TRIVIALITY CRITERIA FOR VECTOR BUNDLES OVER RATIONALLY CONNECTED VARIETIES

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ABSTRACT. Let X be a separably rationally connected smooth projective variety defined over an algebraically closed field K and $E \longrightarrow X$ a vector bundle such that, for every morphism $\gamma : \mathbb{P}^1_K \longrightarrow X$, the pull–back $\gamma^* E$ is trivial. We prove that E is trivial. If $E \longrightarrow X$ is a strongly semistable vector bundle such that $c_1(E)$ and $c_2(E)$ are numerically equivalent to zero, we prove that E is trivial. We also show that X does not admit any nontrivial stratified sheaf. These results are also generalized to principal bundles over X.

1. INTRODUCTION

In [dS], it was shown that given a smooth projective variety X defined over an algebraically closed field of positive characteristic, the F–divided sheaves on X define an affine algebraic group–scheme over the base field. The present work started by trying to show that this group–scheme is trivial if X is separably rationally connected. This led to the following question: if a vector bundle E over a separably rationally connected variety X has the property that the restriction of E to every rational curve is trivial, then is E trivial? Our aim here is to address this question.

Let K be an algebraically closed field. A K–variety X is said to be *separably rationally connected* (“*SRC*” for short) if there exists a K–variety M together with a morphism

$$\theta : M \times \mathbb{P}^1_K \longrightarrow X$$

such that the morphism

$$\theta^{(2)} : M \times \mathbb{P}^1_K \times \mathbb{P}^1_K \longrightarrow X \times X, \qquad (m, t_1, t_2) \longmapsto (\theta(m, t_1), \theta(m, t_2)) \tag{1.1}$$

is dominant and smooth at the generic point [Ko1, p. 199, IV 3.2].

We prove the following theorem:

Theorem 1.1. *Let X be a smooth projective and separably rationally connected (SRC) variety over K. Let E be a vector bundle over X such that, for each morphism $\gamma : \mathbb{P}^1_K \longrightarrow X$, the pull–back $\gamma^* E$ is trivial. Then E itself is trivial.*

The proof of Theorem 1.1 is carried out in Section 2. It is sustained by the existence of a proper morphism to X which trivializes the vector bundle in question (Lemma 2.1) and a technique of “spreading out” which allows us to reduce the problem to the case of

2000 *Mathematics Subject Classification.* 14F05, 14M20.

Key words and phrases. Separably rationally connectedness, principal bundle, stable map, rational curve.

a finite base field. In section 3 we present a simpler proof of Theorem 1.1 communicated to us by the referee. The following theorem is proved using Theorem 1.1.

Theorem 1.2. *Let X be a smooth projective SRC variety over K.*

(1) *There is no nontrivial stratified sheaf over X.*
(2) *Let $E \longrightarrow X$ be a strongly semistable vector bundle such that $c_1(E)$ and $c_2(E)$ are numerically equivalent to zero. Then E is a trivial vector bundle.*
(3) *Let $E \longrightarrow X$ be a vector bundle of rank r such that for every morphism*

$$\gamma : \mathbb{P}^1_K \longrightarrow X ,$$

the pull–back $\gamma^* E$ is semistable. Then there is a line bundle ζ over X such that $E = \zeta^{\oplus r}$.

Theorem 1.2 is proved in Corollary 4.3, Proposition 4.4 and Corollary 4.2.

When the characteristic of K is positive, the first statement in Theorem 1.2 implies that the group–scheme defined by the F–divided sheaves on X is trivial. (The reader is also directed to the remark at the end of section 4.2 for a reference to another proof of this.)

We then turn our attention to principal bundles over the SRC variety X (section 4.4). Let G be an algebraic group defined over the field K (for terminology see section 4.4). There is a short exact sequence of groups

$$e \longrightarrow H \longrightarrow G \longrightarrow A \longrightarrow e ,$$

where H is a linear algebraic group, and A is an abelian variety. Let E_G be a principal G–bundle over a smooth projective SRC variety X over K. We prove the following lemma:

Lemma 1.3. *The principal G–bundle E_G admits a reduction of structure group*

$$E_H \subset E_G$$

to the subgroup H.

If, for each finite dimensional representation V of H over K, the vector bundle $E_H \times^H V$ associated to the principal H–bundle E_H for V is trivial, then E_H is trivial.

Using Lemma 1.3, both Theorem 1.1 and Theorem 1.2 extend to principal G–bundles, see section 4.4.

2. The main criterion for triviality

We will prove Theorem 1.1 in this section.

Let X be a smooth projective and separably rationally connected variety over an algebraically closed field K. Let E be a vector bundle over X such that for each morphism

$$\gamma : \mathbb{P}^1_K \longrightarrow X ,$$

the pull–back $\gamma^* E$ is trivial. The following lemma is crucially used in the proof of Theorem 1.1.

Lemma 2.1. *There exists a proper K–scheme $\overline{Z}$ and a surjective morphism*

$$\overline{\varphi} : \overline{Z} \longrightarrow X$$

such that $\overline{\varphi}^* E$ is trivial.

Proof. Construction of $\overline{Z}$: Let $x \in X(K)$ be a K–rational point. There is a smooth family of rational curves on X

$$\begin{tikzcd} Z \arrow[r, "\varphi"] \arrow[d, "f"] & X \\ T \arrow[u, bend left, "\sigma"] & \end{tikzcd} \tag{2.1}$$

with the following properties:

(1) $Z = T \times \mathbb{P}^1_K$, and f is the projection; T is geometrically irreducible and open in $\mathrm{Mor}(\mathbb{P}^1_K\,, X\,; (0:1) \longmapsto x)$ (hence T is quasiprojective),
(2) for each closed point $t \in T$,
$$\varphi_t : \mathbb{P}^1_{\mathbf{k}(t)} \longrightarrow X \times \operatorname{Spec} \mathbf{k}(t)$$
is non-constant, where $\mathbf{k}(t)$ is the residue field,
(3) $f \circ \sigma = \mathrm{Id}_T$,
(4) φ is dominant, and
(5) $\varphi(\sigma(t)) = x$ for all $t \in T$.

(See [Ko2, Theorem 3].)

