Fundamental groups in algebraic geometry

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text.

This dissertation is not the same as any dissertation that I have submitted for a degree or diploma or other qualification at any other University.

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Abstract

We study analogues of differential equations in algebraic geometry of positive characteristic using the theory of Tannakian reconstruction of Grothendieck and Saavedra. These analogues consist of sheaves of coherent modules over a k-scheme X which, provided that X has a k-rational point, form categories equivalent to the categories of representations of affine group schemes. The case of an abelian variety is closely analyzed.

If the ground field is complete with respect to a non-Archimedean absolute value, we relate the fundamental groups obtained with the fundamental group from rigid geometry.

Motivated by the case of a valued field, we study the problem of existence of local solutions of these differential equations in dimension one and characterize the possible monodromy groups.

Contents

0.1 Introduction

The purpose of this work is to study differential equations in positive characteristic and the philosophy adopted to achieve this goal is that of Riemann and Hilbert: the equations should be regarded as representations of the fundamental group of the ambient space. In the two paragraphs below, we try to explain what me mean by differential equations and fundamental groups.

On the concept of a differential equation — In positive characteristic, the notion of differential equation is not at all connected to geometry as it is in the characteristic zero world because the first derivative does not control the Taylor series. Hence, the definition of differential equation we have adopted here is that of a stratification (introduced by Grothendieck) which roughly means that we have many equations which relate the functions and all higher order derivatives (image under differential operators). We will now be more precise. It is well known and documented that in the complex analytic case, the right concept of a (linear) differential equation is that of a connection on a coherent analytic sheaf over the ambient manifold. Given a complex manifold X and a coherent analytic sheaf $\mathscr{E}$, a connection on $\mathscr{E}$ is a homomorphism of $\mathscr{O}_X$-modules $\nabla : \Theta_X \longrightarrow \mathscr{E}nd_{\mathbb{C}}(\mathscr{E})$ (Θ_X is the tangent sheaf) such that $\nabla(D)(f \cdot e) = D(f) \cdot e + f \cdot \nabla(D)(e)$ for $f \in \mathscr{O}_X$ and $D \in \Theta_X$. Now the module Θ_X is naturally a left $\mathscr{O}_X$-submodule of a (non-commutative) sheaf of $\mathscr{O}_X$-algebras, $\mathscr{D}_X$, the sheaf of differential operators, and is well known that the above connection ∇ will extend to a homomorphism of $\mathscr{O}_X$-algebras (a stratification) if and only if ∇ preserves the Lie bracket (integrability condition). Until now, all the constructions are algebraic and so we can make the same definitions over more general schemes, but the fact that the integrability condition implies the existence of a stratification is no longer true. Nevertheless, the definition of stratification still survives and in this work we study the properties of the category of coherent sheaves with a stratification. That they are a more geometric analogue of differential equations in characteristic zero is a consequence of the fact that, under some natural hypothesis, this category is neutral Tannakian (section 1.3). Also, the concept of stratification allows us to define differential equations over more general schemes (locally noetherian regular) via the Cartier-Katz theorem and F-divided sheaves (section 1.3). F-divided sheaves are tractable objects and most of the time they are used in place of stratifications (see Chapter 3 for example). In summary, the concept of differential equation applied most successfully here is the one provided by the F-divided sheaves.

On the method — The question of what is a "fundamental group" for an arbitrary scheme is a very rich and manifold one and, given some latitude, could be traced back to the discoveries of Galois on solving algebraic equations. Nowadays, the most powerful technique available to tackle this problem with elegance is that if Tannakian categories. This theory was developed by Saavedra (a student of Grothendieck) in his Catégories Tannakiennes and is employed throughout the present work. In a simplified form, the main result of the general theory states that a category endowed with a tensor product and a faithful and exact functor into the categories of finite dimensional vector-spaces over a field is the category of representations of an affine group scheme.

Summary of chapters and results —

Chapter 1: Roughly speaking, this chapter and the next one are products of a through understanding of Tannakian categories [12], Nori's method of construction of universal torsors [33] and the Cartier-Katz construction relating F-divided sheaves and stratified sheaves [16]. Chapter 1 main achievement is to show that the pathology of non-reduced monodromy groups *does not* occur when dealing with the category of stratified sheaves (Theorem 34), contrary to the category of de Rham sheaves (Chapter 2). This is an important fact which contributes to the perspective that stratifications are really the right differential equations to be studied in positive characteristic.

We introduce the notion of F-divided torsors (section 1.4.3) to accompany the notion of F-divided sheaf. This is a useful definition in the theory (as it helps to control Nori's inversion, see Lemma 32 and the remark following it as well as section 1.5). By analyzing closely and expanding Nori's method, we can use the proof of the original Cartier-Katz theorem to produce the equivalence between F-divided torsors and stratified torsors over a smooth k-scheme (section 1.4.3).

In this chapter we also fill some gaps in the literature concerning stratifications on torsors. Let us make a digression on what we mean by gap in the literature. The only reference where the concept of stratification on a torsor is tangentially brought up is [11] (§10), where the characteristic zero case is explored (and even in characteristic zero, we feel that Deligne's exposition lacks a more differential geometric character; there is, for example, no mention to the Atiyah sheaf – probably due to the fact that this topic is lateral to the entire article). Of course, a careful reading of [11] (§5, §6 and §10) and [10] (section 8) and a preparatory work (the analogues of lemmas 2.6–2.8 of [33]) will bring up the results of sections 1.4.1 and 1.4.2 (and even the important Lemma 32), but we could *not* safely cite Deligne and we believe that following Nori's method is more economic and natural. Even though Deligne's view also gives universal torsors associated to fibre functors over general Tannakian categories (section 8 of [10] and §5 of [11]), there is no analogue of [33] lemmas 2.6 and 2.8 in Deligne's writings: these lemmas are very important as they give a precise characterization of faithful and exact tensor functors $\mathrm{Rep}_k(G) \longrightarrow$ (tensor category). Also, we believe that there is a meaningful cognitive gain in sections 1.4.1 and 1.4.2 (and section 2.2.2) as these were elaborated before reading [10], [11].

Of course, the understanding of Deligne's ideas was afterwards important to this work (the notion of algebraic hull in chapter 4 and section 6.2 stems from loc.cit.)

and even though we did not use his methods, we certainly gained a lot from [11]. We thank him heartily for writing such a careful paper as [11].

Chapter 2: We study two important nilpotent Tannakian categories: the category of de Rham sheaves over a smooth k-scheme X, $\mathbf{dR}(X)$, and that of nilpotent stratified (F-divided) sheaves, $\mathfrak{N}\mathbf{str}(X)$. We have given a different and enlightening proof of a result Nori uses *en passant* to show that the category of nilpotent sheaves on a proper, reduced and geometrically connected k-scheme (char. > 0) has a profinite fundamental group scheme. The proof is pure group theory and hence emphasizes once more the importance of the Tannakian approach. Endowed with this result we show that, over proper schemes, the fundamental group schemes of the above mentioned Tannakian categories are profinite (char. > 0).

We chose to make a study of $\mathbf{dR}(X)$ because, in positive characteristic, it is a natural way to obtain a Tannakian category inside the category of modules with integrable connections. We were under the expectation that $\mathbf{dR}(X)$ would preserve some interesting properties of the analogous category in characteristic zero; but that is far from the truth as an easy example in section 2.2.3 shows (the appearance of non-reduced monodromy groups). This is why the study of $\mathbf{dR}(X)$ is somewhat shallow: this category is not as promising in positive characteristic as we expected.

We have also included a discussion – possibly irrelevant to the expert – on the notions of connections on torsors and the factoring of the associated sheaf functor through the category of sheaves with connections (section 2.2.2). Again, Nori's method is used and the same observations made about Deligne's method ([10],[11]) on the review of Chapter 1 above are valid: in Deligne's writings the analogues of [33] lemmas 2.6 and 2.8 is missing, but some preparatory work would bring them up.

Chapter 3: In this chapter we study the category of stratified sheaves $\mathbf{str}(X)$ on an abelian variety X over an algebraically closed field of positive characteristic. We apply the Fourier-Mukai transform to obtain a decomposition of $\mathbf{str}(X)$ into an unipotent and a diagonal part. This resembles the analogous fact in the complex analytic case which is an immediate consequence of the abelianess of the topological fundamental group of a complex torus. Also, using the results of Chapter 2 (Nori's Lemma in section 2.4 and Lemma 46) the unipotent part of the Tannakian fundamental group of an abelian variety if easily characterized and hence we are able to find quite a definite expression for $\Pi^{\mathbf{str}}(X)$. As a corollary, we obtain the relation between the Abelianization of the stratified fundamental group of a curve and the stratified fundamental group of its Jacobian variety (Corollary 59).

Chapter 4: Here the flow of the dissertation inclines to the beautiful theory of rigid analytic geometry; the ground field is now algebraically closed and complete with respect to a non-trivial non-Archimedean absolute value (and of char. > 0). This chapter tries to follow Deligne's version of the Riemann-Hilbert correspondence

([9])

$$\text{representations of } \pi_1 \longrightarrow \text{local systems} \longrightarrow \text{sheaves with integrable connections,}$$

which means, in the present case, that the (purely motivational) goal was to show that any F-divided sheaf on a rigid analytic variety X comes from a representation of the rigid fundamental group.

But then we stumble in the peculiar fact that F-divided sheaves might be trivial on an admissible neighbourhood of any point, whereas the covering so obtained fails to be admissible. Thus we are only able to obtain that the representations of the rigid analytic fundamental group form a Tannakian subcategory of the category of F-divided sheaves on a smooth rigid analytic variety (Theorem 62).

In section 4.3 we use the richness of rigid analytic geometry in order to extend the description of the stratified fundamental group of an abelian variety (case of uniformizable varieties).

This chapter serves as inspiration to the following as it motivates the investigation, in the rigid analytic category, of one of the main ingredients in Deligne's theory of the Riemann-Hilbert correspondence: the Cauchy-Kowalewskaya existence theorem for local solutions of linear differential equations.

Chapter 5: This chapter explores a particular (and already complicated) question raised on Chapter 4 concerning the nature of local solutions to stratified differential equations on small (for the rigid topology) open sets. On Chapter 4 we developed a relation between the (algebraic hull) of the rigid fundamental group of a smooth rigid variety X and the F-divided fundamental group — in order to produce a complete result (equality of these two group schemes) as in Deligne's theory of the Riemann-Hilbert correspondence over $\mathbb{C}$, we are missing two ingredients: (1) given an F-divided $\{M_i\}$ on X, and a $x \in X$, there is an open neighborhood $U_x \subseteq X$ such that $\{M_i\}|U_x$ is trivial and (2) the covering of $\{U_x\}_{x\in X}$ is admissible. Condition (2) is simply not true (as one sees from etale coverings $[l] : \mathbb{G}_m^{\mathrm{an}} \longrightarrow \mathbb{G}_m^{\mathrm{an}}$) and we are left to analyze condition (1). We do that in the one dimensional case or, concretely, over the affinoid disk $\mathbb{D}(\rho)$. Right from the start we can find F-divided modules which do not become trivial even if we let $\rho \longrightarrow 0$ and the question of computing the monodromy groups of the Tannakian category naturally attached to the problem, $\mathscr{T}$, is posed and solved (Theorem 82). This conducts to a result which resembles the statement of the Abhyankar conjecture for the affine line.

As Chapter 5 is almost independent of the previous Chapters, some definitions are repeated for the convenience of the reader. We have also included an Appendix where we explore the notion of differential operators over affinoid domains (over an algebraically closed field of positive characteristic) and show that they have a local nature for the rigid topology (Theorem 98 and Corollary 100).

Chapter 6: This chapter is dedicated to some questions which arose in the course of study and follows Serre's *dictum* that very often the questions asked are more interesting then the results obtained. The origin of these questions is the (quite bold) imitation of [11] (Chapter 10) and [9] in the non-Archimedean world.

Section 6.1: We raise the question of whether it is possible, over projective rigid analytic curves in positive characteristic, to avoid the (spectacular, according to Chapter 5) failure of non-existence of a complete system of local solutions on *any* admissible neighbourhood of a point.

Section 6.2: This section is written to indicate alternatives for future work which would complete Chapter 4, at least in the one dimensional case. The philosophy is to look at various "fundamental groups" (and here we really mean group, not group scheme) for rigid spaces and see if one of them will fit the description we have of $\Pi^{\mathbf{str}}$ for a Tate elliptic curve.

Section 6.3: What happens to regular singular points? Is it possible, in the non-Archimedean world, to extend differential equations to compactifications? From the hierarchy complex analytic $<$ non-Archimedean of char. $0 <$ non-Archimedean of positive char., it seems that any good answer will have to be backed up by a good understanding of the non-Archimedean characteristic zero case (see Question 2) and Riemann's existence theorem.

0.2 Terminology and notation

Unless otherwise stated, the following notations and assumptions are in force throughout this work.

ground field k is a perfect field.

schemes All schemes will be over k and $X \times Y := X \times_{\operatorname{Spec} k} Y$, $\operatorname{Hom}(X, Y) := \operatorname{Hom}_{k-\mathrm{sch}}(X, Y)$.

group schemes Given a k-vector space V, the functor of k-algebras $A \mapsto \mathrm{GL}_A(A \otimes_k V)$ will be denoted by $\mathbb{GL}(V)$. If $V = k^n$, then $\mathbb{GL}(V) =: \mathbb{GL}(n)$.

The standard notations for group schemes are also in force. For example, given a finite group Γ, we denote by $\widetilde{\Gamma}$ the constant group scheme associated to it ([42], 2.3, p. 17). If no confusion is likely, we abandon the $\widetilde{\ }$.

Given an abelian group $\mathbb{X}$, the diagonal group $\operatorname{Diag}(\mathbb{X})$ is the affine group scheme associated to the group functor of k-algebras $A \mapsto \operatorname{Hom}_{\mathrm{groups}}(\mathbb{X}, A^{\times})$.

representations of group schemes If $G = \operatorname{Spec} R$ is a group scheme over k, the category of *finite dimensional* representations of G will be denoted by $\operatorname{Rep}_k(G)$.

The category of all representations (finite or infinite dimensional) will be denoted by $\mathrm{Rep}'_k(G)$. Unless otherwise stated, the term representation means finite dimensional representation. When considering a representation of finite or infinite dimension, we will write so, unless it is obvious from the context. For example, in the sentences

(a) *Let V be a representation of G.*

(b) *The standard left regular representation of $\mathbb{Z}/p\mathbb{Z}$ in $V = \mathscr{O}(\mathbb{G}_a)\dots$*

we mean that in (a) V is of finite dimension and in (b) V is of infinite dimension.

We will also identify representations with their corresponding R-comodules ([42], 3.2) and for an object (V, ρ) of $\mathrm{Rep}'_k(G)$, ρ will denote the comodule map $V \longrightarrow V \otimes_k R$ and the homomorphism $\mathbb{GL}(V)$. (R, ρ_l) (resp. (R, ρ_r)) will be the left (resp. right) regular representation which is suggestively described by $\rho_l(g)(f) : x \mapsto f(g^{-1}x)$ (resp. $\rho_r(g)(f) : x \mapsto f(xg)$) for all $f \in R = \mathrm{Hom}(G, \mathbb{A}^1_k)$.

torsors Given an affine group scheme G a torsor over X is an *affine* and faithfully flat X-scheme P with a *right*-action of G such that: (a) If X is given the trivial action of G, then the structural morphism $P \longrightarrow X$ is equivariant. (b) The natural morphism $P \times_k G \longrightarrow P \times_X P, (q, g) \mapsto (q, q \cdot g)$ is an isomorphism.

differential operators If X is a k-scheme and $\mathscr{E}$ and $\mathscr{F}$ are two $\mathscr{O}_X$-modules, the sheaf of k-linear differential operators of order $\leq m$ from $\mathscr{E}$ to $\mathscr{F}$ will be denoted $\mathscr{D}^{\leq m}_X(\mathscr{E}, \mathscr{F})$. $\mathscr{D}^{<m}_X(\mathscr{E}, \mathscr{F}) = \mathscr{D}^{\leq m-1}_X(\mathscr{E}, \mathscr{F})$. The total module of differential operators

$$\varinjlim_m \mathscr{D}^{\leq m}_X(\mathscr{E}, \mathscr{F})$$

is denoted $\mathscr{D}_X(\mathscr{E}, \mathscr{F})$. (see [3], chap. 1). In the special case $\mathscr{E} = \mathscr{F} = \mathscr{O}_X$: $\mathscr{D}^{\leq m}_X(\mathscr{E}, \mathscr{F}) = \mathscr{D}^{\leq m}_X$. Analogous for the total module of differential operators: $\mathscr{D}_X$. The $\mathscr{O}_X$-ideal of $\mathscr{D}_X$ given by the operators which annihilate the section 1 is denoted by $\mathscr{D}^+_X$ and $\mathscr{D}^{+,\leq m}_X := \mathscr{D}^+_X \cap \mathscr{D}^{\leq m}_X$.

If U is an open subset of X and $(x_1, \dots, x_n) : U \longrightarrow \mathbb{A}^n_k$ is an etale morphism, we follow the usual convention for a basis of $\mathscr{D}^{\leq m}_X|U$: for each $q = (q_1, \dots, q_n) \in \mathbb{N}^n$ with $q_1 + \dots q_n \leq m$, there are canonical differential operators $D_q \in \mathscr{D}^{\leq m}_X(U)$ such that

$$D_q \circ D_{q'} = \begin{pmatrix} q + q' \\ q \end{pmatrix} D_{q+q'}, \quad \mathscr{D}^{\leq m}_X = \bigoplus_{q_1+\dots+q_n \leq m} \mathscr{O}_X D_q.$$

(loc.cit., prop. 2.6).

categorical Let $\psi : k \longrightarrow k$ be a homomorphism and let V, W be k-vector spaces. A homomorphism $f : V \longrightarrow W$ of the underlying groups is ψ-linear when $f(\lambda v) = \psi(\lambda)f(v)$ for all $\lambda \in k$. Given $\mathfrak{A}$ and $\mathfrak{B}$ be k-linear categories, a functor $T : \mathfrak{A} \longrightarrow \mathfrak{B}$ is ψ-linear when the natural maps $\mathrm{Hom}_{\mathfrak{A}}(A, A') \longrightarrow \mathrm{Hom}_{\mathfrak{B}}(TA, TA')$ is ψ-linear.

Chapter 1

Tannakian categories of coherent sheaves

1.1 Introduction

This is a technical chapter. Its existence is justified by the introduction of fundamental concepts as:

(*i*) Locally free tensor categories of coherent sheaves (Definition 2) and methods to show how such categories become neutral Tannakian (existence of a fibre functor). Universal torsors and characterization of certain exact and faithful tensor functors $\mathrm{Rep}_k(G) \longrightarrow$ tensor category. (section 1.2)

(*ii*) The category of F-divided and stratified sheaves, which is certainly the main category studied in this work. (section 1.3)

(*iii*) Stratifications and F-divisions of torsors and the fundamental application of these notions which is to show that the Tannakian category of F-divided sheaves only has reduced monodromy groups. (section 1.4)

(*iv*) The natural construction of a functor going from the category of representations of the etale fundamental group scheme to category of F-divided sheaves based on the fact that the Frobenius morphism on a proetale group scheme is an isomorphism. (section 1.5)

1.2 Fundamental setting

Our main objects of study will be abelian tensor categories ([12], Def. 1.15, p.118) related to the tensor category of coherent sheaves on a connected scheme over k. The following sections (1.2.1–1.2.4) set up the minor technicalities from the theory of tensor categories such as tensor categories of coherent sheaves on a locally noetherian

k-scheme, nilpotent categories and universal torsors. Our intention was to (1) give a simple and direct exposition of the basic facts used to show that a tensor category related to coherent sheaves on a k-scheme is Tannakian (1.2.1) (2) recall the important notion of nilpotent categories (1.2.2) (3) give some fundamental examples of Tannakian categories (1.2.3) and (4) rework Nori's method of universal torsors (1.2.4). Nothing is really new except perhaps the reworking in 1.2.4. These concepts are basic for the rest of the work and hence we deal with them from an early stage.

This section is to be used as reference for latter applications and should be regarded as support for the study of the tensor categories developed from section 1.3 on; we recommend its reading to be done on a need-to-know basis. We assume that the base scheme X is locally noetherian.

1.2.1 Tensor categories of coherent sheaves

For the sake of brevity and clarity, we will adopt the following definitions.

Definition 1. *A k-abtensor category $\mathfrak{A}$ is an abelian and k-linear category with a structure of tensor category $(\mathfrak{A}, \otimes, \varphi, \psi)$ such that $\otimes : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{A}$ k-bilinear. (this equals Def. 1.15, p.118 of loc.cit.).*

For us, the main example of such a category is the category of coherent sheaves, $(\mathrm{coh}(X), \otimes_{\mathscr{O}_X})$, over a connected scheme X/k.

Definition 2. *A locally free tensor category of coherent sheaves on a scheme X/k is a k-abtensor category $(\mathfrak{A}, \otimes, \varphi, \psi, \iota)$ endowed with an exact and faithful k-linear tensor functor*

$$\iota : (\mathfrak{A}, \otimes) \longrightarrow (\mathrm{coh}(X), \otimes_{\mathscr{O}_X}),$$

such that for any object A of $\mathfrak{A}$, $\iota(A)$ is locally free.

For brevity, we shall omit the associativity and commutativity constraints from the notation and say that $(\mathfrak{A}, \otimes, \iota)$ is a locally free tensor category of coherent sheaves.

Lemma 3. *Let $(\mathfrak{A}, \iota)$ be a locally free tensor category of coherent sheaves on the connected k-scheme X. Let K be an extension field of k and let x_0 be a K-rational point of X. Then the tensor functor $\omega := x_0^* \circ \iota : \mathfrak{A} \longrightarrow (K - \mathrm{mod})$ is exact and faithful.*

The proof, which is very easy if one uses the following standard lemma, will be omitted.

Lemma 4. *Let $T : \mathfrak{C} \longrightarrow \mathfrak{D}$ be an additive functor between abelian categories. The following are equivalent:*

i) T is faithful and exact.

ii) T is exact and $TC = 0$ only if $C = 0$.

iii) The sequence $C' \longrightarrow C \longrightarrow C''$ in $\mathfrak{C}$ is exact if and only if $TC' \longrightarrow TC \longrightarrow TC''$ is. □

1.2.2 Nilpotent categories

Here we introduce the notion of nilpotent categories. This concept is useful because, in some cases, it is possible to obtain substantial information from a neutral Tannakian category using its subcategory of nilpotent objects. Even more, nilpotent categories have some unexpectedly nice properties in positive characteristic (Corollary 35, section 2.4 etc).

Let $\mathfrak{A}$ be an abelian category and let A be an object of $\mathfrak{A}$.

Definition 5 ([40])**.** *i) An object N of $\mathfrak{A}$ is called A-nilpotent if there exists a decreasing filtration*

$$0 = F^r N \subseteq F^{r-1} N \subseteq \ldots \subseteq F^0 N = N,$$

and isomorphisms $\mathrm{gr}^i_F \cong A$.

ii) The A-nilpotent subcategory of $\mathfrak{A}$, $\mathfrak{N}_A\mathfrak{A}$, is the the full subcategory of $\mathfrak{A}$ which has the A-nilpotents as objects. The category $\mathfrak{A}$ is A-nilpotent when every object is A-nilpotent.

iii) (Convention) If $\mathfrak{A}$ is also a tensor category, the nilpotent subcategory $\mathfrak{N}\mathfrak{A}$ will be $\mathfrak{N}_{\mathbb{1}}\mathfrak{A}$. A nilpotent tensor category is a tensor category which is $\mathbb{1}$-nilpotent.

Taking the nilpotent category of some abelian category of coherent sheaves (on a locally noetherian scheme) has the advantage to produce flat sheaves as objects. Of course, one can loose the abelianess property. The first structural result for the operation "taking the nilpotent category" addresses this detail:

Lemma 6 ([40], 1.2.1, p. 521)**.** *Let $\mathfrak{A}$ be an abelian tensor category. Then $\mathfrak{N}\mathfrak{A}$ is abelian if $\mathrm{End}_{\mathfrak{A}}(\mathbb{1})$ is a field.* □

From the definition of affine unipotent group schemes as those groups whose representations have always a fixed vector ([42], chap. 8), we obtain from the main theorem of Tannakian duality ([12], Thm. 2.11, p. 130) the following corollary.

Corollary 7. *Let $\mathfrak{A}$ be a neutral Tannakian category ([12], p. 138) with fibre functor $\omega : \mathfrak{A} \longrightarrow (k-\mathrm{mod})$. If $\mathfrak{A}$ is nilpotent, then ω induces an equivalence between $\mathfrak{A}$ and the category of representations of an affine unipotent group scheme.* □

We will now systematize the construction of neutral Tannakian *nilpotent* tensor categories of coherent sheaves.

Let $\mathfrak{A}$ be a k-abtensor category endowed with a faithful and exact k-linear tensor functor $\iota : \mathfrak{A} \longrightarrow \mathrm{coh}(X)$, X locally noetherian and connected. Because every nilpotent sheaf is locally free, given $A \in \mathfrak{A}$ nilpotent, the functor

$$? \otimes \iota(A) : \mathrm{coh}(X) \longrightarrow \mathrm{coh}(X)$$

is exact. By part *iii*) of Lemma 4, $? \otimes A : \mathfrak{A} \longrightarrow \mathfrak{A}$ is also exact. It then follows (from standard linear algebra) that the tensor product of two nilpotent objects of $\mathfrak{A}$ is again nilpotent. Using Lemma 3, we obtain the first part of

Lemma 8. *Let $\mathfrak{A}$ be a k-abtensor category endowed with a k-linear, faithful and exact tensor functor $\iota : \mathfrak{A} \longrightarrow \mathrm{coh}(X)$. Assume that $\mathfrak{N}\mathfrak{A}$ is abelian.*

i) $\mathfrak{N}\mathfrak{A}$ is a tensor category and the restriction of ι to $\mathfrak{N}\mathfrak{A}$ makes it into a locally free tensor category of coherent sheaves on X.

ii) If X has a k-rational point, then $\mathfrak{N}\mathfrak{A}$ is a neutral Tannakian category.

Proof: Part *ii*) is proved following Remark 2.18, p. 137 of [12]. Let x_0 be a k-rational point of X and let $\omega := x_0^* \circ \iota$. Then ω is (Lemma 3) faithful and exact k-linear tensor functor. From the end of the proof of Theorem 2.11, p. 137 in [12], we conclude that $\mathfrak{N}\mathfrak{A}$ is equivalent, via ω, to the category of representations of an affine monoid scheme over k, G , i.e. the spectrum of a k-Hopf algebra without the co-inverse. We claim that $\mathfrak{N}\mathfrak{A}$ is rigid. Let $\rho : G \longrightarrow \mathbb{E}\mathrm{nd}(V)$ be a representation[1] of G and let $\det(\rho) : G \longrightarrow \mathbb{E}\mathrm{nd}(\bigwedge^d V)$ be the determinant representation.

Now, for each one dimensional representation L of G there is a one dimensional representation L^{-1} such that $L \otimes L^{-1} \cong \mathbb{1}$ — in the present case, the only one dimensional representation of G is the trivial one. Hence, ρ factors through $\mathbb{GL}(V)$ and we can form the representation $V^\vee$, which tautologically satisfies the requirements for a dual in the category of representations of G. □

1.2.3 Some examples

Example 9. Let $G = \operatorname{Spec} R$ be an affine group scheme and let $\mathfrak{A}$ be the category $\mathrm{Rep}_k(G)$. Then $\mathfrak{N}\mathfrak{A}$ is neutral Tannakian and the group scheme associated to it via Tannakian duality ([12], Thm. 2.11. p. 130) is the largest prounipotent quotient of G. For example, if G is a diagonal group, then $\mathfrak{N}\mathfrak{A}$ is just the category of direct sums $\mathbb{1}^{\oplus m}$.

[1]Notation: $\mathbb{E}\mathrm{nd}(V)$ is the monoid scheme which associates to each k-algebra the monoid, under composition, of A-linear endomorphisms of $A \otimes_k V$

Example 10 (Monodromy Groups)**.** Let G and H be affine group schemes and let $f : H \longrightarrow G$ be a homomorphism. The *image* group scheme $I := \mathrm{im}(f)$ is the scheme theoretic image of f: it is a closed subscheme $\iota : I \longrightarrow G$ such that $f = \iota \circ f'$ and given any other closed subscheme $j : Y \subseteq G$ with $f : H \longrightarrow G$ factoring through Y, there is a unique $\alpha : I \longrightarrow Y$ such that $j \circ \alpha = \iota$. Because we are dealing with affine schemes, it is easy to see that $\mathrm{im}(f)$ is actually the closed scheme given by the kernel of $f^* : \mathscr{O}(G) \longrightarrow \mathscr{O}(H)$; in particular, it is a quotient group scheme of H. We now give a Tannakian characterization of I, in the case $G = \mathbb{GL}(V)$. That is, we characterize the full subcategory ([12], 2.21, p. 139) $\mathfrak{R} := \mathrm{Rep}_k(I) \subseteq \mathrm{Rep}_k(H)$. Since V is a faithful representation of I, it follows that any object in $\mathfrak{R}$ is a subquotient of some

$$V_{b_1}^{a_1} \oplus \cdots \oplus V_{b_s}^{a_s} =: V_{(b_1,\ldots,b_s)}^{(a_1,\ldots,a_s)}, \qquad V_b^a := V^{\otimes a} \otimes (V^\vee)^{\otimes b}.$$

But by the same proposition of loc.cit, the category $\mathfrak{R}$ is stable under subquotients and hence $\mathfrak{R}$ is the full subcategory of $\mathrm{Rep}_k(H)$ whose objects are subquotients of objects of the form $V_{(b_1,\ldots,b_s)}^{(a_1,\ldots,a_s)}$. This category is denoted by $\langle V \rangle_\otimes$.

