenius kernels of linear algebraic groups. Many among these examples are applications of the following construction principle, which we now explain. We start with a finite abstract group or a restricted Lie algebra, we construct the group ring, respectively the universal enveloping algebra, note that these are in fact cocommutative Hopf algebras (see [A80, Example 2.7, p.62] and [MM65, §6]), and apply the *Hopf dual* functor $(-)^\circ$ [A80, Sw69] to arrive at

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the ring of functions of an group scheme. Thus, we may enquire when $\mathcal{O}(\pi(X,x_0)) \simeq H^\circ$, where H is a "known" cocommutative Hopf algebra. This point of view has the advantage of being more economic and simpler to apply to Tannakian categories; it is easier to deal with modules over an algebra (viz. H) than with comodules over a coalgebra (viz. $\mathcal{O}(\pi(X,x_0))$). It also has the advantage of opening a door to Dieudonné's theory of formal groups [Di73].

As one learns from [Se88], [Bri15] and [SGA1, Exposé IX, §5], introducing non-normal points on a regular variety has the effect of producing certain *accessory groups* on the starting one. More precisely, we assume the existence of a smooth variety Y and a finite morphism

$$\nu : Y \longrightarrow X$$

which allow us to express sheaves on X in terms of sheaves on Y plus some extra data; here we have in mind the process of pinching [Se88, Fe02]. Then, given a certain kind of group associated to a geometric object (see ahead), it is possible to describe the groups associated to X in terms of those of Y *plus* certain *accessory groups*. In the context of Jacobian varieties, the accessory groups are either powers of $\mathbf{G}_a$ or products of Witt groups [Se88, V.13–16]. In the theory of Picard schemes [Bri15], the accessory groups are Weil restrictions, and in [SGA1, IX.5] we encounter amalgamated products $\widehat{\mathbf{Z}} * \cdots * \widehat{\mathbf{Z}}$. It then becomes clear that for the case of the Nori fundamental group scheme an *obviously challenging part is to find analogues of these accessory groups*. Indeed, in our setting, well-known results allow us to rapidly transpose the geometry to a purely algebraic problem controlled by certain associative and cocommutative Hopf algebras $\boldsymbol{H}_D$ (see Section 8). These Hopf algebras (or their Hopf dual) play the role of the accessory groups, and the task is then to see if these objects already have a meaning somewhere else.

This is one point which the present work addresses by bringing to light the "non-commutative infinitesimal Witt group schemes" discovered independently by E. J. Ditters and K. Newman more than 50 years ago [Dit69, Ne74]. (Ditters draws on the outstanding series of papers Dieudonné produced on formal groups in the 1950's and 60's, while Newman is influenced by Sweedler.) Unfortunately, although very interesting, these group schemes have received little attention in overview works. The associative (cocommutative) Hopf algebras discovered by these two mathematicians — *non-commutative exponential group-coalgebras* in [Dit69], $\mathfrak{P}_n$ in [Ne74], but $\mathcal{NW}_{n-1}$ here — are not found in the text-book presentations of Hopf algebras at our disposal and are only tangentially mentioned, in the context of formal groups, at the end of [Haz78, VI.38].

Once these "non-commutative Witt group schemes" are under the light, we see that they have a role similar to the one played by Witt groups in the theory of the Jacobian [Se88, V.16] and are the "accessory groups" we were looking for. Indeed, this is argued, in a different setting, by Ditters and Newman [Dit69, Ne74]. In a nutshell, the "excess" appearing in the Jacobian mentioned by Serre (cf. V.13 and Proposition 9 of V.16 in [Se88]) has a manifestation in the theory of the fundamental group scheme.

Since our objective is to describe as precisely as possible the group scheme $\pi(X,x_0)$, we shall restrict attention to the case where *essentially finite vector bundles on* Y are rather simple, i.e. they are always *trivial*. By doing so, the bulk of the work consists in understanding the accessory group schemes mentioned above. It is not excluded that a more encompassing picture can be reached by allowing a more general Y, but this seems more delicate (see e.g. Example 6.9) and is to be dealt with somewhere else.

Let us now review the remaining sections of the paper. Section 2 has a two-fold function. Firstly, it surveys briefly the paper [DW23] (see Section 2.1) so that we have the theory of fibre-products of tensor categories and amalgamated products of group schemes right from the start. Secondly, it gathers, in Section 2.2, material explaining how the ring of regular functions $\mathcal{O}(G')$ of a certain group scheme G' *associated* to an affine group scheme

$G = \operatorname{Spec} H^\circ$ is more accessible when written as a Hopf dual. This part is employed in Section 7.2, but has some independent interest as well.

In Section 3, we review briefly the process of pinching a closed finite subscheme $D \to Y$ to a smaller finite scheme C to obtain a variety $X = Y \sqcup_D C$. The main reference for the theme here is [Fe02]. This is used mostly to gather notations and terminology. In Section 4 we bring to light Milnor's theorem and its consequences in the description of vector bundles on the schemes previously obtained. We then introduce the categories $\mathcal{T}_{D,C}$, one of the main actors of the paper because of their accessibility, and explain the first decomposition results for these (see Proposition 4.2).

Section 5 continues with the breaking up of the categories $\mathcal{T}_{D,C}$ into smaller pieces, but here the hypothesis are strengthened and C is taken to be *reduced*. In this section, the category $\mathcal{S}_D$, which will produce via Tannakian duality the first "accessory" group scheme Σ_D, appears. Now, if in Section 5 the main categorical decompositions are obtained, it is in Section 6 that the group theoretical results *and* the explicit calculations of $\pi(X)$ appear. In it, by introducing the hypothesis that $\pi(Y) = 0$, we are capable of providing a decomposition of $\pi(X)$ in terms of the group schemes Σ_L, where L is a connected component of D, see Theorem 6.5. Although more can be said about these Σ_L, this section bears already some fruits: we obtain clear examples of schemes X such that $\pi(X)$ is solely local and of a given finite height (see Proposition 6.2). Let us mention that Σ_L is the analogue of Serre's "$V_{(n)}$", which plays a role in describing the Jacobian of a singular curve [Se88, V.13–16].

Section 7 reviews the works of Ditters and Newman and adapts it to our setting. It also provides the group schemes which can be read off from [Dit69, Ne74] a more expressive name, the non-commutative (infinitesimal) Witt group schemes $\boldsymbol{NW}_\ell$, see Definition 7.6. The justification for such a term is offered by Proposition 7.9.

Section 8 makes use of an important theorem of Newman in order to decompose Σ_D, when D is connected, into infinitesimal non-commutative Witt group schemes, see Corollary 7.9. It is a non-commutative analogue of the well-known decomposition of the groups $V_{(n)}$ of [Se88, V.13–16] into Witt groups, cf. [Se88, V.16] This finishes our description of $\pi(X)$ (see Corollary 8.6).

The reader will notice that the text contains many reviews and preliminary material. This is mainly due to the varied nature of the techniques — tensor categories, pinching of schemes, vector bundles and Hopf algebras — employed to obtain a definite result. Requiring the reader to feel confortable with all these branches in order to understand the calculation of a fundamental group scheme seemed a poor decision. Hopefully this shall prove useful in appreciating the points of view of other mathematicians.

Some notations and conventions.

(1) k stands for an algebraically closed field of characteristic $p > 0$, *except* in Section 2, where no assumptions are necessary. The category of finite dimensional vector spaces is denoted by k-**vect**. The scheme $\operatorname{Spec} k$ is denoted by **pt**.
(2) All rings are associative and unital. When discussing algebraic geometric aspects, we shall also assume that the rings in question are commutative.
(3) Given a scheme X, a locally free $\mathcal{O}_X$-module of finite rank is called a vector bundle. The category of vector bundles is denoted by $\mathbf{VB}(X)$.
(4) The Frobenius morphism of a scheme or of a ring is denoted by Fr.
(5) Given an ideal $\mathfrak{a}$ in a commutative ring A of characteristic $p > 0$, we let $\mathfrak{a}^{[p^h]}$ denote the ideal generated by $\mathrm{Fr}^h(\mathfrak{a})$.
(6) Given a vector space V or a set set S, we shall denote the free k-algebra on V, respectively S, by $\boldsymbol{T}(V)$ or $k\{V\}$, respectively $k\{S\}$.

(7) Given an associative algebra R, respectively a coalgebra L, over k, we let R-**mod**, respectively **comod**-L, stand for the category of left R-modules, respectively right L-comodules, which are *finite dimensional over* k.
(8) The category of *Hopf agebras* over k is denoted by **Hpf**. That of cocommutative Hopf algebras is denoted by $\mathbf{Hpf}^{\mathrm{coc}}$. (This differs from the tradition in the theory of formal groups, where this same category is denoted by **GCog** [Di73], or **GCoalg** [Dit69].)
(9) We let **GS** stand for the category of affine group schemes over k. For brevity, objects in **GS** are referred to simply as *group schemes*.
(10) Given an abelian group Λ, we let $\mathrm{Diag}(\Lambda)$ stand for the diagonal group scheme constructed as in [Ja03, Part I, 2.5]: its algebra of functions is the group algebra $k\Lambda$.
(11) We follow the terminology of [DM82, §1] concerning *tensor categories*.
(12) Given an integral and proper k-scheme X, we shall denote by $\mathbf{EF}(X)$ the category of essentially finite vector bundles on X. Given a k-point x_0 of X, we let $\pi(X, x_0)$ stand for the group scheme obtained from $\mathbf{EF}(X)$ via the fibre functor $\bullet|_{x_0} : \mathbf{EF}(X) \to k\text{-}\mathbf{vect}$. See [No76].

2. Subsidiary material

We remind the reader that in this section no assumption on the field k is to be imposed.

2.1. **Fibre products and amalgamated products.** Many of the arguments in this work rely on the notion of fibre product of categories, which we recall briefly, for the convenience of the reader, and to establish notations.

Let $F : \mathcal{A} \to \mathcal{S}$, $G : \mathcal{B} \to \mathcal{S}$ be functors between categories and let

$$\mathcal{P} = \mathcal{A} \times_{\mathcal{S}} \mathcal{B}$$

be the category whose objects are triples

$$(a, b; \gamma) \quad \text{with } a \in \operatorname{Obj} \mathcal{A},\ b \in \operatorname{Obj} \mathcal{B} \text{ and } \gamma : Fa \xrightarrow{\sim} Gb,$$

and whose arrows are defined in the obvious manner, cf. [Fe02, 1.6] or [DW23, 1.1]. Since the case where $\mathcal{S}$ is the category k-**vect** will be mostly used, we will adopt the abbreviation

$$\boxed{\mathcal{A} \times_k \mathcal{B} := \mathcal{A} \times_{k\text{-}\mathbf{vect}} \mathcal{B}.}$$

When dealing with tensor categories [DM82, §1] $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$, *and* tensor functors F and G, the category $\mathcal{P}$ can be endowed with a functor $\mathcal{P} \times \mathcal{P} \to \mathcal{P}$,

$$(A, B; c) \otimes (A', B'; c') = (A \otimes A', B \otimes B'; c \otimes c'),$$

which then turns $\mathcal{P}$ into a tensor category. See [DW23, Lemma 1.5]. In addition, rigidity is preserved in this construction, as is "abelianess" [DW23, Lemma 1.4], provided that F and G are exact.