We now set out to find a "compactification" of the diagram in (2.1). For that we consider the moduli space of stable maps. Note that $Z \longrightarrow T$ is a genus zero prestable curve [BM, Definition 2.1]. Let

$$\beta := \varphi_*[Z] \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(X)\,, \mathbb{Z})$$

be the "homology class" of φ [BM, §2]. Let $\overline{\mathcal{M}}_{0,1}(X, \beta)$ be the (fppf) stack of stable maps from 1–pointed genus zero curves to X of class β; see [BM, Definition 3.2] and [BM, Definition 3.4]. Then diagram (2.1) gives rise to a T–valued point

$$\rho : T \longrightarrow \overline{\mathcal{M}}_{0,1}(X, \beta) \tag{2.2}$$

of $\overline{\mathcal{M}}_{0,1}(X, \beta)$.

Let us now gather some properties of $\overline{\mathcal{M}}_{0,1}(X, \beta)$. First, the diagonal

$$\Delta : \overline{\mathcal{M}}_{0,1}(X, \beta) \longrightarrow \overline{\mathcal{M}}_{0,1}(X, \beta) \times \overline{\mathcal{M}}_{0,1}(X, \beta)$$

is *finite* and in particular *schematic* [LMB, p. 20, 3.9].

Second, $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is a proper algebraic (Artin) stack. The proof of algebraicity, in this generality, is not explicitly stated in [BM], but can be extracted from the paragraph preceding [BM, Corollary 4.8]. Indeed, Behrend and Manin remark that $\overline{\mathcal{M}}_{g,n}(X, \beta)$ admits a faithfully flat morphism from an algebraic stack of finite type; this, together with Artin's theorem [LMB, p. 81, 10.1], suffices to prove algebraicity. Properness is standard and follows from the case of $X = \mathbb{P}^r$ [FP].

By Chow's Lemma [Ol, Thm. 1.1], there exists a projective K–scheme Y together with a proper and surjective morphism

$$\psi : Y \longrightarrow \overline{\mathcal{M}}_{0,1}(X,\beta)\,. \tag{2.3}$$

We have a Cartesian diagram

$$\begin{tikzcd} T_1 \arrow[r, "\rho_1"] \arrow[d, "\psi_1"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\psi"] \\ T \arrow[r, "\rho"'] & \overline{\mathcal{M}}_{0,1}(X,\beta) \end{tikzcd} \tag{2.4}$$

where T_1 is a scheme (due to the fact that the diagonal is schematic and [LMB, p. 21, 3.13]), and the morphism ψ_1 is proper and surjective.

As T is separated (it is open in $\mathrm{Mor}(\mathbb{P}^1_K, X)$) so is T_1. Hence we can apply Nagata's Theorem [Lü, p. 106, Theorem 3.2] to find a proper K–scheme $\overline{T}_1$ together with a schematically dense open immersion

$$i : T_1 \hookrightarrow \overline{T}_1\,. \tag{2.5}$$

Eliminating the "indeterminacy locus" (see Lemma 2.2 of [Lü, pp. 99–100]), we can find a blow–up

$$\delta : \overline{T} \longrightarrow \overline{T}_1\,,$$

whose center is disjoint from T_1, and a morphism

$$\overline{\rho} : \overline{T} \longrightarrow Y \tag{2.6}$$

which extends $\rho_1 : T_1 \longrightarrow Y$ in (2.4). The composition

$$\psi \circ \overline{\rho} : \overline{T} \longrightarrow \overline{\mathcal{M}}_{0,1}(X,\beta)$$

(see (2.3) and (2.6)) represents a family of 1–pointed genus zero stable maps

$$\begin{tikzcd} \overline{Z} \arrow[r, "\overline{\varphi}"] \arrow[d, "\overline{f}"] & X \\ \overline{T} \arrow[u, bend left, "\overline{\sigma}"] & \end{tikzcd} \tag{2.7}$$

whose pull–back via i in (2.5) is the pull–back of the family in (2.1) via the morphism ψ_1 in (2.4). Clearly $\overline{\varphi}$ is dominant (hence surjective), and $\overline{\varphi} \circ \overline{\sigma}$ is a constant map. Note that, without loss of generality, we can assume $\overline{T}$ to be *reduced*.

Triviality of $\overline{\varphi}^* E$. Take any closed point $t \in \overline{T}(K)$. The inverse image $\overline{f}^{-1}(t)$, where $\overline{f}$ is the morphism in (2.7), is a tree of $\mathbb{P}^1_K$ (since $H^1(\mathcal{O}) = 0$). We recall that the pull–back of E by any map from $\mathbb{P}^1_K$ is trivial. So the restriction of

$$\overline{E} := \overline{\varphi}^* E$$

to $\overline{f}^{-1}(t)$ is trivial. Therefore, $\overline{E}$ descends to $\overline{T}$. More precisely, the direct image $\overline{f}_*\overline{E}$ is a vector bundle on $\overline{T}$, and the canonical arrow

$$\overline{f}^*\overline{f}_*\overline{E} \longrightarrow \overline{E} \tag{2.8}$$

is an isomorphism [Mu2, §5]. Indeed, the homomorphism in (2.8) is injective because any section of a trivial vector bundle, over a connected projective scheme, that vanishes at one point actually vanishes identically; the homomorphism in (2.8) is surjective also because $\overline{E}|_{\overline{f}^{-1}(t)}$ is trivial for all t.

Further, the image of (2.8) by $\overline{\sigma}^*$ defines an isomorphism between $\overline{\sigma}^*\overline{E}$ and $\overline{f}_*\overline{E}$. Therefore, using (2.8),

$$\overline{f}^*\overline{\sigma}^*\overline{E} \cong \overline{E}\,. \tag{2.9}$$

Now from the condition that $\overline{\varphi}\circ\overline{\sigma}$ is a constant map it follows immediately that $\overline{\sigma}^*\overline{\varphi}^*E$ $(= \overline{\sigma}^*\overline{E})$ is a trivial vector bundle. Consequently, by (2.9) we conclude that the vector bundle $\overline{\varphi}^*E$ is trivial. This completes the proof of the lemma. $\square$

Remark 2.2. The reader may wonder why we made use of the moduli stack of stable maps to "compactify" when we could have proceeded by taking closures inside some Hilbert scheme. This is due to the failure of the latter argument to preserve good properties of the fibres of $\overline{f}$; we have made definite use of the precise description of the fibres as trees of $\mathbb{P}^1$.