Example 11 (The algebraic hull, reviewing [11], 10.24)**.** Let Γ be an abstract group. The category of abstract representations $\mathrm{Rep}_k(\Gamma)$ is certainly neutral Tannakian and thus is equivalent to the category of representations of an affine group scheme Γ^{alg} called the algebraic hull of Γ (even though Γ^{alg} itself is usually not an algebraic group scheme!) A more constructive description of Γ^{alg} is based on the following. Given a representation $\rho : \Gamma \longrightarrow \mathrm{GL}(V) = \mathbb{GL}(V)(k)$, we obtain a closed, reduced subgroup scheme $G \subseteq \mathbb{GL}(V)$ by taking the Zariski closure of $\mathrm{im}(\rho)$ in $\mathbb{GL}(V)$ with its reduced scheme structure. These groups form a projective system and we only have to take the limit.

More precisely: Consider the category $\mathfrak{R}$ whose objects are pairs (G, f) with G a reduced affine algebraic group scheme and $f : \Gamma \longrightarrow G(k)$ a homomorphism of abstract groups such that the Zariski closure of $f(\Gamma)$ is dense. An arrow between (G, f) and (G', f') is just a homomorphism φ of group schemes which preserves f and f', namely $\varphi(k) \circ f = f'$; note that φ is always a quotient homomorphism. We define Γ^{alg} as the projective limit of the objects of $\mathfrak{R}$.

There is a natural homomorphism $\Phi : \Gamma \longrightarrow \Gamma^{\mathrm{alg}}(k) = \varprojlim G(k)$ and this induces a functor (which preserves the underlying vector spaces) $\alpha : \mathrm{Rep}_k(\Gamma^{\mathrm{alg}}) \longrightarrow \mathrm{Rep}_k(\Gamma)$. On the other hand, any representation $\rho : \Gamma \longrightarrow \mathrm{GL}(V)$ will factor as $f : \Gamma \longrightarrow G(k) \subseteq \mathbb{GL}(V)(k)$ for some $(G, f) \in \mathfrak{R}$; this gives a functor $\beta : \mathrm{Rep}_k(\Gamma) \longrightarrow \mathrm{Rep}_k(\Gamma^{\mathrm{alg}})$. Clearly $\alpha \circ \beta$ is naturally equivalent to the identity. To show that $\beta \circ \alpha \cong \mathrm{id}$, observe that any $V \in \mathrm{Rep}_k(\Gamma^{\mathrm{alg}})$ is the representation induced by a reduced algebraic quotient $G \subseteq \mathbb{GL}(V)$ of Γ^{alg}; such quotients come with a natural homomorphism (obtained via Φ) $f : \Gamma \longrightarrow G(k)$ and $\overline{f(\Gamma)} = G$. It is then obvious that $\beta \circ \alpha$ is naturally equivalent to the identity.

Our preferred example of this situation is the category of complex local systems $\mathbf{LS}(X)$ on a complex manifold X. It is isomorphic to the category of representations of the fundamental group $\pi_1(X)$ and hence the Tannakian fundamental group of $\mathbf{LS}(X)$ is $\pi_1(X)^{\mathrm{alg}}$. We note that even if $\pi_1(X)$ is very simple, e.g. $\mathbb{Z}$, then $\pi_1(X)^{\mathrm{alg}}$ is bigger than a first impression might suggest (look at the inclusion $\mathbb{Z} \longrightarrow \mathbb{G}_a(\mathbb{C})$, for example).

Example 12. Nori considered in chapter IV of [34] the category of nilpotent coherent sheaves $\mathfrak{N}\mathrm{coh}(X)$. Note that if X is locally noetherian, then all nilpotent sheaves, being flat $\mathscr{O}_X$-modules, are locally free. This category is indeed neutral Tannakian if X is connected, has a k-rational point and $\mathrm{H}^0(X, \mathscr{O}_X) = k$. It will be denoted by $\mathbf{nilp}(X)$.

1.2.4 Torsors

(Compare [33], §2; [10], section 8; [11], §5)

This section is a reworking of §2 of Nori's beautiful paper [33] to a more abstract setting. The intent is to show how to construct (quite formally) a torsor in a k-abtensor category $\mathfrak{A}$ starting from a functor $\mathrm{Rep}_k(G) \longrightarrow \mathfrak{A}$. When this category is related to a geometric problem, e.g. $\mathfrak{A}$ is the category of modules on some ringed space (topos), then one can follow Nori to show that the functor (under some extra hypothesis) will be obtained as the associated sheaf construction for this torsor.

Ind-categories

Let $\mathfrak{C}$ be a category. We will follow [11], §4 (who follows SGA 4).

Definition 13. *The category* $\mathrm{Ind}(\mathfrak{C})$ *has objects and arrows as follows:*

Objects *Functors* $X : I \longrightarrow \mathfrak{C}$ *where* I *is a small and filtered category. These objects are denoted by* $\varinjlim_i X_i$.

Arrows $\mathrm{Hom}(\varinjlim_i X_i, \varinjlim_j Y_j) := \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathfrak{C}}(X_i, Y_j)$.

Note that $\mathfrak{C}$ can be seen as a full subcategory of $\mathrm{Ind}(\mathfrak{C})$. This definition of $\mathrm{Ind}(\mathfrak{C})$ satisfies the following universal property: If $T : \mathfrak{C} \longrightarrow \mathfrak{D}$ is a functor to a co-complete category $\mathfrak{D}$ (that is, $\mathfrak{D}$ has arbitrary direct limits for small and filtered directed systems) there is a unique functor $\mathrm{Ind}(T) : \mathrm{Ind}(\mathfrak{C}) \longrightarrow \mathfrak{D}$ such that $\mathrm{Ind}(T) \circ \mathrm{inclusion} = T$. We note that the category $\mathrm{Ind}(\mathfrak{C})$ will be abelian if $\mathfrak{C}$ is and the inclusion $\mathfrak{C} \longrightarrow \mathrm{Ind}(\mathfrak{C})$ is full, faithful and exact (SGA4, exp. I, 8.9.9c, p.115 and 8.8.2, p.101).

Let $\otimes : \mathfrak{C} \times \mathfrak{C} \longrightarrow \mathfrak{C}$ be a functor and let $\otimes' : \mathrm{Ind}(\mathfrak{C}) \times \mathrm{Ind}(\mathfrak{C}) \longrightarrow \mathrm{Ind}(\mathfrak{C})$ be the natural extension of $\otimes$ given by composing

$$\begin{array}{ccc} \mathrm{Ind}(\mathfrak{C}) \times \mathrm{Ind}(\mathfrak{C}) & \longrightarrow & \mathrm{Ind}(\mathfrak{C} \times \mathfrak{C}) \\ & & \\ (\varinjlim_i X_i, \varinjlim_j Y_j) & \longmapsto & \varinjlim_{(i,j)} (X \times Y)_{(i,j)} \end{array}$$

with $\mathrm{Ind}(\otimes)$. If $\otimes$ is the tensor product of a tensor structure for $\mathfrak{C}$, then $\otimes'$ is a tensor product for $\mathrm{Ind}(\mathfrak{C})$. If $\mathfrak{C}$ is k-abtensor then so is $\mathrm{Ind}(\mathfrak{C})$.

Example: For G and affine group scheme, $\mathrm{Ind}(\mathrm{Rep}_k(G)) = \mathrm{Rep}'_k(G)$ ([42], Thm. 3.3, p. 24).

Natural construction of universal torsors

Let $(\mathfrak{A}, \otimes)$ be a k-abtensor category and $G = \mathrm{Spec}\, R$ an affine group scheme over k. Assume that we have an exact and k-linear tensor functor

$$\mathscr{L} : \mathrm{Rep}_k(G) \longrightarrow \mathfrak{A}$$

and let $\mathscr{L}'$ denote the natural extension

$$\mathscr{L}' : \mathrm{Rep}'_k(G) \longrightarrow \mathrm{Ind}(\mathfrak{A}).$$

$\mathscr{L}'$ is a k-linear exact tensor functor between k-abtensor categories. We remind the reader that the notions of algebras, co-algebras, Hopf algebras etc. can be easily translated into the more abstract setting of tensor categories.

Let $\mathfrak{G}$ be the category of affine G-schemes with a right G-action. The global sections functor induces a functor $\Gamma : \mathfrak{G} \longrightarrow \mathrm{Rep}'_k(G)^{\mathrm{op}}$ which takes the direct product $S \times T$ to the tensor product $\Gamma(S) \otimes \Gamma(T)$. Let $(R, \rho_l) = \Gamma(G_{\mathrm{left}})$ be the left regular representation — here G_{left} is the affine scheme G with the right G-action $x \cdot g = g^{-1}x$. Consider the two morphisms in $\mathrm{Rep}'_k(G)$:

$$(R, \rho_l) \otimes (R, \rho_l) \longrightarrow (R, \rho_l), \qquad (R, \rho_l) \longrightarrow (R, \rho_l) \otimes (R, \mathrm{id}_R \otimes 1), \tag{1.1}$$

where the first is just multiplication on the ring R and the second is the image of the group multiplication $G_{\mathrm{left}} \times G_{\mathrm{triv}} \longrightarrow G_{\mathrm{left}}$ under Γ (G_{triv} is G with the trivial action). The first map makes (R, ρ_l) into an algebra in the tensor category $\mathrm{Rep}'_k(G)$ and hence the same can be said of

$$\mathscr{B} := \mathscr{L}'((R, \rho_l)).$$

The second map will give $\mathscr{B}$ the co-action of the Hopf algebra $\mathscr{R}_{\mathfrak{A}} := \mathscr{L}'((R, \mathrm{id}_R \otimes 1))$.

Note also that there is an isomorphism

$$\mathscr{B} \otimes \mathscr{B} \longrightarrow \mathscr{B} \otimes \mathscr{R}_{\mathfrak{A}} \tag{1.2}$$

coming from the isomorphism in $\mathfrak{G}$, $G_{\text{left}} \times G_{\text{triv}} \longrightarrow G_{\text{left}} \times G_{\text{left}},\ (g, h) \mapsto (g, gh)$. If on $\mathscr{B} \otimes \mathscr{R}_{\mathfrak{A}}$ the $\mathscr{R}_{\mathfrak{A}}$-co-action is the one provided by the co-action on $\mathscr{R}_{\mathfrak{A}}$ and on $\mathscr{B} \otimes \mathscr{B}$ it is the one provided by the co-action on the second term of the tensor product, then the isomorphism in (1.2) is also equivariant.

If $(\mathfrak{A}, \iota)$ is a locally free tensor category of coherent sheaves on a scheme X/k, by applying the functor ι to the above constructions, we obtain an action (in the category of X-schemes) of the X-group scheme $\operatorname{Spec} \iota(\mathscr{R}_{\mathfrak{A}}) = G \times X$ on the *flat* X-scheme $P := \operatorname{Spec} \iota(\mathscr{B})$. It is not hard to see (from the exactness of ι) that P is actually faithfully flat. Isomorphism (1.2) shows that P is a G-torsor over X.

Another fundamental construction of Nori [33], Lemma 2.7, p. 33

We shall introduce a very clever construction of Nori which will be used to check that certain functors are naturally isomorphic. Consider two functors

$$\nu_1, \nu_2 : \mathrm{Rep}_k(G) \longrightarrow \mathrm{Rep}'_k(G)$$

given by

$$\nu_1 : (V, \rho) \mapsto (V, \rho) \otimes (R, \rho_l), \qquad \nu_2 : (V, \rho) \mapsto (V, \mathrm{id}_V \otimes 1) \otimes (R, \rho_l).$$

These functors are naturally isomorphic via

$$(\mathrm{id}_V \otimes \text{mult.}) \circ (\mathrm{id}_V \otimes \sigma \otimes \mathrm{id}_R) \circ (\rho \otimes \mathrm{id}_R) : \nu_1(V) \longrightarrow \nu_2(V),$$

where $\sigma : R \longrightarrow R$ represents $g \mapsto g^{-1}$. This natural equivalence is the algebraic analogue of the much more intuitive map: Identify $\nu_1(V)$ and $\nu_2(V)$ with the vector space of morphisms $G \longrightarrow V_a$ and let G act on $\nu_1(V)$ by $gf : x \mapsto \rho(g)f(g^{-1}x)$ and on $\nu_2(V)$ by $gf : x \mapsto f(g^{-1}x)$. Then, the composition giving the isomorphism between the R-comodules is $f \mapsto (g \mapsto \rho(g)^{-1}f(g))$.

Nori discovered that ν_i are actually related to a category finer than $\mathrm{Rep}'_k(G)$: $\mathrm{Rep}''_k(G)$. This category has as objects pairs (W, τ_W), where $W \in \mathrm{Rep}'_k(G)$ and τ_W is an arrow in $\mathrm{Rep}'_k(G)$

$$\tau_W : W \longrightarrow W \otimes (R, \mathrm{id}_R \otimes 1).$$

Both ν_i factor through functors $\overline{\nu}_i : \mathrm{Rep}_k(G) \longrightarrow \mathrm{Rep}''_k(G)$ and the natural isomorphism above is in fact a natural isomorphism $\overline{\nu}_1 \Rightarrow \overline{\nu}_2$.

To define $\overline{\nu}_i$, we have to declare what are the arrows (G-equivariant) $\tau_i : \nu_i(V) \longrightarrow \nu_i(V) \otimes (R, \mathrm{id}_R \otimes 1)$.

τ_1 is the R-comodule map $V \otimes R \longrightarrow V \otimes R \otimes R$ associated to the representation $(V, \mathrm{id}_V \otimes 1) \otimes (R, \rho_r)$ (V has the trivial action and R the right regular action!)

τ_2 is the R-comodule map $V \otimes R \longrightarrow V \otimes R \otimes R$ associated to the representation $(V, \rho) \otimes (R, \rho_r)$.

It is immediate to verify that the above natural isomorphism $\nu_1 \Rightarrow \nu_2$ actually comes from a natural isomorphism $\overline{\nu}_1 \Rightarrow \overline{\nu}_2$. □

Example 14. We shall discuss here the main example:

$$\mathscr{L} = \mathscr{L}_{P/G} : \mathrm{Rep}_k(G) \longrightarrow \mathrm{coh}(X)$$

is the associated sheaf functor obtained from the G-torsor $\psi : P \longrightarrow X$. With this example we fill a little omission of [33], that is, the quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B} = \varinjlim \mathscr{L}(V)$ constructed above is *canonically* isomorphic to $\psi_* \mathscr{O}_P$ and this isomorphism induces an isomorphism of the respective torsors. Let $U = \mathrm{Spec}\, A$ be an affine open of X and let $\psi^{-1}U = \mathrm{Spec}\, B$. By definition of torsor, there is an isomorphism of B-algebras $f : B \otimes_A B \longrightarrow B \otimes_k R$, obtained from the isomorphism of P-schemes $P \times G \longrightarrow P \times_X P$, $(\xi, g) \mapsto (\xi, \xi g)$. We note that $\mathrm{Spec}\,(f)$ is G-equivariant if we give $P \times G$ the right G-action $(\xi, h) \cdot g = (\xi g, g^{-1}h)$ and $P \times_X P$ the right G-action $(\xi, \eta) \cdot g = (\xi g, \eta)$. In particular, if $(V, \rho) \subset (R, \rho_l)$ is a finite dimensional subrepresentation of the left regular representation, $B \otimes_k V \subseteq B \otimes_k R$ with its induced G-action is none other that $\mathrm{Hom}(\psi^{-1}U, V_a)$ with the usual G-action $g \cdot \varphi : \xi \mapsto g \cdot \varphi(\xi g)$. We obtain from f^{-1} an equivariant injection $B \otimes_k V \longrightarrow B \otimes_A B$ and taking G-invariants, we obtain an injective homomorphism of A-modules $\mathscr{L}_{P/G}(V)(U) \longrightarrow \mathscr{O}_P(\psi^{-1}U)$. Passing to the limit and observing that this local construction glues, we obtain an *isomorphism* of $\mathscr{O}_X$-algebras $\mathscr{B} \longrightarrow \psi_* \mathscr{O}_P$. In order to see that this induces an isomorphism of G-torsors $P \longrightarrow \mathrm{Spec}\, \mathscr{B}$ one needs to notice the following. f above makes the diagram (think of the associated spectra)

$$\begin{tikzcd}
B \otimes_k R \arrow[r, "\mathrm{id}_B \otimes \Delta"] & (B \otimes_k R) \otimes_k R \\
B \otimes_A B \arrow[u, "f"] \arrow[r, "\mathrm{id}_B \otimes \mu"'] & (B \otimes_A B) \otimes_k R \arrow[u, "f \otimes \mathrm{id}_R"']
\end{tikzcd}$$

commute (where μ is the co-action and Δ is the co-multiplication). Also, the co-action of R on $B \otimes_k R$ (resp. on $B \otimes_A B$) given on the upper row (resp. lower row) commutes with the co-action of R previously defined (resp. idem). From the definition of the co-action of R on $\mathscr{B}$ (induced from the second arrow in (1.1)), we see that the isomorphism $\mathscr{B} \longrightarrow \psi_*\mathscr{O}_P$ also preserves the co-actions and hence induces an isomorphism of the associated torsors.

1.3 F-divided and stratified sheaves

We introduce here the two main categories of coherent sheaves we will be studying. The category of stratified sheaves is the immediate analogue of the category of linear differential equations on a complex manifold. It was introduced by Grothendieck to study integrable connections and the De Rham cohomology in a more abstract setting.

In section 1.3.4 the theorem of Cartier-Katz is recalled. It relates, on a smooth k-scheme, the categories of F-divided sheaves and stratified sheaves and follows the well-known principle of Cartier that integrable connections with zero p-curvature come from Frobenius pull-backs.

Recall that we have fixed a perfect field k of positive characteristic p. The Frobenius automorphism will be denoted by $\varphi : k \longrightarrow k$.

1.3.1 Preliminaries on the Frobenius

On this section we set up standard material on the Frobenius [18] and its relations to representations. These will be used later on to clarify a basic question we address in this work, which is to interpret, in an abstract neutral Tannakian category, the effect of the Frobenius twist of a representation.

Affine setting

Given a vector space V over k, we denote by $V^{(m)}$ the pull back of V via the automorphism $\operatorname{Spec}\varphi^m$ of $\operatorname{Spec} k$. As an abelian group, $V^{(m)}$ is just V, but the map $k \longrightarrow \operatorname{End}(V^{(m)})$ giving its k-module structure is the composition of φ^{-m} with $k \longrightarrow \operatorname{End}(V)$. Note that $V \mapsto V^{(m)}$ is functor and $\operatorname{Hom}(V, W) \longrightarrow$
$\longrightarrow \operatorname{Hom}(V^{(m)}, W^{(m)})$ is φ^m-linear.

Let A be a commutative k-algebra, $f : k \longrightarrow A$ the structural ring homomorphism. Let $A^{(m)}$ be the k-algebra which, as a ring, is just A but has the k-structure given by $f \circ \varphi^{-m}$. $A^{(m)}$ is, of course, the pull back algebra $(\operatorname{Spec}\varphi^m)^*A$.

Let $F^m : A \longrightarrow A$ ($m \geq 0$) be the φ^m-linear ring homomorphism $a \mapsto a^{p^m}$. Hence we have a k-algebra homomorphism $F^m : A^{(m)} \longrightarrow A$, $a \mapsto a^{p^m}$.

A particular case of interest is that of k-Hopf algebras. If R is such an object, it is easily verified that $R^{(m)}$ itself is a k-Hopf algebra and that $F^m : R^{(m)} \longrightarrow R$ is a homomorphism of k-Hopf algebras. Further properties of this particular case are developed below.

Frobenius twist of representations

Let $G = \operatorname{Spec} R$ be an affine group scheme over k and $F : G \longrightarrow G^{(1)} = \operatorname{Spec} R^{(1)}$ the Frobenius homomorphism. Let (V, ρ) be a representation of $G^{(1)}$, where $\rho : V \longrightarrow V \otimes_k R^{(1)}$ is the comodule map. We can define a representation (V', ρ') of G by taking $V' = V^{(-1)}$ and ρ' the composition

$$V^{(-1)} \xrightarrow{\rho^{(-1)}} (V \otimes_k R^{(1)})^{(-1)} \xrightarrow{\text{canonical} \cong} V^{(-1)} \otimes_k R.$$

This functor $\operatorname{Rep}_k(G^{(1)}) \longrightarrow \operatorname{Rep}_k(G)$ is φ^{-1}-linear and is an equivalence of categories (an inverse being the analogous functor with ${}^{(1)}$ instead of ${}^{(-1)}$).

Let (V, ρ) be a representation of G and let $V^{(1)}$ be the associated representation of $G^{(1)}$.

Definition 15. *The Frobenius twist of* (V, ρ) *is the representation of* G *given by* $\operatorname{Res}(F)(V^{(1)})$. *It is denoted* $(V^{(1)}, \rho^{(1)})$.

In more concrete terms, if $\{v_i\}$ is a basis of V in which $G \longrightarrow \mathbb{GL}(V) = \mathbb{GL}(n)$ in given by the matrix $(a_{ij}) \in \mathrm{GL}(R, n)$, then $(V^{(1)}, \rho^{(1)})$ is given by the matrix (a_{ij}^p). Since $\operatorname{Res}(F)$ is k-linear, the functor $(V, \rho) \mapsto (V^{(1)}, \rho^{(1)})$ is φ-linear. In more pictorial form we have a commutative diagram

$$\begin{tikzcd} \operatorname{Rep}_k(G) \arrow[rr, "\text{Frobenius twist}"] \arrow[dr, "(1)"'] & & \operatorname{Rep}_k(G) \\ & \operatorname{Rep}_k(G^{(1)}). \arrow[ur, "\operatorname{Res}(F)"'] & \end{tikzcd} \tag{1.3}$$

We say that a representation V is *F-divisible* if it is isomorphic to the Frobenius twist of some other representation W.

Geometric setting

Let X be a k-scheme. Let $F : X \longrightarrow X$ be the Frobenius morphism, which on the underlying topological space X is the identity and induces the homomorphism $a \mapsto a^p$ on the sheaf of rings $\mathscr{O}_X$. Consider the commutative diagram with a cartesian square

$$\begin{tikzcd} X \arrow[dr, "F_{\mathrm{geom}}"] \arrow[drr, bend left, "F"] \arrow[ddr] & & \\ & X^{(1)} \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ & \operatorname{Spec} k \arrow[r, "\operatorname{Spec} \varphi"'] & \operatorname{Spec} k, \end{tikzcd} \tag{1.4}$$

where the unmarked arrows are the structural morphisms and F_{geom}, the *geometric Frobenius*, is the morphism obtained from the universal property of the fibred product. Since $\operatorname{Spec}\varphi$ is an isomorphism of schemes, so is π and hence we can identify the topological spaces underlying X and $X^{(1)}$; under this identification, the sheaf of k-algebras $\mathscr{O}_{X^{(1)}}$ defining the k-scheme $X^{(1)}$ is $U \mapsto \mathscr{O}_X(U)^{(1)}$.

We observe that $\pi_* : \mathrm{qcoh}(X^{(1)}) \longrightarrow \mathrm{qcoh}(X)$ is a φ^{-1}-linear equivalence of categories. With the obvious modifications, the same is true for higher powers of the Frobenius.

1.3.2 F-divided sheaves

Let X be a scheme over k. F-divided sheaves were defined by Gieseker in [16]. They were then called flat sheaves. The Cartier-Katz Theorem below states that F-divided sheaves can be seen as "differential equations", but it turns out that F-divided sheaves are much easier to handle than stratifications.

Definition 16. *The category of F-divided sheaves on X, denoted by* $\mathbf{Fdiv}(X)$*, is the k-linear category whose:*

Objects are *families $\{\mathscr{E}_i\}_{i\in\mathbb{N}}$ with $\mathscr{E}_i$ a coherent sheaf over $X^{(i)}$, plus isomorphisms $\sigma_i : F_{\mathrm{geom}}^*\mathscr{E}_i \cong \mathscr{E}_{i-1}$.*

Arrows are *projective systems of arrows $\{f_i \in \operatorname{Hom}_{\mathscr{O}_{X^{(i)}}}(\mathscr{E}_i, \mathscr{F}_i);\ \tau_i \circ F_{\mathrm{geom}}^*(f_i) = f_{i-1} \circ \sigma_i\}_{i\in\mathbb{N}}$.*

The termwise tensor product makes $\mathbf{Fdiv}(X)$ into k-linear tensor category with identity object $\mathbb{1} = \{\mathscr{O}_{X^{(i)}}\}_{i\in\mathbb{N}}$. If X is locally noetherian and regular, $F : X \longrightarrow X$ is faithfully flat ([24], Thm. 23.1, p. 179) and then $\mathbf{Fdiv}(X)$ is k-abtensor (Definition 1): (co)kernels are just the termwise (co)kernels. We define duals term by term. Also, $\iota : \{\mathscr{E}_i\}_{i\in\mathbb{N}} \mapsto \mathscr{E}_0$ is a faithful and exact tensor functor into $\mathrm{coh}(X)$ and $\mathscr{E}_0$ is locally free, as was communicated to me by N. Shepherd-Barron.

Lemma 17. *Assume that all local rings in X are regular noetherian rings. If $\{\mathscr{E}_i\}_{i\in\mathbb{N}}$ is an object of* $\mathbf{Fdiv}(X)$*, then $\iota(\{\mathscr{E}_i\}_{i\in\mathbb{N}}) = \mathscr{E}_0$ is locally free. In particular, ι makes* $\mathbf{Fdiv}(X)$ *into a locally free tensor category of coherent sheaves on X (Definition 2).*

Proof: This is a local question, so we can assume $X = \operatorname{Spec} A$ local. Let $M_i = \Gamma(X^{(i)}, \mathscr{E}_i)$ and let

$$A^n \xrightarrow{\alpha} A^m \longrightarrow M_0 \longrightarrow 0$$

be a finite free presentation of M_0 with m minimal. The r'th Fitting ideals $\Phi_r(\alpha)$ are the ideals generated by the $r \times r$ minors of α ($1 \leq r \leq \min(m, n)$). Also, by convention $\Phi_r = (1)$ for $r \leq 0$ and $\Phi_r = 0$ for $r > \min(m, n)$. The module M_0 is free of rank $m - r$ if and only if (see [5], Prop. 1.4.10, p. 22) $\Phi_{r+1}(\alpha) = 0$ and $\Phi_r(\alpha) = (1)$ holds. The Fitting ideals $\Phi_r(\alpha)$ depend only on the isomorphism class of M_0, that is, for another finite free presentation of M_0

$$A^\nu \xrightarrow{\beta} A^\mu \longrightarrow M_0 \longrightarrow 0$$

we have $\Phi_{\mu-r}(\beta) = \Phi_{m-r}(\alpha)$ for all $r \geq 0$.

Also, they base change nicely (loc.cit., p. 21); thus, given a finite free presentation

$$A^\nu \xrightarrow{\beta} A^\mu \longrightarrow M_i \longrightarrow 0$$

$\Phi_{m-r}(\alpha) = \Phi_{\mu-r}((F^i)^*\beta) = F^i(\Phi_{\mu-r}(\beta))A$. If $\Phi_{m-r}(\alpha) \neq (1)$, then it is contained in $\cap_i \mathrm{rad}(A)^{p^i} = 0$ (by Krull's Intersection Theorem). This shows that M_0 is free. □

Corollary 18. *If X is connected, locally noetherian and regular and has a k-rational point x_0 the functor $x_0^* : \{\mathscr{E}_i\}_{i\in\mathbb{N}} \mapsto x_0^*\mathscr{E}_0$ is a fibre functor and $\mathbf{Fdiv}(X)$ is a neutral Tannakian category.*

Proof: First note that $\iota : \{\mathscr{E}_i\} \mapsto \mathscr{E}_0$ is faithful since F is faithfully flat. It is also exact by definition. To show that x_0^* is exact and faithful one uses Lemma 3. Since all the terms $\mathscr{E}_i$ in an F-division are locally free, it follows that the termwise dual determines a rigid tensor category structure on $\mathbf{Fdiv}(X)$. □

Definition 19. *Assume that X/k is connected, locally noetherian and regular with a $x_0 \in X(k)$. The group scheme associated, via Tannakian duality, to the category $\mathbf{Fdiv}(X)$ with the fibre functor x_0^* is denoted $\Pi^{\mathbf{Fdiv}}(X, x_0)$.*

Since X and x_0 will remain fixed, we abuse notation and denote $\Pi^{\mathbf{Fdiv}}(X, x_0)$ by $\Pi^{\mathbf{Fdiv}}$.