Taking one step further, let us suppose that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$ are *neutralised Tannakian categories*. This means that each one of them is an abelian, k-linear tensor category [DM82, 1.15], which is rigid [DM82, 1.7], and that there exist tensor functors $\xi : \mathcal{A} \to k$-**vect**, $\zeta : \mathcal{B} \to k$-**vect** and $\eta : \mathcal{S} \to k$-**vect** which are faithful, k-linear and exact. Suppose now that the functors F and G preserve these structures: there exist isomorphisms of tensor functors $\xi \overset{\sim}{\Rightarrow} \eta F$ and $\zeta \overset{\sim}{\Rightarrow} \eta G$. Then $\mathcal{P}$ is a k-linear abelian tensor category and comes with an exact and faithful tensor functor $\xi \times \zeta : \mathcal{P} \to k$-**vect** [DW23, Lemma 1.8]. All pieces are in place in order to apply the main existence theorem [DM82, Theorem 2.11].

Definition 2.1. Let G and H be group schemes and endow $\mathbf{Rep}_k(G)$ and $\mathbf{Rep}_k(H)$ with the forgetful functors $\omega_G : \mathbf{Rep}_k(G) \to k$-**vect** and $\omega_H : \mathbf{Rep}_k(H) \to k$-**vect**. We define

$G\star H$ as the group scheme, associated via [DM82, Theorem 2.11], to the couple consisting of the rigid abelian tensor category

$$\mathbf{Rep}_k(G)\times_k\mathbf{Rep}_k(H)$$

and the functor

$$\omega_G\times\omega_H:\mathbf{Rep}_k(G)\times_k\mathbf{Rep}_k(H)\longrightarrow k\text{-}\mathbf{vect}.$$

From [DW23, Corollary 1.12], it follows that $G\star H$ is the *co-product, in the category of group schemes* **GS**, of G and H. (The co-product will also be called the amalgamated product.)

2.2. **On the Hopf dual.** Let $R\in\mathbf{Hpf}^{\rm coc}$; the Hopf algebra R° [A80, Sw69] is commutative and $G:=\operatorname{Spec}R^\circ$ then comes with the structure of a group scheme. Let now $\mathbf{A}$ be a full subcategory of **GS** for which it is possible to construct the *largest quotient in* $\mathbf{A}$ [HdS18, Section 2.2]; let us agree to denote this quotient by $G\to G^{\mathbf{A}}$. We may then enquire if $G^{\mathbf{A}}$ can be described in terms of R exclusively. This is a useful entreprise for the present work and, in addition, has a certain public utility, so that we have enough reason to include the following lines dealing with the case where $\mathbf{A}$ is either the category of abelian or unipotent group schemes. It should be noticed that the specific case of group algebras $R=k\Gamma$, where Γ is an abstract group and $\operatorname{char}k=0$, is much used in Quillen's theory of the Malcev completion, cf. [Ha93, Section 3].

Let **Ab**, respectively **Un**, be the categories of abelian, respectively unipotent [Wa79, Chapter 8], group schemes. According to general category theory [HdS18, Section 2.2], the construction of the largest quotient is possible in these.

Let $\varepsilon:R\to k$ be the co-unit of R and $\mathfrak{a}$ its kernel, the augmentation ideal. Let R-**mod** stand for the category of all left R-modules whose dimension as a k-space is finite; in R-**mod** we have one preferred object, $\mathbf{I}:=R/\mathfrak{a}$, called the unit object. This terminology is coherent because the fact that R is a cocommutative Hopf algebra allows us to endow R-**mod** with a tensor structure and, in this case, $\mathbf{I}$ is a unit object. This is well explained in [Sch95, Proposition 1.1]. Similarly, let **comod**-R° stand for the category of right R°-comodules having finite dimension over k. Using [Mo93, 2.1.3] and [Sw69, 1.6.4], the identity functor on k-**vect** gives rise to an isomorphism of k-linear tensor categories

$$\Theta:R\text{-}\mathbf{mod}\stackrel{\sim}{\longrightarrow}\mathbf{comod}\text{-}R^\circ=\mathbf{Rep}_k\,G.$$

In particular $\mathbf{I}$ corresponds to the trivial representation of G.

Let

$$R^\dagger=\{f\in R^\circ\,:\,f(\mathfrak{a}^n)=0\text{ for some }n\geq1\}$$

and $R^{\rm ab}=R/\mathfrak{C}$, where $\mathfrak{C}$ is the two-sided ideal generated by the commutators in R.

Lemma 2.2. *(1) The subspace $R^\dagger\subset R^\circ$ is a Hopf subalgebra.*

(2) The k-algebra $R^{\rm ab}$ has a unique structure of Hopf algebra such that $R\to R^{\rm ab}$ is a morphism of Hopf algebras.

(3) The natural morphism $G=\operatorname{Spec}R^\circ\to\operatorname{Spec}R^{{\rm ab},\circ}$ is the largest abelian quotient of G.

(4) The natural morphism $G=\operatorname{Spec}R^\circ\to\operatorname{Spec}R'$ is the largest unipotent quotient of G.

Proof. (1) This is [Mo93, Lemma 9.2.1].

(2) We only need to observe that $\mathfrak{C}$ is a Hopf ideal and apply standard theory [A80, 4.2.1].

(4) Let R-**modu** be the full subcategory of R-**mod** consisting of those M for which the Jordan-Hölder factors are all isomorphic to $\mathbf{I}$. It is not difficult, using simple linear algebra, to see that under the isomorphism Θ mentioned above, the category R-**modu** corresponds to the category of *unipotent* comodules **ucomod**-R°, that is, those R°-comodules whose Jordan-Hölder factors are all isomorphic to $\mathbb{1}$. We contend that $M\in R$-**modu** if and only

if some power of $\mathfrak{a}$ annihilates M. Suppose then that $M \in R$-**modu** and let $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ be its Jordan-Hölder filtration where $M_i/M_{i-1} \simeq \mathbf{I}$. Then, for each $a \in \mathfrak{a}$, it follows that $a \cdot M_i \subset M_{i-1}$, and hence $a_1 \cdots a_n M = 0$ whenever $a_1, \ldots, a_n \in \mathfrak{a}$, i.e. $\mathfrak{a}^n M = 0$. Conversely, let $M \in R$-**mod** be annihilated by $\mathfrak{a}^n$ Then $\mathfrak{a}^n$ annihilates any simple module S appearing in a Jordan-Hölder filtration of M. Now, $\mathfrak{p} := \{r \in R : rS = 0\}$ is a two-sided *prime ideal* [GW04, 3.12, p.56] containing $\mathfrak{a}^n$ and hence $\mathfrak{p} \supset \mathfrak{a}$. This implies that $\mathfrak{p} = \mathfrak{a}$ and $S \simeq \mathbf{I}$.

In conclusion, only the unit representation of $G^\dagger := \operatorname{Spec} R^\dagger$ is simple and is hence $G^\dagger$ is unipotent. In addition, if $f : G \to U$ is a morphism to an unipotent group scheme U, then $\mathbf{Rep}_k\, U \to \mathbf{Rep}_k\, G \simeq R$-**mod** must have its image in R-**modu** and therefore, Tannakian duality leads to a factorization of f into $G \to G^\dagger \to U$.

(3) The proof is simple and omitted. □

3. Pinching a modulus

In this section, we explain the construction of the main space studied in the paper. Notations shall also be employed in other sections.

Let Y be an integral and *projective* scheme over k. Inspired by Serre [Se88, III.1], we define a *modulus* on Y as any closed immersion $\theta : D \to Y$ of a *finite* k-scheme D. Let C be another finite k-scheme and let $\mu : D \to C$ be a schematically dominant morphism, i.e. $\mu^\# : \mathcal{O}(C) \to \mathcal{O}(D)$ is injective. From [Fe02, Theorem 5.4], there exists a scheme X (see Remark 3.2 below) and a *co-cartesian diagram*

$$\begin{array}{ccc} D & \overset{\theta}{\hookrightarrow} & Y \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ C & \underset{\sigma}{\hookrightarrow} & X; \end{array} \tag{1}$$

the scheme X shall be referred to as *the pinching of D to C in Y*. In this case, σ is a closed immersion, ν is a finite morphism and

$$\nu : Y \setminus D \longrightarrow X \setminus C$$

is an isomorphism (in particular ν is also surjective). As Ferrand explains on [Fe02, 6.1], the scheme X is proper over k as well. The case when C is a point appears in [Se88, IV.4]; it was recently employed in [Da24].

As we assume Y to be projective, the same can be said about X. (Here we rely on the finiteness of D, cf. [Fe02, Section 6].) Indeed, let $\mathcal{L}$ be an ample invertible sheaf on Y. Since D is finite, the locally free $\mathcal{O}_D$-module $\theta^*\mathcal{L}$ is *free* of rank one. Consequently, using the equivalence $\mathbf{VB}(X) \simeq \mathbf{VB}(C) \times_{\mathbf{VB}(D)} \mathbf{VB}(Y)$ (this shall be explained in below), it follows that $\mathcal{L}$ descends to an invertible sheaf on X, which must then be ample by the criterion of [SP, tag 0B5V].

It is important to note that by allowing D, C and μ to vary substantially, this construction can be used to obtain projective varieties from their normalisations. Loosely speaking, the above process can be reversed. Let V be any projective variety over k and let S be the closed subset of non-normal points of V. Let $n : W \to V$ be its normalisation; this is a finite morphism and $n : W \setminus n^{-1}(S) \to V \setminus S$ is an isomorphism. In addition, W is a normal and projective variety. Note that the coherent $\mathcal{O}_V$-module $n_*\mathcal{O}_W/\mathcal{O}_V$ is supported at S. Let $\mathcal{C} = \operatorname{Ann}_{\mathcal{O}_V}(n_*\mathcal{O}_W/\mathcal{O}_V)$ be the conductor; it is an ideal of $\mathcal{O}_V$ whose image $n^\#\mathcal{C} \subset n_*(\mathcal{O}_W)$ is also an ideal and hence $\mathcal{C}$ gives rise to a closed subscheme $F \subset W$, a closed subscheme $E \subset V$ and a schematically dominant morphism $F \to E$.

More importantly, the following diagram of rings on V

$$\begin{array}{ccc} \mathcal{O}_V & \overset{n^\#}{\hookrightarrow} & n_*(\mathcal{O}_W) \\ \downarrow & & \downarrow \\ \mathcal{O}_V/\mathcal{C} & \hookrightarrow & n_*(\mathcal{O}_W)/n^\#\mathcal{C} \end{array}$$

is *cartesian* by the "conductor square" (see pp. 556-7 in [Fe02]). From this and Scholium 4.3 of [Fe02], it follows that $V = W \sqcup_F E$ in the category of ringed spaces and hence also in the category of schemes. Finally, it should be noticed that $E \subset V$ is supported on S but may well fail to be reduced.

Example 3.1. Let $X \subset \operatorname{Spec} k[x, y] = \mathbf{A}^2$ be the singular curve cut out by $y^2 - x^5 = 0$. Let $\mathbf{A}^1 = \operatorname{Spec} k[t]$ and define $\nu : \mathbf{A}^1 \to X$ by $a \mapsto (a^2, a^5)$. This gives rise to an isomorphism $\mathbf{A}^1 \setminus \{0\} \xrightarrow{\sim} X \setminus \{0\}$. Hence,

$$\mathcal{O}(X) \simeq k[t^2, t^5] \simeq k[t] \underset{k[t^2]/(t^4)}{\times} k[t]/(t^4).$$

So, X is obtained from $\mathbf{A}^1$ by pinching the modulus $\operatorname{Spec} k[t]/(t^4)$ to $\operatorname{Spec} k[t^2]/(t^4)$. Clearly the *conductor* is (t^4).

Remark 3.2. There is point in the phrasing of [Fe02] which can lead to confusion: he first constructs co-product $Y \sqcup_D C$ in the category **RS** of ringed spaces and then shows that $Y \sqcup_D C$ is a scheme. Now, the category of schemes is *not* a full subcategory of **RS**! Hence, the possibility that a morphism $Y \sqcup_D C \to S$ in **RS** obtained from morphisms of *schemes* $Y \to S$ and $C \to S$ fails to be a morphism of *locally* ringed spaces must be avoided. Luckily this is the case.