Proof of Theorem 1.1. Let

$$\varphi\,:\,Z \longrightarrow X \tag{2.10}$$

be a proper and surjective morphism such that φ^*E is trivial; it exists by Lemma 2.1. From this it follows that for all $i \geq 1$, the Chern class $c_i(E)$ is numerically equivalent to zero.

The proof of the theorem is divided into three steps.

First step ("Spreading out"):

Let Λ be the set of all subrings of K which are of finite type over $\mathbb{Z}$, so that, after ordering Λ by inclusion, we have

$$\operatorname{Spec} K \;=\; \varprojlim_{\lambda\in\Lambda} \operatorname{Spec} A_\lambda.$$

By [EGA IV_3, 8.8.2], we know that there exists an element $\alpha\in\Lambda$ and a (unique) morphism

$$\varphi_\alpha\,:\,Z_\alpha \longrightarrow X_\alpha$$

of A_α–schemes which gives back φ in (2.10) by the base change $A_\alpha \longrightarrow K$. Using [EGA IV_3, 8.10.5], we can also assume that

- φ_α is surjective,
- X_α is A_α–projective and
- Z_α is A_α–proper.

Further, letting X_λ denote $X_\alpha \otimes_{A_\alpha} A_\lambda$ for each $\lambda \geq \alpha$, Proposition 8.2.5 of [EGA IV_3] assures that the obvious morphisms

$$u_\lambda : X \longrightarrow X_\lambda, \qquad (\lambda \geq \alpha),$$

allow us to identify X with the projective limit

$$X \cong \varprojlim_{\lambda \geq \alpha} X_\lambda \tag{2.11}$$

in the category of all schemes.

Now we can apply [EGA IV_3, 8.5.2(ii)], and then [EGA IV_3, 8.5.5], to conclude that the coherent and locally free sheaf E on X is of the form $u_\mu^* E_\mu$, where E_μ is a coherent and locally free $\mathcal{O}_{X_\mu}$–module, and $\mu \geq \alpha$. Since $\varphi^* E$ is trivial, [EGA IV_3, 8.5.2.5] lets us suppose that $\varphi_\mu^*(E_\mu)$ is trivial. Finally, smoothness of X over K implies the existence of a $\kappa \geq \mu$ for which $X_\kappa \longrightarrow \operatorname{Spec} A_\kappa$ is smooth [EGA IV_4, 17.7.8(ii)].

We now recapitulate and "reset notation". We have obtained the following:

(1) a subring A_α of K which is of finite type over $\mathbb{Z}$,
(2) two A_α–schemes X_α and Z_α, where X_α is smooth projective, and Z_α is proper,
(3) a surjective A_α–morphism

$$\varphi_\alpha : Z_\alpha \longrightarrow X_\alpha$$

which induces $\varphi : Z \longrightarrow X$ under the base change $A_\alpha \longrightarrow K$, and
(4) a coherent and locally free $\mathcal{O}_{X_\alpha}$–module E_α such that $\varphi_\alpha^* E_\alpha$ is trivial, and E_α produces E by base change $A_\alpha \longrightarrow K$.

Now we spread out the properties related to rational connectedness; this consists of finding a subfield $Q \subseteq K$ which

- contains A_α,
- is of finite type over the prime field, and
- enjoys the property that $X_\alpha \otimes Q$ is SRC.

(N.B.: All we do at this stage is to give a complete answer to Exercise 3.2.5 of [Ko1, IV].)

We write

$$\operatorname{Spec} K = \varprojlim_{i \in I} \operatorname{Spec} K_i \,,$$

where K_i is a field of finite type over the prime field and $A_\alpha \subseteq K_i$. This will be technically useful since the morphisms $\operatorname{Spec} K \longrightarrow \operatorname{Spec} K_i$ are all faithfully flat.

By the definition of a separably rational connected variety [Ko1, IV, 3.2], there exists a variety M and a morphism

$$\theta : M \times \mathbb{P}^1_K \longrightarrow X$$

such that the corresponding morphism

$$\theta^{(2)} : M \times \mathbb{P}^1_K \times \mathbb{P}^1_K \longrightarrow X \times X$$

(see (1.1)) is dominant and smooth at the generic point, which is same as being smooth at *some point*. Another application of [EGA IV_3, 8.8.2] and [EGA IV_3, 8.10.5] allows us to find an element $i \in I$, and a separated K_i–scheme of finite type M_i which gives back M after the base-change $K_i \longrightarrow K$. Moreover, 8.8.2 of EGA ensures the existence of a morphism

$$\theta_i : M_i \times_{K_i} \mathbb{P}^1_{K_i} \longrightarrow X_\alpha \otimes_{A_\alpha} K_i$$

that induces θ after the base–change $K_i \longrightarrow K$. Since X and M are integral, so are $X_\alpha \otimes K_j$ and $M_i \otimes K_j$ for each $j \geq i$ [EGA IV_2, 2.1.13]. Now [EGA IV_4, 17.7.8(i)] guarantees the existence of a $j \geq i$ such that

$$\theta_j^{(2)} : (M_i \otimes K_j) \underset{K_j}{\times} \mathbb{P}^1_{K_j} \underset{K_j}{\times} \mathbb{P}^1_{K_j} \longrightarrow (X_\alpha \otimes K_j) \underset{K_j}{\times} (X_\alpha \otimes K_j)$$

is smooth at some non–empty open subset of the domain. In conclusion, $X_\alpha \otimes K_j$ is separably rationally connected.

Let us collect our findings:

(i) There exists a subring $A \subset K$ which is of finite type over $\mathbb{Z}$.
(ii) There are two A–schemes $\mathcal{X}$ and $\mathcal{Z}$, where $\mathcal{X}$ smooth and projective, $\mathcal{Z}$ is proper, and the generic fiber of $\mathcal{X}$ is a separably rationally connected variety.
(iii) There is a surjective A–morphism

$$\Phi : \mathcal{Z} \longrightarrow \mathcal{X}$$

which induces $\varphi : Z \longrightarrow X$ under the base change $A \longrightarrow K$.
(iv) There is a coherent and locally free $\mathcal{O}_\mathcal{X}$–module $\mathcal{E}$ which induces E, and $\Phi^*\mathcal{E}$ is trivial.