For later reference, we will state and prove a lemma of Gieseker [16], prop. 1.7, p. 6. To simplify notation, we work with the absolute Frobenius.

Lemma 20. *Let X/k be as before. If $\{\mathscr{E}_i\}_{i\in\mathbb{N}}$ and $\{\mathscr{F}_i\}_{i\in\mathbb{N}}$ are two objects of $\mathbf{Fdiv}(X)$ such each $\mathscr{E}_i$ is isomorphic as an $\mathscr{O}_X$-module to $\mathscr{F}_i$ and $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}_i, \mathscr{E}_i)$ is finite dimensional, then $\{\mathscr{E}_i\}_{i\in\mathbb{N}} \cong \{\mathscr{F}_i\}_{i\in\mathbb{N}}$ in $\mathbf{Fdiv}(X)$.*

Proof: Because of the finite dimensionality assumption, there exists an $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$ $F^* : \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}_{i+1}, \mathscr{E}_{i+1}) \longrightarrow \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}_i, \mathscr{E}_i)$ is bijective since it is always injective (and φ-linear). Take $i \geq i_0$. Let $f_i \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}_i, \mathscr{F}_i)$ and $f_{i+1} \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}_{i+1}, \mathscr{F}_{i+1})$ be isomorphisms. There exists an automorphism $g_{i+1} \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}_{i+1}, \mathscr{E}_{i+1})$ such that $F^*(g_{i+1}) = F^*(f_{i+1})^{-1} \circ f_i \circ F^*(f_{i+1})$ and hence $f'_{i+1} := f_{i+1} \circ g_{i+1} \circ f_{i+1}^{-1}$ is an isomorphism and is taken by F^* to f_i. By induction we can construct an isomorphism between $\{\mathscr{E}_i\}$ and $\{\mathscr{F}_i\}$ in $\mathbf{Fdiv}(X)$. □

1.3.3 Stratified sheaves

Here we define stratified sheaves on smooth schemes. They are the correct analogue of linear differential equations over a complex manifold since over a field of characteristic zero, the sheaf of differential operators is generated by the vector fields ([3], Theorem 2.15).

Let $\mathscr{D}_X$ be the sheaf of k-linear differential operators on a smooth scheme X/k.

Definition 21. *A stratified sheaf on X, $(\mathscr{E}, \nabla)$, is the data of a coherent sheaf $\mathscr{E}$ on X and a $\mathscr{O}_X$-linear ring homomorphism $\nabla : \mathscr{D}_X \longrightarrow \mathscr{E}nd_k(\mathscr{E})$, called a stratification of $\mathscr{E}$. The category of stratified sheaves on X, $\mathbf{str}(X)$, has stratified sheaves as objects and $\mathscr{D}_X$-modules homomorphisms as arrows. The arrows are also called horizontal maps.*

We observe that a homomorphism ∇ as above takes $\mathscr{D}_X^{\leq m}$ into $\mathscr{D}_X^{\leq m}(\mathscr{E})$. The reason for this is the following criterion for differential operators given in EGA IV$_4$, 16.8.8, p. 42. A k-linear endomorphism of $\mathscr{E}$ is a differential operator of order $\leq m$ if and only if for every open set $U \subseteq X$ and every $a \in \mathscr{O}_X$, the operator $\mathrm{ad}_a(D) \in \mathscr{E}nd_k(\mathscr{E}|U)$ defined by

$$\mathrm{ad}_a(D)(s) = aD(s) - D(as), \quad s \in \mathscr{E}(V),\ V \subseteq U$$

is a differential operator of order $\leq m - 1$ on U.

The category $\mathbf{str}(X)$ is k-linear and abelian and has a tensor product, which makes the functor "forget the stratification" a tensor functor. In order to elaborate on the stratification of a tensor product, we have to use Grothendieck's notion of stratification. Let $\mathscr{P}$ denote the sheaf $\mathscr{O}_X \otimes_k \mathscr{O}_X$ on X and let $\mathscr{I}$ denote the ideal sheaf of $\mathscr{P}$ given as the kernel of the natural multiplication map $\mathscr{P} \longrightarrow \mathscr{O}_X$. Write $\mathscr{P}^n$ for the sheaf of k-algebras $\mathscr{P}/\mathscr{I}^{n+1}$; note that $\mathscr{P}^n$ is also a sheaf of $\mathscr{O}_X$-algebras in two distinct ways: via $d_0 : a \mapsto a \otimes 1$ and $d_1 : a \mapsto 1 \otimes a$. Given a sheaf $\mathscr{E}$ of $\mathscr{O}_X$-modules on X, Proposition 2.11 of [3] states that the above definition of a stratification on $\mathscr{E}$ is the same as the data of $\mathscr{P}^n$-linear maps

$$\varepsilon_n : \mathscr{E} \otimes_{\mathscr{O}_X, d_1} \mathscr{P}^n \longrightarrow \mathscr{E} \otimes_{\mathscr{O}, d_0} \mathscr{P}^n$$

satisfying a set of conditions which we leave to Definition 2.10 of [3]. It is also instructive to observe that if we let $\mathfrak{X}$ be the formal completion of $X \times X$ along the diagonal and denote by $p_1, p_2 : \mathfrak{X} \longrightarrow X$ the natural projections, then the ε_n above are the data of an isomorphism

$$\varepsilon : p_2^* \mathscr{E} \longrightarrow p_1^* \mathscr{E},$$

which satisfies the cocycle condition $p_{13}^*(\varepsilon) = p_{12}^*(\varepsilon) \circ p_{23}^*(\varepsilon)$.

Hence, given two sheaves $\mathscr{E}$ and $\mathscr{F}$ with stratifications $\{\varepsilon_n\}$ and $\{\delta_n\}$ respectively, we obtain the stratification on the tensor product $\mathscr{E} \otimes \mathscr{F}$ by defining the $\mathscr{P}^n$-linear maps

$$\varphi_n : (\mathscr{E} \otimes \mathscr{F}) \otimes_{\mathscr{O}_X, d_1} \mathscr{P}^n \longrightarrow (\mathscr{E} \otimes \mathscr{F}) \otimes_{\mathscr{O}_X, d_0} \mathscr{P}^n$$

as being the obvious ones.

Recall that the action of a differential operator $D \in \mathscr{H}om_{\mathscr{O}_X}(\mathscr{P}^n, \mathscr{O}_X)$ of order $\leq n$ on a stratified sheaf $\{\mathscr{E}, \varepsilon_i\}$ is given by the composition

$$\mathscr{E} \longrightarrow \mathscr{E} \otimes_{\mathscr{O}_X, d_1} \mathscr{P}^n \xrightarrow{\varepsilon_n} \mathscr{E} \otimes_{\mathscr{O}_X, d_0} \mathscr{P}^n \xrightarrow{\mathrm{id}_{\mathscr{E}} \otimes D} \mathscr{E}$$

(see [3], Prop. 2.11 on page 2.14). In fact, the coherence assumption on $\mathscr{E}$ and $\mathscr{F}$ can be dropped, since the result of loc.cit. holds for arbitrary $\mathscr{O}_X$-modules. In this way, we obtain a stratification on the tensor product of any two $\mathscr{O}_X$-modules with stratifications.

Using the definition of tensor product stratification given in terms of the transition homomorphisms ε_n, we have the convenient result:

Lemma 22. *Given stratified sheaves (not assumed to be coherent, only $\mathscr{O}_X$-modules) $(\mathscr{E}, \nabla)$ and $(\overline{\mathscr{E}}, \overline{\nabla})$ and an open set $U \subseteq X$ with etale coordinates $(x_1, \ldots, x_n) : U \longrightarrow \mathbb{A}_k^n$, the action of D_q is*

$$\nabla_{\mathscr{E} \otimes \overline{\mathscr{E}}}(D_q)(e \otimes \overline{e}) = \sum_{q'+q''=q} \nabla(D_{q'})(e) \otimes \overline{\nabla}(D_{q''})(\overline{e}), \quad e \in \mathscr{E}(U), \overline{e} \in \overline{\mathscr{E}}(U). \tag{1.5}$$

$\square$

Returning to the case $\mathscr{E}$ coherent, the same formalism allows us to define a stratification on the dual $\mathscr{E}^\vee$: since $\mathscr{P}^n$ is a flat $\mathscr{O}_X$-algebra ([3], Prop. 2.2), we get from the $\mathscr{P}^n$-linear maps ε_n other $\mathscr{P}^n$-linear maps $\varepsilon_n^* : \mathscr{E}^\vee \otimes_{d_1} \mathscr{P}^n \longrightarrow \mathscr{E}^\vee \otimes_{d_0} \mathscr{P}^n$ satisfying the cocycle condition.

From [3], Prop. 2.16, follows that every stratified sheaf is locally free. Hence, given a k-rational point $x_0 \in X(k)$, the functor $x_0^* : \mathbf{str}(X) \longrightarrow (k - \mathrm{mod})$ turns $\mathbf{str}(X)$ into a neutral Tannakian category (Lemma 3).

Definition 23. *The group scheme associated to the Tannakian category $\mathbf{str}(X)$ via the fibre functor x_0^* is denoted $\Pi^{\mathbf{str}}(X, x_0)$.*

1.3.4 The Cartier-Katz Theorem

This theorem was proved by Katz and appeared in Gieseker, [16], Theorem 1.3, p. 4. It says that, *over a smooth X, the categories of F-divided sheaves and stratified* *sheaves are equivalent* and gives an explicit construction of the equivalence. We describe here this construction. See loc.cit for the proofs. The same construction will be used another time in (1.4.3).

Let $(\mathscr{E}, \nabla)$ be a stratified sheaf and let $\mathscr{E}_i$ be the sheaf

$$U \mapsto \{e \in \mathscr{E}(U);\ \nabla(D)(e_x) = 0, \forall D \in \mathscr{D}_{X,x}^{+,<p^i}, \forall x \in U\}.$$

Let $\mathscr{O}_{(i)}$ denote the sheaf $U \mapsto \mathscr{O}_X(U)^{(i)}$ (see section 1.3.1) on the topological space of X — this sheaf defines $X^{(i)}$. The geometric Frobenius $F^i : \mathscr{O}_{(i)} \longrightarrow \mathscr{O}_X$ makes $\mathscr{E}_i$ into a sheaf of $\mathscr{O}_{(i)}$-modules and we have a natural $\mathscr{O}_{(i)}$-linear map $\mathscr{E}_{i+1} \otimes_{\mathscr{O}_{(i+1)}} \mathscr{O}_{(i)} \longrightarrow \mathscr{E}_i$. This is an isomorphism and thus we have constructed one side of the equivalence $\mathbf{str}(X) \longrightarrow \mathbf{Fdiv}(X)$. The main point in the proof is, of course, showing that $\mathscr{E}_{i+1} \otimes_{\mathscr{O}_{(i+1)}} \mathscr{O}_{(i)} \longrightarrow \mathscr{E}_i$ is really an isomorphism. This is an ingenious application of Cartier's original result (as exposed in [19]): working locally, we give $\mathscr{E}_i$ a *connection* by letting $\frac{\partial}{\partial x_j}$ act as $\nabla(D_{e_j})$, where $e_j \in \mathbb{N}^n$ has p^i on the jth coordinate and zero on the remaining.

The inverse equivalence is given as follows. Let $\{\mathscr{E}_i\}_{i \in \mathbb{N}}$ be an F-divided sheaf. We can define the stratification ∇ on $\mathscr{E}_0$ by letting $\mathscr{E}_i$ be the sheaf of operators annihilated by $\mathscr{D}_X^{+,<p^i}$. Precisely, let $e \in \mathscr{E}_0(U)$ be of the form $\sum_\nu f_\nu \otimes e_\nu$ where $f_\nu \in \mathscr{O}_{(i)}(U)$ and $e_\nu \in \mathscr{E}_i(U)$. Then, for a $D \in \mathscr{D}_X^{+,<p^i}(U)$,

$$\nabla(D)e = \sum_\nu D(f_\nu) \otimes e_\nu.$$

Because the $\mathscr{E}_i$ are locally free, this is well defined and gives the equivalence between $\mathbf{str}(X)$ and $\mathbf{Fdiv}(X)$ (we remark that Gieseker [16] forgot to mention that F-divided sheaves are locally free, so there is a little ambiguity – corrected by local freeness – in loc.cit., compare *i*) of Proposition 31). It is immediate to check that these functors preserve the tensor products (using formula (1.5)). That the identity object is taken to an identity object by the functor $\mathbf{Fdiv} \longrightarrow \mathbf{str}$ follows easily from Lemma 24 below.

Scholium: When is an element killed by all the derivations a p-power? We do not know the answer to this question in general, but provided we know that the completions in question are nice enough, then we can shed some more light on this question. Let A be a noetherian regular local k-algebra and let $S \subseteq \mathrm{Der}_k(A)$. We want to look at rings A which are of a geometric nature in the following sense. The completion $\hat{A}$ is isomorphic to $K[[x_1, \ldots, x_n]]$ where K/k is a finite separable extension of k (as all field extensions of k are). Assume that all $\sigma \in S$ map the maximal ideal into itself and hence induce continuous derivations on $\hat{A}$. Assume that the canonical

$\frac{\partial}{\partial x_i}$ are among these induced derivations. Then an element $f \in A$ which is killed by all $\partial \in S$ will be a p-power in $\hat{A}$. So the question is: *what is $\hat{A}^p \cap A$?*

Lemma 24. *Assume furthermore that A is a G-ring. Then $\hat{A}^p \cap A = A^p$.*

Proof: This Lemma is an exercise once the notion of G-ring is brought up. We learned this from a paper of Z. Robinson on p-powers and of rigid analytic functions (to appear in J. Number Theory 116 (2006), no. 2, 474–482). We reproduce the proof given there. Assume the existence of $g \in \hat{A}^p \cap A - A^p$ and let R be the finite A-algebra $A[t]/(t^p - g)$ which naturally *embeds* in $\hat{A}$ (A is an UFD by Auslander-Buchsbaum, [24], Thm. 20.3, p. 163). Note that R is also a G-ring (it is a hard theorem of Grothendieck that every algebra of finite type over a G-ring is a G-ring). As all the localizations $R_{\mathfrak{p}}$ are reduced so are all the completions $\hat{R}_{\mathfrak{p}}$ ([24], Cor. to Thm 23.9, p. 184). The completion of R with respect to the the rad(A)-adic topology is

$$\hat{A}[t]/(t-\gamma)^p\hat{A}[t] = \prod_{\mathfrak{m}\in\mathrm{Max}(R)} \hat{R}_{\mathfrak{m}}$$

and hence some $\hat{R}_{\mathfrak{m}}$ is not reduced, which is a contradiction. It follows that $A^p = \hat{A}^p \cap A$ □

From now on, if X is smooth, we shall use the terms stratified and F-divided interchangeably, as well as the notations $\mathbf{str}(X)$ *and* $\mathbf{Fdiv}(X)$. *We also abandon the distinction between geometric and absolute Frobenius when dealing with F-divided sheaves*

1.4 Stratifications and F-divisions on torsors

We will introduce the concepts of a stratification and an F-division on a torsor and derive the analogous Cartier-Katz Theorem on stratifications and Frobenius pullbacks. The differential side of this formalism is well known from differential geometry (connections on principal bundles); unfortunately, none of the available references is suitable for our purposes. In fact, this extra care pays off, since a minor subtlety appears: the data of a section of the canonical map from the Atiyah sheaf to the tangent sheaf does not seem to be enough information to give the right definition of a stratification. Also, we are interested in the interplay (see the next paragraph) between the differential and the F-divided manifestations and hence the presentation below seems reasonable. The discussion of the aforementioned topics will occupy sections 1.4.1—1.4.3.

Given an F-divided sheaf $\{\mathscr{E}_i\}_{i\in\mathbb{N}}$ we can consider the shifted F-divided sheaf $\{F^*\mathscr{E}_i\}$ — the functor obtained from this association is an equivalence and we are interested in the group-theoretic counterpart of this equivalence. One is immediately led to believe that this shift is just the Frobenius twist of representations (section 1.3.1) and one of the goals of the study to follow is to establish this claim. The idea behind is quite simple: the Frobenius twist is taken under the associate sheaf functor (for some torsor) to the Frobenius pull-back of the sheaf. This is discussed in section 1.4.4, where the first structure result for $\Pi^{\mathbf{Fdiv}}$ is given (Theorem 34).

1.4.1 Stratifications

From section 1.4.1 to 1.4.3, X/k is smooth and connected, $G = \operatorname{Spec} R$ is an affine group scheme and $\psi : P \longrightarrow X$ is a G-torsor.

The action of G on P induces (and in fact is given by) an $\mathscr{O}_X$-linear homomorphism

$$\mu : \psi_*\mathscr{O}_P \longrightarrow \psi_*\mathscr{O}_P \otimes_k R = (\psi_*\mathscr{O}_P) \otimes_{\mathscr{O}_X} (\mathscr{O}_X \otimes_k R),$$

which is a co-action of the $\mathscr{O}_X$-Hopf algebra $\mathscr{O}_X \otimes_k R$ on $\psi_*\mathscr{O}_P$. Consider the $\mathscr{O}_X$-linear homomorphism $\theta : \mathscr{E}nd_k(\psi_*\mathscr{O}_P) \longrightarrow \mathscr{E}nd_k(\psi_*\mathscr{O}_P, \psi_*\mathscr{O}_P \otimes_k R)$ which over an open $U \subseteq X$ is given by

$$\begin{aligned} \mathscr{E}nd_k(\psi_*\mathscr{O}_P|U) &\longrightarrow \mathscr{E}nd_k(\psi_*\mathscr{O}_P|U, \psi_*\mathscr{O}_P|U \otimes_k R) \\ T &\longmapsto (T \otimes_k \mathrm{id}_R) \circ \mu(U) - \mu(U) \circ T. \end{aligned}$$

The kernel of θ is an $\mathscr{O}_X$-algebra called the sheaf of invariant operators of P.

Let $\mathscr{D}_X(\psi_*\mathscr{O}_P)$ be the sheaf of k-linear differential operators on $\psi_*\mathscr{O}_P$ (as a module over $\mathscr{O}_X$) and let the *sheaf of invariant differential operators*, $\mathscr{I}\mathscr{D}_X(P)$, be the intersection of $\mathscr{D}_X(\psi_*\mathscr{O}_P)$ with the sheaf of invariant operators on P. Note that invariant operators send $\mathscr{O}_X \subseteq \psi_*\mathscr{O}_P$ into itself and hence there is a natural $\mathscr{O}_X$-linear homomorphism

$$\mathscr{I}\mathscr{D}_X(P) \longrightarrow \mathscr{D}_X. \tag{1.6}$$

Definition 25. *A stratification on P is an $\mathscr{O}_X$-algebra homomorphism*

$$\nabla : \mathscr{D}_X \longrightarrow \mathscr{I}\mathscr{D}_X(P)$$

such that:

1. *Composing ∇ with the canonical homomorphism (1.6) gives the identity of $\mathscr{D}_X$.*
2. *If we give $\psi_*(\mathscr{O}_P) \otimes_{\mathscr{O}_X} \psi_*(\mathscr{O}_P)$ the tensor product stratification, then multiplication is horizontal.*

Remark: Condition 2 of the above definition is technical. We were not able to find an adequate substitute for the Atiyah sheaf of P in the case of stratifications (that is, an $\mathscr{O}_X$-module such that the existence of a section already gives a stratification on the torsor; see section 2.2.2).

1.4.2 Factorizations of $\mathscr{L}_{P/G}$ and stratifications

We wish to prove the analogous of Prop. 2.9, p. 34 of [33] for the case of stratifications (see Proposition 26 below).

Assume that:

(FCT) $\mathscr{L}_{P/G}$ is the composition of a (exact) tensor functor $\mathscr{L} : \mathrm{Rep}_k(G) \longrightarrow \mathbf{str}(X)$ with the natural inclusion.

From section 1.2.4, we obtain an algebra $\mathscr{B}$ on the tensor category $\mathrm{Ind}(\mathbf{str}(X))$ which has a co-action of the Hopf algebra $\mathscr{R}_{\mathbf{str}(X)}$ (see section 1.2.4 for notation):

$$\mu : \mathscr{B} \longrightarrow \mathscr{B} \otimes \mathscr{R}_{\mathbf{str}(X)}.$$

Let ∇ denote the stratification of the $\mathscr{O}_X$*-module* $\mathscr{B}$. Since μ is horizontal (recall that $\mathscr{R} = \mathscr{O}_X \otimes_k R$ has the trivial stratification) follows that ∇ takes values in $\mathscr{I}\mathscr{D}_X(P)$. Here we are identifying the torsors $\mathrm{Spec}\,\mathscr{B}$ and P using Example 14.

Moreover, because $\mathscr{B}$ is an algebra in $\mathrm{Ind}(\mathbf{str}(X))$, condition 2. of Definition 25 is verified. Condition 1. is also verified because $\mathscr{L}$ takes the trivial representation to the trivial module with the canonical stratification. So ∇ is a stratification of the torsor P.

Let $V \in \mathrm{Rep}_k(G)$. The sheaf $\mathscr{L}_{P/G}(V)$ has two stratifications: The one coming from (FCT), denoted $\overline{\nabla}^V$, and the one associated to the stratification ∇, denoted ∇^V. Let us describe this last one more closely. Take $U = \mathrm{Spec}\,A$ and affine open subset of X and let $\psi^{-1}U = \mathrm{Spec}\,B = \mathscr{B}(U)$. Let $\{v_i\}$ be a basis of V, $s = \sum_i b_i \otimes v_i \in (B \otimes_k V)^G$ a section of $\mathscr{L}_{P/G}(V)$ over U and $D \in \mathscr{D}_X(U)$ a differential operator. Then

$$\nabla^V(D)s := \sum_i \nabla(D) b_i \otimes v_i. \tag{1.7}$$

This is obviously independent of the basis $\{v_i\}$ chosen and gives a factorization (FCT) starting from a stratification on the torsor.

Proposition 26. *i) The functors $\mathscr{L}$ and $V \mapsto (\mathscr{L}_{P/G}(V), \nabla^V)$ are naturally isomorphic tensor functors.*

ii) There is a bijection between isomorphism classes of stratifications on P and isomorphism classes of factorizations of $\mathscr{L}_{P/G}$ as (FCT) above.

Proof: It is obvious that *ii*) follows from *i*), on whose proof we now concentrate. This follows from Nori's fundamental construction (see section 1.2.4), but we will repeat the important steps in the proof for the convenience of the reader. Consider the R-comodules $W_1 := (V, \rho) \otimes (R, \rho_l)$ and $W_2 := (V, \mathrm{id}_V \otimes 1) \otimes (R, \rho_l)$. They are canonically isomorphic via the composition

$$(\mathrm{id}_V \otimes \mathrm{mult.}) \circ (\mathrm{id}_V \otimes \sigma \otimes \mathrm{id}_R) \circ (\rho \otimes \mathrm{id}_R) : W_1 \longrightarrow W_2.$$

If $\mathscr{L}'$ denotes the natural extension of $\mathscr{L}$ to $\mathrm{Rep}'_k(G)$, then the quasi-coherent $\mathscr{O}_X$-modules with stratifications

$$\mathscr{L}'(W_1) = \mathscr{L}(V) \otimes_{\mathscr{O}_X} \mathscr{B} \text{ (tensor product stratification)}$$

and

$$\mathscr{L}'(W_2) = V \otimes_k \mathscr{B} \cong (\mathscr{B}^{\oplus \dim V}, \nabla^{\oplus \dim V})$$

are horizontally isomorphic. This isomorphism is natural because it is none other than the composition of $\mathscr{L}'$ with the natural transformation between the functors

$$\begin{array}{ccc} \nu_1 : \mathrm{Rep}_k(G) \longrightarrow \mathrm{Rep}'_k(G) & \qquad & \nu_2 : \mathrm{Rep}_k(G) \longrightarrow \mathrm{Rep}'_k(G) \\ V \longrightarrow V \otimes (R, \rho_l) & \qquad & V \longrightarrow (V, \mathrm{id}_V \otimes 1) \otimes (R, \rho_l). \end{array} \tag{1.8}$$

Note that $\mathscr{L}'(W_1)$ and $\mathscr{L}'(W_2)$ are G-sheaves; the action of G on $\mathscr{L}'(W_1)$ is via $\mathscr{B}$ and the action on $\mathscr{L}'(W_2)$ is the tensor product of the action on each factor (note that this description immediately shows the functoriality of the G-sheaf structure). In [33], Lemma 2.8, p. 34, it is also proved that the isomorphism above preserves the structure of G-sheaves on them. Hence, if $\sum_i b_i \otimes v_i \in (B \otimes_k V)^G$ corresponds to $s \otimes 1 \in (\mathscr{L}(V) \otimes_{\mathscr{O}_X} \mathscr{B})^G(U) = \mathscr{L}(V)(U)$, then $\overline{\nabla}^V(D)(s) \otimes 1$ corresponds to $\sum_i \nabla(D) b_i \otimes v_i$ and we obtain the required natural isomorphism in the stratified category. $\square$

Let X be a smooth and connected scheme with a k-rational point x_0. Let $\omega := x_0^*$ be a fibre functor for $\mathbf{str}(X)$ giving an equivalence of $\mathbf{str}(X)$ with $\mathrm{Rep}_k(\Pi^{\mathbf{str}})$ (1.3.3). The set $\mathrm{Hom}(\Pi^{\mathbf{str}}, ?)$ is the solution to a classification problem. Let $\mathscr{L} : \mathrm{Rep}_k(\Pi^{\mathbf{str}}) \longrightarrow \mathbf{str}(X)$ be an inverse equivalence to ω preserving the fibre functors and let $X^{\mathbf{str}}$ be the $\Pi^{\mathbf{str}}$-torsor over X with a stratification $\nabla_{\mathbf{str}}$ obtained from the factorization (FCT). This particular choice of $\mathscr{L}$ gives a k-rational point $x_{\mathbf{str}}$ above x_0.

A pointed stratified torsor P over X is a stratified torsor with a k-rational point above x_0. Let $\mathrm{StrTors}^*(G/X)$ denote the set of all pointed stratified G-torsors over X (modulo isomorphisms). Proceeding as in [33], Prop. 3.1, p. 40, we obtain

Theorem 27. *The map from* $\mathrm{Hom}(\Pi^{\mathbf{str}}, G)$ *to* $\mathrm{StrTors}^*(G/X)$ *which associates to* $\theta : \Pi^{\mathbf{str}} \longrightarrow G$ *the pointed torsor with stratification* $X^{\mathbf{str}} \times^{\Pi^{\mathbf{str}}} G$ *is a bijection.* $\square$

1.4.3 The Cartier-Katz Theorem for torsors

The goal of this section is to establish

> *an equivalence between the category of torsors with a stratification and the category of F-divided torsors (definition below) – or the Cartier-Katz Theorem for torsors.*

The functors allowing the equivalence are constructed below (Proposition 29 and i) of Proposition 31) — they are just the obvious analogues of Katz's original construction. That the composition of these functors is naturally the identity, follows from part ii) of Proposition 29 and part iv) of Proposition 31.

Definition 28. *An F-division of a G-torsor P is a family of G-torsors $\psi_i : P_i \longrightarrow X^{(i)}$, with $P_0 = P$, and isomorphisms of G-torsors between P_i and the geometric Frobenius pull-back of P_{i+1}. If an F-division $\{P_i\}$ of P is fixed, we call it F-divided.*

A morphism between F-divided torsors $\{P_i\}$ and $\{Q_i\}$ is a sequence of morphisms of G-torsors $f_i : P_i \longrightarrow Q_i$ such that $F^*_{\mathrm{geom}}(f_{i+1}) = f_i$.

Stratifications to F-divisions

We will use the notation of section 1.3.1. Let $\mathscr{O}_{(i)}$ denote the sheaf of k-algebras on the topological space X giving it the k-scheme structure of $X^{(i)}$. Because X is smooth over k and hence the Frobenius is faithfully flat, we can also identify $\mathscr{O}_{(i)}$ with the sheaf of k-subalgebras of $\mathscr{O}_X$ given by the local p^i-powers. On what follows we will make no notational distinction between the geometric and absolute Frobenius.

Consider a stratification $\nabla : \mathscr{D}_X \longrightarrow \mathscr{I}\mathscr{D}_X(P)$ of P and let $\mathscr{B}_i$ be the sheaf

$$U \mapsto \{f \in \mathscr{O}_P(\psi^{-1}U);\ \nabla(D)f_x = 0\ \forall D \in \mathscr{D}_{X,x}^{+,<p^i}, \forall x \in U\}.$$

We remind the reader that $\mathscr{D}_X^{+,<m}$ is the sheaf of differential operators of order $< m$ which annihilate 1. Clearly $\mathscr{B}_i$ is a sheaf of $\mathscr{O}_{(i)}$-modules: For $a \in \mathscr{O}_{(i)}(U)$ and $b \in \mathscr{B}_i(U)$, $a \cdot b := b \cdot F^i a$. Moreover, from formula (1.5) it actually follows that multiplication of sections of $\mathscr{B}_i$ is still a section of $\mathscr{B}_i$, i.e. $\mathscr{B}_i$ is an $\mathscr{O}_{(i)}$-algebra. Because $\mathscr{D}_X^{\leq m}$ is coherent and F^i is finite, follows that $\mathscr{B}_i$ is quasi-coherent.