Remark 3.3 (Rosenlicht-Serre curves)**.** Let us add to the assumptions made on Y that it is regular and of dimension one, i.e. Y is a regular and proper curve. Then, under the assumption that C is reduced, the curve $X = Y \sqcup_D C$ is of a certain special kind. S. Das, in [Da24, Definition 2.1], calls these "Rosenlicht-Serre curves" because of its use in [Se88].

4. Vector bundles on pinched schemes

We shall keep the notations and setting introduced in section 3, specially those in diagram (1).

4.1. **The category $\mathcal{T}_{D,C}$ and Milnor's Theorem.** Theorem 2.2(iv) in [Fe02], due to Milnor, allows us to describe $\mathbf{VB}(X)$ from $\mathbf{VB}(Y)$ and $\mathbf{VB}(D)$. To be more precise, we shall employ the categorical fibre product $\mathbf{VB}(C) \times_{\mathbf{VB}(D)} \mathbf{VB}(Y)$ constructed by employing the functors $\theta^* : \mathbf{VB}(Y) \to \mathbf{VB}(D)$ and $\mu^* : \mathbf{VB}(C) \to \mathbf{VB}(D)$ (cf. Section 2.1). Milnor's Theorem can then be applied to show that the natural functor

$$\mathbf{VB}(X) \longrightarrow \mathbf{VB}(C) \times_{\mathbf{VB}(D)} \mathbf{VB}(Y),$$

described in terms of objects by

$$\mathcal{E} \longmapsto (\sigma^*\mathcal{E}, \nu^*\mathcal{E}; \mu^*\sigma^*\mathcal{E} \xrightarrow{\text{can}} \theta^*\nu^*\mathcal{E}),$$

is an equivalence. (In [Fe02, Theorem 2.2(iv)], the author presents a proof in the case of affine schemes; the transposition to the general case is simple as D lies in an affine subscheme of Y. See [Ho12, Theorem 3.13] for details.)

Since our objective is to describe $\pi(X, x_0)$ as precisely as possible — and as explained in Section 1, little is known about $\pi(Y)$ in general — we shall concentrate not on $\mathbf{VB}(X)$ itself, but on the full subcategory

$$\mathbf{VB}(C) \underset{\mathbf{VB}(D)}{\times} \mathbf{VB}^{\mathrm{tr}}(Y), \tag{2}$$

of $\mathbf{VB}(C) \times_{\mathbf{VB}(D)} \mathbf{VB}(Y)$ consisting of those objects whose "second component" is trivial. This is done with the intention of imposing $\pi(Y) = 0$ further ahead (see Section 6.2). To ease notation, we make the following definition.

Definition 4.1. Let $\mathcal{T}_{D,C}$ be the category which has as

objects those triples $(V, W; \beta)$, where $V \in \mathbf{VB}(C)$, $W \in k$-**vect**, and $\beta : W \otimes_k \mathcal{O}_D \to \mu^* V$ is an isomorphism. And an

arrow from $(V, W; \beta)$ to $(V', W'; \beta')$ is a couple $(f, g) \in \mathrm{Hom}_{\mathcal{O}_C}(V, V') \times \mathrm{Hom}(W, W')$ rendering commutative the diagram

$$\begin{array}{ccc} \mu^* V & \xrightarrow{\mu^* f} & \mu^* V' \\ {\scriptstyle \beta}\uparrow & & \uparrow{\scriptstyle \beta'} \\ W \otimes_k \mathcal{O}_D & \xrightarrow{g \otimes \mathrm{id}} & W' \otimes_k \mathcal{O}_D. \end{array}$$

The category $\mathcal{T}_{D,C}$ comes with a k-bilinear functor

$$\otimes : \mathcal{T}_{D,C} \times \mathcal{T}_{D,C} \longrightarrow \mathcal{T}_{D,C}$$

defined by

$$(V, W; \beta) \otimes (V', W'; \beta') = (V \otimes_{\mathcal{O}_C} V', W \otimes_k W'; \tau(\beta, \beta')),$$

where $\tau(\beta, \beta')$ renders commutative the obvious diagram. This turns $\mathcal{T}_{C,D}$ into a tensor category. We let

$$\omega_{D,C} : \mathcal{T}_{D,C} \longrightarrow k\text{-}\mathbf{vect}, \tag{3}$$

be defined on the level of objects by $(V, W; \beta) \mapsto W$. Note that we have an equality $\otimes_k \circ (\omega_{D,C} \times \omega_{D,C}) = \omega_{D,C} \circ \otimes$. In what follows, the functor $\omega_{D,C}$ shall be referred to as *the fibre functor* of $\mathcal{T}_{D,C}$.

Finally, we observe that

$$\mathcal{T}_{D,C} \longrightarrow \mathbf{VB}(C) \underset{\mathbf{VB}(D)}{\times} \mathbf{VB}^{\mathrm{tr}}(Y), \qquad (V, W; \beta) \longmapsto (V, W \otimes_k \mathcal{O}_Y; \beta^{-1}) \tag{4}$$

gives rise to an equivalence since $H^0(Y, -) : \mathbf{VB}^{\mathrm{tr}}(Y) \to k$-**vect** is an equivalence. In what follows, $\mathcal{T}_{D,C}$ is the object of our studies. Our method consists in breaking up $\mathcal{T}_{D,C}$ into fibered products over k-**vect** and we begin by the simplest one in Section 4.2 below.

4.2. **The case where C is a disjoint union.** We now suppose that C can be written as a disjoint union of two closed and open subschemes, that is, for $i = 1, 2$, we have closed immersions $\varphi_i : C_i \to C$ such that $C_1 \sqcup C_2 \simeq C$. We adopt the notations implied in the following *cartesian diagram:*

$$\begin{array}{ccc} D_i & \xrightarrow{\psi_i} & D \\ {\scriptstyle \mu_i}\downarrow & & \downarrow{\scriptstyle \mu} \\ C_i & \xrightarrow[\varphi_i]{} & C. \end{array}$$

We then define functors

$$F_i : \mathcal{T}_{D,C} \longrightarrow \mathcal{T}_{D_i,C_i}$$

as follows. Let $(V, W; \beta) \in \operatorname{Obj} \mathcal{T}_{D,C}$ and write $F_i(V, W; \beta) = (\varphi_i^* V, W; \beta_i)$, where β_i is obtained from the commutative diagram

$$\begin{array}{ccccc} \psi_i^*(W \otimes_k \mathcal{O}_D) & \xrightarrow{\psi_i^*(\beta)} & \psi_i^* \mu^* V & \xrightarrow{\text{can.}} & \mu_i^*(\varphi_i^* V) \\ \uparrow{\scriptstyle \text{can.}} & & & \nearrow_{\beta_i} & \\ W \otimes_k \mathcal{O}_{D_i}. & & & & \end{array}$$

Note that the composition

$$\mathcal{T}_{D,C} \xrightarrow{F_i} \mathcal{T}_{D_i,C_i} \xrightarrow{\omega_{D_i,C_i}} k\text{-}\mathbf{vect}$$

is just $\omega_{D,C}$. Using the notion of 2-commutative diagram (see [DW23, Section 1]) we then arrive at a functor

$$G : \mathcal{T}_{D,C} \longrightarrow \mathcal{T}_{D_1,C_1} \times_k \mathcal{T}_{D_2,C_2},$$

through [DW23, Lemma 1.3].

Proposition 4.2. *The functor G is an equivalence of categories.*

Proof. We make first a general remark. Let $\boldsymbol{V} = (V, W; \beta) \in \mathcal{T}_{D,C}$ and $a : W' \to W$ be an isomorphism. Define $\beta' : W \otimes_k \mathcal{O}_D \to \mu^* V$ as the composition

$$\mathcal{O}_D \otimes W' \xrightarrow{\quad \mathrm{id} \otimes a \quad} \mathcal{O}_D \otimes_k W \xrightarrow{\ \beta\ } \mu^* V.$$

We have therefore a new object $\boldsymbol{V'} = (V, W'; \beta')$, and $(\mathrm{id}_V, a) : \boldsymbol{V'} \to \boldsymbol{V}$ is isomorphism. Consequently, in the isomorphism class of an object

$$\boldsymbol{P} = ((V_1, W_1; \beta_1), (V_2, W_2; \beta_2); a : W_1 \xrightarrow{\sim} W_2)$$

in $\mathcal{T}_{D_1,C_1} \times_k \mathcal{T}_{D_2,C_2}$, we always can find and object of the specific form

$$\boldsymbol{P'} = ((V_1, W_1; \beta_1), (V_2, W_1; \beta_2'); \mathrm{id}_{W_1}).$$

This observation, together with the fact that $C \simeq C_1 \sqcup C_2$, proves that G is essentially surjective. □

5. Description of the category $\mathcal{T}_{D,C}$ when C is reduced

We shall keep the assumptions and notations of Section 4. We assume further that C is *reduced* and set out to describe in more $\mathcal{T}_{D,C}$. This restriction on C is an important one, but produces a clear picture: we shall soon see that the basic building blocks of our analysis of $\mathcal{T}_{D,C}$ are certain categories $\mathcal{S}_\bullet$ associated to connected components of D (see Definition 5.1). These simpler categories will then produce by Tannakian duality the group schemes $\Sigma_\bullet$ (see Definition 6.1), which appeared, in a different form, in [Dit69, Dit75, Ne74]. The case when C is not reduced should be dealt with in future work.

Let $C = C_1 \sqcup \cdots \sqcup C_m$, where $C_i = \mathbf{pt} = \operatorname{Spec} k$. As in Section 4.2, let D_i stand for the inverse image of C_i inside D. From Proposition 4.2, we conclude that

$$\mathcal{T}_{D,C} \simeq \mathcal{T}_{D_1,\mathbf{pt}} \times_k \cdots \times_k \mathcal{T}_{D_m,\mathbf{pt}}.$$

It is therefore necessary to pay attention to the case where $C = \mathbf{pt}$.

5.1. **Description of $\mathcal{T}_{D,\mathbf{pt}}$ for D connected.** We assume in this section that D is connected, i.e. $\mathcal{O}(D)$ is a local ring, and let $s : \mathbf{pt} \to D$ be its unique point. Given an isomorphism $f : W \to W'$ of vectors spaces, we have an isomorphism

$$(\mathrm{id}_V, f) : (V, W; \beta) \xrightarrow{\sim} (V, W'; \beta \circ (f^{-1} \otimes_k \mathrm{id}_{\mathcal{O}_D})) \tag{5}$$

and, in particular, each $(V, W; \beta) \in \mathrm{Obj}\, \mathcal{T}_{D,\mathbf{pt}}$ is isomorphic to some (V, V, β') by letting $f = s^*(\beta)$. This shows how to break up $\mathcal{T}_{D,\mathbf{pt}}$ into smaller pieces with the help of:

Definition 5.1. Define $\mathcal{S}_D^+$ as the category whose

objects are (V, β), with $V \in k$-**vect** and $\beta : V \otimes \mathcal{O}_D \to V \otimes \mathcal{O}_D$ an isomorphism, and

arrows from (V, β) to (V', β') are maps $f \in \mathrm{Hom}_k(V, V')$ rendering

$$\begin{array}{ccc} V \otimes \mathcal{O}_D & \xrightarrow{f \otimes \mathrm{id}} & V' \otimes \mathcal{O}_D \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \beta'} \\ V \otimes \mathcal{O}_D & \xrightarrow{f \otimes \mathrm{id}} & V' \otimes \mathcal{O}_D \end{array}$$

commutative.