Using the fact that the set of all points $s \in \operatorname{Spec} A$ for which $\mathcal{X}_s$ is geometrically integral is open [EGA IV_3, 12.2.4] together with Theorem 3.11 of [Ko1, IV], we can furthermore assume the following:

(v) for each $s \in \operatorname{Spec} A$, the fiber $\mathcal{X}_s$ is geometrically integral (or just connected, since $\mathcal{X}_s$ is smooth).
(vi) For each $s \in \operatorname{Spec} A$, the fiber $\mathcal{X}_s$ is SRC.

Second step (vector bundles on rationally connected varieties over finite fields):

Our goal is to show that the $\mathcal{O}_\mathcal{X}$–module $\mathcal{E}$ constructed above is trivial on the pre–image of some non–empty open subset of $\operatorname{Spec}(A)$; this will be accomplished in the third and final step. In this step, we will show that for each closed point $s \in \operatorname{Spec}(A)$, the restriction $\mathcal{E}_s := \mathcal{E}|\mathcal{X}_s$ is essentially finite, meaning that it is given by a representation of the fundamental group scheme of Nori [No1, §3]. More precisely, let $s \in \operatorname{Spec}(A)$ be a closed point, so that the residue field $\mathbf{k}(s)$ is finite, and let F_s be the absolute Frobenius morphism of $\mathcal{X}_s$. We will show the existence of a pair of distinct positive integers a, b (depending on s) such that

$$(F_s^a)^*\mathcal{E}_s \cong (F_s^b)^*\mathcal{E}_s\,, \tag{2.12}$$

and then conclude that $\mathcal{E}_s$ is essentially finite using [Bi2, Corollary 1.2], see also [LS, Kor. 1.6].

So let $s \in \operatorname{Spec} A$ be a closed point, $\mathbf{k}(s)$ the (finite) residue field and

$$F := F_s : \mathcal{X}_s \longrightarrow \mathcal{X}_s$$

the absolute Frobenius morphism. Let $\Phi_s = \Phi|\mathcal{Z}_s$ be the restriction. Since $\Phi_s^* F^{n*}\mathcal{E}_s$ is trivial, it follows that $c_i(F^{n*}\mathcal{E}_s)$ is numerically trivial for all $i \geq 1$. We now show that $F^{n*}\mathcal{E}_s$ is (slope) semistable with respect to any chosen polarization. For this purpose we can assume that $\mathbf{k}(s)$ is algebraically closed and we set out to prove that for each $\mathbf{k}(s)$–smooth irreducible curve $C \hookrightarrow \mathcal{X}_s$, the restriction $F^{n*}\mathcal{E}_s|_C$ is semistable.

Let

$$C' \hookrightarrow \mathcal{Z}_s$$

be an irreducible curve such that $\Phi_s(C') = C$; the curve C' can be constructed as the closure of a closed point of the generic fiber of $\Phi_s^{-1}(C) \longrightarrow C$. Since the pull–back of $F^{n*}\mathcal{E}_s|_C$ to C' is trivial, so also is the pull–back of $F^{n*}\mathcal{E}_s|_C$ to the normalization of C'. Consequently, the vector bundle $F^{n*}\mathcal{E}_s|_C$ is semistable.

The isomorphism classes of (slope) strongly semistable sheaves with numerically trivial positive Chern classes is finite, because family of such sheaves is bounded [La1] while the field $\mathbf{k}(s)$ is finite. This establishes the existence of a pair of distinct positive integers a and b such that (2.12) holds, and lets us conclude that $\mathcal{E}_s$ is essentially finite for each closed point $s \in \operatorname{Spec} A$.

Third step (triviality of Nori's fundamental group scheme):

We will now show that $\mathcal{E}$ is trivial (after shrinking $\operatorname{Spec}(A)$ is necessary), and for that it is enough to show that for each closed point $s \in \operatorname{Spec}(A)$, the restriction $\mathcal{E}_s := \mathcal{E}|\mathcal{X}_s$ is trivial; indeed, if this is the case, we can proceed as in [Mu2, §5] because $s \longmapsto \dim H^0(\mathcal{X}_s,\, \mathcal{E}_s)$ is a constant function on $\operatorname{Spec} A$.

Since for each $s \in \operatorname{Spec} A$ the fiber $\mathcal{X}_s$ is a separably rationally connected variety (see the first step), the same can be said of $\mathcal{X} \otimes_A \overline{\mathbf{k}(s)}$. Now the main result in [Bi1, Theorem 2.1], whose proof will be sketched below, states that all essentially finite vector bundles on $\mathcal{X} \otimes_A \overline{\mathbf{k}(s)}$ are trivial. Using [No2, p. 89, Proposition 5], we immediately obtain the triviality of $\mathcal{E}_s$.

For the convenience of the reader, and also to isolate a remark which stems from the proof, we shall sketch how to obtain the above mentioned [Bi1, Theorem 2.1]. In fact, we will give the proof of the following lemma and let the reader look into [Bi1] and in [De2, Corollary 3.6] (see also Remark 2.5) to see that any SRC variety fulfills the hypothesis in the lemma (i.e., there are no global one forms and no étale coverings on a smooth projective SRC variety except the trivial ones).

Lemma 2.3. *Let k be an algebraically closed field of positive characteristic and M a smooth connected k–scheme. Assume that*

$$H^0(M,\, \Omega^1_{M/k}) = 0\,.$$

Let V be a coherent $\mathcal{O}_M$–module such that F_M^*V is trivial, where $F_M : M \longrightarrow M$ is the absolute Frobenius morphism. Then V is trivial. In particular, if Γ is a finite and local group scheme over k, then any principal Γ–bundle over M is trivial. If, furthermore, there are no non-trivial étale coverings of M, then each essentially finite vector bundle on M is trivial.