Proposition 29. *i) The natural homomorphism of $\mathscr{O}_{(i)}$-algebras $\mathscr{B}_{i+1}\otimes_{\mathscr{O}_{(i+1)},F}\mathscr{O}_{(i)} \longrightarrow \mathscr{B}_i$ obtained from the inclusion $\mathscr{B}_{i+1} \subseteq \mathscr{B}_{i+1}$ is an isomorphism.*

ii) Let U be an open subset of X and $\varphi \in \mathscr{B}_0(U)$ correspond, under item i), to $f \otimes a \in (\mathscr{B}_i \otimes_{\mathscr{O}_{(i)}} \mathscr{O}_X)(U)$. Then, for any $D \in \mathscr{D}_X^{+,<p^i}(U)$, $\nabla(D)(\varphi)$ corresponds to $f \otimes\ D(a)$.

iii) Let ∇ be a stratification of P and let $\mathscr{B}_i$ be the $\mathscr{O}_{(i)}$-algebras of sections of $\psi_*\mathscr{O}_P$ annihilated by $\mathscr{D}_X^{+,<p^i}$. Then $\{\operatorname{Spec}\mathscr{B}_i\}_{i\in\mathbb{N}}$ is an F-division of P.

Proof: The proof of $i)$ is very much the same as that of theorem 1.3, p. 4 of [16]. The idea is to use Cartier's Theorem ([19], 5.1, p. 190 — which is valid without the coherence assumption in [16]) to obtain from ∇ an integrable connection ∇_i on $\mathscr{B}_i$ of p-curvature zero which will have $\mathscr{B}_{i+1}$ as its sheaf of horizontal sections and the result follows from that of Cartier. The construction of ∇_i is very natural if we work locally (this is a local problem, of course). Just let $\nabla_i(\partial/\partial x_j)$ act as $\nabla(D_{(0,\dots,p^i,\dots,0)})$. The details are checked in loc.cit.

$ii)$ Standard.

$iii)$ Because $\nabla(D)$ is an invariant operator, follows that the co-action $\mu : \mathscr{B}_0 \longrightarrow \mathscr{B}_0\otimes_k R$ induces co-actions on each $\mathscr{B}_i$. Hence, $\operatorname{Spec}\mathscr{B}_i$ is an $X^{(i)}$-scheme with a right G-action which makes the structural morphism to $X^{(i)}$ equivariant ($X^{(i)}$ with the trivial action). From $i)$, follows that the pull-back of $\operatorname{Spec}\mathscr{B}_i$ through Frobenius is isomorphic to $\operatorname{Spec}\mathscr{B}_{i-1}$ and it is immediate to verify that this isomorphism preserves the actions of G. Using Lemma 30 below, it follows that $\operatorname{Spec}\mathscr{B}_i$ is a G-torsor.

Lemma 30. *Let $f : V \longrightarrow W$ be a faithfully flat and quasi-compact morphism of schemes. Let $\alpha : Q \longrightarrow W$ be an affine W-scheme with a G-action such that, giving W the trivial action, α becomes equivariant. If f^*Q is a G-torsor, then so is Q.*

Proof: α is faithfully flat because f and the projection $f^*Q \longrightarrow V$ are so. The morphism $f^*Q \times G \longrightarrow f^*Q \times_V f^*Q, (q, g) \mapsto (q, qg)$ is the f-pull-back of the analogous morphism for Q. Hence it is an isomorphism by fpqc descent. □

F-divisions to stratifications

Let $\{P_i\}_{i\in\mathbb{N}}$ be an F-division of P and let $\mathscr{B}_i$ denote the quasi-coherent $\mathscr{O}_{(i)}$-algebras $\psi_{i*}\mathscr{O}_{P_i}$. We shall define a stratification of P using the $\mathscr{B}_i$. Let $U \subseteq X$ be an open set, $D \in \mathscr{D}_X^{<p^i}(U)$ and $\sum_\nu f_\nu \otimes a_\nu \in (\mathscr{B}_i \otimes_{\mathscr{O}_{(i)}} \mathscr{O}_X)(U)$. Regarding $\sum_\nu f_\nu \otimes a_\nu$ as an element of $\mathscr{B}_0(U)$, we define

$$\nabla(D)\sum_\nu f_\nu \otimes a_\nu := \sum_\nu f_\nu \otimes D(a_\nu). \tag{1.9}$$

Proposition 31. *i) The definition (1.9) is independent of the choices made and gives a stratification of the quasi-coherent $\mathscr{O}_X$-module $\mathscr{B}_0$.*

ii) Let $m : \mathscr{B}_0 \otimes_{\mathscr{O}_X} \mathscr{B}_0 \longrightarrow \mathscr{B}_0$ be the multiplication of $\mathscr{B}_0$. If we let $\mathscr{B}_0 \otimes_{\mathscr{O}_X} \mathscr{B}_0$ have the tensor product stratification, then m is horizontal.

iii) $\nabla : \mathscr{D}_X \longrightarrow \mathscr{E}nd_k(\mathscr{B}_0)$ has image in the sheaf of G-invariant operators.

iv) Let $\sigma_i : \mathscr{B}_i \longrightarrow \mathscr{B}_0$ be the injection of sheaves of rings provided by the isomorphism $\mathscr{B}_i \otimes_{\mathscr{O}_{(i)}} \mathscr{O}_X \cong \mathscr{B}_0$. Then the image of σ_i is the sheaf

$$U \longmapsto \{f \in \mathscr{B}_0(U);\ \nabla(D)f_x = 0\ , \forall x \in U,\ \forall D \in \mathscr{D}^{+,<p^i}_{X,x}(U)\}.$$

Proof: *i*). We have to show that if $\sum_\nu f_\nu \otimes a_\nu = \sum f'_\nu \otimes a'_\nu$, then $\sum_\nu f_\nu \otimes D(a_\nu) = \sum_\nu f'_\nu \otimes D(a'_\nu)$. Because $\mathscr{B}_i$ is the algebra associated to a G-torsor, it is the direct limit of locally free coherent $\mathscr{O}_{(i)}$-modules and the homomorphisms in the direct system are all injective. Consider $x \in U$ and to keep notation simple, denote the germs of f_ν, a_ν, etc on the corresponding stalks by the same characters. It follows that there exist $\mathscr{O}_{(i),x}$-linearly independent elements $\{\beta_j\}$ in $\mathscr{B}_{i,x}$ (because $\mathscr{B}_{i,x}$ is the increasing union of free modules) such that $f_\nu = \sum_j g_{\nu j}\beta_j$ and $f'_\nu = \sum_j g'_{\nu j}\beta_j$. Hence, by flatness of F,

$$\sum_\nu (g_{\nu j})^{p^i} a_\nu = \sum_\nu (g'_{\nu j})^{p^i} a'_\nu.$$

In particular, $\sum_\nu (g_{\nu j})^{p^i} D(a_\nu) = \sum_\nu (g'_{\nu j})^{p^i} D(a'_\nu)$, and thus

$$\begin{aligned} \textstyle\sum_\nu f_\nu \otimes D(a_\nu) &= \textstyle\sum_{\nu,j} \beta_j \otimes (g_{\nu j})^{p^i} D(a_\nu) \\ &= \textstyle\sum_j \beta_j \otimes \Big(\sum_\nu (g'_{\nu j})^{p^i} D(a'_\nu)\Big) \\ &= \textstyle\sum_\nu f'_\nu \otimes D(a'_\nu). \end{aligned}$$

This proves that ∇ is well defined. It is immediate to check the second assertion of *i*).

ii). Let $x \in X$ and $D \in \mathscr{D}_{X,x}$ be the restriction of D_q with $|q| < p^i$. Let $\sum_\nu f_\nu \otimes a_\nu$ and $\sum_\mu f'_\mu \otimes a'_\mu$ be elements of $\mathscr{B}_{i,x} \otimes_{\mathscr{O}_{(i),x}} \mathscr{O}_{X,x}$. Then

$$\begin{aligned} \nabla(D) \textstyle\sum_{\mu,\nu} f_\nu f'_\mu \otimes a_\nu a'_\mu &= \textstyle\sum_{\nu,\mu} f_\nu f'_\mu \otimes D(a_\nu a'_\mu) \\ &= \textstyle\sum_{\mu,\nu} f_\nu f'_\mu \otimes \sum_{q'+q''=q} D_{q'}(a_\nu) D_{q''}(a'_\mu) \\ &= \textstyle\sum_{q'+q''=q} \nabla(D_{q'}) \left(\sum_\nu f_\nu \otimes a_\nu\right) \cdot \nabla(D_{q''}) \Big(\sum_\mu f'_\mu \otimes a'_\mu\Big). \end{aligned}$$

This proves *ii*) in view of formula (1.5) above.

iii). Because the isomorphism $\mathscr{B}_i \otimes_{\mathscr{O}_{(i)}} \mathscr{O}_X \longrightarrow \mathscr{B}_0$ respects the co-actions of R, follows that we only need to verify that the diagram

$$\begin{array}{ccc}
\mathscr{O}_X(U)\otimes_{\mathscr{O}_{(i)}(U)}\mathscr{B}_i(U) & \xrightarrow{\mathrm{id}\otimes\mu} & (\mathscr{O}_X(U)\otimes_{\mathscr{O}_{(i)}(U)}\mathscr{B}_i(U))\otimes_k R \\
\Big\downarrow{\scriptstyle\nabla(D)} & & \Big\downarrow{\scriptstyle\nabla(D)\otimes\mathrm{id}_R} \\
\mathscr{O}_X(U)\otimes_{\mathscr{O}_{(i)}(U)}\mathscr{B}_i(U) & \xrightarrow[\mathrm{id}\otimes\mu]{} & (\mathscr{O}_X(U)\otimes_{\mathscr{O}_{(i)}(U)}\mathscr{B}_i(U))\otimes_k R
\end{array}$$

(μ is the co-action) is commutative for any affine open $U \subseteq X$ and any differential operator $D \in \mathscr{D}_X^{<p^i}(U)$. This is obvious from equation (1.9).

iv). Let x be a closed point of X and let $f = \sum_\nu f_\nu \otimes a_\nu \in \mathscr{B}_{i,x} \otimes_{\mathscr{O}_{(i),x}} \mathscr{O}_{X,x}$ be annihilated by $\mathscr{D}_{X,x}^{+,<p^i}$. Let $\{\beta_j\}$ and $g_{\nu j}$ be as in the proof of *i*). Then $\sum_\nu (g_{\nu j})^{p^i} a_\nu$ is annihilated by $\mathscr{D}_{X,x}^{+,<p^i}$ and hence there exist $a'_j \in \mathscr{O}_{X,x}$ such that $(a'_j)^{p^i} = \sum_\nu (g_{\nu j})^{p^i} a_\nu$ (this is a consequence, for instance, of Lemma 24). It then follows that $f = \sum_j a'_j \beta_j \otimes 1$. □

Remark: There are many compatibilities to be checked, all of them trivial. We will call the reader's attention to the following, which will enable us to understand better $\Pi^{\mathbf{Fdiv}}$. Let ∇ be a stratification on P and let $\{P_i\}_{i\in\mathbb{N}}$ be the associated F-division. Given a representation V of G, $\mathscr{E} = \mathscr{L}_{P/G}(V)$ has a stratification via formula (1.7). Also, $\mathscr{E}$ has an F-division provided by $\mathscr{E}_i = \mathscr{L}_{P_i}(V)$ (use the description of $\mathscr{L}$ as the sheaf of sections of a geometric vector bundle in [18], p. 89); the isomorphisms $F^*\mathscr{E}_{i+1} \cong \mathscr{E}_i$ are those coming from $F^*P_{i+1} \cong P_i$. It is immediate to see that the Cartier-Katz construction of the F-division of $\mathscr{E}$ is none other than $\{\mathscr{E}_i\}_{i\in\mathbb{N}}$. To repeat, if CK is the Cartier-Katz functor that takes a stratification to an F-division, then $\mathrm{CK} \circ \mathscr{L}_{P/G}(?) = \{\mathscr{L}_{P_i}(?)\}$.

1.4.4 The Frobenius on $\Pi^{\mathbf{Fdiv}}$

As was remarked above, for a given smooth and connected k-scheme X, we can obtain a tensor equivalence $\mathscr{L} : \mathrm{Rep}_k(\Pi^{\mathbf{str}}) \longrightarrow \mathbf{str}(X)$ which, when identifying $\mathbf{str}(X)$ with $\mathbf{Fdiv}(X)$, is given by $\{\mathscr{L}_{P_i}\}$ for an F-divided torsor $\{P_i\}_i$. It is convenient to remove the smoothness assumption and work with F-divided sheaves over regular, locally noetherian and connected k-schemes. The process is parallel with the one adopted above and follows Nori's construction of a fundamental torsor.

We use this to prove that the group scheme $\Pi^{\mathbf{Fdiv}}$ is *reduced* (in fact, that the Frobenius is an isomorphism).

Let X/k be *connected, locally noetherian and regular.* Let

$$\mathscr{L} : \mathrm{Rep}_k(\Pi^{\mathbf{Fdiv}}) \longrightarrow \mathbf{Fdiv}(X)$$

be a k-linear tensor equivalence. This gives rise to a family of faithful exact k-linear tensor functors $\mathscr{L}_i : \mathrm{Rep}_k(\Pi^{\mathbf{Fdiv}}) \longrightarrow \mathrm{coh}(X^{(i)})$. Since $\mathscr{L}$ is a tensor functor, there is a natural isomorphism of *tensor functors* (see [12], Def. 1.12, p. 116 for terminology) $\sigma_i : F^*\mathscr{L}_{i+1} \Rightarrow \mathscr{L}_i$.

From section 1.2.4, we obtain $\Pi^{\mathbf{Fdiv}}$-torsors $\psi_i : P_i \longrightarrow X^{(i)}$ given by $\mathscr{B}_i := \psi_{i*}\mathscr{O}_{P_i} = \mathscr{L}_i'((R, \rho_l))$; in particular, there are isomorphisms of *torsors* $\sigma_i : F^*P_{i+1} \cong P_i$. This gives another k-linear tensor functor $V \mapsto \{\mathscr{L}_{P_i}(V)\}_i$, where the structural isomorphisms $F^*\mathscr{L}_{P_{i+1}}(V) \longrightarrow \mathscr{L}_{P_i}(V)$ are the ones induced by the σ_i – to see that these are really isomorphisms one can use the characterization of $\mathscr{L}_P$ as sections of a geometric vector-bundle, [18], p. 89 or [33], Prop. 2.9, (a), p. 34.

Lemma 32. *The tensor functor $\{\mathscr{L}_{P_i}\}_i$ is naturally isomorphic to $\mathscr{L}$.*

Proof: This is just an analogue of Proposition 26 and we borrow notation from there. Let V be a representation of $\Pi = \Pi^{\mathbf{Fdiv}} = \operatorname{Spec} R$ and consider $W_1 := (V, \rho) \otimes (R, \rho_l)$, $W_2 = (R, \rho_l)^{\oplus \dim V}$. There is a canonical isomorphism of R-comodules $W_1 \longrightarrow W_2$ which will induce an isomorphism $\mathscr{L}'(W_1) \cong \mathscr{L}'(W_2)$; this isomorphism gives a sequence of isomorphisms of quasi-coherent $\mathscr{O}_{X^{(i)}}$-modules

$$\lambda_i : \mathscr{L}_i(V) \otimes \mathscr{B}_i \longrightarrow V \otimes_k \mathscr{B}_i \tag{1.10}$$

which satisfy $F^*(\lambda_{i+1}) = \lambda_i$. Also, each sheaf on eq. (1.10) has an action of Π and λ_i preserves it (since this action is induced by a Π-sheaf structure on each of W_i, see the proof of Proposition 26 above for more details). We note that the sheaf of Π invariants of $\mathscr{L}_i(V) \otimes \mathscr{B}_i$ (resp. $V \otimes_k \mathscr{B}_i$) is $\mathscr{L}_i(V)$ (resp. $\mathscr{L}_{P_i}(V)$). Since F^* is an exact functor, it commutes with the functor "taking Π invariants" and consequently $\{\lambda_i^{\Pi}\}_i$ gives a natural isomorphism $\mathscr{L}_i \Rightarrow \{\mathscr{L}_{P_i}\}_i$. $\square$

Remark: Obviously, Lemma 32 holds for any k-linear, faithful and exact tensor functor $\mathscr{L} : \mathrm{Rep}_k(G) \longrightarrow \mathbf{Fdiv}(X)$ (analogous to section 1.4.2).

Consider

$$\mathbf{Fdiv}(X) \xrightarrow{\quad\Phi\quad} \mathbf{Fdiv}(X)$$

$$\{\mathscr{E}_i; \sigma_i\}_{i\in\mathbb{N}} \longmapsto \{F^*(\mathscr{E}_i); F^*(\sigma_i)\}_{i\in\mathbb{N}}.$$

the shifting functor. It is obviously an equivalence.

Lemma 33. *The diagram*

$$\begin{tikzcd}
\mathbf{Fdiv}(X) \arrow[rr, "\Phi"] & & \mathbf{Fdiv}(X) \\
\mathrm{Rep}_k(\Pi^{\mathbf{Fdiv}}) \arrow[u, "\mathscr{L}"] \arrow[rr, "\textit{Frobenius twist}"'] & & \mathrm{Rep}_k(\Pi^{\mathbf{Fdiv}}) \arrow[u, "\mathscr{L}"']
\end{tikzcd}$$

is commutative up to natural isomorphism.

Proof: We recall that given any group scheme G and any G-torsor P over X, there is a natural isomorphisms of $\mathscr{O}_X$-modules $\lambda : F^*\mathscr{L}_{P/G}(V) \longrightarrow \mathscr{L}_{P/G}(V^{(1)})$. If $U = \operatorname{Spec} A$ is an affine open of X and $\operatorname{Spec} B$ is the inverse image of U in P, then $\lambda(U)$ is

$$A \otimes_{F,A} (B \otimes V)^G \longrightarrow (B \otimes V^{(1)})^G$$

$$a \otimes \sum_j b_j \otimes v_j \longmapsto \sum_j ab_j^p \otimes v_j.$$

Thus, for any representation V of $\Pi^{\mathbf{Fdiv}}$, there are natural isomorphisms

$$\lambda_i : F^*\mathscr{L}_{P_i}(V) \longrightarrow \mathscr{L}_{P_i}(V^{(1)}).$$

All we have to do is check that $\{\lambda_i\} : \{F^*\mathscr{L}_{P_i}(V)\} \longrightarrow \{\mathscr{L}_{P_i}(V^{(1)})\}$ is an arrow in $\mathbf{Fdiv}(X)$, i.e., the diagrams

$$\begin{tikzcd}
F^*_{\mathrm{geom}}(F^*\mathscr{L}_{P_{i+1}}(V)) \arrow[r, "F^*_{\mathrm{geom}}(\lambda_{i+1})"] \arrow[d, "F^*(\sigma_{i+1})"'] & F^*_{\mathrm{geom}}(\mathscr{L}_{P_{i+1}}(V^{(1)})) \arrow[d, "\sigma_{i+1}"] \\
F^*\mathscr{L}_{P_i}(V) \arrow[r, "\lambda_i"'] & \mathscr{L}_{P_i}(V^{(1)}).
\end{tikzcd} \tag{1.11}$$

are commutative. The question is then local on $X^{(i)}$ and we can use the local expression of λ_i above. It is now a tedious but straightforward algebraic manipulation to check the commutativity of the above diagram. □

Theorem 34. *Let X be connected regular noetherian scheme over k with a k-rational point. Then the Frobenius homomorphism $F : \Pi^{\mathbf{Fdiv}} \longrightarrow (\Pi^{\mathbf{Fdiv}})^{(1)}$ is an isomorphism.*

Proof: From Lemma 33 above, follows that taking the Frobenius twist of a representation induces an equivalence $\mathrm{Rep}_k(\Pi^{\mathbf{Fdiv}}) \longrightarrow \mathrm{Rep}_k(\Pi^{\mathbf{Fdiv}})$. Now we use diagram (1.3) to conclude that $\mathrm{Res}(F)$ is an equivalence of categories. By general Tannakian duality ([12], 2.21, p.139) we are done. □

Corollary 35. *i)* $\mathrm{Lie}(\Pi^{\mathbf{Fdiv}}) = 0$.

ii) If $\Pi^{\mathbf{Fdiv}}$ *is finite or smooth it is etale.*

iii) Any profinite quotient of $\Pi^{\mathbf{Fdiv}}$ *is proetale.*

Proof: *i*). Because the relative Frobenius is a closed embedding, the differential at the identity $dF_e = 0$ is injective ([42], 12.2, p. 94, Corollary).

ii). An affine algebraic group scheme is smooth if and only if its dimension coincides with the dimension of its Lie algebra ([42], 12.2, p. 94, Corollary). If $\Pi^{\mathbf{Fdiv}}$ is finite, part *i*) shows that it is smooth, hence etale. Again, part *i*) shows that if it is smooth then it has dimension zero.

iii). By definition, if G is a quotient of $\Pi^{\mathbf{Fdiv}}$ then $\mathscr{O}(G)$ is a sub-Hopf-algebra of the Hopf-algebra of $\Pi^{\mathbf{Fdiv}}$. It follows that $\mathscr{O}(G)$ is reduced and if it is finite dimensional it is etale over k ([42], Thm. 6.2, p. 46). $\square$

1.5 The relation between $\Pi^{\mathbf{Fdiv}}$ and the etale fundamental group

The goal of this section is to prove and discuss the hypothesis (see the remark after the proof) of the very natural proposition below which relates the F-divided fundamental group with the (geometric) etale fundamental group. It generalizes and brings to the right context Prop. 1.9 on p. 7 of [16].

We take X/k connected, locally noetherian and regular with a k-rational point x_0 defining $\Pi^{\mathbf{Fdiv}}(X, x_0)$. Perhaps the best way to encapsulate this discussion is the slogan: "etale coverings are differential equations with finite monodromy".

Proposition 36. *Let k be algebraically closed. There is a natural quotient homomorphism of group schemes $\nu : \Pi^{\mathbf{Fdiv}}(X, x_0) \longrightarrow \Pi^{\mathrm{et}}(X, x_0)$ which identifies $\pi_0(\Pi^{\mathbf{Fdiv}})$ with Π^{et}.*

Proof: We will abuse notation and write π to denote $\Pi^{\mathrm{et}}(X, x_0)$ as well as its group of k-points (such abuse is justified by Theorem 6.4, p. 49 of [42]). For any proetale group scheme G, the Frobenius twist of representations is an equivalence and hence, to every representation V of G, there exist unique representations V_i such that $V_{i+1}^{(1)} = V_i$ and $V_0 = V$. Also, if $P \longrightarrow X$ is a G-torsor, we obtain a tensor functor

$$\varphi_P : \mathrm{Rep}_k(G) \longrightarrow \mathbf{Fdiv}(X), \qquad \varphi_P(V) := \{\mathscr{L}_{P/G}(V_i)\}. \tag{1.12}$$

If P has a k-rational point above x_0 (as it does, since k is algebraically closed), then $x_0^*\mathscr{L}_{P/G}$ is canonically isomorphic to the forgetful functor. By Tannakian duality, we obtain a homomorphism $\Pi^{\mathbf{Fdiv}} \longrightarrow G$. If $\pi = \varprojlim_i \pi_i$ with π_i a quotient, there are connected pointed π_i-torsors $E_i \longrightarrow X$, such that $E_i \times^{\pi_i} \pi_{i-1} = E_{i-1}$ as *pointed* torsors. It follows easily that these torsors give rise to a homomorphism $\nu : \Pi^{\mathbf{Fdiv}} \longrightarrow \pi$; such a homomorphism factors through $\overline{\nu} : \pi_0(\Pi^{\mathbf{Fdiv}}) \longrightarrow \pi$. We will prove that $\overline{\nu}$ is an isomorphism by showing that $\mathrm{Res}(\overline{\nu})$ is an equivalence of categories.

$\mathrm{Res}(\overline{\nu})$ **is full and faithful** This amounts to show that, given a connected G-torsor P, G-etale, φ_P is full and faithful (as the subcategory of $\mathbf{Fdiv}(X)$ corresponding to $\pi_0(\Pi^{\mathbf{Fdiv}})$ is full). Faithfulness is obvious and we concentrate on the fullness. By linear algebra, it is enough to show that any arrow $\theta : \mathbb{1} \longrightarrow \varphi_P(V)$ in $\mathbf{Fdiv}(X)$ comes from a $v \in V^G$. But such a θ is given by a sequence of morphisms $s_i : P \longrightarrow V_i$ such that the natural composition

$$\mathscr{L}_{P/G}(V_i)(X) \longrightarrow \mathscr{L}_{P/G}(V_{i-1})(X)$$

takes s_i to s_{i-1}. The above composition is just (after giving V a basis)

$$(f_1, \dots, f_d) \mapsto (f_1^p, \dots, f_d^p)$$

and hence $s_0 \in k^{\oplus d}$, as the scheme P is regular (locally noetherian), has a k-rational point and is connected.

$\mathrm{Res}(\overline{\nu})$ **is essentially surjective** Let $\{\mathscr{E}_i\}$ be an object of $\mathbf{Fdiv}(X)$ which is in the subcategory corresponding to $\pi_0(\Pi^{\mathbf{Fdiv}})$. There exists an etale group scheme G and an F-divided pointed G-torsor $\{P = P_0, P_1, \dots\}$ such that $\{\mathscr{E}_i\} = \{\mathscr{L}_{P_i/G}(V)\}$ as objects of $\mathbf{Fdiv}(X)$.

From the claim below, it will follow that $\{\mathscr{L}_{P_i/G}(?)\}_i$ and $\varphi_P(?)$ are naturally isomorphic tensor functors. Now, $P = E_j \times^{\pi_j} G$ for some j and hence $\{\mathscr{E}_i\} \cong \varphi_{E_j}(\mathrm{Res}^G_{\pi_j}(V))$ thereby proving that $\mathrm{Res}(\overline{\nu})$ is essentially surjective.

Claim: Let G be an etale group scheme and let $\mathrm{Tors}(G/X)$ be the category of G-torsors over X. Then the functor $F^* : \mathrm{Tors}(G/X^{(1)}) \longrightarrow \mathrm{Tors}(G/X)$ (pull-back by Frobenius) is an equivalence.

Proof: Recall that the group scheme G is, as a functor on the category of k-schemes, given by $G(S) = \mathrm{Map}(\pi_0(S), G(k))$. Hence, $G(f)$ will be an isomorphism for any $f : S' \longrightarrow S$ which induces a bijection on the underlying topological spaces. By EGA I, 3.5.2 (ii) (p. 115), 3.5.7 (ii) (p. 116), 3.5.5 (p. 115) and 3.5.11 (p. 117), it follows that given a morphism $U \longrightarrow X^{(1)}$, $G(\mathrm{pr}) : G(U) \longrightarrow G(F^*U)$ is an isomorphism (where $\mathrm{pr} : F^*U \longrightarrow U$ is the natural projection). Now the category of G-torsors over X is isomorphic to the groupoid $[Z^1(X, G)/B^1(X, G)]$: Objects are cocycles $\{g_{ij}\} \in G(U_i \times_X U_j) = G(U_{ij})$, for some etale covering $(U_i \longrightarrow X)_{i \in I}$ (in the sense of [31], that is, U_i is finite over X and $\#I < \infty)^2$, and an arrow $\{g_{ij}\} \longrightarrow \{g'_{ij}\}$ is an element $\{h_i\} \in \prod G(U_i)$ such that $g'_{ij} = (h_i|U_{ij}) \cdot g_{ij} \cdot (h_j|U_{ij})^{-1}$. The pull-back functor $F^* : \mathrm{Tors}(G/X^{(1)}) \longrightarrow \mathrm{Tors}(G/X)$ takes the cocycle $\{g_{ij}\} \in \prod G(U_{ij})$ (resp. the coboundary $\{h_i\} \in \prod G(U_i)$) to the cocycle $\{G(\mathrm{pr})(g_{ij})\} \in \prod G(F^*U_{ij})$ (resp. the coboundary $\{G(\mathrm{pr})(h_i)\} \in \prod G(F^*U_i)$) and hence is full and faithful. That F^* is essentially surjective follows from the fact that faithfully flat, universally injective and quasi-compact morphisms give equivalences on the categories of etale coverings ([31], Prop. 7.2.2, p. 146). This finishes the proof of the claim.