Analogously, we define $\mathcal{S}_D$ as being the full subcategory of $\mathcal{S}_D^+$ whose objects are the couples (V, β) where, in addition, $s^*(\beta) = \mathrm{id}_V$.

Not surprisingly, $\mathcal{S}_D^+$ and $\mathcal{S}_D$ have structures of tensor categories, where

$$(V, \beta) \otimes (V', \beta') = (V \otimes_k V, \tau(\beta, \beta'))$$

and $\tau(\beta, \beta')$ renders commutative the diagram

$$\begin{array}{ccc} (V \otimes_k V') \underset{k}{\otimes} \mathcal{O}_D & \xrightarrow{\tau(\beta,\beta')} & (V \otimes_k V') \underset{k}{\otimes} \mathcal{O}_D \\ {\scriptstyle \mathrm{can.}}\downarrow & & \downarrow{\scriptstyle \mathrm{can.}} \\ (V \otimes_k \mathcal{O}_D) \underset{\mathcal{O}_D}{\otimes} (V' \otimes_k \mathcal{O}_D) & \xrightarrow[\beta \otimes \beta']{} & (V \otimes_k \mathcal{O}_D) \underset{\mathcal{O}_D}{\otimes} (V' \otimes_k \mathcal{O}_D). \end{array}$$

We remark that the forgetful functor

$$\sigma_D : \mathcal{S}_D^+ \longrightarrow k\text{-}\mathbf{vect} \tag{6}$$

is a faithful and k-linear tensor functor. We shall soon see that $\mathcal{S}_D^+$ plays a supporting role with respect to $\mathcal{S}_D$ (Proposition 5.3).

There is an obvious functor

$$F : \mathcal{S}_D \longrightarrow \mathcal{T}_{D,\mathbf{pt}} \tag{7}$$

defined on the level of objects by $(V, \beta) \mapsto (V, V; \beta)$ and on the level of arrows by

$$\mathrm{Hom}((V, \beta), (V', \beta')) \longrightarrow \mathrm{Hom}(F(V, \beta), F(V', \beta')), \qquad f \longmapsto (f, f).$$

From the isomorphism (5) above, it follows that F is essentially surjective. A simple verification shows that F is also fully faithful and hence the proposition below holds true:

Proposition 5.2. *The functor $F : \mathcal{S}_D \to \mathcal{T}_{D,\mathbf{pt}}$ of eq. (7) is an equivalence of k-linear categories.* □

We now study the relation between $\mathcal{S}_D^+$ and $\mathcal{S}_D$. Let

$$\mathcal{I} := \mathbf{Rep}_k(\mathbf{Z}).$$

For each $\boldsymbol{V} := [(V, \beta), (W, g); \varphi] \in \mathcal{S}_D \times_k \mathcal{I}$, we let

$$G\boldsymbol{V} = (V, \beta(\varphi^{-1} g \varphi)_D) \in \mathrm{Obj}\, \mathcal{S}_D^+;$$

here and below, we abbreviate $\xi\otimes\mathrm{id}_{\mathcal{O}_D}$ to ξ_D. Also, define for each arrow (f,h) in $\mathcal{S}_D\times_k\mathcal{I}$, the arrow $G(f,h)=f$. This gives rise to a functor $G:\mathcal{S}_D\times_k\mathcal{I}\to\mathcal{S}_D^+$.

Proposition 5.3. *The above defined functor $G:\mathcal{S}_D\times_k\mathcal{I}\to\mathcal{S}_D^+$ is an equivalence of tensor categories.*

Proof. This is a sequence of simple verifications. Let $(V,\gamma)\in\operatorname{Obj}\mathcal{S}_D^+$. Then, $(V,\gamma\circ(s^*(\gamma)^{-1})_D)$ is an object of $\mathcal{S}_D$ and $(V,s^*\gamma)$ is an object of $\mathcal{I}$. If we put

$$\boldsymbol{V}:=(V,\gamma(s^*(\gamma)^{-1})_D\,,\,(V,s^*\gamma)\,;\,\mathrm{id}_V]\in\operatorname{Obj}\mathcal{S}_D\times_k\mathcal{I},$$

then $G\boldsymbol{V}=(V,\gamma)$, and G is essentially surjective.

We now check that G is fully faithful. Each $[(V,\beta),(W,g);\varphi]$ is isomorphic, via $(\mathrm{id},\varphi^{-1})$, to $[(V,\beta),(V,\varphi^{-1}g\varphi);\mathrm{id}_V]$ and we may then work solely with objects of this simplified kind. Let $\boldsymbol{V}=[(V,\beta),(V,g);\mathrm{id}]$ and $\boldsymbol{V}'=[(V',\beta'),(V',g');\mathrm{id}]$ be given in $\mathcal{S}_D\times_k\mathcal{I}$, and let $\varphi:V\to V'$ satisfy

$$\varphi_D\beta g_D=\beta' g'_D\varphi_D; \tag{8}$$

i.e. $\varphi\in\operatorname{Hom}_{\mathcal{S}_D^+}(G\boldsymbol{V},G\boldsymbol{V}')$. Applying s^* to eq. (8), we have

$$\varphi g=g'\varphi, \tag{9}$$

i.e. $\varphi\in\operatorname{Hom}_{\mathcal{I}}((V,g),(V',g'))$. From eqs. (8) and (9), we conclude $\varphi_D\beta g_D=\beta'\varphi_D g_D$ and thus $\beta'\varphi_D=\varphi_D\beta$. This says that $\varphi\in\operatorname{Hom}_{\mathcal{S}_D}((V,\beta),(V',\beta'))$ and (φ,φ) is an arrow $\boldsymbol{V}\to\boldsymbol{V}'$ inducing $\varphi\in\operatorname{Hom}_{\mathcal{S}_D^+}(G\boldsymbol{V},G\boldsymbol{V}')$. Consequently, $G_{\boldsymbol{V},\boldsymbol{V}'}:\operatorname{Hom}(\boldsymbol{V},\boldsymbol{V}')\to\operatorname{Hom}(G\boldsymbol{V},G\boldsymbol{V}')$ is surjective. Injectivity is even simpler and we do not write it down. Nor shall we write down the verifications concerning the final statement, and we conclude here the proof. □

5.2. **Determination of $\mathcal{T}_{D,\mathbf{pt}}$ for arbitrary D.** We now assume that $D=D_1\sqcup\cdots\sqcup D_m$, where each D_i is connected and $m\geq 2$. Let $\tilde{D}:=D_2\sqcup\cdots\sqcup D_m$ so that $D=D_1\sqcup\tilde{D}$. We identify $\mathbf{VB}(D)$ with $\mathbf{VB}(D_1)\times\mathbf{VB}(\tilde{D})$; in particular, maps between $\mathcal{O}_D$-modules shall be expressed as *pairs of maps*. We then pick a point $s:\mathbf{pt}\to\tilde{D}$ of $\tilde{D}$. Finally, let $d:D\to\mathbf{pt}$, $d_1:D_1\to\mathbf{pt}$ and $\tilde{d}:\tilde{D}\to\mathbf{pt}$ be the structural morphisms.

The reader is asked to bear in mind that $\mathcal{T}_{\tilde{D},\mathbf{pt}}$ and $\mathcal{S}_{D_1}^+$ come equipped with functors to k-**vect** (cf. eq. (3) and eq. (6)), which allow us to define $\mathcal{S}_{D_1}^+\times_k\mathcal{T}_{\tilde{D},\mathbf{pt}}$. We set out to show that $\mathcal{T}_{D,\mathbf{pt}}\simeq\mathcal{S}_{D_1}^+\times_k\mathcal{T}_{\tilde{D},\mathbf{pt}}$.

Given an object $[(U,\alpha),(V,W;\tilde{\beta});\varphi]$ of $\mathcal{S}_{D_1}^+\times_k\mathcal{T}_{\tilde{D},\mathbf{pt}}$, we wish to associate to it an object of $\mathcal{T}_{D,\mathbf{pt}}$. Note that we are already in possession of an isomorphism $\tilde{\beta}:\tilde{d}^*W\xrightarrow{\sim}\tilde{d}^*V$ and we need an isomorphism of $\mathcal{O}_{D_1}$-modules $\beta_1:d_1^*W\xrightarrow{\sim}d_1^*V$. Now,

$$U\xrightarrow{\ \varphi\ }W\xrightarrow{\ s^*(\tilde{\beta})\ }V$$

is an isomorphism, and hence it is only natural that we define β_1 by rendering

$$\begin{array}{ccc} d_1^*(U) & \xrightarrow{\ d_1^*(\varphi)\ } & d_1^*(W) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta_1} \\ d_1^*(U) & \xrightarrow[\ d_1^*(s^*(\tilde{\beta})\varphi)\ ]{} & d_1^*(V) \end{array}$$

commutative, i.e.

$$\beta_1=d_1^*(s^*(\tilde{\beta})\varphi)\circ\alpha\circ d_1^*(\varphi^{-1}).$$

Therefore, for each $[(U,\alpha),(V,W;\tilde\beta);\varphi]$ we introduce the object

$$H[(U,\alpha),(V,W;\tilde\beta);\varphi]=(V,W\,;\,d_1^*(s^*(\tilde\beta)\varphi)\circ\alpha\circ d_1^*(\varphi^{-1}),\tilde\beta)$$

of $\mathcal{T}_{D,\mathbf{pt}}$. Also, given any arrow in $\mathcal{S}^+_{D_1}\times_k\mathcal{T}_{\tilde D,\mathbf{pt}}$,

$$(f,g,h):[(U,\alpha),(V,W;\tilde\beta);\varphi]\longrightarrow[(U',\alpha'),(V',W';\tilde\beta');\varphi'],$$

an intricate but straightforward computation shows that

$$(g,h):(V,W;\beta_1,\tilde\beta)\longrightarrow(V',W';\beta_1',\tilde\beta')$$

is an arrow in $\mathcal{T}_{D,\mathbf{pt}}$ and in this way we arrive at a *faithful* and k-linear functor

$$H:\mathcal{S}^+_{D_1}\times_k\mathcal{T}_{\tilde D,\mathbf{pt}}\longrightarrow\mathcal{T}_{D,\mathbf{pt}}.$$

Proposition 5.4. *The functor H defined above is an equivalence of neutralized tensor categories.*