Proof. Let r be the rank of V. Any two connections on $\mathcal{O}_M^{\oplus r}$ actually coincide, because they differ by an element in $H^0(M, \, End(\mathcal{O}_M^{\oplus r}) \otimes \Omega^1)$, and

$$H^0(M, \, End(\mathcal{O}_M^{\oplus r}) \otimes \Omega^1) \, = \, H^0(M, \, \Omega^1)^{\oplus r^2} \, = \, 0 \, .$$

Cartier's celebrated theorem on the p–curvature [Ka, p. 190, Theorem 5.1] states that for each quasi–coherent $\mathcal{O}_M$–module W on M, the pull–back F_M^*W is endowed with a canonical integrable connection with vanishing p–curvature, and furthermore, this construction establishes an equivalence between the Frobenius pull–back of quasicoherent sheaves and the integrable connections with vanishing p–curvature. Since

$$F_M^*\mathcal{O}_M^{\oplus r} \, \cong \, \mathcal{O}_M^{\oplus r} \cong F_M^*V \, ,$$

and there is only one connection on $F^*\mathcal{O}_M^{\oplus r}$, from the theorem of Cartier it follows that $V \, \cong \, \mathcal{O}_M^{\oplus r}$. □

This also finishes the proof of Theorem 1.1. □

Remark 2.4. Lemma 2.3 is of independent interest. For example, it allows one to conclude that K3 surfaces have trivial fundamental group scheme. Indeed, a theorem of Rudakov and Shafarevich states that there are no global one forms on a K3 surface [RS1], [RS2], and it is well known that there are no nontrivial étale coverings of a K3 surface.

Remark 2.5. In the above proof we used the triviality of the etale fundamental group of a SRC variety over an algebraically closed field. The proof in the reference we gave [De2, Corollary 3.6] contains an imprecision: in order to use the theorem of de Jong and Starr, one needs to assume that the variety $\mathcal{Y}$ is regular, which is not clear in the case $\ell = p$ (notations of [De2]).

The fact that there are no non-trivial etale Galois coverings of a smooth SRC variety Y can be proved using [Cr, Corollary 1.7], the fact that $H^i(Y_{\text{et}}, \mathbb{Q}_p)$ is the part of slope zero in $H^i_{\text{cris}}(Y)$ [I], and the main theorem of [Es]. All this was communicated to us by H. Esnault.

3. A simpler proof of Theorem 1.1

We now present a simpler proof of Theorem 1.1 kindly suggested to us by the referee. Notations are those admitted in the beginning of the proof; $\overline{\varphi} : \overline{Z} \longrightarrow X$ is a morphism from a proper variety which trivializes E. (The referee also points out that the technique to prove the existence of $\overline{\varphi}$ is standard [BlS].) In addition to the referee's suggestions, we can introduce one further simplification: if

$$\overline{Z} \xrightarrow{\overline{\varphi}_1} \overline{Z}_0 \xrightarrow{\overline{\varphi}_0} X$$

is the Stein factorization of $\overline{\varphi}$, then $\overline{\varphi}_0^* E$ is trivial, since $(\overline{\varphi}_1)_* \mathcal{O}_{\overline{Z}} = \mathcal{O}_{\overline{Z}_0}$. Henceforth, we assume that $\overline{\varphi}$ is finite. In particular, the generic smoothness of $\overline{\varphi}$ now becomes generic etaleness.

We view E as a GL_r–bundle; let $\pi : E \longrightarrow X$ be the projection. Triviality of $\overline{\varphi}^* E$ is then synonymous to the existence of a morphism $\psi : \overline{Z} \longrightarrow E$ such that $\overline{\varphi} = \pi \circ \psi$. We now show that ψ is constant on the fibers of $\overline{\varphi}$. As already pointed out above, there exists an open dense subset $U \subseteq X$ over which $\overline{\varphi}$ is etale. The main improvement now is to use [Ko2, Theorem 3] which guarantees the existence of a family

$$\gamma_w : \mathbb{P}^1 \longrightarrow X$$

with dense image and such that the fiber product

$$C_w := \overline{Z} \underset{\overline{\varphi}, X, \gamma_w}{\times} \mathbb{P}^1$$

is irreducible. By further restricting the family $\{\gamma_w\}$, we can even suppose that $\gamma_w(\mathbb{P}^1) \cap U \neq \emptyset$, in which case C_w is also integral. The pull–back of E to C_w has two a priori two distinct trivializations, one coming from $\overline{\varphi}$ and one from γ_w (due to the hypothesis!). Since C_w is integral and proper over K, these two trivializations must agree, so that we obtain a commutative diagram:

$$\begin{tikzcd}
 & & X \\
\mathbb{P}^1 \arrow[urr, "\gamma_w"] \arrow[r] & E \arrow[ur, "\pi"'] & \\
C_w \arrow[u] \arrow[r, "u"'] & \overline{Z} \arrow[u, "\psi"'] \arrow[uur, bend right, "\overline{\varphi}"'] &
\end{tikzcd}$$

Now, if $z, z' \in \overline{Z}$ are closed points in the image of u, then $\psi(z) = \psi(z')$. By the definition of C_w as a fiber product, we see that if $\overline{\varphi}(z)$ belongs to the image of γ_w, then the whole fiber $\overline{\varphi}^{-1}(\overline{\varphi}(z))$ belongs to the image of u. Hence, the set

$$\left\{ x \in X \text{ closed such that } \psi \text{ is constant on } \overline{\varphi}^{-1}(x) \right\}$$

is dense in X. It is not hard to show that the above set is closed (here we work only with closed points) hence equals the whole of X. Let now $\overline{\Gamma} \subset \overline{Z} \times E$ be the graph of ψ and let Γ be its image in $X \times E$ (we give it the reduced structure).

$$\begin{tikzcd}
\overline{\Gamma} \subseteq \overline{Z} \times E \arrow[r, "\mathrm{pr}"] \arrow[d, "\overline{\varphi} \times \mathrm{id}"'] & \overline{Z} \arrow[d, "\overline{\varphi}"] \\
\Gamma \subseteq X \times E \arrow[r, "\mathrm{pr}"'] & X
\end{tikzcd}$$

Since the projection $\Gamma \longrightarrow X$ is a bijection on closed points and induces a separable extension of function fields, it has to be an isomorphism and we obtain the desired section to $\pi : E \longrightarrow X$.

4. Applications of Theorem 1.1

4.1. **Restriction and semistability.** The aim in this section is to prove Corollary 4.2 (part (3) of Theorem 1.2). Let X be a smooth projective SRC variety over K.

Proposition 4.1. *Let $E \longrightarrow X$ be a vector bundle of rank r such that for every morphism*

$$\gamma : \mathbb{P}^1_K \longrightarrow X ,$$

the pull–back $\gamma^* E$ is isomorphic to $L(\gamma)^{\oplus r}$ for some line bundle $L(\gamma) \longrightarrow \mathbb{P}^1_K$. Then there is a line bundle ζ over X such that $E = \zeta^{\oplus r}$.