Now we show how to obtain that $\{\mathscr{L}_{P_i/G}(?)\}_i \cong \varphi_P(?)$. Clearly φ_P is the functor associated to some F-divided G-torsor $\{Q_i\}$ such that $Q_0 = P = P_0$ (compare Lemma 32). From the Claim, the F-division $\{P_i\}$ is isomorphic to the F-division $\{Q_i\}$ and we conclude that the two functors $\varphi_P = \{\mathscr{L}_{Q_i/G}\}$ and $\{\mathscr{L}_{P_i/G}\}$ are naturally isomorphic. $\square$

Remark: The above proposition cannot hold true if k is not algebraically closed. The reason is quite obvious, as the category $\mathbf{Fdiv}(\operatorname{Spec} k)$ is certainly trivial; that is, $\Pi^{\mathbf{Fdiv}}$ is insensitive to the arithmetic etale fundamental group.

[2] In fact, objects are equivalence classes of cocycles under the usual process of taking a finer covering. We will overlook this technicality. Also, it is always possible to take $\#I = 1$

Chapter 2

The categories $\mathbf{dR}(X)$ and $\mathfrak{N}\mathbf{str}(X)$

2.1 Introduction

We will consider nilpotent tensor categories of sheaves on a smooth scheme X: the category of de Rham sheaves $\mathbf{dR}(X)$ and the category of nilpotent stratified sheaves $\mathfrak{N}\mathbf{str}(X)$. By the fundamental theorem of Tannakian duality ([12], Theorem 2.11, p. 130), the categories here considered will be equivalent, once fixed a k-rational point of X, to categories of representations of unipotent affine group schemes ([42], Ch. 8).

With the aid of Nori's Lemma (Lemma 50), we will be able to prove, in all the considered cases (see theorems 40 and 46), that these groups are *profinite* if X is proper. By using Corollary 35, the category $\mathfrak{N}\mathbf{str}(X)$ will be controlled by the etale fundamental group (Corollary 47). All of this is an application of the insight provided by Tannakian duality theory, which allows the cooperation of (elementary) group theory and algebraic geometry.

One important topic is the study of the de Rham fundamental group in positive characteristic. We run through the usual argument relating connections on torsors and tensor functors from the category of representations of a group scheme to the category of coherent sheaves with connections (section 2.2.2). Because of the similarity with section 1.4.2 we will be brief in section 2.2.2 (also, after a careful reading, the results of that section can be extracted from [11] (specially 10.26-10.30, pp. 194-96), but we feel that Nori's method is more natural and allow us to use a more differential geometric language).

It can be said that the study of the de Rham fundamental group presented here is intended to disqualify this category as a category of differential equations in positive characteristic (of course, the characteristic zero case is much more promising, [40]); this is what the example in section 2.2.3 does.

In section 2.4, we give a straightforward and enlightening proof of Nori's Lemma based entirely on easy group theoretical methods.

2.2 The de Rham category

We have fixed X/k smooth and connected in all that follows.

In this section we will discuss coherent modules with connections. For details on the theory of connections see [19]. Let $\mathrm{DE}(X/k)$ be the category of k-linear differential equations on X: Objects are $(\mathscr{E}, \nabla)$, where $\mathscr{E}$ is a coherent $\mathscr{O}_X$-module and $\nabla : \Theta_X \longrightarrow \mathscr{E}nd_k(\mathscr{E})$ is a k-linear integrable connection on $\mathscr{E}$ ([19], 1.0, pp. 178-79). Morphisms are the horizontal maps (loc.cit., 1.1, p. 180).

The category $\mathrm{DE}(X/k)$ is a k-linear abelian tensor category (loc.cit. 1.1, p. 180) and in characteristic 0, the inclusion functor into $\mathrm{coh}(X)$ is a structure of locally free tensor category of coherent sheaves on X (Definition 1), as a theorem of Cartier shows (loc.cit. Prop. 8.8, p. 206). In positive characteristic this is not true.

Definition 37. *The de Rham category of X, $\mathbf{dR}(X)$, is the the nilpotent category $\mathfrak{N}\mathrm{DE}(X/k)$. Objects of $\mathbf{dR}(X)$ will be called de Rham sheaves.*

Lemma 38. *Assume that X is connected, has a k-rational point x_0 and $\mathrm{H}^0_{\mathrm{dR}}(X/k) = k$. Then $x_0^* : \mathbf{dR}(X) \longrightarrow (k-\mathrm{mod})$ makes $\mathbf{dR}(X)$ into a neutral tannakian category.*

Proof: Use Lemma 6 to show that $\mathbf{dR}(X)$ is abelian and then apply Lemma 8 with $\mathfrak{A} = \mathrm{DE}(X/k)$ and ι =the forgetful functor. □

Definition 39. *Let $x_0 \in X(k)$. The de Rham fundamental group $\Pi^{\mathbf{dR}}(X, x_0)$ is the group scheme associated, via Tannakian duality ([12], Thm. 2.11, p. 130), to the fibre functor $x_0^* : \mathbf{dR}(X) \longrightarrow (k - \mathrm{mod})$.*

As we shall keep X and $x_0 \in X(k)$ fixed, we will abuse notation and denote $\Pi^{\mathbf{dR}}(X, x_0)$ by $\Pi^{\mathbf{dR}}$. The above definitions and Lemma are due to [11] and [40].

2.2.1 Profiniteness

Take X connected and $\mathrm{H}^0_{\mathrm{dR}}(X/k) = k$. Let $x_0 \in X(k)$ define the group $\Pi^{\mathbf{dR}}$.

Theorem 40. *If $\mathrm{H}^1_{\mathrm{dR}}(X/k)$ is finite dimensional (for example X/k proper) and k is of positive characteristic, then $\Pi^{\mathbf{dR}}$ is profinite.*

Proof: Note that for any group scheme G over k, the k-vector space $\mathrm{Hom}(G, \mathbb{G}_a)$ is isomorphic to $\mathrm{Ext}^1_G(\mathbb{1}, \mathbb{1})$, since $\mathbb{G}_a$ can be seen as the subgroup of upper triangular matrices in $\mathbb{GL}(2)$. If $G = \Pi^{\mathbf{dR}}$, then this last vector space is $\mathrm{Ext}^1_{\mathrm{DE}}(\mathbb{1}, \mathbb{1})$. But the functors $(\mathscr{E}, \nabla) \mapsto \mathrm{H}^*_{\mathrm{dR}}((\mathscr{E}, \nabla))$ from $\mathrm{DE}(X/k)$ to $(k - \mathrm{mod})$ are the right derived functors of $\Gamma(\mathscr{E})^\nabla$; by general theory of abelian categories (with enough injectives), $\mathrm{Ext}^1_{\mathrm{DE}}(\mathbb{1},\mathbb{1}) = R^1(\mathrm{Hom}_{\mathrm{DE}}(\mathbb{1},?))(\mathbb{1}) = \mathrm{H}^1_{\mathrm{dR}}(X/k)$. The theorem follows from Lemma 50. $\square$

2.2.2 Connections on torsors and $\mathbf{dR}(X)$

We will use the setting of section 1.2.4. This section is entirely analogous to section 1.4.2.

Let $G = \operatorname{Spec} R$ be an affine group scheme and $\psi : P \longrightarrow X$ be a G-torsor. The associated sheaf construction gives us an exact tensor functor

$$\mathscr{L}_{P/G} : \mathrm{Rep}_k(G) \longrightarrow \mathrm{coh}(X),$$

and we will be interested in the case where $\mathscr{L}_{P/G}$ factors through $\mathrm{DE}(X/k)$. More precisely, assume that

(FCT) $\mathscr{L}_{P/G}$ is the composition of a (exact) tensor functor $\mathscr{L} : \mathrm{Rep}_k(G) \longrightarrow \mathrm{DE}(X/k)$ with the natural inclusion.

For a representation V of G, we will denote the connection on $\mathscr{L}_{P/G}(V)$ by ∇^V. Recall (Example 14) that the quasi-coherent $\mathscr{O}_X$-algebra $\psi_*\mathscr{O}_P$ is canonically isomorphic to $\mathscr{L}'((R,\rho_l))$ (notations from section 1.2.4), which is an object in the category of quasi-coherent modules with integrable connections. Let $\nabla : \Theta_X \longrightarrow \mathscr{E}nd_k(\mathscr{B})$ denote the connection on $\mathscr{B}$. The fact that $\mathscr{B}$ is an *algebra* in the category of $\mathscr{O}_X$-modules with an integrable connection means that given an open $U \subseteq X$, the diagram

$$\begin{tikzcd}
\mathscr{B}|U \otimes \mathscr{B}|U \arrow[r, "m"] \arrow[d, "\nabla(D)\otimes\mathrm{id}+\mathrm{id}\otimes\nabla(D)"'] & \mathscr{B}|U \arrow[d, "\nabla(D)"] \\
\mathscr{B}|U \otimes \mathscr{B}|U \arrow[r, "m"'] & \mathscr{B}|U
\end{tikzcd}$$

is commutative for any $D \in \Theta_X(U)$. In particular, for any $a, b \in \mathscr{B}(U)$ and $D \in \Theta_X(U)$, we have

$$\nabla(D)(ab) = a\nabla(D)(b) + b\nabla(D)(a). \tag{2.1}$$

Also, if $a \in \mathscr{O}_X(U)$, then

$$\nabla(D)(a) = D(a) \tag{2.2}$$

because $\mathscr{L}$ takes the trivial representation to $\mathscr{O}_X$ with the canonical connection.

Definition 41. *The sub-$\mathscr{O}_X$-module of $\psi_*\Theta_P$ consisting of invariant operators (1.4.1) is called the Atiyah sheaf of P (notation: $\mathscr{A}t(P)$).*

Since the map $\mu : \mathscr{B} \longrightarrow \mathscr{B} \otimes_k R$ giving the action of G on P is horizontal, follows that ∇ takes values in the sheaf of invariant operators. Together with equations (2.1) and (2.2) this implies that $\nabla : \Theta_X \longrightarrow \mathscr{A}t(P)$ is an $\mathscr{O}_X$-module section of the natural restriction $\mathscr{A}t(P) \longrightarrow \Theta_X$. Such a section is usually called a *connection on the principal bundle* P. Note that a connection automatically makes multiplication $\psi_*\mathscr{O}_P \otimes_{\mathscr{O}} \psi_*\mathscr{O}_P \longrightarrow \psi_*\mathscr{O}_P$ horizontal.

A connection is said to be integrable whenever it is a morphism of sheaves of $\mathscr{O}_X$-Lie algebras. Hence, the factorization (FCT) above gives an integrable connection on the G-torsor P.

Using equation (1.7) above, we also see how to obtain connections on each $\mathscr{L}_{P/G}(V)$ starting from a connection ∇ on P — call it ∇^V. Of course, the analogue of Proposition 26 holds.

Proposition 42. *i) The functors $\mathscr{L}$ and $V \mapsto (\mathscr{L}_{P/G}(V), \nabla^V)$ are naturally isomorphic tensor functors.*

ii) There is a bijection between isomorphism classes of connections on P and isomorphism classes of factorizations of $\mathscr{L}_{P/G}$ as (FCT) above.

$\square$

Applying the construction of a connection from a factorization (FCT) to a suitable equivalence inverse for the fibre functor $\omega := x_0^* : \mathbf{dR}(X) \longrightarrow \mathrm{Rep}_k(\Pi^{\mathbf{dR}})$, we obtain a $\Pi^{\mathbf{dR}}$-torsor $X^{\mathbf{dR}}$ over X with an integrable connection $\nabla_{\mathbf{dR}}$ and a canonical $x_{\mathbf{dR}} \in X^{\mathbf{dR}}(k)$ above x_0.

Finally, we are able to give a description of $\mathrm{Hom}(\Pi^{\mathbf{dR}}, ?)$ as a solution to a classification problem. A pointed G-torsor with connection is a G-torsor with a connection plus a k-rational point above x_0. Let $\mathrm{ICTors}^*(G/X)$ be the set of all pointed G-torsors with an integrable connection (modulo isomorphisms). The analogue of Theorem 27 is:

Theorem 43. *The map from $\mathrm{Hom}(\Pi^{\mathbf{dR}}, G)$ to $\mathrm{ICTors}^*(G/X)$ which associates to $\theta : \Pi^{\mathbf{dR}} \longrightarrow G$ the pointed torsor with integrable connection $X^{\mathbf{dR}} \times^{\Pi^{\mathbf{dR}}} G$ is a bijection.*

$\square$

Remark: Ends meet: We know that $\mathrm{Hom}(\Pi^{\mathbf{dR}}, \mathbb{G}_a) = \mathrm{Ext}^1_{\mathbf{dR}(X)} \cong \mathrm{H}^1_{\mathrm{dR}}(X/k)$ and from the Hodge to de Rham spectral sequence follows that $\ker(\mathrm{H}^1_{\mathrm{dR}}(X/k) \longrightarrow \mathrm{H}^1(\mathscr{O}_X))$ is the space of closed regular 1-forms on X. Let $\mathbf{nilp}(X)$ be the category of nilpotent sheaves on X. Consider the canonical homomorphism of vector spaces $\alpha : \mathrm{Hom}(\Pi^{\mathbf{dR}}, \mathbb{G}_a) \longrightarrow \mathrm{Hom}(\Pi^{\mathbf{nilp}}, \mathbb{G}_a)$ obtained from the inclusion $\iota : \mathbf{dR}(X) \longrightarrow \mathbf{nilp}(X)$. Any θ in $\ker(f)$ will correspond to an integrable connection on the trivial bundle $X \times \mathbb{G}_a$ and vice-versa. For G smooth and algebraic,

$$\mathscr{A}t(P) \longrightarrow \Theta_X$$

has kernel $\mathscr{L}_{P/G}(\mathrm{Lie}(G))$ (action is by the adjoint representation), which in the case $P = X \times \mathbb{G}_a$ is $\mathscr{O}_X$; in fact $\mathscr{A}t(X \times \mathbb{G}_a) = \mathscr{O}_X \oplus \Theta_X$. Thus, the space of connections on $X \times \mathbb{G}_a$ is $\mathrm{Hom}_{\mathscr{O}_X}(\Theta_X, \mathscr{O}_X) = \mathrm{H}^0(X, \Omega^1_{X/k})$ and the integrability condition is simply the condition that the corresponding 1-form is closed.

2.2.3 Connections and Frobenius pull-back

Take k of positive characteristic p.

Assume that the torsor P has a connection ∇. We want to describe the connection on $\mathscr{L}(V^{(1)})$ in terms of data in $\mathrm{DE}(X/k)$ (Lemma 44) and then relate this to the p-curvature (see [19], 5, pp. 189-94 for the basic properties of the p-curvature). This will give us examples of de Rham sheaves whose monodromy group is not etale (see the example below) and hence exclude the category $\mathbf{dR}(X)$ of the study of "true" differential equations.

Recall that for any quasi-coherent sheaf $\mathscr{E}$ on X, the Frobenius pull-back $F^*\mathscr{E}$ has a canonical integrable connection ∇_{can} and that this connection has p-curvature zero [19]; it is characterized by making all the sections $1 \otimes e \in \mathscr{O}_X \otimes_{F,\mathscr{O}_X} \mathscr{E}$ ∇_{can}-horizontal.

Also recall that given any quasi-coherent $\mathscr{O}_X$-module $(\mathscr{E}, \nabla)$ on X with ∇ an integrable connection of p-curvature zero, the $\mathscr{O}_{X^{(1)}}$-module $\mathscr{F} := \mathscr{E}^{\nabla}$ is quasi-coherent and $(F^*\mathscr{F}, \nabla_{\mathrm{can}})$ is *horizontally* isomorphic to $(\mathscr{E}, \nabla)$ (loc.cit., Thm 5.1, p. 190).

Lemma 44. *Let V be a representation of G and let $V^{(1)}$ be its Frobenius twist. Then the canonical isomorphism of $\mathscr{O}_X$-modules*

$$\theta : F^*\mathscr{L}_{P/G}(V) \longrightarrow \mathscr{L}_{P/G}(V^{(1)})$$

is horizontal.

Proof: Let $U = \operatorname{Spec} A$ be an affine open subset of X and let $\psi^{-1}U = \operatorname{Spec} B$. Over U, θ is given by

$$A \otimes_{F,A} (B \otimes_k V)^G \longrightarrow (B \otimes_k V^{(1)})^G$$

$$1 \otimes \textstyle\sum_i b_i \otimes v_i \longmapsto \textstyle\sum_i b_i^p \otimes v_i.$$

The sections $1 \otimes \sum_i b_i \otimes v_i \in F^*\mathscr{L}_{P/G}(U)$ are ∇_{can}-horizontal. Since the connection on $\mathscr{L}_{P/G}(V^{(1)})$ is the one induced by the connection on P, the sections $\sum_i b_i^p \otimes v_i \in \mathscr{L}_{P/G}(V^{(1)})(U)$ are horizontal. If we shrink U so that $\mathscr{L}_{P/G}(U)$ becomes a free A-module, we can assume that the A-module $A \otimes_{F,A} (B \otimes_k V)^G$ is free with a basis of the form $\{1 \otimes s_j\}$. Hence θ is horizontal. $\square$

From Lemma 44, follows that if every representation V of $\Pi^{\mathbf{dR}}$ is F-divisible (see subsection 1.3.1 for terminology), then every $(\mathscr{E},\nabla)\in\mathbf{dR}(X)$ has p-curvature zero. This is very far from the truth (see below) and hence $\mathbf{dR}(X)$ will not have enough properties to make it an interesting (and by interesting we mean similar to the complex analytic case) category of differential equations.

Example: Let X be an elliptic curve over k algebraically closed and let ω be a generator of the vector space $\mathrm{H}^0(X,\Omega^1_{X/k})$. Consider the de Rham sheaf $(\mathscr{E},\nabla)$, with $\mathscr{E}=\mathscr{O}_X^{\oplus 2}$ as an $\mathscr{O}_X$-module and

$$\nabla(D)(\alpha,\beta)=(D\alpha+\beta\omega(D),D\beta),\qquad \alpha,\beta\in\mathscr{O}_X(U),\quad D\in\Theta_X(U).$$

By Cartier's Theorem ([19], Thm. 5.1, p. 190) if $\mathscr{E}$ had zero p-curvature, it would be locally generated, as an $\mathscr{O}_X$-module, by the horizontal sections. Then, we would be able to obtain an open cover $\{U_i\}$ and invertible functions $\alpha_i,\beta_i\in\mathscr{O}_X^\times(U_i)$ such that

$$d\alpha_i+\beta_i\omega=d\beta_i=0.$$

Let K denote the function field of X. The expression above will show that

$$\omega=-\frac{d\alpha_i}{\gamma_i^p},\qquad \gamma_i\in K$$

in $\Omega^1_{K/k}$. Computing the Hasse invariant as originally defined by Hasse (see S. Lang's *Elliptic Functions*), we conclude that X is supersingular. In particular, when X is ordinary, the monodromy group $\Pi^{\mathbf{dR}}_{\mathrm{mono}}(\mathscr{E})=\mathrm{im}(\Pi^{\mathbf{dR}}\longrightarrow\mathrm{GL}(x_0^*\mathscr{E}))$ is not etale.

2.3 The category $\mathfrak{Nstr}(X)$

Take k of positive characteristic p. Let x_0 be a k-rational point of X noetherian, regular, separated and connected.

Definition 45. *The nilpotent stratified fundamental group of X, $\Pi^{\mathbf{nstr}}$, is the group scheme associated to the neutral Tannakian category $\mathfrak{Nstr}(X)$ via the fibre functor x_0^*.*

2.3.1 Profiniteness

Theorem 46. *Assume that $\mathrm{H}^1(X,\mathscr{O}_X)$ and $\mathrm{H}^0(X,\mathscr{O}_X)$ are finite dimensional. Then $\Pi^{\mathbf{nstr}}$ is profinite.*

Proof: As in Theorem 40, we need to show that $\mathrm{Ext}^1_{\mathbf{str}}(\mathbb{1},\mathbb{1}) = \mathrm{Hom}_{\mathbf{str}}(\Pi^{\mathbf{nstr}}, \mathbb{G}_a)$ is a finite dimensional vector space. An extension of $\mathbb{1}$ by $\mathbb{1}$ in $\mathbf{str}(X)$ is given by an F-divided sheaf $\{\mathscr{E}_i\}_{i\in\mathbb{N}}$ such that each $\mathscr{E}_i$ is an extension of $\mathscr{O}_{X^{(i)}}$ by itself and the diagram

$$\begin{tikzcd} 0 \arrow[r] & F^*\mathscr{O}_{X^{(i+1)}} \arrow[r] \arrow[d] & F^*\mathscr{E}_{i+1} \arrow[r] \arrow[d, "\cong"'] & F^*\mathscr{O}_{X^{(i+1)}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{O}_{X^{(i)}} \arrow[r] & \mathscr{E}_i \arrow[r] & \mathscr{O}_{X^{(i)}} \arrow[r] & 0 \end{tikzcd}$$

commutes. Hence we obtain a homomorphism from $\mathrm{Ext}^1_{\mathbf{str}}(\mathbb{1},\mathbb{1})$ to

$$\varprojlim \mathrm{Ext}^1_{\mathscr{O}}(\mathscr{O},\mathscr{O}) = \varprojlim \mathrm{H}^1(X^{(i)}, \mathscr{O}_{X^{(i)}}),$$

with F^* being used to form the projective limit. We claim that this homomorphism is actually bijective. Surjectivity is obvious and we prove injectivity. Let $\{\mathscr{E}_i\}_{i\in\mathbb{N}}$ be an extension such that $\mathscr{E}_i \cong \mathscr{O}^{\oplus 2}_{X^{(i)}}$ for each i. Since $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X^{\oplus 2}, \mathscr{O}_X^{\oplus 2})$ is finite dimensional, we can apply Gieseker's result (Lemma 20) to prove that $\{\mathscr{E}_i\}_{i\in\mathbb{N}} \cong \mathbb{1}\oplus\mathbb{1}$ in $\mathbf{str}(X)$. It is easy to verify (thinking of representations of groups) that, in this case, the extension $\{\mathscr{E}_i\}$ is equivalent to the trivial extension.

Let $V := \mathrm{H}^1(X, \mathscr{O}_X)$. By flat base change, the vector spaces $\mathrm{H}^1(X^{(i)}, \mathscr{O}_{X^{(i)}})$ is canonically isomorphic to the vector space $V^{(i)}$ (see section 1.3.1 for notation). So, $\mathrm{Ext}^1_{\mathbf{str}}(\mathbb{1},\mathbb{1})$ is the vector space $\{(v_i)_{i\in\mathbb{N}} \in V;\ F^*(v_{i+1}) = v_i\}$ with calibrated multiplication by k: $\lambda\cdot(v_n) = (\lambda^{p^{-n}} v_n)$.

Assuming for the moment that k is algebraically closed, we can use the corollary on page 143, §14 of [30] to decompose V as $V_{\mathrm{s}} \oplus V_{\mathrm{n}}$; each summand is stable under F^* and $F^{*r}|V_{\mathrm{n}} = 0$, for some r sufficiently large, while $F^*|V_{\mathrm{s}}$ is bijective. It follows immediately that the first projection induces an isomorphism between $\mathrm{Ext}^1_{\mathbf{str}}(\mathbb{1},\mathbb{1})$ and V_{s}.

In general, we note that, for an extension field K/k, $\varprojlim_i V^{(i)} \subseteq \varprojlim (V^{(i)} \otimes_k K)$ and that the later K-vector space is of finite dimension for some finite extension K/k. It follows that $\varprojlim_i V^{(i)}$ is also of finite dimension over k. $\square$

Corollary 47. *If X is as above and k is algebraically closed, then $\Pi^{\mathbf{nstr}}$ is proetale. In fact, it is the largest unipotent quotient of the etale fundamental group (scheme).*

Proof: Use Corollary 35, Proposition 36 and the theorem above. $\square$

$\mathrm{Ext}^1_{\mathbf{str}}(\mathbb{1},\mathbb{1})$, another perspective

Using the Cartier-Katz Theorem for torsors, and the equivalence of the categories of stratified and F-divided sheaves, one can translate Theorem 27 in terms of F-divided torsors. Define a pointed F-divided G-torsor as the data of an F-divided torsor $\{P_i\}_{i\in\mathbb{N}}$ with k-rational points $p_i \in P_i(k)$ above the points $F^i(x_0) \in X^{(i)}(k)$ plus the requirement that the isomorphisms $F^*P_{i+1} \cong P_i$ take p_{i+1} to p_i.

Let X be smooth and connected with $x_0 \in X(k)$. We can reobtain, in the case $\mathrm{H}^0(X, \mathscr{O}_X) = k$, the description of $\mathrm{Hom}(\Pi^{\mathbf{str}}, \mathbb{G}_a) = \mathrm{Ext}^1_{\mathbf{str}}(\mathbb{1}, \mathbb{1})$ as the projective limit $\varprojlim \mathrm{H}^1(X, \mathscr{O}_X)$ (Frobenius pull-back giving the transition maps in the projective system) from the universal property of $\Pi^{\mathbf{str}}$. Recall that $\mathrm{H}^1(X, \mathbb{G}_a)$ is the same for the Zariski, etale or flat topologies on X. For every pointed F-divided $\mathbb{G}_a$-torsor $\{P_i, p_i\}_{i\in\mathbb{N}}$ we can naturally associate an element of $\varprojlim \mathrm{H}^1(X^{(i)}, \mathscr{O}_{X^{(i)}})$ given by the class of P_i in each H^1. We claim that this map is bijective. Surjectivity follows from the fact that every $\mathbb{G}_a$-torsor over X has a k-rational point above x_0 (since it is locally trivial for the Zariski topology). To prove injectivity, assume that for two pointed F-divided torsors $\{P_i, p_i\}$ and $\{Q_i, q_i\}$, P_i and Q_i are isomorphic as torsors. Since $\mathbb{G}_a$ is abelian, follows that we can arrange for the isomorphisms of torsors between P_i and Q_i to take p_i to q_i. Call this isomorphism α_i. The lemma below proves that there is only *one* such isomorphism and hence $F^*(\alpha_{i+1}) = \alpha_i$. So $\{P_i, p_i\}$ and $\{Q_i, q_i\}$ are isomorphic pointed F-divided torsors.

Lemma 48. *Let G be an abelian affine group scheme and let P be a G-torsor over X. Suppose that all morphisms of X to G are constant.*[1] *Let $\alpha, \beta : P \longrightarrow P$ be morphisms of torsors which coincide on some k-rational point of P. Then $\alpha = \beta$.*

Proof: The proof is an exercise in the theory of torsors. Let $U_i \longrightarrow X$ be an fpqc covering of X which trivializes P and let $\psi_i : U_i \times G \longrightarrow P|U_i$ be isomorphisms of torsors ($P|U_i$ is the base change of P to U_i). Let $\psi_{ij} \in G(U_{ij})$ be the transition functions given by $\psi_j^{-1} \circ \psi_i = \mathrm{id} \times \psi_{ij}$. On this covering, let $\alpha(u, g) = (u, \alpha_i(u) \cdot g)$. Then $\alpha_j|U_{ij} = \psi_{ij} \cdot \alpha_i|U_{ij} \cdot \psi_{ij}^{-1} = \alpha_i|U_{ij}$. Follows (by fpqc descent) that $\alpha_i = f|U_i$ for some $f : X \longrightarrow G$; the analogous being true for β. By the assumption made on the morphisms $X \longrightarrow G$, $\alpha_i(u) = c_\alpha$ and $\beta_i(u) = c_\beta$ for c_α, c_β k-rational points of G. Because α, β coincide on some k-rational point of P, follows that $c_\alpha = c_\beta$ and, again by fpqc descent, $\alpha = \beta$. $\square$

2.4 A lemma of Nori

Lemma 50 below can be extracted from chapter IV of Nori's thesis, [34]. Nori uses this result to obtain the profiniteness of the nilpotent fundamental group in positive characteristic. We will give a different and more direct proof.

[1] That means: For every $f : X \longrightarrow G$, there exists a k-rational point $g_0 \in G(k)$ such that for all k-algebras $f_A : X(A) \longrightarrow G(A)$ is just the image of g_0 in $G(A)$

For the sake of elegance, we shall use a well known elementary result from the theory of additive polynomials.

Recall that an additive polynomial[2] $P \in k[x] - \{0\}$ is a polynomial which satisfies the following identity in $k[x_1, x_2]$:

$$P(x_1 + x_2) = P(x_1) + P(x_2).$$

Obviously, addition and *composition* of additive polynomials results in an additive polynomial. The ring of additive polynomials (multiplication = composition) together with the zero polynomial is none other than $\mathrm{Hom}_{k-\mathrm{group}}(\mathbb{G}_a, \mathbb{G}_a)$ which, in positive characteristic p, is the twisted polynomial ring $k\{F\}$ with $\lambda^p F = F\lambda$, where F is the Frobenius endomorphism of $\mathbb{G}_a$ ([42], 8.4, Theorem, p. 65).

Lemma 49 (existence of lcm)**.** *Given additive polynomials P and Q, there exists an additive polynomial L which is right divisible by P and Q.*

□

Lemma 50. *Let $G = \mathrm{Spec}\, R$ be an unipotent affine group scheme over k. If $\mathrm{Hom}(G, \mathbb{G}_a)$ is finite dimensional, then G is profinite.*

Proof: Because G is a projective limit of unipotent algebraic quotients ([42], 3.3, Corollary, p. 24 and exercise 3. of Chapter 8, p. 66), we can assume that G itself is algebraic and we shall prove, under this assumption, that G is finite.