Proof. This is again a series of straightforward long verifications. *Essential surjectivity.* Let $\boldsymbol{V}=(V,W;\beta_1,\tilde\beta)\in\operatorname{Obj}\mathcal{T}_{D,\mathbf{pt}}$ where, as before, $\beta_1:d_1^*W\to d_1^*V$ and $\tilde\beta:\tilde d^*W\to\tilde d^*V$ are isomorphisms. Let $\boldsymbol{V}_1=(W,d_1^*s^*(\tilde\beta^{-1})\circ\beta_1)\in\operatorname{Obj}\mathcal{S}^+_{D_1}$ and $\tilde{\boldsymbol{V}}=(V,W;\tilde\beta)\in\operatorname{Obj}\mathcal{T}_{\tilde D,\mathbf{pt}}$. Then

$$H(\boldsymbol{V}_1,\tilde{\boldsymbol{V}};\mathrm{id}_W)=\boldsymbol{V}.$$

Fully faithfulness. Faithfulness has already been justified above. Each $\boldsymbol{V}\in\operatorname{Obj}\mathcal{S}^+_{D_1}\times_k\mathcal{T}_{\tilde D,\mathbf{pt}}$ is isomorphic to an object of the form $[(W,\alpha),(V,W;\tilde\beta);\mathrm{id}_W]$ so that in what follows we can restrict attention to these. Let $\boldsymbol{V}=[(W,\alpha),(V,W;\tilde\beta);\mathrm{id}_W]$ and $\boldsymbol{V}'=[(W',\alpha'),(V',W';\tilde\beta');\mathrm{id}_{W'}]$ be objects of $\mathcal{S}^+_{D_1}\times_k\mathcal{T}_{\tilde D,\mathbf{pt}}$ and let

$$(g,h):H\boldsymbol{V}\longrightarrow H\boldsymbol{V}'$$

be an arrow of $\mathcal{T}_{D,\mathbf{pt}}$. Hence,

$$d_1^*(g)\circ d_1^*(s^*(\tilde\beta))\circ\alpha=d_1^*(s^*(\tilde\beta'))\circ\alpha'\circ d_1^*(h)\tag{10}$$

and

$$\tilde d^*(g)\circ\tilde\beta=\tilde\beta'\circ\tilde d^*(h).\tag{11}$$

From eq. (11) we conclude that $g\circ s^*(\tilde\beta)=s^*(\tilde\beta')\circ h$, which together with eq. (10) shows that $d_1^*(h)\circ\alpha=\alpha'\circ d_1^*(h)$. But this means that $h:W\to W'$ defines an arrow of $\mathcal{S}^+_{D_1}$ from (W,α) to (W',α') and we conclude that (g,h) comes from an arrow $\boldsymbol{V}\to\boldsymbol{V}'$. □

Using Proposition 5.4 and then Proposition 5.3, we have:

Corollary 5.5. *If we decompose D into connected components, $D=D_1\sqcup\cdots\sqcup D_m$, then the tensor category $\mathcal{T}_{D,\mathbf{pt}}$ is equivalent to $\mathcal{S}_{D_1}$ if $m=1$ and to*

$$(\mathcal{I}\times_k\mathcal{S}_{D_1})\times_k\cdots\times_k\left(\mathcal{I}\times_k\mathcal{S}_{D_{m-1}}\right)\times_k\mathcal{S}_{D_m}$$

*if $m\geq 2$. In addition, the equivalences preserve the canonical functors to k-**vect**.* □

6. Description of $\pi(X)$ when C is reduced

We write the findings of Section 5 in terms of group schemes in order to express $\pi(X)$ as an amalgamated product. One of the pieces in this product — the group scheme Σ_D obtained from $\mathcal{S}_D$ when D is connected — turns out to be a fundamental block and will then be analysed in Section 8.

6.1. **Translation in terms of Tannakian group schemes.** We assume that D is connected, except for Theorem 6.5 below.

Definition 6.1. We let Σ_D be the group scheme obtained from $\mathcal{S}_D$ through the fibre functor $\sigma_D : \mathcal{S}_D \to k\text{-}\mathbf{vect}$ and Tannakian duality [DM82, Theorem 2.11]. Similarly, Σ_D^+ is the group scheme obtained from $\mathcal{S}_D^+$.

Let $\mathbf{Z}^{\mathrm{alg}}$ be the group scheme associated to $\mathcal{I} = \mathbf{Rep}_k(\mathbf{Z})$ via the forgetful functor. Letting $\mathbf{Z}_p$ be the pro-finite group scheme $\varprojlim_\ell \mathbf{Z}/p^\ell$, it is not difficult to show that $\mathbf{Z}^{\mathrm{alg}} = \mathrm{Diag}(k^*) \times \mathbf{Z}_p$. More precisely, denote by $[m] : k^* \to k^*$ the map $a \mapsto a^m$. Then, under the identification $\mathrm{Diag}(k^*)(k) = \mathrm{Hom}_{\mathbf{Grp}}(k^*, k^*)$ [Ja03, Part I, 2.5], it follows that the morphism of groups

$$\mathbf{Z} \longrightarrow \mathrm{Diag}(k^*)(k) \times \mathbf{Z}_p, \qquad m \longmapsto ([m], m)$$

induces an equivalence between representation categories. (It should be noted that $\mathbf{Z}^{\mathrm{alg}}$ is not necessarily pro-finite.) From [DW23, Corollary 1.11] and Proposition 5.3 we conclude that

$$\Sigma_D^+ \simeq \Sigma_D \star (\mathrm{Diag}(k^*) \times \mathbf{Z}_p)\,. \tag{12}$$

Let us note the following result, whose proof shall be given in Section 8.1. (It rests on the fact that $\mathcal{O}(\Sigma_D)$ can be very easily described as a Hopf dual.) The reader is asked to bear in mind our blanket assumption that D is connected.

Proposition 6.2. *The group scheme Σ_D is local. In fact, let $\mathfrak{m}$ stand for the maximal ideal of $\mathcal{O}(D)$ and let $h \geq 1$ be such that $\mathfrak{m}^{[p^h]} = 0$ while $\mathfrak{m}^{[p^{h-1}]} \neq 0$. Then Σ_D is annihilated by Fr^h but not by Fr^{h-1}, i.e. is of height h* [DG70, II.7.1.4–6]. □

Because each local group scheme is pro-fintie (cf. Remark 6.6) we have.

Corollary 6.3. *The group scheme Σ_D is pro-finite.* □

With a view towards the computation of $\pi(X)$ in Section 6.2 below, let us now determine the *largest* pro-finite quotient $(\Sigma_D^+)^{\mathrm{pf}}$ of Σ_D^+. (For this notion, see Section 2.2.) Being a *left-adjoint*, the functor $(-)^{\mathrm{pf}}$ must commute with colimits [Mac98, V.5] and hence eq. (12) jointly with Corollary 6.3 give us

$$\begin{aligned}\left(\Sigma_D^+\right)^{\mathrm{pf}} &\simeq \Sigma_D^{\mathrm{pf}} \star (\mathrm{Diag}(k^*) \times \mathbf{Z}_p)^{\mathrm{pf}} \\ &\simeq \Sigma_D \star (\mathrm{Diag}(k^*) \times \mathbf{Z}_p)^{\mathrm{pf}}\,.\end{aligned}$$

Now, $(k^*)_{\mathrm{tors}} \simeq \varinjlim_m \mathbf{Z}/m$, the limit being taken over all positive integers which are not divisible by p, so that

$$(\mathrm{Diag}(k^*) \times \mathbf{Z}_p)^{\mathrm{pf}} \simeq \widehat{\mathbf{Z}} = \varprojlim_m \mathbf{Z}/m.$$

In a nutshell:

Proposition 6.4. *We have an isomorphism of group schemes $\left(\Sigma_D^+\right)^{\mathrm{pf}} \simeq \Sigma_D \star \widehat{\mathbf{Z}}$.* □

Let us now *drop the assumption that D is connected.* Using again that the functor $(-)^{\mathrm{pf}}$ commutes with co-products, we derive from Corollary 5.5 the following consequence.

Theorem 6.5. *We decompose D into connected components, $D = D_1 \sqcup \cdots \sqcup D_m$. Then the pro-finite group scheme $\pi(\mathcal{T}_{D,\mathbf{pt}})^{\mathrm{pf}}$ is isomorphic to*

$$\widehat{\mathbf{Z}}^{\star(m-1)} \star \Sigma_{D_1} \star \cdots \star \Sigma_{D_m},$$

where we adopt the convention that $\widehat{\mathbf{Z}}^{\star 0} = \{e\}$.

Remark 6.6. We briefly remove all assumptions on the field k. Let G be a group scheme which is local, that is, G only has one closed point. Let $G = \varprojlim_i G_i$, where the canonical morphisms $G \to G_i$ are all faithfully flat and where G_i is algebraic over k, cf. [Wa79, 3.3 and 14.1]. Then G_i is also local and being of finite type, must be finite over k [EGA I, 6.4.4].

6.2. **Conclusion: Determination of $\pi(X)$ when C is reduced.** Let us now remove the assumption that $C = \mathbf{pt}$ and that D is connected; we suppose only that $C = C_1 \sqcup \cdots \sqcup C_\ell$, where $C_i = \mathbf{pt}$. Then, for each $i \in \{1, \ldots, \ell\}$, we let D_i be the inverse image of C_i in D, and write $D_i = D_{i,1} \sqcup \cdots \sqcup D_{i,m_i}$, where each $D_{i,j}$ is a finite and local scheme. On the other hand, we add the assumption that *all essentially finite vector bundles on Y are in fact trivial*, i.e. $\pi(Y) = 0$.

It then follows that under the equivalence $\mathbf{VB}(X) \stackrel{\sim}{\to} \mathbf{VB}(C) \times_{\mathbf{VB}(D)} \mathbf{VB}(Y)$, each $\mathcal{E} \in \mathbf{EF}(X)$ has its image in $\mathbf{VB}(C) \times_{\mathbf{VB}(D)} \mathbf{VB}^{\rm tr}(Y) \stackrel{\sim}{\to} \mathcal{T}_{D,C}$. We then arrive at a *faithfully flat* morphism of group schemes

$$\pi(\mathcal{T}_{D,C}) \longrightarrow \pi(X), \tag{13}$$

cf. [BHdS21, Lemma 2.1]. As $\pi(X)$ is pro-finite, this morphism shall factor as $\pi(\mathcal{T}_{D,C}) \to \pi(\mathcal{T}_{D,C})^{\rm pf} \to \pi(X)$. It is not difficult to see that the resulting morphism

$$\pi(\mathcal{T}_{D,C})^{\rm pf} \longrightarrow \pi(X)$$

is in fact an *isomorphism*. Indeed, a finite representation of $\pi(\mathcal{T}_{D,C})$ must come from an object of $\mathbf{EF}(X)$. We can now express our findings in the following synthetic form.

Corollary 6.7. *Suppose that $\pi(Y) = \{0\}$. Then*

$$\begin{aligned}\pi(X) &\simeq \pi(\mathcal{T}_{D_1,C_1})^{\rm pf} \star \cdots \star \pi(\mathcal{T}_{D_\ell,C_\ell})^{\rm pf} \\ &\simeq \bigstar_{i=1}^{\ell} \widehat{\mathbf{Z}}^{\star(m_i-1)} \star \Sigma_{D_{i,1}} \star \cdots \star \Sigma_{D_{i,m_i}}.\end{aligned}$$

□

From Corollary 6.7, we see that the heart of the computation of $\pi(X)$ is the determination of Σ_D for a finite and local D. This shall occupy the rest of the paper.

Remark 6.8. If $\mathrm{Pic}(X)$ contains a copy of k^*, then the morphism in eq. (13) fails to be an isomorphism: pick an invertible sheaf on X which becomes trivial on Y, and such that its class in $k^* \subset \mathrm{Pic}(X)$ is not of finite oder.

6.3. **A cautionary example.** It is natural to inquire if $\pi(X, x_0)$ can be described as efficiently as it can in the etale case even by *dropping the assumption* $\pi(Y) = 0$ made in Corollary 6.7. More precisely, we have in mind the analogous situation of Theorem 4.12 and Corollary 5.4 in [SGA1, Exp. IX] which we briefly recall. To simplify, let us suppose that D is either $\mathbf{pt} \sqcup \mathbf{pt}$ or $\mathrm{Spec}\, k[t]/(t^2)$, and that $C = \mathbf{pt}$. Then, according to [SGA1], we have an isomorphism $\pi^{\rm et}(Y) \star \widehat{\mathbf{Z}} \simeq \pi^{\rm et}(X)$ in the first case and $\pi^{\rm et}(Y) \simeq \pi^{\rm et}(X)$ in the second. (Here, the amalgamated product is taken in the category of pro-finite groups.) The following example gives some limits to these speculations.