Proof. The given condition on $\gamma^* E$ and Theorem 1.1 ensure that the vector bundle $End(E)$ is trivial. This implies that for any $x_0 \in X(K)$, the evaluation map

$$H^0(X, End(E)) \longrightarrow \mathrm{End}_K(E(x_0)) \tag{4.1}$$

is as isomorphism. Let $A : E \longrightarrow E$ be an isomorphism such that all the eigenvalues $\lambda_1 , \dots , \lambda_r$ of $A(x_0)$ are distinct. As the eigenvalues of $A(x)$ are independent of $x \in X$, it follows that E is isomorphic to the direct sum of the line subbundles

$$\mathcal{L}_i := \mathrm{kernel}(\lambda_i - A) \subseteq E ,$$

$1 \leq i \leq r$.

Since the evaluation map in (4.1) is injective, we have

$$\dim H^0(X, \mathcal{L}_i \otimes \mathcal{L}_j^*) \leq 1$$

for all $i , j \in [1 , r]$ because if $\dim H^0(X, \mathcal{L}_i \otimes \mathcal{L}_j^*) \geq 2$, then given any point x_0, there is a section of $\mathcal{L}_i \otimes \mathcal{L}_j^* \subset End(E)$ that vanishes at x_0. Next, note that if $H^0(X, \mathcal{L}_i \otimes \mathcal{L}_j^*) = 0$ for some i , j, then

$$\dim H^0(X, End(E)) < r^2 ,$$

which contradicts the fact that $End(E)$ is trivial.

For $s_{ij} \in H^0(X, \mathcal{L}_i \otimes \mathcal{L}_j^*) \setminus \{0\}$, $i , j \in [1 , r]$, the composition $s_{ij} \circ s_{ji}$ is an automorphism of $\mathcal{L}_i$, hence each s_{ij} is an isomorphism. This completes the proof of the proposition. □

A theorem due to Grothendieck says that any vector bundle over $\mathbb{P}^1_K$ decomposes into a direct sum of line bundles [Ha, p. 384, V, Ex. 2.6]. Therefore, Proposition 4.1 has the following corollary:

Corollary 4.2. *Let $E \longrightarrow X$ be a vector bundle of rank r such that for every morphism*

$$\gamma : \mathbb{P}^1_K \longrightarrow X ,$$

the vector bundle $\gamma^* E$ is semistable. Then there is a line bundle $\zeta \longrightarrow X$ such that $E = \zeta^{\oplus r}$.

4.2. D–**modules.** Let K is an algebraically closed field of characteristic $p > 0$. Let F_X be the absolute Frobenius morphism of a smooth projective SRC variety X/K.

Let $D_{X/K}$ denote the sheaf of all differential operators on X as constructed in [EGA IV_4, 16.8] or in [BO, §2] (the notation in these references is "$\mathrm{Diff}_{X/K}$").

Let E be a coherent $\mathcal{O}_X$–module endowed with an $\mathcal{O}_X$–linear homomorphism of sheaves of rings

$$\nabla : D_{X/K} \longrightarrow \mathcal{E}nd_K(E).$$

Such pairs (E, ∇) are called *stratified sheaves.* (In [BO], this is not the definition, but an equivalent form of it, see [BO, Proposition 2.11]). Berthelot and Ogus also do not require the $\mathcal{O}_X$–module E to be coherent.) The stratified sheaves are the objects of a category whose morphisms are simply homomorphisms of $D_{X/K}$–modules. A theorem of Katz [Gi, Theorem 1.3] states that there is an equivalence of between the category of stratified sheaves and the category of F–divided sheaves, which we now define.

A F–divided sheaf consists of a family $\{E_i, \sigma_i\}_{i\in\mathbb{N}}$, where E_i is a coherent $\mathcal{O}_X$–module and $\sigma_i : F_X^* E_{i+1} \longrightarrow E_i$ is an isomorphism between E_i and $F_X^* E_{i+1}$. Arrows

$$\{E_i, \sigma_i\} \longrightarrow \{E_i', \sigma_i'\}$$

are simply compatible systems of $\mathcal{O}_X$–linear maps $f_i : E_i \longrightarrow E_i'$. (We note that in [Gi, Definition 1.1], F-divided sheaves were called *flat.*)

Corollary 4.3. *Let (E, ∇) be a stratified sheaf on X. Then there exists an isomorphism $\mathcal{O}_X^{\oplus r} \longrightarrow E$ that takes the action of $D_{X/K}$ on E to the action of $D_{X/K}$ on $\mathcal{O}_X^{\oplus r}$ defined by the natural action of $D_{X/K}$ on $\mathcal{O}_X$. Put differently, all stratified sheaves on X are "trivial".*

Proof. It is known that the sheaf E is locally free [BO, Proposition 2.16]. Let $\{E_i, \sigma_i\}_{i\in\mathbb{N}}$ be the F–divided sheaf associated to the stratified sheaf (E, ∇). Due to [Gi, Proposition 1.7], it suffices to show that each E_i is trivial. But this is an immediate consequence of Theorem 1.1 and the fact that all stratified sheaves on $\mathbb{P}^1_K$ are trivial [Gi, Theorem 2.2]. □

Important remark: Recently, H. Esnault and V. Mehta proved [EM] that a smooth and projective variety in positive characteristic which has a trivial étale fundamental group has no non-trivial D–modules (as conjectured by Gieseker in [Gi]). This result, together with Kollár's theorem on the triviality of the étale fundamental group, implies Corollary 4.3 above.

4.3. **The S–fundamental group–scheme of** [BPS, §5]**,** [La2]**.** Let X/K be a smooth projective SRC variety, where K is an algebraically closed field. We fix a very ample line bundle ξ on X (so that semistability is well defined). If the characteristic of K is zero, by a strongly semistable vector bundle on X we will mean simply a semistable vector bundle.

Let d be the dimension of X. For any zero–cycle $Z \in \mathrm{CH}^d(X)$, let

$$[Z] \in \mathbb{Z}$$

denote its degree. For a vector bundle $E \longrightarrow X$, let

$$ch_2(E) := \frac{1}{2}c_1(E)^2 - c_2(E) \in \mathrm{CH}^2(X) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{4.2}$$

be the second Chern character.