By the Lie-Kolchin Theorem ([42], 8.3, Theorem, p. 64) we can assume that G is a closed subgroup of $\mathbb{U}_n$. Let x_{ij} denote the restriction to G of the coordinate functions on $\mathbb{A}^{n^2}$. If $\Delta : R \longrightarrow R \otimes R$ is the co-multiplication we have, for $i < j$,

$$\Delta(x_{ij}) = \begin{cases} x_{ij} \otimes 1 + 1 \otimes x_{ij}, & \text{if } j = i+1, \\ x_{ij} \otimes 1 + 1 \otimes x_{ij} + \sum_{i<l<j} x_{il} \otimes x_{lj}, & \text{ij } j > i+1. \end{cases} \tag{2.3}$$

Thus, $k[x] \longrightarrow R, x \mapsto x_{i,i+1}$ determines a homomorphism $G \longrightarrow \mathbb{G}_a$. By assumption, the vector space $V_{i,i+1} := \mathrm{span}\{x_{i,i+1}^{p^m};\ m \in \mathbb{N}\}$ is finite dimensional. In consequence, there exist additive polynomials $\widetilde{P}_i$ such that $\widetilde{P}_i(x_{i,i+1}) = 0$. By the existence of lcm, there exists an additive polynomial P_1 with $P_1(x_{i,i+1}) = 0$ for all $i = 1, \ldots, n-1$.

Assume that we have proved, for $r > 1$, that

There are additive polynomials $P_1, \ldots, P_{r-1}$ such that $P_j(x_{i,i+j}) = 0$. (†)

[2]We have removed the 0 polynomial from the definition of additive polynomial for convenience.

We want to prove that (†) holds for r and this will be sufficient to prove the lemma. Let $V_{ij} := \mathrm{span}\{x_{ij}^{p^m}\,;\ m \in \mathbb{N}\}$. From the expressions in (2.3),

$$\Delta(x_{i,i+r}^{p^m}) - x_{i,i+r}^{p^m} \otimes 1 - 1 \otimes x_{i,i+r}^{p^m} \in \sum_{l=i+1}^{i+r-1} V_{il} \otimes V_{l,i+r}, \qquad (m \in \mathbb{N}).$$

Since

$$\sum_{l=i+1}^{i+r-1} V_{il} \otimes V_{l,i+r}$$

is finite dimensional, there is a non-trivial relation

$$\sum_m \lambda_{im} \Delta(x_{i,i+r}^{p^m}) - \lambda_{im}(x_{i,i+r}^{p^m} \otimes 1) - \lambda_{im}(1 \otimes x_{i,i+r}^{p^m}) = 0, \qquad i = 1, \dots, n-r.$$

Hence there exist additive polynomials $\widetilde{Q}_i$ such that $\widetilde{Q}_i(x_{i,i+r}) : G \longrightarrow \mathbb{G}_a$ defines a homomorphism and from the hypothesis of the lemma there exist additive polynomials Q_i such that $Q_i(x_{i,i+r}) = 0$. The existence of the lcm now completes the induction in (†). $\square$

Chapter 3

The stratified fundamental group of an abelian variety

3.1 Introduction

Let X be an abelian variety of dimension g and take k algebraically closed of positive characteristic p. The base point x_0, giving the fibre functors, will be the identity element of $X(k)$.

We shall be interested in *computing* the stratified fundamental group scheme of X (Theorem 58). This will be achieved using the previous description of the unipotent part, a description of the character group and a decomposition result.

The insight provided by Tannakian duality is crucial for the study of these objects. Indeed, even though there exist decomposition theorems for homogeneous sheaves (see remark at the end of chapter), none of them will give a satisfactory decomposition result for $\Pi^{\mathbf{str}}(X, x_0)$ if we do not let group theory guide us in the formulation of the theorems. But once we set out in this direction the results follow smoothly.

3.2 Stratified sheaves of rank one

Notation: Let G be an abelian abstract group and let $[p] : G \longrightarrow G$ be multiplication by p. We shall denote by $G\langle p\rangle$ the group

$$\varprojlim \left(\cdots \xrightarrow{[p]} G \xrightarrow{[p]} G \xrightarrow{[p]} \cdots \right).$$

Lemma 51. *Let Y be a proper and smooth scheme over k such that $\mathrm{NS}(Y)$ is free. Then the character group $\mathbb{X}(\Pi^{\mathbf{str}}_Y)$ is isomorphic to $\mathrm{Pic}^0(Y)\langle p\rangle$. In particular, this holds for curves and abelian varieties.*

Proof: Let $\mathscr{L} = \{\mathscr{L}_i\}_{i\in\mathbb{N}}$ be an F-divided invertible sheaf, which corresponds to a character of $\Pi^{\mathbf{str}}$. Taking the isomorphism classes of $\mathscr{L}_i$ in $\mathrm{Pic}(Y)$, we clearly obtain a surjective homomorphism $\mathbb{X}(\Pi^{\mathbf{str}}) \longrightarrow \mathrm{Pic}(Y)\langle p\rangle$. If $\mathscr{L}_i \cong \mathscr{O}_Y$ for each i, then Lemma 20 shows that $\{\mathscr{L}_i\}_i$ is isomorphic in $\mathbf{str}(Y)$ to the identity element. Now $\mathrm{Pic}(Y)\langle p\rangle = \mathrm{Pic}^0(Y)\langle p\rangle$ because the Néron-Severi group is free (of finite rank). That the Néron-Severi group of an abelian variety is free follows from Cor. 2, p. 178 of [30]. $\square$

Of course, the description of the character group does not, in general, give much information about the group. In the present case though, it will be useful since we shall obtain a decomposition of $\Pi^{\mathbf{str}}$ as a direct product of a *diagonal* group and an *unipotent* group (Theorem 58).

Example 52. Computation of the monodromy. Let $\mathscr{L} = \{\mathscr{L}_i\}_{i\in\mathbb{N}}$ be an invertible stratified sheaf and let G be the monodromy group $\mathrm{im}((\Pi^{\mathbf{str}}) \longrightarrow \mathrm{GL}(e^*\mathscr{L}) = \mathbb{G}_m)$. This group is either μ_m or $\mathbb{G}_m$. μ_m will occur whenever $\mathscr{L}$ has order m in $\mathbb{X}(\Pi^{\mathbf{str}})$. In this case m, is prime to p and each $\mathscr{L}_i$ has order m in $X^\vee(k)$. If $G \cong \mathbb{G}_m$, then either $\mathscr{L}_0$ has infinite order in $X^\vee(k)$ or $\mathscr{L}_i$ has finite order divisible by p for some $i \geq 0$. If every $\xi \in X^\vee(k)$ is of finite order, then

$$\mathbb{X} \cong V_p(X^\vee) \oplus \left(\bigoplus_{l\neq p} (\mathbb{Q}_l/\mathbb{Z}_l)^{\oplus 2g} \right).$$

For another computation of this group of characters, the reader is directed to section 4.3.2.

3.3 Main theorem 58

We are interested in Theorem 58. We start by recalling the following result of Gieseker.

Theorem 53 ([16], thm. 2.6, p. 10)**.** *Every simple representation V of $\Pi^{\mathbf{str}}$ is one dimensional.* $\square$

This result says that $\Pi^{\mathbf{str}}$ is a projective limit of triangularizable groups and that every $\mathscr{E} \in \mathrm{Ob}\,\mathbf{str}(X)$ has a filtration in $\mathbf{str}(X)$

$$0 \subseteq \mathscr{E}_0 \subseteq \cdots \subseteq \mathscr{E}_r = \mathscr{E}, \quad \mathscr{E}_{i+1}/\mathscr{E}_i = \{\mathscr{L}_0, \mathscr{L}_1, \ldots\} \tag{3.1}$$

with $\mathscr{L}_i \in \mathrm{Pic}^0(X/k)$. In particular (see *ii*) of corollary 57) we have:

Corollary 54. *The terms $\mathscr{E}_i$ of an object $\{\mathscr{E}_i\}$ in $\mathbf{str}(X)$ are homogeneous.* $\square$

Theorem 55. *Every stratified sheaf $\mathscr{E}$ is isomorphic in $\mathbf{str}(X)$ to a direct sum*

$$\bigoplus_i \mathscr{L}_i \otimes \mathscr{N}_i,$$

where $\mathscr{L}_i$ is invertible and stratified and $\mathscr{N}_i \in \mathfrak{N}\mathbf{str}(X)$.

The proof will make use of the Fourier-Mukai transform ([28]). We digress to make a brief introduction to this marvelous object.

Digression: Let Y and Z be k-schemes and let $\mathscr{P}$ be an invertible $\mathscr{O}_{Y\times Z}$-module. Denote by p_Y and p_Z the projections from $Y \times Z$. Consider the functor $\mathrm{Mod}(Y) \longrightarrow \mathrm{Mod}(Z)$ given by $\mathscr{M} \mapsto p_{Z*}(p_Y^*\mathscr{M} \otimes \mathscr{P})$ and let

$$\Phi_{\mathscr{P}} : D^-(Y) \longrightarrow D^-(Z)$$

be its right derived functor. Because $\mathscr{P}$ is $\mathscr{O}_X$-flat, for any $\mathscr{O}_X$-module $\mathscr{E}$, $\Phi_{\mathscr{P}}(\mathscr{E}) = \mathbf{R}p_{Z,*}(\mathscr{P} \otimes p_Y^*\mathscr{E})$.

The theory works beautifully for the abelian variety X. Let Φ_X be $\Phi_{\mathscr{P}}$ where $\mathscr{P}$ is a normalized[1] Poincaré sheaf of $X \times X^\vee$. Analogous notation for $X^\vee$ is in force. Here is the main theorem followed by two the very useful working properties. Let x, y, etc denote points of $X(k)$ and α, β, etc points of $X^\vee(k)$. Let P_α denote $\mathscr{P}|X \times \alpha$ and P_x denote $\mathscr{P}|x \times X^\vee$.

Theorem 56. *i) The functors Φ_X and $\Phi_{X^\vee}$ can be extended to $D(X)$ and $D(X^\vee)$ respectively and there are natural isomorphisms*

$$\Phi_X \circ \Phi_{X^\vee} = (-1)^* \circ [-g], \qquad \Phi_{X^\vee} \circ \Phi_X = (-1)^* \circ [-g],$$

where $[-g]$ denotes the shift of the complex g places to the left. (See [28], thm. 2.2, p. 156. or [35], Thm 11.6, p. 140).

ii) Let $x \in X(k)$ and $\alpha \in X^\vee(k)$. There are natural isomorphisms of functors

$$\Phi_X \circ (? \otimes P_\alpha) \cong t_\alpha^* \circ \Phi_X(?), \qquad \Phi_{X^\vee} \circ t_\alpha^*(?) \cong \Phi_{X^\vee}(?) \otimes P_{-\alpha}.$$

(See [28], 3.1, p. 158 or or [35], 11.3.1-2, p. 140)

iii) If $f : X \longrightarrow Y$ is an isogeny, then the diagrams

$$\begin{array}{ccc} D(X) & \xrightarrow{\Phi_X} & D(X^\vee) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle (f^\vee)_*} \\ D(Y) & \xrightarrow[\Phi_Y]{} & D(Y^\vee). \end{array}$$

commute up to natural isomorphism ([28], 3.4, p. 159, or [35], 11.3.5, p. 142).

[1] Normalized means that $\mathscr{P}|X \times e \cong \mathscr{O}_X$ and $\mathscr{P}|e \times X^\vee \cong \mathscr{O}_{X^\vee}$

Let $k(\alpha)$ denote the skyscraper sheaf $\alpha_*(k)$ of $X^\vee$.

Corollary 57. *i)* $\Phi_X(P_\alpha) \cong k(-\alpha)[-g]$.

ii) If $\mathscr{E} \in \mathrm{coh}(X)$ *has a filtration*

$$\mathscr{E} = \mathscr{E}_r \supseteq \cdots \supseteq \mathscr{E}_0 = 0, \qquad \mathscr{E}_{i+1}/\mathscr{E}_i \cong P_{\alpha_{i+1}},$$

then $\Phi_X(\mathscr{E}) \cong R^g p_{X^\vee,*}(\mathscr{P} \otimes p_X^*\mathscr{E})[-g]$ *and has a filtration*

$$R^g p_{X^\vee,*}(\mathscr{P} \otimes p_X^*\mathscr{E}) = V_r \supseteq \cdots \supseteq V_0 = 0, \qquad V_{i+1}/V_i \cong k(-\alpha_{i+1}).$$

In particular, $R^g p_{X^\vee,*}(\mathscr{P} \otimes p_X^*\mathscr{E})$ *is skyscraper and* $\mathscr{E}$ *is homogeneous.*

iii) The functor $T = R^g p_{X^\vee,*}(\mathscr{P} \otimes p_X^* ?)$ *induces an equivalence between homogeneous sheaves on* X *and skyscraper coherent sheaves on* $X^\vee$.

Proof: *i*) follows from *ii*) of Theorem 56 and the general properties of the cohomology of the Poincaré sheaf.

ii) follows from *i*) (the homogeneity of $\mathscr{E}$ is a consequence of the fact that $\Phi_X(\mathscr{E}) \otimes P_x \cong \Phi_X(\mathscr{E})$ for all $x \in X(k)$ plus part *ii*) of Theorem 56).

iii) is in [28], Ex. 3.2, p. 158. □

End of digression.

Proof of Theorem 55: We have to show that if $\mathscr{E} \in \mathrm{Ob}\,\mathbf{str}(X)$ is indecomposable, then the invertible stratified sheaves $\mathscr{L}^i = \{\mathscr{L}^i_\nu\}_{\nu \in \mathbb{N}}$ in the filtration provided by (3.1) are all isomorphic (in $\mathbf{str}(X)$ of course). Suppose, by absurd, that for some $i \neq j$ we have $\mathscr{L}^i \ncong \mathscr{L}^j$ and hence, there is a $\nu_0 \in \mathbb{N}$ such that $\mathscr{L}^i_{\nu_0} \ncong \mathscr{L}^j_{\nu_0}$ (as $\mathscr{O}_X$-modules). Because $\mathscr{E} = \{\mathscr{E}_0, \mathscr{E}_1, \ldots\}$ is indecomposable, so is $\mathscr{E}' = \{\mathscr{E}_{\nu_0}, \mathscr{E}_{\nu_0+1}, \ldots\}$ and hence we can assume $\nu_0 = 0$. Let $\{T\mathscr{E}_i\} := \{E_i\}$. By *iii*) of Theorem 56 (and the usual manipulation of Frobenius), $(F^\vee)_*(E_{i+1}) \cong E_i$.

From *ii*) of Corollary 57, $\mathrm{Supp}(E_0) = S'_0 \sqcup S''_0$ (disjoint union); thus

$$\mathrm{Supp}(E_{i+1}) = S'_{i+1} \sqcup S''_{i+1}$$

with $F^\vee(S'_{i+1}) = S'_i$ and $F^\vee(S''_{i+1}) = S''_i$. It results that there are decompositions $E_i = E'_i \oplus E''_i$ with $(F^\vee)_*(E'_{i+1}) \cong E'_i$ and $(F^\vee)_*(E''_{i+1}) \cong E''_i$. Hence there exist stratified sheaves $\{\mathscr{E}'_i\}$ and $\{\mathscr{E}''_i\}$ with $\mathscr{E}_i \cong \mathscr{E}'_i \oplus \mathscr{E}''_i$ (as $\mathscr{O}_X$-modules). Another application of Lemma 20 gives a contradiction with the indecomposability of $\{\mathscr{E}_i\}$ and consequently the theorem is proved. □

We now translate Theorem 55 into the terminology of tensor product of Tannakian categories. Recall that given two k-linear abelian categories $\mathfrak{A}_1$ and $\mathfrak{A}_2$, the *tensor* *product* is a k-linear abelian category $\mathfrak{A}$ with a k-bilinear right-exact-in-each-variable functor

$$\otimes : \mathfrak{A}_1 \times \mathfrak{A}_2 \longrightarrow \mathfrak{A} \tag{3.2}$$

which is universal for all k-bilinear and right-exact-in-each-variable $T : \mathfrak{A}_1 \times \mathfrak{A}_2 \longrightarrow \mathfrak{C}$, that is, given such T, there is a k-linear and right exact $T' : \mathfrak{A}_1 \otimes \mathfrak{A}_2 \longrightarrow \mathfrak{C}$ such that $T' \circ \otimes = T$. Moreover, this T' is unique up to natural isomorphism.

The tensor product $\mathfrak{A}$ of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ exists if each $\mathfrak{A}_i$ is artinian and $\mathrm{Hom}_{\mathfrak{A}_i}(X, Y)$ is a finite dimensional k-space. Also, there is a natural isomorphism obtained from (3.2)

$$\mathrm{Hom}_{\mathfrak{A}_1}(X_1, Y_1) \otimes_k \mathrm{Hom}_{\mathfrak{A}_2}(X_2, Y_2) \xrightarrow{\cong} \mathrm{Hom}_{\mathfrak{A}}(X_1 \otimes X_2, Y_1 \otimes Y_2). \tag{3.3}$$

Theorem 58. *There is a natural isomorphism*

$$\Pi^{\mathbf{str}}(X, x_0) \xrightarrow{\cong} T_p(X) \times \mathrm{Diag}(\mathrm{Pic}^0\langle p \rangle),$$

where $T_p(X)$ is the p-adic Tate module (of k-rational points).

Proof: Let $\mathbf{lstr}(X)$ be the full neutral Tannakian subcategory of $\mathbf{str}(X)$ whose objects are direct sums of invertible and stratified $\{\mathscr{L}_i\}$. Restriction of the tensor product

$$\otimes : \mathfrak{N}\mathbf{str}(X) \times \mathbf{lstr}(X) \longrightarrow \mathbf{str}(X)$$

will give, according to Theorem 55, an *essentially surjective* k-linear tensor functor

$$\underline{\otimes} : \mathfrak{N}\mathbf{str}(X) \otimes \mathbf{lstr}(X) \longrightarrow \mathbf{str}(X).$$

Write $\mathfrak{A}_1 = \mathfrak{N}\mathbf{str}(X)$ and $\mathfrak{A}_2 = \mathbf{lstr}(X)$. Then formula (3.3) is easily verified for $Y_1 = \mathbb{1}$ and $X_2 = \mathbb{1}$. By rigidity we obtain the general case; this proves that $\underline{\otimes}$ is full and faithful and consequently an equivalence.

Let $\Pi^{\mathbf{lstr}}$ and $\Pi^{\mathbf{nstr}}$ denote the fundamental groups associated to $\mathbf{lstr}(X)$ and to $\mathfrak{N}\mathbf{str}(X)$ via the fibre functor x_0^*. In [10], 6.21, p. 164 (or 5.18, p. 151) its is proved that, for affine group schemes G_1, G_2, the tensor product

$$\mathrm{Rep}_k(G_1) \times \mathrm{Rep}_k(G_2) \longrightarrow \mathrm{Rep}_k(G_1 \times G_2), \quad (V_1, V_2) \mapsto \mathrm{Res}(\mathrm{pr}_1)(V_1) \otimes \mathrm{Res}(\mathrm{pr}_2)(V_2)$$

makes $\mathrm{Rep}_k(G_1 \times G_2)$ into the tensor product category $\mathrm{Rep}_k(G_1) \otimes \mathrm{Rep}_k(G_2)$. So the tensor equivalence $\underline{\otimes}$ corresponds to an isomorphism $\Pi^{\mathbf{str}} \longrightarrow \Pi^{\mathbf{nstr}} \times \Pi^{\mathbf{lstr}}$.

All representations of $\Pi^{\mathbf{lstr}}$ are direct sums of one dimensional representations and hence this group is diagonalizable ([42], 2.2, p. 15 for the terminology): $\Pi^{\mathbf{lstr}} =$

$\mathrm{Diag}(\mathrm{Pic}^0(X)\langle p\rangle)$, by Lemma 51 above. By Corollary 47, the group $\Pi^{\mathbf{nstr}}$ is the largest unipotent quotient of the etale fundamental group $\Pi^{\mathbf{et}}$ of X

$$\Pi^{\mathbf{et}} \cong \prod_{l \text{ prime}} T_l(X), \quad \text{canonically by [30], §18, p. 171.}$$

For $l \neq p$, the group schemes $T_l(X)$ are non-canonically isomorphic to $\mathbb{Z}_l^{2g}$ — which is diagonalizable, and $T_p(X)$ is non-canonically isomorphic to $\mathbb{Z}_p^r$ — which is unipotent ([42], Theorem 8.5, p. 66). $\square$

Let C be a smooth and projective curve over k with a k-rational point P. Let J be its Jacobian and let $f : C \longrightarrow J$ be the natural morphism given by the universal property of J; in particular it sends P to 0.

Given any affine algebraic group scheme G, we can form the largest abelian quotient $G^{\mathrm{ab}} := G/[G,G]$ which has the expected universal property: any homomorphism $G \longrightarrow A$ with A abelian group scheme factors through G^{ab} ([42], ex. 1, p. 125). For a general group scheme $G = \varprojlim_\alpha G_\alpha$ with $\mathscr{O}(G_\alpha) \subseteq \mathscr{O}(G)$ of finite type, we can form G^{ab} as the limit $\varprojlim_\alpha G_\alpha^{\mathrm{ab}}$; obviously we have the same universal property as before. The next corollary gives the expected version in positive characteristic of the known complex analytic analogue.

Corollary 59. *The natural homomorphism* $\Pi^{\mathbf{str}}(C,P) \longrightarrow \Pi^{\mathbf{str}}(J,0)$ *induces an isomorphism* $\Pi^{\mathbf{str}}(C,P)^{\mathrm{ab}} \longrightarrow \Pi^{\mathbf{str}}(J,0)$.

Proof: Because $\Pi^{\mathbf{str}}(C,P)^{\mathrm{ab}}$ is abelian it can be decomposed into a diagonal and an unipotent part. The diagonal part is controlled by the character group $\mathrm{Pic}^0(C)\langle p\rangle$ while the unipotent part is controlled by the largest etale quotient of $\Pi^{\mathbf{nstr}}(C,P)^{\mathrm{ab}}$. Now the corollary follows from the fact that $f^* : \mathrm{Pic}^0(J) \longrightarrow \mathrm{Pic}^0(C)$ and $f^* : \Pi^{\mathbf{et}}(C,P)^{\mathrm{ab}} \longrightarrow \Pi^{\mathbf{et}}(J,0)$ are isomorphisms ([27], 9.3, p. 196 and 9.1, p. 195). $\square$

Remark: $\Pi^{\mathbf{str}}$ will not base change correctly (at least in a functorial way). That can be seen as follows. First note that given a diagonal group G over k, the natural map $\mathbb{X}(G) \longrightarrow \mathbb{X}(G_K)$ is bijective for any extension field $K \supseteq k$. Let K be an algebraically closed field containing the function field of $X^\vee$. Then $X^\vee(k) \longrightarrow (X_K)^\vee(K) = X^\vee(K)$ will *not* be bijective (the point corresponding to the inclusion in K of the function field will not be in the image of $X^\vee(k)$). This is a different picture from the etale case because characters of $\Pi^{\mathbf{et}}$ lie in $X^\vee(k)_{\mathrm{tors}} \cong X^\vee(K)_{\mathrm{tors}}$.

Remark: A. Scholl and N. Shepherd-Barron pointed out that the proof of Theorem 55 is a simple transposition of known results.

Let $\mathscr{E} = \{\mathscr{E}_i\} \in \mathbf{str}(X)$ be indecomposable. By the Krull-Schimdt property of vector-bundles, we can assume that $\mathscr{E}_i$ is indecomposable for $i \gg 0$. The existence of Gieseker's filtration $\mathscr{E} = \mathscr{E}^{(r)} \supset \mathscr{E}^{(r-1)} \cdots \supset \mathscr{E}^{(0)} = 0$ (Theorem 53) implies that $\mathscr{E}_i$ is indecomposable only if it is a successive extension of *the same* invertible sheaf $\mathscr{L}_i$ (see Thm 2.3 of M. Miyanishi's paper in Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki, Kinokuniya, Tokyo, 1973).

Our contribution to the study of $\Pi^{\mathbf{str}}$ relies on finding the correct definition ($\Pi^{\mathbf{nstr}}$) for the complementary summand of $\Pi^{\mathbf{lstr}}$ in terms of *group theoretical data*, which in turn allowed to describe precisely $\Pi^{\mathbf{str}}$ from our previous results. Only by paying attention to group theory we were led to a satisfactory result.

Chapter 4

A link with rigid geometry

4.1 Introduction

In the complex analytic world, the topological fundamental group controls the category of stratified (complex analytic coherent) sheaves [9]. The GAGA theorems of Serre show that in the projective case, this control is extended even to coherent sheaves on algebraic varieties. Motivated by this beautiful fact, we investigate in this chapter the part played by the rigid fundamental group in classifying stratified (F-divided) sheaves on a smooth rigid analytic variety. The relation between the rigid and stratified fundamental groups (section 4.2) is presented by a natural tensor functor between the representations categories and was used in [17] for the first (and only, as far as we know) time. It turns out that the rigid fundamental group is far from giving complete information about the stratified sheaves (see Theorem 62). In remark (a) after the proof of Theorem 62, the reader will find one reason behind this deficiency: admissible covers are rather limited set-theoretical covers. This phenomenon is already well known from the theory of p-adic differential equations. A more disturbing reason is the topic of Chapter 5.

In section 4.3 we examine the case of abelian varieties whose analytification is a torus. It turns out that the structural results obtained in the previous chapters (proetaleness of the unipotent stratified fundamental group and decomposition into diagonal and unipotent parts) can be successfully used to understand the discrepancy between the topological fundamental group and the stratified fundamental group (Proposition 66 and Corollary 68). See also section 6.2.

It is important to point out that the results here are far from definitive and present only a tentative of understanding, in positive characteristic non-Archimedean geometry, what are "differential equations" and what part is played by the rigid topology in solving them.

4.2 A natural relation between the rigid analytic and stratified fundamental groups

We will be dealing with rigid analytic varieties. Let k be algebraically closed of characteristic $p > 0$, endowed with a non-Archimedean metric $|\cdot| : k \longrightarrow \mathbb{R}_{\geq 0}$ with respect to which it is complete. We will follow [4] and work with rigid analytic varieties with the strong topology – the reader should always bear in mind that affinoid domains form a basis for this G-topology. A rigid analytic variety is a triple of a set X, a Grothendieck topology[1] on the category of subsets of X (a G-topological space) and a sheaf of k-algebras $\mathscr{O}_X$ for this Grothendieck topology such that (a) all the fibres $\mathscr{O}_{X,x}$ are local rings and (b) for some admissible cover $X = \cup X_\alpha$, $(X_\alpha, \mathscr{O}_X|X_\alpha)$ is isomorphic, as a locally ringed G-space, to the maximal spectrum of an affinoid algebra endowed with its strong topology and structural sheaf.

Given such an object X, the Frobenius morphism $F : X \longrightarrow X$ is a morphism of locally ringed G-topological spaces which is the identity on the categories giving the G-topology and has the absolute Frobenius $a \mapsto a^p$ as defining homomorphism $\mathscr{O}_X \longrightarrow \mathscr{O}_X = F_*\mathscr{O}_X$. If the local rings in X are all regular, then the $F^{-1}\mathscr{O}_X$-algebra $\mathscr{O}_X$ is coherent and faithfully flat.

We now state and prove a lemma which will be used throughout what follows. Recall ([4], p. 337) that a rigid analytic variety Y is connected if there is no admissible covering $\{U_i;\ i \in I\}$ of Y such that

$$I = I_1 \sqcup I_2, \quad \bigcup_{i\in I_1} U_i \cap \bigcup_{i\in I_2} U_i = \emptyset \quad \text{and} \quad \bigcup_{i\in I_1} U_i \neq \emptyset \neq \bigcup_{i\in I_2} U_i.$$

Lemma 60 (Analytic continuation)**.** *Let Y be a connected rigid analytic variety whose local rings are domains.*

i) If for some $y \in Y$ we have $f_y = 0$ (the element of $\mathscr{O}_y$ induced by f), then $f = 0$.

ii) If $f \in \mathscr{O}(Y)$ is such that, for some $y \in Y$, $f_y \in \mathscr{O}_{Y,y}^{p^r}$ for every $r \in \mathbb{N}$, then f is constant.

iii) Let $\mathscr{O}_{(n)} \subseteq \mathscr{O}_Y$ be the sheaf $\operatorname{im}(F^n : \mathscr{O}_Y \longrightarrow \mathscr{O}_Y)$. Then $\cap_n \mathscr{O}_{(n)}$ is the constant sheaf $\widetilde{k}$.