Example 6.9. Let us suppose that $C = \mathbf{pt}$ and that $\theta : D \to Y$ is a tangent vector at a point $y \in Y(k)$, i.e. $D = \mathrm{Spec}\, k[t]/(t^2)$. (Since we assume that D is a closed subscheme, this tangent vector cannot vanish.) In addition, let us assume the existence of a finite group scheme G and a principal G-bundle $Z \to Y$ such that, for any $z \in Z(k)$ of image y, the derivative $T_z Z \to T_y Y$ vanishes. This is the case, for example, when Y is an abelian variety, G is the kernel of multiplication by p, and $Z = Y \to Y$ is multiplication by p.

We contend that in this situation *it is not* possible to obtain an isomorphism

$$\pi(Y) \star S \simeq \pi(X)$$

where S is some pro-fintie group scheme. More precisely, let $i : \pi(Y) \to \pi(Y) \star S$ be the canonical morphism and suppose that there exists a pro-finite group scheme S and an isomorphism $\varphi : \pi(Y) \star S \stackrel{\sim}{\to} \pi(X)$ such that $\nu_\# = \varphi \circ i$. Let now $\psi : \pi(X) \to \pi(Y)$ be the morphism defined by means of the universal property of $\pi(Y) \star S$ applied to $\mathrm{id} : \pi(Y) \to \pi(Y)$ and the trivial morphism $S \to \pi(Y)$. Then, $\psi\nu_\# = \mathrm{id}_{\pi(Y)}$ and we arrive at a functor $\Psi : \mathbf{EF}(Y) \to \mathbf{EF}(X)$ such that $\nu^* \circ \Psi : \mathbf{EF}(Y) \to \mathbf{EF}(Y)$ is naturally isomorphic, as a tensor functor, to $\mathrm{id}_{\mathbf{EF}(Y)}$. Hence, the tensor functors

$$F : \mathbf{EF}(Y) \longrightarrow \mathcal{O}(D)\text{-}\mathbf{mod}, \quad \mathcal{E} \longmapsto \Gamma(D, \mathcal{E}|_D)$$

and

$$G : \mathbf{EF}(Y) \longrightarrow \mathcal{O}(D)\text{-}\mathbf{mod}, \quad \mathcal{E} \longmapsto \mathcal{E}|_y \otimes \mathcal{O}(D)$$

are isomorphic because $F \circ \nu^*$ and $G \circ \nu^*$ are likewise.

Then $Z|_D \to D$ is isomorphic to the principal G-bundle $Z|_y \times D \to D$ so that $\theta : D \to Y$ lifts to $\widetilde{\theta} : D \to Z$, hence obtaining a contradiction.

7. Non-commutative Witt groups: The works of Ditters and Newman

As we mentioned in the Introduction, one of the difficulties behind the effective calculation of fundamental group schemes lies in the fact that we lack enough identifiable local group schemes. In this section, we make a brief presentations on a theory which allows us to obtain very relevant local group schemes and which seems to have drawn, unfortunately, little attention: this is the work of Ditters and Newmann [Dit69, Ne74]. This section makes no use of constructions and notations of Sections 3 to 5 and the geometry of pinched schemes.

7.1. **The algebra $\mathcal{Z}$ of Ditters.** We fix m a positive integer. Let $\mathcal{Z}$, respectively $\mathcal{Z}(m)$, be the associative algebra (over k) on the variables $\{Z_i\}_{i=1}^{\infty}$, respectively $\{Z_i\}_{i=1}^{m}$. Clearly, we regard $\mathcal{Z}(m)$ as a subalgebra of $\mathcal{Z}$. By convenience let us write $Z_0 = 1$. On $\mathcal{Z}$, we then introduce the structure of bialgebra by decreeing that comultiplication Δ satisfies

$$\Delta Z_h = \sum_{i+j=h} Z_i \otimes Z_j,$$

and that the co-unit maps $Z_1, Z_2, \ldots$ to 0. The bialgebra $\mathcal{Z}$ is sometimes referred to as the *Leibniz algebra* or the *Hopf algebra of non-commutative symmetric functions*, or the *universal non-commutative group-coalgebra (UNG)*.

In [Dit69, 1.2.7], it is proved that once we define elements $\{S_i\}_{i=1}^{\infty}$ by the equations $S_0 = 1$ and $\sum_{i=0}^{n} S_i Z_{n-i} = 0$ for $n \geq 1$, then the morphism of algebras $S : \mathcal{Z} \to \mathcal{Z}$ sending Z_i to S_i endows $\mathcal{Z}$ with the structure of a *cocommutative Hopf-algebra.* In addition, $\mathcal{Z}(m)$ is also a sub-Hopf-algebra.

A final piece of structure on $\mathcal{Z}$ is its grading, which is obtained by decreeing each Z_i to be homogeneous of degree i.

Notation 7.1. For each integer $n > 0$, let $\|n\|$ stand for $[\log_p n]$, that is, $p^{\|n\|} \leq n < p^{\|n\|+1}$.

Definition 7.2 ([Dit69, 1.4.8])**.** Let $H \in \mathbf{Hpf}^{\mathrm{coc}}$. Given $\ell \in \mathbf{N}^* \cup \{\infty\}$, a curve of length ℓ on H is a sequence of elements $(c_1, \ldots, c_\ell) \in H^{\times \ell}$ such that, after fixing $c_0 = 1$, we have $\Delta c_j = \sum_{i=0}^{j} c_i \otimes c_{j-i}$, for every $1 \leq j \leq \ell$. A curve of length ℓ is also called a divided power sequence of length ℓ, cf. [Ne74, 1.23]. The set of curves of length ℓ on H shall be denoted by $\mathrm{Curv}_\ell(H)$.

Definition 7.3. A curve $(c_i)_{i=1}^{\ell}$ is said to be minimal if $k\{c_1,\ldots,c_i\} = k\{c_1,\ldots,c_{p^{\|i\|}}\}$ for $i = 1,\ldots,\ell$.

Note that, if $(c_i)_{i=1}^{\ell}$ is a minimal curve, then in fact $k\{c_1,\ldots,c_i\}$ can be generated simply by the elements indexed by power of p, i.e. $k\{c_1,\ldots,c_i\} = k\{c_1,c_p,\ldots,c_{p^{\|i\|}}\}$.

Remark 7.4. In [Dit75, Definition 6.1], Ditters puts forward the notion of a *pure* curve in $\mathcal{Z}$. Our definition of "minimal" seeks to imitate this in a different setting, viz. for general Hopf algebras and without mention of weights.

To every arrow $\varphi : H \to H'$ in $\mathbf{Hpf}^{\mathrm{coc}}$ we have an associated map $\mathrm{Curv}_\ell(H) \to \mathrm{Curv}_\ell(H')$ and in this way $H \mapsto \mathrm{Curv}_\ell(H)$ becomes a functor $\mathbf{Hpf}^{\mathrm{coc}} \to \mathbf{Set}$. It is clear that Curv_ℓ is then represented by $\mathcal{Z}(\ell)$.

Theorem 7.5 (Minimal curves [Dit69, Theorem 2.1.4])**.** *There exists a minimal curve $(E_i)_{i=1}^{\infty}$ in $\mathcal{Z}$ with the additional following properties.*
(a) Each E_i is homogeneous of degree i.
(b) For each $i \geq 0$, we have $\deg(E_{p^i} - Z_{p^i}) < p^i$.

Let us now fix a curve $(E_i)_{i\geq 1}$ as in Theorem 7.5. It is clear that $k\{E_1,\ldots,E_{p^\ell}\} \subset \mathcal{Z}(p^\ell)$ is a sub-Hopf-algebra of $\mathcal{Z}(p^\ell)$.

Definition 7.6. For each $\ell \geq 0$, we define

$$\mathcal{NW}_{\ell+1} = k\{E_{p^0},\ldots,E_{p^\ell}\} = k\{E_{p^0},E_p,\ldots,E_{p^\ell}\}.$$

This shall be called the *non-commutative Witt Hopf-algebra* of length $\ell+1$. (See Section 7.2 for an explanation concerning the name.) The associated group scheme $\mathrm{Spec}\,\mathcal{NW}_{\ell+1}^{\circ}$ shall be denoted by $\boldsymbol{NW}_{\ell+1}$, and will be called the *non-commutative infinitesimal Witt group scheme.*

Note that it is a priori not clear that $\mathcal{NW}_{\ell+1}$ does not depend on the minimal curve chosen. This shall be cleared below.

An obvious property of $\boldsymbol{NW}_{\ell+1}$ is the following. (For the concept of height, see [DG70, II.7.1.4–6].)

Lemma 7.7. *The group scheme $\boldsymbol{NW}_{\ell+1}$ is local and of height $\leq \ell+1$.*

Proof. Let Ver stand for the Verschiebung morphism of the cocommutative coalgebra $\mathcal{NW}_{\ell+1}$ [A80, Ch. 2, Section 5.3, 112ff]. Since $(E_i)_{i=1}^{\infty}$ is a curve in $\mathcal{Z}$, then

$$\mathrm{Ver}^{i+1}(E_{p^i}) = 0,$$

see [A80, Lemma 2.5.8]. Hence, $\mathrm{Ver}^{\ell+1}(\mathcal{NW}_{\ell+1}) = k$. From the canonical formula

$$f(\mathrm{Ver}(a)) = f^p(a)^{1/p}, \qquad \text{for each } f \in \mathcal{NW}_{\ell+1}^{*} \text{ and } a \in \mathcal{NW}_{\ell+1},$$

we conclude that the augmentation ideal $(k1)^{\perp} \subset \mathcal{NW}_{\ell+1}^{\circ}$ is annihilated by $\mathrm{Fr}^{\ell+1}$, which implies that $\mathcal{NW}_{\ell+1}^{\circ}$ is local of height at most $\ell+1$. □

The curve $(E_i)_{i=1}^{\infty}$ is not the unique one enjoying the properties of Theorem 7.5; on the other hand, given any minimal curve $(E_i')_{i=1}^{\infty}$ satisfying (a) and (b) of Theorem 7.5, the following can be said. If

$$\varphi : \mathcal{Z} \longrightarrow k\{E_i' \,:\, i \geq 1\}$$

is defined by $\varphi(Z_i) = E_i'$, then

$$k\{E_i \,:\, i \geq 1\} \xrightarrow{\varphi} k\{E_i' \,:\, i \geq 1\}$$

is an isomorphism. See [Dit69, Lemma 3.1.2, p. 51].

Remark 7.8. In [Dit69, 3.1.3, p.52], the Hopf algebra $k\{E_i, i \geq 1\}$ is called the "non-commutative exponential". In [Ne74], what we called $\mathcal{NW}_{\ell+1}$ is called $\mathfrak{P}_\ell$. We have adopted the shifted index in order to accord with the notation for Witt vectors.

7.2. **Relation to the Witt group scheme.** Let ℓ be a positive integer. In order to highlight the importance of the Hopf algebra $\mathcal{NW}_\ell$ and the group scheme $\operatorname{Spec}\mathcal{NW}_\ell^\circ$, we relate it to the "usual Witt Hopf algebra." This was already remarked by both Ditters and Newman in a different setting. In order to provide reliable arguments, we require a certain number of "well-known" results on the duality theory of Hopf algebras, which are explained in Section 2.2.