Proposition 4.4. *Let $E \longrightarrow X$ be a strongly semistable vector bundle such that*

(1) $[c_1(E) \cdot c_1(\xi)^{d-1}] = 0$ *and*
(2) $[ch_2(E) \cdot c_1(\xi)^{d-2}] = 0$ *(see* (4.2)*).*

Then E is trivial.

Proof. The case of characteristic zero follows from the work of Simpson and Campana. We may assume that $K = \mathbb{C}$. Using [Si, p. 40, Corollary 3.10] the vector bundle E has a flat connection, while $\pi_1(X) = 1$, see [Ca, p. 545, Theorem 3.5] or [De1, p. 97, Corollary 4.18]. Hence E is trivial if the characteristic of K is zero.

Assume that K is of positive characteristic. Let $F_X : X \longrightarrow X$ be the absolute Frobenius morphism. Let

$$\gamma : \mathbb{P}^1_K \longrightarrow X$$

be a morphism; we want to show that $\gamma^* E$ is trivial in order to apply Theorem 1.1.

We start by noting that the family of vector bundles

$$\{(F^i_X)^* E\,;\, i \in \mathbb{N}\}$$

is bounded in the sense that there exists a K–scheme of finite type T (the parameter space) together with a coherent sheaf $\mathcal{E}$ (the bounding sheaf) over $X \times T$, such that each $(F^i_X)^* E$ is isomorphic to $\mathcal{E}|X_{t_i}$ for some K–point t_i of T (see [La1, Theorem 4.4]). Since

$$F_X \circ \gamma = \gamma \circ F_{\mathbb{P}^1_K}\,,$$

where $F_{\mathbb{P}^1_K}$ is the Frobenius morphism of $\mathbb{P}^1_K$,

$$\{(F^i_{\mathbb{P}^1_K})^* \gamma^* E\,;\, i \in \mathbb{N}\}$$

is a bounded family of sheaves on $\mathbb{P}^1_K$; the parameter space is again T and the bounding sheaf is $(\gamma \times \mathrm{Id}_T)^* \mathcal{E}$.

We will show that $\gamma^* E$ is semistable of degree zero.

Since $\{(F^i_{\mathbb{P}^1_K})^* \gamma^* E\,;\, i \in \mathbb{N}\}$ is a bounded family, it follows that the sequence

$$\{\mathrm{degree}((F^i_{\mathbb{P}^1_K})^* \gamma^* E)\,;\, i \in \mathbb{N}\}$$

is bounded (the method of proof is recalled below). On the other hand,

$$\mathrm{degree}((F^i_{\mathbb{P}^1_K})^* \gamma^* E) = p^i \cdot \mathrm{degree}(\gamma^* E)\,,$$

where p is the characteristic of K. Hence we conclude that

$$\mathrm{degree}(\gamma^* E) = 0\,.$$

We will now show that $\gamma^* E$ is semistable.

Assume that $\gamma^* E$ is not semistable. Let

$$S \hookrightarrow \gamma^* E$$

be a subbundle of positive degree. Then

$$\begin{aligned} h^0((F^i_{\mathbb{P}^1_K})^* \gamma^* E) &\geq h^0((F^i_{\mathbb{P}^1_K})^* S) \\ &\geq \chi((F^i_{\mathbb{P}^1_K})^* S) \\ &= p^i \cdot \mathrm{degree}(S) + \mathrm{rank}(S)\,. \end{aligned}$$

It follows that the sequence $h^0((F^i_{\mathbb{P}^1_K})^*(\gamma^* E))$ tends to infinity. On the other hand, by choosing a flattening stratification

$$T = T_1 \sqcup \cdots \sqcup T_a$$

of T [Mu1, Lecture 8], and applying the theorem on semicontinuity of cohomology [Mu2, p. 50], we see that the sequence $\{h^0(F^i_{\mathbb{P}^1_K})^* \gamma^* E)\,;\, i \in \mathbb{N}\}$ is bounded (this is a standard argument).

Hence we have proved that $\gamma^* E$ is semistable of degree zero. Consequently, $\gamma^* E$ is trivial (using again [Ha, p. 384, V, Ex. 2.6]). Hence E is trivial (Theorem 1.1). ☐

4.4. **Principal bundles.** *Terminology:* Following [Co], we will call a smooth and connected (respectively, affine) group scheme over K simply an algebraic group (respectively linear algebraic group).

Let G be an algebraic group defined over the field K. A theorem due the Chevalley says that G fits in a short exact sequence of groups

$$e \longrightarrow H \stackrel{\iota}{\longrightarrow} G \longrightarrow A \longrightarrow e\,, \tag{4.3}$$

where H is a linear algebraic group, and A is an abelian variety [Ch] (see also [Co]).

As before, let X be a smooth projective SRC K–variety. Let

$$E_G \longrightarrow X$$

be a principal G–bundle.

Lemma 4.5. (i) *The principal G–bundle E_G admits a reduction of structure group*

$$E_H \subset E_G$$

to the subgroup H in (4.3).

(ii) *If for each finite dimensional representation V of H over K, the associated vector bundle $E_H \times^H V \longrightarrow X$ is trivial, then the principal H–bundle E_H is trivial. If, in addition, H is reductive and $H \hookrightarrow \mathrm{GL}(V)$ is faithful, then E_H is trivial whenever $E_H \times^H V$ is likewise.*

Proof. (i) Using the exact sequence of pointed sets [DG, p. 373, III, §4, 4.6]

$$H^1(X_{\mathrm{fppf}},\, H) \longrightarrow H^1(X_{\mathrm{fppf}},\, G) \longrightarrow H^1(X_{\mathrm{fppf}},\, A) \tag{4.4}$$

associated to the short exact sequence in (4.3), we only need to show that

$$E_A := E_G \times^G A$$

is trivial. Note that E_A is, a priori, only an fppf sheaf which is an A–torsor over X. Since X is regular, [Ra, XIII 2.6(ii), p. 197] shows that E_A is represented by a projective X–scheme — see also pages 193 and 194 of [Ra] for relevant terminology. Due to [Ra, XIII 2.3(ii), p. 195], the class of E_A in $H^1(X_{\text{fppf}}, A)$ is annihilated by some integer m; applying this fact to the exact sequence of cohomology [DG, p. 373, III, §4, 4.6] we obtain

$$E_A \cong E_\Gamma \times^\Gamma A,$$

where $\Gamma = A[m]$ and $E_\Gamma \longrightarrow X$ is a Γ–bundle. By [Bi1, Theorem 2.1], E_A is trivial.