Proof: *i*). This is folklore. Let $\mathfrak{U} = \{U_\alpha;\ \alpha \in I\}$ be an admissible covering of Y by affinoids. Refine the covering $\mathfrak{U}$ by an admissible covering $\mathfrak{V} = \{V_\beta, \beta \in J\}$ where each V_β is an affinoid and $\mathscr{O}(V_\beta)$ is a *domain.* The reason for the existence of such a refinement is as follows. Let A be the ring of analytic functions on some U_α. Given $z \in U_\alpha = \operatorname{Max}(A)$, the natural map $A_z \longrightarrow \mathscr{O}_z$ is injective ([15], Prop. 4.6.1, p. 92)

[1] This topology should satisfy certain basic axioms omitted here but described in [4].

and it follows that all local rings of A are domains. Hence (by standard commutative algebra),

$$\mathrm{Max}(A) = \bigsqcup_{i=1}^{r} \mathrm{Max}(A_i),$$

where the A_i are integral domains; note that there exists a function $g_j \in A$ which is zero on all $\mathrm{Max}(A_i)$ with $i \neq j$ and 1 on $\mathrm{Max}(A_j)$. This implies, in particular, that the above covering is an (admissible) affinoid covering. Using once more the injectivity of $\mathscr{O}(V_\beta)_z \longrightarrow \mathscr{O}_{Y,z}$ we obtain that $f|V_\beta = 0$ whenever $f_z = 0$ for some $z \in V_\beta$. In particular, defining $J_0 := \{\beta \in J;\ f|V_\beta = 0\}$, it follows that $\cup_{\beta\in J_0} V_\beta$ will not intersect $\cup_{\beta\notin J_0} V_\beta$. The connectedness of Y now shows that $Y = \cup_{\beta\in J_0} V_\beta$ and we are done.

ii). There is a $c \in k$ such that $f_y - c \in \cap_r \mathrm{rad}(\mathscr{O}_y)^{p^r} = 0$. By item *i*), $f - c$ is globally zero.

iii). We have a natural inclusion $\widetilde{k} \subseteq \cap_n \mathscr{O}_{(n)}$ and we will show that it is an isomorphism on each $U \subseteq X$ open admissible affinoid which is connected. Since taking the "sheaf associated to the presheaf" preserves fibres, $\mathscr{O}_{(n),y} = \mathscr{O}_y^{p^n} \subseteq \mathscr{O}_y$. Let U be admissible connected of Y and let $f \in \cap_n \mathscr{O}_{(n)}(U)$. Given $y \in U$ it follows that f_y is a p^r-power for every r. Using *ii*), $f \in k$ and we are done. ☐

We recall the notion of analytic covering and refer to loc.cit. for the basic properties; also, we note that B. Conrad has elaborated on the notion of connected components (more generally, irreducible components) in rigid geometry [7], an important aspect of the theory which was overlooked by both [4] and [15].

Definition 61 ([7], p. 492)**.** *Let Y be a rigid analytic variety and let $y \in Y$. Consider the set $C(y) \subseteq Y$ of all $y' \in Y$ such that there exists $Y_1, \ldots, Y_n$ affinoid opens of Y with $Y_i \cap Y_{i+1} \neq \emptyset$ and $y \in Y_1, y' \in Y_n$. Then*

1. *The set $C(y)$ is admissible.*
2. *If $C(y) \cap C(z) \neq \emptyset$, then $C(y) = C(z)$.*
3. *Given a set $S \subseteq Y$, the subset $C = \cup_{s\in S} C(s)$ is admissible and $\{C(s)\}_{s\in S}$ is an admissible covering of C.*

$C(y)$ is called the connected component of the point y in the rigid analytic variety Y.

We say that a morphisms of rigid analytic varieties $f : Z \longrightarrow Y$ is a trivial covering if every connected component of Z is taken isomorphically to Y. In general,

f is said to be a (rigid analytic, or rigid) *covering* if there is an *admissible* covering $\{Y_i\}$ of Y with each Y_i connected and such that $f^{-1}(Y_i) \longrightarrow Y_i$ is a trivial covering. As remarked below, the constraint of admissibility of $\{Y_i\}$ is what makes the theory more interesting.

For convenience, any open $U \subseteq Y$ such that $f^{-1}(U) \longrightarrow U$ is a trivial covering is called a *distinguished* set (open, admissible). A universal analytic covering $\Omega \longrightarrow Y$ is a simply connected analytic covering. The group of automorphisms $\mathrm{Aut}_Y(\Omega)$ is called the *rigid fundamental group*. See section 5.7 of [15] for the basic properties of Ω and $\mathrm{Aut}_Y(\Omega)$ (which are identical to the well known properties of the topological case).

Let X_0 be a smooth connected variety over k. Let X denote the analytification of X_0 (X is automatically connected, see 5.1.3, p. 531 of [7]) and assume that X has a universal analytic covering $\pi : \Omega \longrightarrow X$ with fundamental group $\Lambda = \mathrm{Aut}_X(\Omega)$ (here we are choosing a point $x_0 \in X_0(k)$). We will construct (following Gieseker) a tensor functor from $\mathrm{Rep}_k(\Lambda)$ to $\mathbf{str}(X_0)$ and then show that it identifies (non-canonically) the algebraic hull Λ^{alg} with a quotient of $\Pi^{\mathbf{str}}(X_0)$ in the case where X_0 is projective. In fact, we will only work with the category of F-divided sheaves on a smooth rigid variety (definition is analogous to the classical case; this category will be denoted by the usual $\mathbf{str}(?)$) and then pass to $\mathbf{str}(X_0)$ using rigid GAGA. The method is the obvious analogue of the usual construction in complex analytic geometry, where the Tannakian group scheme obtained from the category of local systems is the algebraic hull of the topological fundamental group (the complex case is in [11]). The reader should observe that the proof of Lemma 17 above shows that for $\{M_i\} \in \mathbf{str}(X)$, the M_i are all locally free ([15], Def. 4.5.1, p. 87) and hence we have a neutral Tannakian category with fibre functor x_0^*.

Construction of a functor Let $\rho : \Lambda \longrightarrow \mathrm{GL}(V, k)$ be a finite dimensional representation of Λ and consider the associated sheaf on X, $\mathfrak{L}(V)$. Recall the definition of $\mathfrak{L}$. For an open $U \subseteq X$, the open $\pi^{-1}(U)$ is Λ-invariant. Let $\lambda \in \Lambda$ act on $V \otimes_k \mathscr{O}_\Omega(\pi^{-1}(U))$ by

$$\lambda \cdot (\sum_i v_i \otimes f_i) = \sum_i \rho(\lambda) v_i \otimes f \circ \lambda^{-1}.$$

Then $\mathfrak{L}(V)(U)$ is the $\mathscr{O}_\Omega(\pi^{-1}(U))^\Lambda = \mathscr{O}_X(U)$-module of all invariant elements of $V \otimes \mathscr{O}_\Omega(\pi^{-1}(U))$. It is easy to see that $\mathfrak{L}(V)$ is always a coherent analytic sheaf on X. We also note that the natural map $\pi^*\mathfrak{L}(V) \longrightarrow V \otimes_k \mathscr{O}_\Omega$ is an isomorphism.

Because k is perfect, every representation V_0 of Λ can be F-divided: there exists a representation V_1 of Λ such that the Frobenius twist of V_1 is isomorphic to V_0.

Moreover, such a representation is unique because fixing a basis of V_0 with respect to which $\rho : \Lambda \longrightarrow \mathrm{GL}(V)$ is given by the matrices $(a_{ij}(\lambda))$, the representation V_1 will be the one given by the matrices $(a_{ij}(\lambda)^{p^{-1}})$. More precisely, the Frobenius twist is an *equivalence* of the category of representations of Λ. For a representation $\rho : \Lambda \longrightarrow \mathrm{GL}(V_0)$ of Λ, we let $\rho_1 : \Lambda \longrightarrow \mathrm{GL}(V_1)$ be the representation of Λ which as a vector-space is $V_0^{(-1)}$ and we let Λ act on it just as it acts on V_1 (since as additive groups V_1 and V_0 are the same). Inductively, we let V_i be the representation obtained in the same way from V_{i-1}. Note that there is a natural p-linear homomorphism

$$\mathfrak{L}(V_{i+1})(U) \longrightarrow \mathfrak{L}(V_i)(U), \quad (f_1, \ldots, f_d) \mapsto (f_1^p, \ldots, f_d^p),$$

which induces an isomorphism $F^*\mathfrak{L}(V_{i+1}) \cong \mathfrak{L}(V_i)$. Hence, $\mathfrak{L}$ (abusing notation) is naturally an exact tensor functor from the category of representations of Λ to the category of F-divided coherent sheaves on X.

Theorem 62. *The functor $\mathfrak{L}$ identifies $\mathrm{Rep}_k(\Lambda)$ with a tensor subcategory of $\mathbf{str}(X)$. That is, (a) $\mathfrak{L}$ is full and faithful and (b) any sub-object $M \subseteq \mathfrak{L}(V)$ is the image of a subobject $W \subseteq V$.*

Proof: We start by proving (a). It is sufficient to show that a horizontal arrow $\mathbb{1} \longrightarrow \mathfrak{L}(V)$ is induced by an element of V^Λ. Such an arrow is given as a compatible system of global sections $s_i \in \mathfrak{L}(V_i)(X)$; compatible, of course, means that under the natural isomorphism $\mathfrak{L}(V_n) \otimes_{\mathscr{O},F^n} \mathscr{O}_X \longrightarrow \mathfrak{L}(V_0)$ the global section $s_n \in \mathfrak{L}(V_n)(X)$ is taken to s_0 via

$$\mathfrak{L}(V_n)(X) \longrightarrow \mathfrak{L}(V_n)(X) \otimes_{\mathscr{O}(X),F^n} \mathscr{O}(X) \longrightarrow (\mathfrak{L}(V_n) \otimes_{\mathscr{O},F^n} \mathscr{O})(X) \cong \mathfrak{L}(V_0)(X).$$

But a global section s_n of $\mathfrak{L}(V_n)(X)$ is just a Λ-invariant d-uple ($d = \dim V_n$) of analytic functions $(f_1, \ldots, f_d)$ on Ω. Because the composition map above is just $(f_1, \ldots, f_d) \mapsto (f_1^{p^n}, \ldots, f_d^{p^n})$, it follows from $iii)$ of Lemma 60 that $(f_1, \ldots, f_d)$ are in V_n^Λ. This proves that the functor $\mathfrak{L}$ is full; faithfulness is immediate and we have proved (a).

The proof of (b) runs through the usual argument. That is, if $M \subseteq \mathfrak{L}(V)$ is a subobject in $\mathbf{str}(X)$, then there is an admissible cover of $X = \cup_\alpha U_\alpha$ such that the restriction $M|U_\alpha$ is trivial (as F-divided sheaf on U_α). Hence there is a subrepresentation of $W \subseteq V$ such that the inclusion $M \cong \mathfrak{L}(W)$. In order to elaborate on this, we will need the technical results below.

Define the category $\mathbf{LCstr}(X)$ as the full subcategory of $\mathbf{str}(X)$ with the following class of objects: $\{M_i\}$ is an F-divided sheaf such that for some *admissible* cover by affinoids $\{U_\alpha\}$ of X, the restriction of $\{M_i\}$ to U_α is trivial.

Since the local rings are all regular, we can assume that the U_α are in fact connected and $\mathscr{O}_X(U_\alpha)$ are regular domains (see the proof of $i)$ in Lemma 60).

Lemma 63. *i)* $\mathbf{LCstr}(X)$ *is stable under tensor products, duals, direct sums, kernels and cokernels.*

ii) The functor $\mathfrak{L} : \mathrm{Rep}_k(\Lambda) \longrightarrow \mathbf{str}(X)$ *factors through* $\mathbf{LCstr}(X)$*.*

Proof: The only non-obvious claim in $i)$ is the one about kernels and cokernels and it follows from the fact that choosing a point in each U_α makes the category $\mathbf{str}(U_\alpha)$ neutral Tannakian; hence subquotients of trivial objects are trivial. To prove $ii)$ we note that $\mathfrak{L}(V)|U$ is trivial if $\pi^{-1}(U) \longrightarrow U$ is a trivial covering. $\square$

Let $V \in \mathrm{Rep}_k(\Lambda)$ and let $\mathbb{L}(V)$ be the locally constant sheaf[2] of k-vector spaces associated to V;

$$\mathbb{L}(V) : U \mapsto \{\sum_j f_j \otimes v_j \in \mathscr{O}_\Omega(\pi^{-1}U)^\Lambda;\ f_j \text{ is locally constant}\}.$$

We observe that, if we give $\mathbb{L}(V) \otimes_k \mathscr{O}_X$ the F-division provided by the submodules $\{\mathbb{L}(V) \otimes_k \mathscr{O}_{(i)}\}_i$, then $\mathbb{L}(V) \otimes_k \mathscr{O}_X = \mathfrak{L}(V)$ as elements of $\mathbf{str}(X)$.

Abuse notation and denote by M_i the image sheaf of M_i in M_0 — it is a sheaf of $\mathscr{O}_{(i)}$-modules which generates M_0 (as $\mathscr{O}_X$-module).

The idea to prove part (b) of Theorem 62 is to consider the *locally constant* (by part i) of Lemma 63 and iii) of Lemma 60) sheaf of finite dimensional k-spaces $\mathbb{W} := \cap_n M_n \subseteq \cap_n \mathfrak{L}(V_n) = \mathbb{L}(V)$ and show that this comes from a subrepresentation of V, i.e. $\mathbb{W} = \mathbb{L}(W)$. Note that $\mathbb{W}$ is constant on each connected distinguished admissible.

Let $\mathbb{V} := \mathbb{L}(V)$. Denote by $\iota : \pi^{-1}\mathbb{V} \longrightarrow \widetilde{V}$ the canonical inclusion into the constant sheaf $\widetilde{V}$. It is obviously an isomorphism. Let $Y \subset \Omega$ be a connected admissible such that $Y' := \pi^{-1}(\pi(Y)) = \sqcup_{\lambda\in\Lambda} \lambda Y$ and $\pi|Y : Y \longrightarrow \pi(Y)$ is an isomorphism. Over Y, we have

$$\pi^{-1}(\mathbb{V})(Y) = \left(\prod_\lambda \widetilde{V}(\lambda Y)\right)^\Lambda \tag{4.1}$$

and $\iota(Y)$ is just the projection onto the coordinate Y: $\iota(\{v_\lambda\}) = v_e$. In particular, the inverse of $\iota(Y)$ is the map that takes $v \in \widetilde{V}(Y) = V$ to the *unique* element in the right-hand-side of (4.1) which, on the coordinate associated to Y, is v. That is,

$$\iota(Y)^{-1}(v) = (\rho(\lambda) \cdot v)_{\lambda Y}. \tag{4.2}$$

[2] A sheaf F on X is locally constant when there exists an admissible covering $\{U_\alpha\}$ of X with $F|U_\alpha$ constant.

In particular, the composition

$$V = \widetilde{V}(\Omega) \xrightarrow{\text{res}} \widetilde{V}(Y) \xrightarrow{\iota(Y)^{-1}} \mathbb{V}(\pi Y) \xrightarrow{\iota(\lambda Y)} \widetilde{V}(\lambda Y) \xrightarrow{\text{res}^{-1}} \widetilde{V}(\Omega) = V \tag{4.3}$$

is just $v \mapsto \rho(\lambda) \cdot v$.

Now we have a subspace $W \subseteq V$ and a commutative diagram (since Ω is simply connected and thus any locally constant sheaf is constant)

$$\begin{tikzcd} \pi^{-1}\mathbb{W} \arrow[r, "\cong"] \arrow[d] & \widetilde{W} \arrow[d] \\ \pi^{-1}\mathbb{V} \arrow[r, "\iota"'] & \widetilde{V}. \end{tikzcd} \tag{4.4}$$

Using the composition in (4.3), we define an action of Λ on $W = \widetilde{W}(\Omega)$ which makes the inclusion $W \subseteq V$ Λ-equivariant. Hence, we have a natural inclusion $\mathbb{L}(W) \subseteq \mathbb{V} = \mathbb{L}(V)$. It is obvious that this inclusion factors through $\mathbb{W}$ and thus we have an isomorphism $\mathbb{L}(W) \longrightarrow \mathbb{W}$, since on connected distinguished affinoids the dimension of the vector space of sections are the same.

Completion of the proof of Theorem 62: We keep identifying M_i with its image in M_0 and each $\mathfrak{L}(V_i)$ with its image in $\mathfrak{L}(V)$ so that the inclusion $M_0 \longrightarrow \mathfrak{L}(V)$ preserves these subsheaves. It is easy to see that the natural map $\mathfrak{L}(W) = \mathbb{W} \otimes_k \mathscr{O}_X = (\cap_n M_n) \otimes_k \mathscr{O}_X \longrightarrow M_0$ defines an isomorphism which takes the subsheaves $\mathbb{W} \otimes \mathscr{O}_{(i)}$ to the subsheaves M_i and hence is an isomorphism in $\mathbf{str}(X)$. □

Translating in terms of fundamental groups for the corresponding Tannakian categories and using rigid GAGA, we have

Corollary 64. *assume further that X_0 is projective. Then the algebraic hull of Λ is a quotient of $\Pi^{\mathbf{str}}(X_0)$.* □

That the identification of Λ^{alg} with a quotient of $\Pi^{\mathbf{str}}(X_0)$ is non-canonical follows from the fact that we are choosing an inverse tensor functor to the analytification tensor equivalence an : $\text{coh}(X_0) \longrightarrow \text{coh}(X)$ ([12], 1.11, p. 116) and a point of the universal covering above the base point of X.

Remarks: *a*) Of course, $\Pi^{\mathbf{str}}(X_0)$ and Λ^{alg} cannot be isomorphic because, for example, there are finite etale coverings of X_0 which do not arise from rigid coverings of X; $\Omega = (\mathbb{G}_m^{\text{an}})^g$ has no rigid analytic coverings but has plenty of etale coverings. This pathology is caused by the requirement that all admissible coverings must have, in some sense, finite sub-coverings and thus etale coverings like $[l] : (\mathbb{G}_m^{\mathrm{an}})^g \longrightarrow (\mathbb{G}_m^{\mathrm{an}})^g$ are disregarded as rigid analytic coverings. One wonders if, given any $\mathscr{E}$ in $\mathbf{str}(X_0)$, it is possible to find a covering $\{U_i\}$ (not admissible!) of X by affinoids such that $\mathscr{E}^{\mathrm{an}}|U_i$ is trivial (this is true in characteristic zero). The understanding of this question is the motivating reason for the work to follow (see chapter 5 and also section 6.2).

b) The construction of the functor $\mathfrak{L}$ above is due to Gieseker [17]. The goal in that paper was to show the existence of sheaves arising from representations of the rigid fundamental group Λ of a Mumford curve X which are not semistable. Later on, Faltings, in [14], showed how to construct, from a semistable sheaf $\mathscr{E}$ (of degree zero) on X, a Φ-bounded representation $V_{\mathscr{E}}$ of Λ (see §4 of [14] for the notion of Φ-boundedness) such that $\mathfrak{L}(V_E) \cong \mathscr{E}$, see [37], Thm 5.1, p. 594 where Falting's hypothesis of a discretely valued ground field was removed. Also, the functor $\mathscr{E} \mapsto V_{\mathscr{E}}$ is full and faithful (same theorem of loc.cit.) and establishes an equivalence from the category of semistable sheaves to the category of Φ-bounded representations. Of course, the drawback is that the constructions of linear algebra *do not* preserve the category of Φ-bounded representations. Analogous results were obtained in [38] for uniformized abelian varieties (rigid analytic tori) defined over certain complete fields.

Theorem 65 (Faltings, Reversat–van der Put)**.** *Let k be an algebraically closed field of positive characteristic and complete with respect to a non-Archimedean absolute value. Any semistable vector bundle $\mathscr{E}$ on a Mumford curve X over k admits an F-division. If the field k is (isometrically) contained in the completion of the algebraic closure of $\mathbb{F}_p((t))$, then the same result holds for rigid analytic tori (uniformizable abelian varieties in our terminology).* □

We finish this remark by noting that if an abelian variety X/k is such that X^{an} can be uniformized, then X is ordinary. Using rigid GAGA and the above result, it follows that any semistable (of degree zero) sheaf on X can be F-divided. Shepherd-Barron points out that this last result can be derived using Fourier-Mukai and Remark 2.5 of [26].

4.3 Scholia in the case of uniformizable abelian varieties (conditions for the existence of a true fundamental group)

Our hope that the rigid fundamental group will describe all the stratified sheaves on a smooth projective variety is certainly unfounded. But that is only the beginning of a more subtle investigation. So, in order to coin a group whose algebraic hull (or at least, *continuous algebraic hull*, see section 6.2 below for the definition) would be isomorphic to the stratified fundamental group, we need to describe this last object more precisely. This is very pleasant in the case of uniformizable abelian varieties. The ground field k is as before.

Let X_0 be an abelian variety over k such that $X = X_0^{\mathrm{an}}$ can be uniformized[3], $X = (\mathbb{G}_m^{\mathrm{an}})^g/\Lambda$ (as group objects also) and $\Lambda = \mathbb{Z}^g$. Let $\mathscr{L}$ be the composition of $\mathfrak{L}$ with an inverse equivalence for $(\cdot)^{\mathrm{an}}$ which preserves the tensor structures.

4.3.1 The nilpotent part

Any unipotent representation of Λ, $\rho : \Lambda \longrightarrow \mathbb{U}_n(k)$, will factor through $(\mathbb{Z}/p^t\mathbb{Z})^g$ (for some t) as every element of $\mathbb{U}_n(k)$ has order a power of p. Hence it is just natural to expect that the unipotent part of the algebraic hull of Λ and the unipotent part of $\Pi^{\mathbf{str}}(X_0)$ (which is just $\varprojlim_i (\mathbb{Z}/p^i\mathbb{Z})^g$, see Theorem 58) will be isomorphic. This is the content of the next Proposition.

Proposition 66. *$\mathscr{L}$ induces an equivalence between $\mathfrak{N}\mathrm{Rep}_k(\Lambda)$ and $\mathfrak{N}\mathbf{str}(X_0)$.*

Proof: First note that these categories have proetale Tannakian fundamental groups isomorphic to $\varprojlim_i (\mathbb{Z}/p^i\mathbb{Z})^g$ (X_0 has maximal p-rank g). Hence, any object of $\mathscr{E} \in \mathfrak{N}\mathbf{str}(X_0)$ is the sheaf associated to a representation of $(\mathbb{Z}/p^n\mathbb{Z})^g$ (for some n) via the obvious $(\mathbb{Z}/p^n\mathbb{Z})^g$-torsor $\pi : X_0/(\ker[p^n])^\circ \longrightarrow X_0$; that is, $\mathscr{E} = \varphi(V)$ with $V \in \mathrm{Rep}_k((\mathbb{Z}/p^n\mathbb{Z})^g)$ (the notation is as in Proposition 36, see eq. (1.12)).

So, in order to show that $\mathscr{L}$ is essentially surjective, we have to prove that π is the analytification of some analytic covering $Y \longrightarrow X$. By rigid GAGA, the analytification functor induces an equivalence between the categories of finite schemes over X_0, $\mathbf{FSCH}/X_0$, and the category of finite rigid analytic varieties over X, $\mathbf{FRA}/X$. We claim π^{an} is the natural analytic covering $\nu : T/p^n\Lambda \longrightarrow T/\Lambda$, where $T = (\mathbb{G}_m^{\mathrm{an}})^g$. This will finish the proof.

Proof of $\pi^{\mathrm{an}} = \nu$: This is a consequence of the uniqueness of the factorization of an isogeny into separable and purely inseparable isogenies. We have a commutative diagram in $\mathbf{FRA}/X$

$$\begin{tikzcd} X = T/\Lambda \arrow[r, "{[p^n]}"] \arrow[d] & X = T/\Lambda \\ T/p^n\Lambda, \arrow[ur, "\nu"'] & \end{tikzcd}$$

[3]The notion of uniformization used in this brief discussion is not the correct one as we are working under the assumption that the universal covering is just a rigid torus; in general, a uniformization means that X is a quotient $G/\mathbb{Z}^h$, where G is a certain extension of an abelian variety by a torus of dimension h, see [15], 6.7.3. Nevertheless, $G \longrightarrow X$ is still a universal covering (loc.cit, Cor. 6.7.9).

and via $(\cdot)^{\mathrm{an}}$, this corresponds to a commutative diagram in $\mathbf{FSCH}/X_0$

$$\begin{tikzcd} X_0 \arrow[r, "{[p^n]}"] \arrow[d, "\mu"'] & X_0 \\ Y_0, \arrow[ur, "\nu_0"'] & \end{tikzcd}$$

where ν_0 is an *etale* map and $(\nu_0)^{\mathrm{an}} = \nu$. Hence, by Lang-Serre, Y_0 is an abelian variety and ν_0 is a separable isogeny of degree p^{ng}. Note that, choosing the identity of Y_0 to be $\mu(e_{X_0})$, μ also becomes an isogeny. Since p^{ng} is also the separable degree of $[p^n] : X_0 \longrightarrow X_0$, we can use the uniqueness result mentioned earlier to conclude that $\nu_0 = \pi$.

□

4.3.2 Another visit to the character group

We concentrate on elliptic curves in order to fix ideas. So from now on $X = \mathbb{G}_m^{\mathrm{an}}/q^{\mathbb{Z}}$ is just a Tate curve.

We saw above that the unipotent part of $\Pi^{\mathbf{str}}(X_0)$ is well behaved; now we will see that the diagonal part is not — the character group $\mathbb{X} = \mathbb{X}(\Pi^{\mathbf{str}})$ is quite big. From Lemma 51 we have $\mathbb{X} = \mathrm{Pic}^0(X_0)\langle p\rangle$. Since X_0 is self dual, $\mathrm{Pic}^0(X_0) = X_0^{\vee}(k) \cong X_0(k) \cong X(k) \cong k^\times/q^{\mathbb{Z}}$.

Proposition 67. *There is a non-canonical isomorphism of abelian groups* $\mathbb{X} \cong k^\times \oplus \mathbb{Z}_p/\mathbb{Z}$.

Proof: We have to compute the limit

$$\varprojlim \left(\cdots \xrightarrow{[p]} k^\times/q^{\mathbb{Z}} \xrightarrow{[p]} k^\times/q^{\mathbb{Z}} \xrightarrow{[p]} \cdots \right)$$

To do that, we consider the projective systems of abelian groups

$$\begin{aligned} A_\bullet : &\quad \cdots \xrightarrow{[p]} \mathbb{Z} \xrightarrow{[p]} \mathbb{Z} \xrightarrow{[p]} \cdots \\ B_\bullet : &\quad \cdots \xrightarrow{[p]} k^\times \xrightarrow{[p]} k^\times \xrightarrow{[p]} \cdots \\ C_\bullet : &\quad \cdots \xrightarrow{[p]} k^\times/q^{\mathbb{Z}} \xrightarrow{[p]} k^\times/q^{\mathbb{Z}} \xrightarrow{[p]} \cdots . \end{aligned}$$

These fit into an exact sequence (in the category of projective systems of abelian groups)

$0 \longrightarrow A_\bullet \longrightarrow B_\bullet \longrightarrow C_\bullet \longrightarrow 0$ so that we will have the long exact sequence

$$1 \longrightarrow k^\times \longrightarrow (k^\times/q^{\mathbb{Z}})\langle p\rangle \longrightarrow R^1 \varprojlim(A_\bullet) \longrightarrow R^1 \varprojlim(B_\bullet).$$

Using Prop. 3.5.7 of [43] on p. 83, it follows that $R^1 \varprojlim(B_\bullet)$ vanishes and hence $(k^\times/q^{\mathbb{Z}})\langle p\rangle$ is an extension of $R^1 \varprojlim(A_\bullet)$ by $k^\times$. Now $k^\times$ is divisible and consequently is an injective $\mathbb{Z}$-module (loc.cit., 2.3.2 on p. 39). It follows that the extension above splits and hence there is an isomorphism of abelian groups $(k^\times/q^{\mathbb{Z}})\langle p\rangle \cong k^\times \oplus R^1 \varprojlim(A_\bullet)$. We are now left with the computation of $R^1 \varprojlim(A_\bullet)$. Using that the projective system $A_\bullet$ is isomorphic to the projective system

$$\cdots \subset p^{n+1}\mathbb{Z} \subset p^n\mathbb{Z} \subset \cdots,$$

the proposition follows from the computation made in loc.cit., example 3.5.5, p. 82. □

Putting together propositions 66 and 67, we obtain

Corollary 68. *Let X_0/k be an elliptic curve with $|j(X_0)| > 1$ (Tate curve) and let Λ be the rigid fundamental group of X_0^{an} (isomorphic to $\mathbb{Z}$). There is an exact sequence of group schemes*

$$0 \longrightarrow \mathrm{Diag}(\mathbb{Z}_p/\mathbb{Z}) \longrightarrow \Pi^{\mathbf{str}}(X_0) \longrightarrow \Lambda^{\mathrm{alg}} \longrightarrow 0.$$

□

Remarks:(a) We can topologize $\mathbb{X}$ in such a way that it becomes a Hausdorff group. This is done by noting that

$$\frac{k^\times \oplus \mathbb{Z}_p}{(q,1)^{\mathbb{Z}}}$$

contains $k^\times$ as $k^\times \oplus \{0\}$ and the quotient is $\mathbb{Z}_p/\mathbb{Z}$. Since there is only one extension of $\mathbb{Z}_p/\mathbb{Z}$ by $k^\times$ up to isomorphism, follows that $\mathbb{X} \cong (k^\times \oplus \mathbb{Z}_p)/(q,1)^{\mathbb{Z}}$. I thank A. Scholl for pointing this out to me.