Let $\mathcal{W}_\ell := \mathcal{NW}_\ell^{\mathrm{ab}}$ be the cocommutative and commutative Hopf algebra constructed in Section 2.2. We shall explain its relation to the Hopf algebra of the Witt group scheme $\boldsymbol{W}_\ell$ [DG70, V.1.6, p.543]. More precisely, let $S_j \in \mathbf{F}_p[x_0, \ldots, x_j\,;\, y_0, \ldots, y_j]$ stand for the additive Witt polynomials [DG70, V.1.4, p.541-2] and let $\mathcal{O}(\boldsymbol{W}_\ell) := k[x_0, \ldots, x_{\ell-1}]$ be given the structure of Hopf algebra coming from the Witt group scheme $\boldsymbol{W}_\ell$ of dimension ℓ over k, i.e. comultiplication is

$$X_j \longmapsto S_j(x_0 \otimes 1, \ldots, x_j \otimes 1\,;\, 1 \otimes x_0, \ldots, 1 \otimes x_j).$$

Then, either from the work of Dieudonné [Dit69, 3.1.5-8] or from the appendix of [Ne74], it follows that

$$\mathcal{O}(\boldsymbol{W}_\ell) \simeq \mathcal{W}_\ell.$$

(We note that in [Ne74, Appendix], Newman does not follow the standard convention concerning Witt vectors.) It is important to analyse not only $\mathcal{W}_\ell$ but also the Hopf dual $\mathcal{W}_\ell^\circ$, since the pro-finite group scheme $\boldsymbol{NW}_\ell$ will have $\operatorname{Spec}\mathcal{W}_\ell^\circ$ as its largest *abelian* quotient (see Lemma 2.2). The structure of the group scheme $\operatorname{Spec}\mathcal{O}(\boldsymbol{W}_\ell)^\circ$ should be known, but we find it hard to give a clear reference.

Let us begin by employing [Wa79, Theorem 9.5] to obtain an isomorphism

$$\operatorname{Spec}\mathcal{O}(\boldsymbol{W}_\ell)^\circ \simeq \operatorname{Diag}(\Lambda) \times U,$$

where Λ is an abelian group and U is an unipotent group scheme. Clearly Λ is just the group of characters of $\operatorname{Spec}\mathcal{O}(\boldsymbol{W}_\ell)^\circ$, which is the group of group-like elements of $\mathcal{O}(\boldsymbol{W}_\ell)^\circ$. From [A80, p. 129], Λ is isomorphic to the group of k-points of $\boldsymbol{W}_\ell$, i.e. $\Lambda \simeq \boldsymbol{W}_\ell(k)$.

Employing Lemma 2.2, we conclude that $\mathcal{O}(U)$ is isomorphic to the Hopf subalgebra $\mathcal{O}(\boldsymbol{W}_\ell)^\dagger$ of $\mathcal{O}(\boldsymbol{W}_\ell)^\circ$. Now, we make use of the fact that we are in positive characteristic; letting $\mathfrak{A}_\ell$ stand for the augmentation ideal of $\mathcal{O}(\boldsymbol{W}_\ell)$, it follows that

$$\mathcal{O}(\boldsymbol{W}_\ell)^\dagger = \varinjlim_h \left(\mathcal{O}(\boldsymbol{W}_\ell)/\mathfrak{A}_\ell^{[p^h]}\right)^*.$$

The determination of Hopf algebra $\left(\mathcal{O}(\boldsymbol{W}_\ell)/\mathfrak{A}_\ell^{[p^h]}\right)^*$ is well-known as it enters the context of Cartier duality of finite group schemes. Indeed, let $\boldsymbol{I}_h$ stand for functor associating to a group scheme its hth Frobenius kernel; we have $\mathcal{O}(\boldsymbol{I}_h\boldsymbol{W}_\ell) = \mathcal{O}(\boldsymbol{W}_\ell)/\mathfrak{A}_\ell^{[p^h]}$. Then [De72, Theorem, p.61] says that there exists a canonical isomorphism

$$\boldsymbol{I}_h\boldsymbol{W}_\ell \simeq \boldsymbol{I}_\ell\boldsymbol{W}_h.$$

We then conclude that $U \simeq \varprojlim_h \boldsymbol{I}_\ell\boldsymbol{W}_h$, i.e.

$$U \simeq \boldsymbol{I}_\ell\boldsymbol{W}.$$

In conclusion:

Proposition 7.9. *The abelian group scheme* $\boldsymbol{NW}_\ell^{\mathrm{ab}}$ *is isomorphic to* $\operatorname{Diag}(\boldsymbol{W}_\ell(k)) \times \boldsymbol{I}_\ell\boldsymbol{W}$. □

8. The group scheme Σ_D and the Hopf algebra $\boldsymbol{H}_D$ when D is local

We now regain the setting and notations of Sections 3 to 5, and in addition suppose that $A := \mathcal{O}(D)$ is a local ring with maximal ideal $\mathfrak{m} \neq 0$. We find convenient to write $\mathcal{S}_A$ instead of $\mathcal{S}_D$ in what follows. In order to gain knowledge about Σ_D (see Definition 6.1), we shall find an equivalence

$$\mathcal{S}_A \xrightarrow{\sim} \boldsymbol{H}_A\text{-}\mathbf{mod},$$

for a certain *associative algebra* $\boldsymbol{H}_A$, introduce on $\boldsymbol{H}_A$ a bialgebra structure and then relate it to the *tensor product on* $\mathcal{S}_A$. Moving on, we relate $\boldsymbol{H}_A$ and the non-commutative Witt Hopf-algebras of Definition 7.6 by means of an encompassing structure result of [Ne74]. (Newman's result is also in the direction of the "Campbell-Hausdorff-Dieudonné" Theorem in [Dit69, Theorem 3.5.4].)

8.1. **The bialgebra $\boldsymbol{H}_A$ and the category $\mathcal{S}_A$.** Using the decomposition $A = k1 \oplus \mathfrak{m}$, we shall identify $\mathfrak{m}^*$ with a subspace of A^* in what follows.

Using the obvious bijections

$$\begin{aligned}\mathrm{Hom}_A(V \otimes A, V \otimes A) &\xrightarrow{\sim} \mathrm{Hom}_k(V, V \otimes A)\\ &\xrightarrow{\sim} \mathrm{Hom}_k(A^*, \mathrm{End}_k(V)),\end{aligned}$$

and the restriction map $\mathrm{Hom}_k(A^*, \mathrm{End}_k(V)) \to \mathrm{Hom}_k(\mathfrak{m}^*, \mathrm{End}_k(V))$, we can associate to each $(V, \varphi) \in \mathrm{Obj}\, \mathcal{S}_A$ an element of $\mathrm{Hom}_k(\mathfrak{m}^*, \mathrm{End}_k(V)) = \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(\boldsymbol{T}(\mathfrak{m}^*), \mathrm{End}_k(V))$, that is, a structure of $\boldsymbol{T}(\mathfrak{m}^*)$-module on V. Letting $\eta \in A^*$ stand for the canonical map $A \to A/\mathfrak{m} = k$, we observe that $\varphi \in \mathrm{Hom}_k(A^*, \mathrm{End}_k(V))$ gives rise to an object of $\mathcal{S}_A$ if and only if $\varphi(\eta) = \mathrm{id}_V$. From this, the identity functor of k-**vect** gives rise to an equivalence of k-linear categories

$$\Omega : \mathcal{S}_A \xrightarrow{\sim} \boldsymbol{T}(\mathfrak{m}^*)\text{-}\mathbf{mod}. \tag{14}$$

In down-to-earth terms: Let $\{a_i\}_{i=1}^n$ be a basis of $\mathfrak{m}$ with dual basis $\{e_i\}_{i=1}^n$. Given $(V, \varphi) \in \mathrm{Obj}\, \mathcal{S}_A$, we write $\varphi \in \mathrm{Hom}_A(V \otimes A, V \otimes A)$ as $\varphi(v \otimes 1) = v \otimes 1 + \sum_{i=1}^n \varphi_i(v) \otimes a_i$. Then $\Omega(V, \varphi)$ is the $\boldsymbol{T}(\mathfrak{m}^*)$-module defined by the morphism of k-algebras

$$\boldsymbol{T}(\mathfrak{m}^*) \longrightarrow \mathrm{End}_k(V), \quad e_i \longmapsto \varphi_i.$$

We now define

$$\boxed{\boldsymbol{H}_A = \boldsymbol{T}(\mathfrak{m}^*).}$$

We are now required to endow $\boldsymbol{H}_A$ with the structure of a cocommutative bialgebra. Let $\varepsilon : \boldsymbol{H}_A \to k$ be the map of rings defined by $\varepsilon(1) = 1$ and $\varepsilon(\mathfrak{m}^*) = \{0\}$. Let $\Delta_0 : \mathfrak{m}^* \to \mathfrak{m}^* \otimes \mathfrak{m}^*$ be the transpose of multiplication $\mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ and let $\Delta : \boldsymbol{T}(\mathfrak{m}^*) \to \boldsymbol{T}(\mathfrak{m}^*) \otimes \boldsymbol{T}(\mathfrak{m}^*)$ be the map of k-algebras associated to the linear map

$$\Delta_0 + \mathrm{id} \otimes 1 + 1 \otimes \mathrm{id} : \mathfrak{m}^* \longrightarrow \boldsymbol{T}(\mathfrak{m}^*) \otimes \boldsymbol{T}(\mathfrak{m}^*).$$

Explicitly, this says the following. If c_{ij}^h are the structure constants of multiplication $\mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$, i.e. $a_i a_j = \sum_{h=1}^n c_{ij}^h a_h$, then $\varepsilon(e_i) = 0$ and

$$\Delta(e_h) = e_h \otimes 1 + 1 \otimes e_h + \sum_{i,j} c_{ij}^h\, e_i \otimes e_j.$$

We write $A^* = k \cdot \eta \oplus \mathfrak{m}^*$ and let

$$f : A^* \longrightarrow \boldsymbol{H}_A \tag{15}$$

be the linear map which sends η to 1 and $\mathfrak{m}^*$ identically to the canonical copy of $\mathfrak{m}^*$ in $\boldsymbol{T}(\mathfrak{m}^*)$. If δ is the comultiplication and $\mathrm{ev}_1 : A^* \to k$ is the co-unit of A^*, then $\mathrm{ev}_1 = \varepsilon f$ and $f \otimes f \circ \delta = \Delta \circ f$. These formulas, together with the fact that $\mathfrak{m}^*$ generates $\boldsymbol{H}_A$, allow us to show that Δ and ε endow $\boldsymbol{H}_A$ with the structure of a coalgebra, and $f : A^* \to \boldsymbol{H}_A$ is a morphism of such. Since ε and Δ are maps of algebras, $\boldsymbol{H}_A$ *is now a cocommutative bi-algebra.*

These very natural bialgebras are unfortunately not well documented in the literature on the theme, except in the theory of the "non-commutative symmetric functions", as we see in the following.

Example 8.1. Assume that $A = k[t]/(t^{m+1})$, where $m \geq 1$. Letting $a_i = t^i$, so that $\{a_i\}_{i=1}^m$ is a basis of $\mathfrak{m}$, it follows that

$$c_{ij}^h = \begin{cases} 0, & h \neq i+j, \\ 1, & h = i+j. \end{cases}$$

Then $\Delta e_h = e_h \otimes 1 + 1 \otimes e_h + \sum_{i=1}^{h-1} e_i \otimes e_{h-i}$, and $\boldsymbol{H}_A$ is just the truncated algebra of non-commutative symmetric functions $\mathcal{Z}(m)$, cf. Section 7.1.