(ii) Let us start in more generality. Let $j : B \hookrightarrow C$ be a closed embedding of linear algebraic groups. Let Q denote the quotient C/B: it is the *K–scheme* representing the fppf sheaf associated to the pre-sheaf which to every K–scheme S attaches the set $C(S)/B(S)$. (For more details see [DG, III §3, 5.4].) Consider the exact sequence of *pointed sets* [DG, p. 373, III, §4, 4.6]

$$C(X) \longrightarrow Q(X) \longrightarrow H^1(X_{\text{fppf}}, B) \xrightarrow{H^1(j)} H^1(X_{\text{fppf}}, C) \tag{4.5}$$

We are essentially interested in knowing when the arrow $H^1(j)$ is injective. Using the properness of X, this will be the case whenever Q is *quasi-affine*, that is, open in an affine K–scheme. Closed embeddings enjoying this property are called *observable* — see Theorems 1.2 and 2.1 of [GH] — and the following properties on B always produce observable embeddings:

(a) B is reductive (this is due to the combination of theorems due to Nagata and Haboush, see Corollary 2.4 and Corollary 4.6 of [GH]);
(b) B is unipotent (this can be proved with the help of [DG, p. 344, III §3, 5.9(iv)]).

These considerations have the following consequences:

C1: If B is a reductive group and $\rho : B \hookrightarrow \mathrm{GL}(W)$ is a faithful representation, then a B–bundle P over X is trivial if and only if $P \times^B W$ is likewise.
C2: Same as **C1**, but under the assumption B unipotent.

We now return to the specific problem of showing that E_H is a trivial H–bundle. Let $R \trianglelefteq H$ be the *unipotent radical*, so that $H/R =: Q$ is reductive. Take $\rho : Q \hookrightarrow \mathrm{GL}(W)$ a faithful representation. From **C1** above, the triviality of

$$E_H \times^H W \cong [E_H \times^H Q] \times^Q W$$

shows that $E_H \times^H Q$ is trivial, which means that E_H is induced by an R–bundle E_R. Since R is unipotent, conclusion **C2** above shows, with the same argument as before, that E_R is trivial and this finishes the proof of (ii). □

Theorem 1.1 and Lemma 4.5 together give the following:

Corollary 4.6. *Let E_G be a principal G–bundle over X such that for each morphism $\gamma : \mathbb{P}^1_K \longrightarrow X$, the pull–back $\gamma^* E_G$ is trivial. Then E_G itself is trivial.*

Lemma 4.5 and Corollary 4.3 together give the following:

Corollary 4.7. *There is no nontrivial F–divided principal G–bundle over X.*

We maintain the assumption that X is a smooth projective and d–dimensional SRC variety over K (§1). Let H be a *reductive* linear algebraic group over K (for terminology see the beginning of this section),

$$E_H \longrightarrow X$$

a principal H–bundle, and

$$\mathrm{ad}(E_H) = E_H \times^H \mathrm{Lie}(H) \longrightarrow X$$

the adjoint vector bundle.

Fix a very ample line bundle ξ over X.

Proposition 4.8. *Assume that the following three conditions hold:*

(1) *E_H is strongly semistable,*
(2) *$[c_2(\mathrm{ad}(E_H)) \cdot c_1(\xi)^{d-2}] = 0$, and*
(3) *for each character χ of H, the associated line bundle*

$$L_\chi := E_H \times^\chi K \longrightarrow X$$

is trivial.

Then the principal H–bundle E_H is trivial.

Proof. Since E_H is strongly semistable, the vector bundle $\mathrm{ad}(E_H)$ is also strongly semistable (this follows from [RR, p. 285, Theorem 3.18] if the characteristic of K is zero, and from [RR, p. 288, Theorem 3.23] if the characteristic of K is positive). Applying the third condition in the proposition to the H–module

$$\bigwedge^{\mathrm{top}} \mathrm{Lie}(H)$$

we conclude that the line bundle $\wedge^{\mathrm{top}}\mathrm{ad}(E_H)$ is trivial. Hence from Proposition 4.4 and condition (2) above it follows that $\mathrm{ad}(E_H)$ is trivial.

Let $Z(H)$ denote the center of H, so that $\mathrm{Lie}(H)$ is a faithful $H/Z(H)$–module for the conjugation action. The quotient

$$E_H/Z(H) = E \times^H (H/Z(H))$$

is a principal $H/Z(H)$–bundle over X.

Since $\mathrm{Lie}(H)$ is a faithful $H/Z(H)$–module, and the vector bundle $\mathrm{ad}(E_H)$ is trivial, it follows from (ii) of Lemma 4.5 that the principal $H/Z(H)$–bundle $E_H/Z(H)$ is trivial. Consequently, the principal H–bundle E_H admits a reduction of structure group to the subgroup–scheme $Z(H)$. Using the fact that $Z(H)$ is a direct product of a finite group scheme and a torus Z_0 [SGA 3 XII, 4.11], we can conclude, with the help of [Bi1, Theorem 2.1], that E_H admits a reduction of structure group to Z_0:

$$E_{Z_0} \subset E_H.$$

The torus Z_0 fits in a short exact sequence

$$e \longrightarrow \Gamma \longrightarrow Z_0 \longrightarrow H/[H,H] \longrightarrow e\,,$$

where Γ is a finite group–scheme. Using the third condition in the statement and Lemma 4.5, we infer that the principal $H/[H,H]$–bundle obtained by extending the structure group of E_{Z_0} to $H/[H,H]$ is trivial. Also, as noted earlier, any Γ–bundle on X is trivial. Therefore, combining these we conclude that E_{Z_0} is trivial. This completes the proof of the proposition. $\square$

The condition in Proposition 4.8 that L_χ is trivial can be replaced by the weaker condition that $c_1(L_\chi)$ is numerically equivalent to zero. To see this the reader should remark that L_χ is always strongly semistable [RR] and then apply Proposition 4.4 to obtain the triviality of L_χ.

Acknowledgments. We thank R. Lodh for pointing out [Ol], H. Esnault for pertinent remarks and the referee for the simpler proof presented in section3.

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