The above constructions are related to the tensor structure on $\mathcal{S}_A$. Indeed, let (V, φ) and (W, ψ) be objects of $\mathcal{S}_A$, where $\varphi \in \mathrm{Hom}_k(V, V \otimes A)$ and $\psi \in \mathrm{Hom}_k(W, W \otimes A)$. The tensor product in $\mathcal{S}_A$, $(V, \varphi) \otimes (W, \psi) = (V \otimes W, \varphi \boxtimes \psi)$, is obtained by fixing

$$\varphi \boxtimes \psi \in \mathrm{Hom}_k(V \otimes W, V \otimes W \otimes A)$$

as the composition

$$V \otimes W \xrightarrow{\varphi\otimes\psi} V \otimes A \otimes W \otimes A = V \otimes W \otimes A \otimes A \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{mult}} V \otimes W \otimes A\ . \tag{16}$$

In explicit terms, if $\varphi = \mathrm{id}_V \otimes 1 + \sum_i \varphi_i \otimes a_i$ and $\psi = \mathrm{id}_W \otimes 1 + \sum_i \psi_i \otimes a_i$, then

$$\varphi \boxtimes \psi = \mathrm{id}_{V\otimes W} \otimes 1 + \sum_h \left(\mathrm{id}_V \otimes \psi_h + \varphi_h \otimes \mathrm{id}_W + \sum_{i,j} c_{ij}^h \, \varphi_i \otimes \psi_j \right) \otimes a_h.$$

Hence, the $\boldsymbol{H}_A$-module $\Omega[(V, \varphi) \otimes (W, \psi)]$ is given by the composition

$$\boldsymbol{H}_A \xrightarrow{\Delta} \boldsymbol{H}_A \otimes \boldsymbol{H}_A \longrightarrow \mathrm{End}(V) \otimes \mathrm{End}(W) \xrightarrow{\text{natural}} \mathrm{End}(V \otimes W)\ .$$

It is also clear that, letting $\mathbf{I}$ stand for the unit object of $\mathcal{S}_A$, then $\Omega(\mathbf{I})$ is the $\boldsymbol{H}_A$-module defined by $\varepsilon : \boldsymbol{H}_A \to k = \mathrm{End}_k(k)$. This means that if we endow $\boldsymbol{H}_A$**-mod** with its tensor structure obtained from Δ and ε, cf. [Ka95, III.5], then Ω is a tensor functor.

Consequently, because of the equivalence of tensor categories $\boldsymbol{H}_A$**-mod** $\simeq$ **comod-**$\boldsymbol{H}_A^\circ$, we conclude that

$$\mathcal{O}(\Sigma_D) \simeq \boldsymbol{H}_A^\circ.$$

8.2. **Pointed irreducible and cocommutative bialgebras.** The bialgebra $\boldsymbol{H}_A$ has already appeared in [Ne74]. For the sake of the reader with little experience in the theory of coalgebras, let us make a brief introduction to some basic material from [Ne74].

Recall from [A80, p. 80] that a coalgebra L is said to be irreducible if its coradical $\mathrm{corad}\, L$ is a simple coalgebra. As the Jacobson radical of A is $\mathfrak{m}$, it follows that $(\mathrm{corad}\, A^*)^\perp = \mathfrak{m}$ [A80, 2.3.9(i), p.84]. Consequently, $\mathrm{corad}\, A^* = k\eta$ and A^* *is irreducible.* It is in addition *pointed* [A80, p. 80].

Following [Ne74], we abbreviate "pointed, irreducible and cocommutative" to "PIC", so that A^* is a PIC coalgebra. Using the map f of eq. (15), we regard A^* as a subcoalgebra of $\boldsymbol{H}_A$ (and the element $\eta \in A^*$ is the unit of $\boldsymbol{H}_A$). Let $\boldsymbol{H}_A(1)$ be the irreducible component of $\boldsymbol{H}_A$ containing 1 [A80, p.97]: from [A80, Theorem 2.4.5(ii), p.96], $\boldsymbol{H}_A(1)$ is the sum of all irreducible subcoalgebras of $\boldsymbol{H}_A$ containing 1. In particular, $A^* \subset \boldsymbol{H}_A(1)$ and $\boldsymbol{H}_A(1)$ contains algebra generators of $\boldsymbol{H}_A$. Since $\boldsymbol{H}_A(1)$ is a *subalgebra* also [A80, Theorem 2.4.27p.105], it follows that $\boldsymbol{H}_A = \boldsymbol{H}_A(1)$. In summary, $\boldsymbol{H}_A$ is also PIC. A fundamental result in the theory now says that $\boldsymbol{H}_A$ already *comes with an antipode*, see p. 71 and Proposition 9.2.5 of [Sw69], and hence is a *Hopf algebra*. (Note that no particular property about the field k is employed here.)

As noted in [Ne74, Remark, p.4], $\boldsymbol{H}_A$ is in fact the solution to a certain universal problem in the following sense. Let the category of PIC coalgebras, respectively Hopf algebras, be denoted by **picCog**, respectively **picHpf**.

Proposition 8.2. *The forgetful functor* **picHpf** $\to$ **picCog** *has a left adjoint* $\mathcal{F}$: **picCog** $\to$ **picHpf** *and* $\mathcal{F}(A^*) = \boldsymbol{H}_A$. □

(We note that Newman uses "F" instead of "$\mathcal{F}$"; we made the change to avoid confusion with the Frobenius morphisms.) The above characterisation of $\boldsymbol{H}_A$ allowed Newman to prove a very useful structure result.

8.3. **The structure of $\boldsymbol{H}_A$: Newman's Theorem.** We shall require the theory of the Verschiebung morphism of cocommutative coalgebras [A80, Chapter 2, Section 5.3, 112ff]. Let

$$h = \min\{i \in \mathbf{N} \,:\, \mathrm{Fr}^i(\mathfrak{m}) = 0\}.$$

(By assumption $h \geq 1$.) From the duality equation for $\mathrm{Ver} : A^* \to A^*$,

$$\lambda(\mathrm{Ver}(a)) = \lambda^p(a)^{1/p}, \qquad \text{for each } \lambda \in (A^*)^* \text{ and } a \in A^*,$$

we have

$$\left(\mathrm{Fr}^i(A^{**})\right)^{\perp} = \mathrm{Ker}\,\mathrm{Ver}^i.$$

(This also holds for the cocommutative algebra $\boldsymbol{H}_A$.) This gives rise to a filtration

$$0 = K_0 \subset \underbrace{\mathrm{Ker}\,\mathrm{Ver}}_{K_1} \subset \cdots \subset \underbrace{\mathrm{Ker}\,\mathrm{Ver}^h}_{K_h} = (A^*)^+, \tag{17}$$

where we have followed custom and written $(-)^+$ for the augmentation ideal of a coalgebra. Because $\boldsymbol{H}_A$ is generated, as a k-algebra by $f[(A^*)^+]$, the equality $\mathrm{Ver} \circ f = f \circ \mathrm{Ver}$ holds true, and $\mathrm{Ver} : \boldsymbol{H}_A \to \boldsymbol{H}_A$ is a morphism of rings, we derive claims (i)–(iii) of the following:

Lemma 8.3. *(i) The Verschiebung morphism* $\mathrm{Ver}^h : \boldsymbol{H}_A \to \boldsymbol{H}_A$ *annihilates the augmentation ideal* $(\boldsymbol{H}_A)^+$*, (ii) while* Ver^{h-1} *does not. In particular, (iii)* Fr^h *annihilates* $(\boldsymbol{H}_A^\circ)^+ = \{\varphi \,:\, \varphi(1) = 0\}$*. (iv) The morphism* Fr^{h-1} *does not annihilate* $(\boldsymbol{H}_A^\circ)^+$.

Proof. Only (iv) remains to be checked, and this follows from the fact that the map of algebras $\boldsymbol{H}_A^\circ \to (A^*)^* = A$ is surjective since $\boldsymbol{H}_A = \boldsymbol{T}(\mathfrak{m}^*)$. □

Another consequence of the filtration (17) is that A^* comes equipped with a *regular basis* in the sense of [Ne74, Definition 2.1]. We recall what this means. A basis $\mathcal{B}$ of A^* is regular when (i) $\mathrm{Ver}(\mathcal{B}) \subset \mathcal{B} \cup \{0\}$, (ii) if $\mathrm{Ver}(\lambda) = \mathrm{Ver}(\lambda')$, then either $\lambda = \lambda'$ or $\mathrm{Ver}(\lambda) = 0$ and (iii) $\mathcal{B} \setminus (A^*)^+ \subset \{\eta\}$. Now, to construct a regular basis, we only need to employ the filtration $\{K_i\}$ and proceed as in the construction of a Jordan basis for a nilpotent linear map.

From Theorem [Ne74, Theorem 2.16] and Proposition 8.2, we can assure that in the category **Hpf**, the Hopf algebra $\boldsymbol{H}_A$ is a co-product of non-commutative Witt Hopf algebras, that is,

$$\boldsymbol{H}_A \simeq \mathcal{NW}_{\ell_1} \sqcup \cdots \sqcup \mathcal{NW}_{\ell_m}. \tag{18}$$

In addition, $\max_i\{\ell_i\} \leq h$ and for $\ell \geq 1$, the number of copies of $\mathcal{NW}_\ell$ in the above decomposition can be determined in the following way. (The reader is asked to bear in mind the difference between our convention and that of [Ne74], see Remark 7.8.) A regular sequence in $\mathcal{B}$ is a sequence $(b_0, \ldots, b_r)$ of elements in $\mathcal{B}$ such that $\mathrm{Ver}(b_i) = b_{i-1}$ for

$i = 1, \dots, r$, and $\mathrm{Ver}(b_0) = 0$. Then, the number of copies of $\mathcal{NW}_\ell$ in the decomposition (18) above is the number of maximal regular sequences of ℓ *elements.*

Corollary 8.4. *If $h = 1$, then $\boldsymbol{H}_A$ is a coproduct of copies of $\mathcal{NW}_1 \simeq k[E_1]$ and hence is the universal enveloping algebra of a free Lie algebra on $\dim \mathfrak{m}$ generators.*

Proof. In this case, $\mathrm{Ver}(e_i) = 0$ for all $1 \leq i \leq n$ and $(e_1), \dots, (e_n)$ are maximal regular sequences. □

Let us now translate these findings in terms of the group scheme Σ_D. Since the functor $(-)^\circ : \mathbf{Hpf} \to \mathbf{Hpf}^{\mathrm{opp}}$ is left-adjoint to $(-)^\circ : \mathbf{Hpf}^{\mathrm{opp}} \to \mathbf{Hpf}$, cf. the paragraph following [A80, 2.3.14], we conclude from general nonsense [Mac98, V.5] that $(-)^\circ$ sends co-products in $\mathbf{Hpf}$ to products in $\mathbf{Hpf}$. Hence, since $\mathcal{O}(\Sigma_D) = \boldsymbol{H}_A^\circ$, we conclude from eq. (18) that $\mathcal{O}(\Sigma_D) \simeq \mathcal{NW}_{\ell_1}^\circ \sqcap \cdots \sqcap \mathcal{NW}_{\ell_m}^\circ$, where we use the notation $\sqcap$ for the product in $\mathbf{Hpf}$. More significantly, we then have.

Corollary 8.5. *The group scheme $\Sigma_D \simeq \operatorname{Spec} \boldsymbol{H}_A^\circ$ is isomorphic to the amalgamated product*

$$\boldsymbol{NW}_{\ell_1} \star \cdots \star \boldsymbol{NW}_{\ell_m}.$$

The number of copies of $\boldsymbol{NW}_\ell$ equals $2 \dim K_\ell - \dim K_{\ell+1} - \dim K_{\ell-1}$, where $K_i = \mathrm{Ker}\,\mathrm{Ver}^i$.

Let us end with a summary result. For that, we abandon the assumption made on D, but refrain from making explicit the components of $\boldsymbol{NW}_\ell$ appearing in the decomposition.

Corollary 8.6. *Let D, Y, X and C be as in Section 3, and assume that C is reduced while $\pi(Y) = 0$. Then $\pi(X)$ is isomorphic to the amalgamated product of copies of $\widehat{\mathbf{Z}}$ and of $\boldsymbol{NW}_\ell$.*

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