(b) The general case of an uniformizable abelian variety is analogous, but notationally cumbersome.

Chapter 5

Local solutions to differential equations in the positive characteristic non-Archimedean case

5.1 Introduction

A long time ago, Cauchy established the existence and uniqueness of (a system of) local solutions for ordinary differential equations in the complex domain. Now-a-days, the usual proof of this theorem makes use of some contraction principle in complete metric spaces, but Cauchy's proof is, by far, the most interesting one for the algebraist. It consists of (1) solving the equations formally in power series and (2) taking care of convergence in a neighborhood of the point in question (see, for example, the last chapter of [6]). Once that has been done, the general theory of complex analysis assures (3) convergence in bigger domains. This last statement is the first one to fail in the characteristic zero non-Archimedean setting but, nevertheless, (1) and (2) can still be carried ([13], III, 5.). The goal of this work is to understand better (2) in the case of a an algebraically closed base field k of positive characteristic and complete with respect to a non-trivial non-Archimedean absolute value $|\cdot| : k^* \longrightarrow \mathbb{R}_{>0}$.

The reader might interject to point out that in the theory differential equations of positive characteristic, (2) above (existence of convergent local solutions) is governed by the p-curvature (Cartier's Theorem). But this result is a "first order" result and the concrete analog of differential equations in positive characteristic are modules having the action of *all differential operators*. So, for these modules one is presented with the very basic question of *convergence* of local solutions.

That is, let R be a ring of power series $\sum_0^\infty a_i x^i$ with some boundedness condition on the $|a_i|$ (for example, that given in eq. (5.2) below). Let ∂_n be the operators analogous to

$$\frac{1}{n!}\frac{d^n}{dx^n}$$

on R. A linear stratified differential equation of rank μ is a *system* of equations,

$$\begin{aligned} \partial_n y_1 &= b_{11}^{(n)} y_1 + \ldots b_{1\mu}^{(n)} y_\mu \\ &\vdots \qquad\qquad\qquad\qquad (n \in \mathbb{N}) \\ \partial_n y_\mu &= b_{\mu 1}^{(n)} y_1 + \ldots + b_{\mu\mu}^{(n)} y_\mu, \end{aligned} \tag{5.1}$$

where the $b_{ij}^{(n)}$ are in R and the matrices $(b_{ij}^{(n)})$ satisfy some set of relations. These relations reflect the relations between the operators ∂_n, such as $(\partial_1)^p = 0$ and $\partial_0 = \mathrm{id}$. If we were in characteristic zero, the constraints would restrict the $(b_{ij}^{(n)})$ in such a way that $(b_{ij}^{(1)})$ is already sufficient information to determine the other matrices; so we are really dealing with a generalization of differential equations. It is quite easy to see that these equations will always have solutions $(y_1, \ldots, y_\mu)$ in the ring $k[[x]]$, in fact, there exists a $Y = (y_{ij}) \in GL_\mu(k[[x]])$ whose columns are solutions. Moreover, because the space of solutions will have k-dimension $\leq \mu$, follows that any other $\mu \times \mu$ invertible matrix whose columns are solutions to (5.1) is of the form $Y \cdot A,\ A \in GL_\mu(k)$.

Formally the problem is resolved, so we ask about convergence of the solutions. We will see below (the example after Lemma 76) that it is not always possible to find Y with entries in

$$k\{x\} := \left\{ \sum_{i=0}^{\infty} a_i x^i;\ \text{limsup}\ \sqrt[i]{|a_i|} < +\infty \right\}.$$

Presented with such an impossibility, we then want to understand how far we are from solving these equations with convergent functions (in $k\{x\}$). At this point, it becomes convenient to introduce the local fundamental group scheme $\Pi^{\mathbf{loc}}$ (Definition 77) and the monodromy groups = algebraic quotients of $\Pi^{\mathbf{loc}}$. The introduction of these concepts is a precious technical tool which helps us to formulate the right questions pretty much the same way we learned to study Galois groups rather than try to solve algebraic equations. This point of view is made manifested in Theorem 82 which is modeled in the similar one proved in [25].

Finally, we make a topological consideration. Convergence itself can be thought as a topological problem, i.e. find "small neighborhoods" in which all solutions exist as analytic functions. Here we take "neighborhoods" and "analytic" in the sense Grothendieck taught us to: regular functions of the structure ring of some site. The existence of formal solutions is just the existence question examined in the *smooth* topology (this is a consequence of the Artin-Néron-Popescu-Rotthaus desingularization: see [2] for example). So this work studies convergence in the rigid topology (which again was an idea of Tate influenced by Grothendieck's notion of topology).

Throughout this work, we will let k be an algebraically closed field of positive characteristic p complete with respect to a non-trivial non-Archimedean absolute value

$$|\cdot| : k^\times \longrightarrow \mathbb{R}_{>0}.$$

By a group or a group scheme we will mean an *affine* group scheme (Hopf algebra).

5.2 Premiss

Given a $\rho \in |k^*|$, let

$$\mathscr{O}(\rho) = \left\{ \sum_{i=0}^{\infty} a_i x^i \in k[[x]];\ \lim_i |a_i|\rho^i = 0 \right\}. \tag{5.2}$$

denote the affinoid algebra of analytic functions on the disk $\mathbb{D}(\rho) = \{z \in k;\ 0 \leq |z| \leq \rho\}$.

Since ρ will be fixed in what follows, we will omit it from the notations and write $R := \mathscr{O}(\rho)$ and $\mathbb{D} = \mathbb{D}(\rho)$. On R, there are k-linear homomorphisms ∂_n defined by

$$\partial_n \left(\sum_{i \geq 0} a_i x^i \right) = \sum_{i \geq n} \begin{pmatrix} i \\ n \end{pmatrix} \cdot a_i x^{i-n}, \qquad (n \geq 0),$$

these operators are the formal equivalent of $\frac{1}{n!}\frac{d^n}{dx^n}$ in positive characteristic. Note that

1. $\partial_0 = \mathrm{id}$,
2. $\partial_n(fg) = \sum \partial_r(f)\partial_s(g)$, where the sum is over all $r, s \geq 0$ with $r + s = n$.
3. $\partial_m \circ \partial_n = \begin{pmatrix} m+n \\ m \end{pmatrix} \partial_{m+n}$.

The subring of $\mathrm{End}_k(R)$ generated over R by all these operators is called *the ring of differential operators*[1] and is denoted by $\mathscr{D}$.

Definition 69. *Let M be finite R module. A stratification on M is a homomorphism of R-algebras*

$$\nabla : \mathscr{D} \longrightarrow \mathrm{End}_k(M).$$

This amounts to a family of k-linear homomorphisms $\{D_n\}_{n \in \mathbb{N}}$ satisfying

[1] A more indicative name would be the ring of finite differential operators.

1. $D_0 = \mathrm{id}$

2. $D_n(f \cdot e) = \sum \partial_r(f) D_s(e)$ *for* $f \in R$ *and* $e \in M$*; the sum is over all* $r, s \geq 0$ *with* $r + s = n$.

3. $D_m \circ D_n = \left(\begin{array}{c} m+n \\ m \end{array} \right) D_{m+n}$.

The reader is directed to sections 5.8.2 and 5.8.3 (specially Corollary 100) of the appendix for a more sheaf theoretical definition of stratified modules.

The category of modules with a stratification forms an abelian tensor category as one can see by defining the tensor product stratification module $(M, D_n^M) \otimes (N, D_n^N) := (M \otimes_R N, D_n^{M \otimes N})$ where

$$D_n^{M \otimes N}(s \otimes t) = \sum D_i^M(s) \otimes D_j^N(t), \quad \text{sum over all } i, j \geq 0 \text{ such that } i + j = n.$$

In order to obtain a fibre functor, we need to show that sheaves with a stratification are locally free. This can be done by copying the proof of Proposition 8.9 in [19], but we will prove this using the F-divided structure on a module with a stratification. For now, we assume this fact (see section 5.8.1 of the Appendix).

It follows in particular that the category of stratified modules over $\mathbb{D}$ is neutral Tannakian, if one takes the fibre at 0 as fibre functor ([12], Def. 2.19, p. 138). This category is denoted by $\mathbf{str}(\mathbb{D})$.

Another basic result on stratified modules is the Cartier-Katz Theorem. In the definition below $F : R \longrightarrow R$ is the absolute Frobenius.

Definition 70. *The category of* F*-divided sheaves (or modules over* R*) on* $\mathbb{D}$, $\mathbf{Fdiv}(\mathbb{D})$ *has objects* $\{M_n, \varphi_n\}_{n \in \mathbb{N}}$*, where* M_n *are finite* R*-modules and* $\varphi_n : F^* M_{n+1} \longrightarrow M_n$ *is an isomorphism of* R*-modules. An arrow* $\theta_\bullet : \{M_n, \varphi_n\} \longrightarrow \{N_n, \psi_n\}$ *is a sequence of* R*-linear homomorphisms* $\theta_n : M_n \longrightarrow N_n$ *such that* $\psi_n \circ F^*(\theta_{n+1}) = \theta_n \circ \varphi_n$.

Obviously the category of F-divided sheaves is a tensor category, using the faithful flatness of F, it is also abelian with kernels and cokernels defined termwise. By Lemma 17, if $\{M_i\}$ is an F-divided module, then all the M_i are locally free over R.

Theorem 71 (Cartier-Katz)**.** *The categories* $\mathbf{Fdiv}(\mathbb{D})$ *and* $\mathbf{str}(\mathbb{D})$ *are naturally equivalent* tensor categories. □

See the Appendix for a discussion of this theorem.

Remark: If the reader is familiar with the Galois theory of linear differential equations, it is obvious how to connect the notion of a stratification and a system of differential equations as in (5.1) above. If the reader is not familiar with this, the following might be useful.

Let M be a free module over R on the basis $e_1, \ldots, e_\mu$ and consider a connection $D : M \longrightarrow M$: $D(fs) = \frac{df}{dx}s + f \cdot D(s)$. Let $D(e_j) = \sum_i a_{ij}e_i$. The equation

$$D(y_1e_1 + \ldots + y_\mu e_\mu) = 0 \tag{5.3}$$

will be translated into the system of differential equations for the functions $(y_1, \ldots, y_\mu)$:

$$\frac{dy_j}{dx} + a_{j1}y_1 + \ldots + a_{j\mu}y_\mu = 0, \quad j = 1, \ldots, \mu. \tag{5.4}$$

And conversely, given such a system of differential equations, we obtain a connection on the free module R^μ.

The generalization of this to systems of systems of equations like (5.1) above follows the same reasoning — the only additional difficulty being of a notational nature. So, stratified modules are really a generalization of linear differential equations and the elements of a stratified module which are killed by all the differential operators correspond to solutions to the associated system.

5.3 Fundamental matrices and monodromy groups

In this section we study two aspects of stratified modules. The first (section 5.3.1) is the problem of finding *formal* solutions to a system like in eq. (5.1) — in the terminology of section 5.2 we are trying to find, for each stratified module (M, ∇), a trivialization of $(M \otimes k[[x]], \nabla)$. As remarked in the introduction, this step goes back to Cauchy's philosophy that analytic differential equations should be solved formally and then made convergent. This is conceptually useful because when looking for convergent solutions we only have to look for formal solutions that converge. In our particular case, the F-division plays a central role: we are able to find a basis for the solution space using a special x-adic limit of matrices, called a fundamental matrix. The second aspect analyzed (section 5.3.2) are the stratified modules introduced by H. Matzat and M. van der Put in [25] and their monodromy groups.

We keep the notations of the section 5.2: $\rho \in |k^*|$, R is the ring of analytic functions on $\mathbb{D} = \mathbb{D}(\rho)$. Also, let $R_n = R \cap k[[x^{p^n}]] = R_n \subset R$ be the image of R under the absolute Frobenius $F : R \longrightarrow R$.

5.3.1 Fundamental matrices

Since R is a principal ideal domain, stratified modules are always free. Take M a free R-module of rank μ with a stratification ∇ and let $\{M = M_0 \supset M_1 \supset \ldots\}$ be the F-

division of M obtained via the Cartier-Katz Theorem; M_n is the subspace of elements in M killed by all $\nabla(\partial_\nu)$ with $0 < \nu < p^n$. We see each M_n as a subgroup of M which is invariant under multiplication by R_n and the natural inclusion $M_{n+1} \subset M_n$ induces an isomorphism $M_{n+1} \otimes_{R_{n+1}} R_n \longrightarrow M_n$ of R_n-modules. Take $\mathbf{e}^{(0)} = (e_1^{(0)}, \dots, e_\mu^{(0)})$ a basis of $M = M_0$ and let $\mathbf{e}^{(n)}$ be a basis of M_n over R_n. If we agree to write the column vectors of $\mathbf{e}^{(n)}$ in the basis $\mathbf{e}^{(0)}$, then $\mathbf{e}^{(n)}$ is a matrix in $GL_\mu(R)$. Let φ_n be the isomorphism $M_{n+1} \otimes_{R_{n+1}} R_n \longrightarrow M_n$ and identify it with the invertible matrix with coefficients in R_n representing the lower horizontal arrow in the diagram below

$$\begin{tikzcd}
M_{n+1} \otimes_{R_{n+1}} R_n \arrow[r, "\varphi_n"] \arrow[d, "\text{via } \mathbf{e}^{(n+1)}"'] & M_n \arrow[d, "\text{via } \mathbf{e}^{(n)}"] \\
R_n^{\oplus\mu} \arrow[r] & R_n^{\oplus\mu}.
\end{tikzcd}$$

Again, in matricial terms, we have $\mathbf{e}^{(n+1)} = \mathbf{e}^{(n)} \cdot \varphi_n$. If we let $f_n := \varphi_0 \cdots \varphi_{n-1}$, then the operators $D_n := \nabla(\partial_n)$ will be given by (following the recipe given by the Cartier-Katz Theorem)

$$D_\nu(\mathbf{e}^{(0)}) := (D_\nu e_1^{(0)}, \dots, D_\nu e_\mu^{(0)}) = \mathbf{e}^{(0)} \cdot f_n \cdot \partial_\nu(f_n^{-1}), \quad 0 < \nu < p^n.$$

If we take $\varphi_n(0) = \mathrm{I}$ (which is always possible if we pick $\mathbf{e}^{(n)}$ conveniently), the sequence $\{f_n\}$ will converge to some element Φ of $GL_\mu(k[[x]])$ (in the x-adic topology!) and the columns of Φ are all killed by D_n for $n > 0$. To see this, it is sufficient to prove that for any $n > 0$,

$$D_n(\mathbf{e}^{(0)} \cdot \Phi) \in \left(\bigcap_{m \geq 0} x^m \hat{M}\right)^\mu = 0,$$

where $\hat{M} \supset M$ is the x-adic completion of M. But if ν is large enough, there is a matrix γ with power series entries such that $\Phi - f_\nu = x^{p^n} \cdot \gamma$ and hence

$$D_n(\mathbf{e}^{(0)} \cdot \Phi) = D_n(\mathbf{e}^{(0)} \cdot (\Phi - f_\nu)) = x^{p^n} D_n(\mathbf{e}^{(0)} \cdot \gamma).$$

Also note that *any* $s \in \hat{M}$ *which is killed by all* D_n *with* $n > 0$ *is a* k*-linear combination of the columns of* $\mathbf{e}^{(0)} \cdot \Phi$, *or, in more symbolic terms*

$$\mathrm{Hom}_{\mathbf{str}(\mathbb{D}(\rho))}(\mathbb{1}, M) = \{\alpha \in k^\mu;\ \Phi \cdot \alpha \in M\}.$$

Definition 72. *Assume all basis* $\mathbf{e}^{(n)}$ *have been chosen to satisfy* $\varphi_n(0) = \mathrm{I}$. *The matrix* Φ *constructed above is a called a fundamental matrix of* (M, ∇). *If all the choices are made explicit, we will say that* Φ *is* the *fundamental matrix.*

5.3.2 The stratified modules of Matzat and van der Put

Definition 73 (compare [25]). *Given a k-vector space V of finite dimension and a sequence $\{\varphi_n \in GL(V \otimes R_n)\}_{n \in \mathbb{N}}$ with $\varphi_n(0) = \mathrm{I}$, we obtain an F-divided (stratified) module $M(\varphi_\bullet)$ by setting $M(\varphi_\bullet)_n = V \otimes_k R_n$ and using the φ_n as transition isomorphisms.*

Moreover, this association preserves the constructions of linear algebra: $M((\varphi \otimes \psi)_\bullet) = M(\varphi_\bullet) \otimes M(\psi_\bullet)$, $M((\varphi^\vee)_\bullet) = M(\varphi_\bullet)^\vee$ etc.

The fundamental matrix for $M(\varphi_\bullet)$ is just the x-adic limit $\lim_n \varphi_0 \cdots \varphi_n$.

If we take the matrices φ_n in $G(R_n) \subseteq \mathbb{GL}(V \otimes R_n)$, for some algebraic subgroup G of $\mathbb{GL}(V)$, then the monodromy group of $M(\varphi_\bullet)$ in $\mathbf{str}(\mathbb{D})$ will be naturally a subgroup of G (Lemma 75). First the formal definition of monodromy (Galois group).

Definition 74. *Let $\mathcal{G}$ be a group scheme and let V be a representation of it. Let $\langle V \rangle_\otimes$ be the full subcategory of $\mathrm{Rep}_k(\mathcal{G})$ having as objects the sub-quotients (=quotients of sub-objects) of objects of the form*

$$V_{(b_1,\dots,b_s)}^{(a_1,\dots,a_s)} := V_{b_1}^{a_1} \oplus \cdots \oplus V_{b_s}^{a_s}, \qquad V_b^a := V^{\otimes a} \otimes (V^\vee)^{\otimes b}, \tag{5.5}$$

where s runs over the non-negative integers and $(a_1, \dots, a_s), (b_1, \dots, b_s)$ run over all the s-uples of non-negative integers. The monodromy group of V is the group scheme associated, via Tannakian duality ([12], 2.11), to the category $\langle V \rangle_\otimes$. If $\mathscr{C}$ is a Tannakian category which is equivalent to the category of representations of a group scheme $\mathcal{G}$, then the monodromy group of an object $V \in \mathscr{C}$ is just the monodromy group of the corresponding representation in $\mathrm{Rep}_k(\mathcal{G})$. Notations: $G(V, \mathscr{C})$ or $G_{\mathrm{mono}}(V, \mathscr{C})$.

Returning to the discussion of the monodromy group of $M(\varphi_\bullet)$ started above, note that for any representation

$$\theta : G \longrightarrow \mathbb{GL}(W)$$

we can associate another stratified module using the matrices $\theta(\varphi_n) \in GL(W \otimes R_n)$; moreover, this defines a tensor functor τ from $\mathrm{Rep}_k(G)$ to the category of stratified modules over $\mathbb{D}$. Since every representation of G is a sub-quotient of some representation as in (5.5), follows that τ takes values in the Tannakian subcategory $\langle M(\varphi_\bullet) \rangle_\otimes$ which is equivalent to $\mathrm{Rep}_k(G_{\mathrm{mono}}(M(\varphi_\bullet), \mathbf{str}))$. Because $\tau(V) = M(\varphi_\bullet)$, the group homomorphism $\pi \longrightarrow G$ obtained from τ is a closed embedding ([12], 2.21, p. 139). This shows

Lemma 75 (compare [25], Prop. 5.3). *If the matrices φ_n defining the stratified module $M(\varphi_\bullet)$ belong to $G(R_n)$, $G \subseteq \mathbb{GL}(V)$ a closed subgroup, then the monodromy group in the category of stratified modules over $\mathbb{D}$ of $M(\varphi_\bullet)$ is a closed subgroup of*

*G. More precisely, the tensor functor $\tau : \mathrm{Rep}_k(G) \longrightarrow$ **str** defined (completely) by $\tau(V) = M(\varphi_\bullet)$ induces a closed embedding.* $\square$

Remark: (a) One should notice that Kedlaya proves in [21] that every locally free module over a disk in any dimension is free (the analogous of the Quillen-Suslin Theorem). So, one can control stratified modules over such affinoids by the method of fundamental matrices.

(b) Given a stratified module M over R, we have produced a stratified module over $k[[x]]$ which is trivialized using the fundamental matrix (after choice of a basis). The reader should notice that the fundamental matrix obtained here is just the fundamental matrix, in the sense of [25] Def. 3.3, p. 7–8, for the module $M \otimes k[[x]]$.

5.4 One dimensional modules

Keep $\rho \in |k^*|$. Let again $\mathbb{D}$ denote the disk $\{z \in k;\ 0 \leq |z| \leq \rho\}$ and $R = \mathscr{O}(\rho)$ its ring of analytic functions. For a stratified module (M, ∇) over $\mathbb{D}$, we let Φ denote the fundamental matrix obtained in Definition 72 (given a choice of the $\mathbf{e}^{(n)}$).

Let (M, ∇) be a rank one stratified module. Write $\varphi_n = 1 - x^{p^n}\gamma_n$.

Lemma 76. *The radius of convergence of Φ is ρ. In particular, if $r \in |k^*|$ is strictly smaller than ρ, then M restricted to $\mathbb{D}(r)$ is trivial.*

Proof: Let $r < \rho$ be in $|k^*|$ and let $\|\cdot\|_r$ denote the spectral norm of the disk $\mathbb{D}(r)$. It is given by

$$\left\| \sum_{i \geq 0} a_i x^i \right\|_r = \sup_i |a_i| r^i = \sup_{z \in \mathbb{D}(r)} \left| \sum_{i \geq 0} a_i z^i \right|,$$

and hence is well defined as a function $k[[x]] \longrightarrow \mathbb{R} \cup \{+\infty\}$. We have the estimates:

1. $\|\varphi_n\|_r = 1$.

2. $\lim_n \|\varphi_n - 1\|_r = 0$.

The lemma is a consequence of 1. and 2. since $\|\cdot\|_r$ is multiplicative on $\mathscr{O}(r)$ and this algebra is complete with respect to it ([4], 6.1.5, p. 234). Both 1. and 2. will be a consequence of Prop. 5.1.3/1 on p. 193 of [4]; this proposition states that $f \in \mathscr{O}(1) - \{0\}$ is invertible if and only if $\|f\|_1 = |f(0)|$ and $\|f - f(0)\|_1 < \|f\|_1$. Because $\rho \in |k^*|$, $\mathscr{O}(\rho)$ is isometrically isomorphic to $\mathscr{O}(1)$; it follows that

$$\|x^{p^n}\gamma_n\|_\rho < 1.$$

Now 1. follows from the fact that $\|x^{p^n}\gamma_n\|_r \leq \|x^{p^n}\gamma_n\|_\rho < 1$ and $\|\cdot\|_r$ is non-Archimedean. Item 2. follows from

$$\|\varphi_n - 1\|_r = \|x^{p^n}\|_r \cdot \|\gamma_n\|_r \leq r^{p^n} \cdot \|\gamma_n\|_\rho < r^{p^n}\rho^{p^{-n}} = (r/\rho)^{p^n}.$$

□

Example: If we try to adapt the same proof to the case of higher rank, we will find an obstruction in the existence of nilpotents in the algebra $\mathrm{End}_k(R^{\oplus m})$. The entries of an invertible matrix in $\mathrm{End}(R^{\oplus m})$ might have large spectral norms: take $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ for example. If we pick

$$\varphi_n = \begin{pmatrix} 1 & a_n x^{p^n} \\ 0 & 1 \end{pmatrix},$$

then

$$\Phi = \lim_n \varphi_0 \cdots \varphi_n = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}, \qquad \theta = \sum_{i=0}^{\infty} a_i x^{p^i}.$$

Choosing the a_i conveniently, this gives an example of a stratified module over $\mathbb{D}$ which, even if we shrink $\mathbb{D}$, is not trivial. One can say that the presence of unipotent matrices is an obstruction to the convergence. Nevertheless, the stratified module given by the matrices φ_n above has one solution (horizontal vector) given by $(1, 0)^t$. Note that this vector is also fixed by $\varphi_n(c)$ where n runs over $\mathbb{N}$ and c over k. To produce stratified modules with no solutions at all (even in a small disk), we will avoid the existence of such a fixed vector. See the example of given of Lemma 83.

5.5 Local monodromy groups

We now introduce the formalism of Tannakian categories ([12]) to this situation. As we are really interested in *convergence in some neighbourhood of the origin*, $\mathbf{str}(\mathbb{D}(\rho))$ is not enough as, in contrast to the complex analytic case, the non-Archimedean world does not allow solutions to be prolonged to larger disks in the domain of definition (there is no reason for the restriction $\mathbf{str}(\mathbb{D}(\rho)) \longrightarrow \mathbf{str}(\mathbb{D}(\rho'))$ to be an isomorphism). Hence the correct setting is the category given in Definition 77 below.

Definition 77. *$\mathscr{T}$ is the Tannakian category*

$$\varinjlim_{\rho} \mathbf{str}(\mathbb{D}(\rho)), \qquad \rho \in |k^*|.$$

That is, the class of objects $\mathrm{Ob}\,\mathscr{T}$ *is just the union of the class of objects in* $\mathbf{str}(\mathbb{D}(\rho))$ *for all* $\rho \in |k^*|$ *and the arrows between* $M \in \mathbf{str}(\mathbb{D}(\rho_1))$ *and* $N \in \mathbf{str}(\mathbb{D}(\rho_2))$ *are*

$$\varinjlim \mathrm{Hom}_{\mathbf{str}(\mathbb{D}(r))}(M|\mathbb{D}(r), N|\mathbb{D}(r)), \quad r < \min(\rho_1, \rho_2).$$

$\Pi^{\mathbf{loc}}$ *is the fundamental group scheme associated to it via the fibre functor* $0^* : \mathscr{T} \longrightarrow (k - \mathrm{mod})$ *(see [12], Thm. 2.11).*

We are interested in the algebraic quotients of $\Pi^{\mathbf{loc}}$, which are just the monodromy groups of its objects. The reader should notice that *the monodromy group* $G(?, \mathscr{T})$ *(Definition 74) is the obstruction to finding a fundamental matrix which converges in a neighbourhood of the origin.*

We give a adaptation of Lemma 75 to this situation.

Lemma 78. *Same notation of Lemma 75. The monodromy group of* $M(\varphi_\bullet)$ *in the category* $\mathscr{T}$ *is a closed subgroup of the monodromy group of* $M(\varphi_\bullet)$ *in* $\mathbf{str}(\mathbb{D}(\rho))$*. In particular it is a closed subgroup of* G*.*

Moreover, let $\rho' < \rho$ *be in* $|k^*|$ *and let* M *(resp.* M'*) denote an object of* $\mathbf{str}(\mathbb{D}(\rho))$ *(resp. its restriction to* $\mathbb{D}(\rho')$*). Then there exists a natural closed embedding*

$$\iota : G(M', \mathbf{str}(\mathbb{D}(\rho'))) \hookrightarrow G(M, \mathbf{str}(\mathbb{D}(\rho)))$$

of the monodromy groups and under this homomorphism the representation corresponding to M *restricts to the representation corresponding to* M'*.*

Proof: The monodromy group of $M(\varphi_\bullet)$ in $\mathbf{str}(\mathbb{D}(\rho))$ is the Tannakian fundamental group associated to the category of all sub quotients in $\mathbf{str}(\mathbb{D}(\rho))$ of $M(\varphi_\bullet)^{(a_1,\dots,a_s)}_{(b,\dots,b_s)}$ (notation is that of (5.5)). The monodromy of $M(\varphi_\bullet)$ in $\mathscr{T}$ is the analogous and the Lemma is just an application of [12], Prop. 2.21, p. 139.

The second part is just as easy (and uses again the same proposition in loc.cit.). $\square$

Corollary 79. *Let* M *be an object of* $\mathscr{T}$*. Then there is a* $\rho \in |k^*|$ *and an* M_ρ *such that* M *is the image of* M_ρ *under the functor* $\mathbf{str}(\mathbb{D}(\rho)) \longrightarrow \mathscr{T}$ *and the natural homomorphism* $G(M, \mathscr{T}) \longrightarrow G(M_\rho, \mathbf{str}(\mathbb{D}(\rho)))$ *is an isomorphism.*

Proof: This is a consequence of Lemma 78 and the fact that all monodromy groups in $\mathbf{str}(\mathbb{D}(r))$ are reduced (Theorem 34).

Choose some $r \in |k^*|$ such that M is induced by $M_r \in \mathbf{str}(\mathbb{D}(r))$. The category $\langle M \rangle_\otimes$ is the direct limit

$$\varinjlim_{|r'|<|r|} \langle M_{r'} \rangle_\otimes \qquad (M_{r'} \text{ is the restriction of } M_r)$$

and so $G(M, \mathscr{T})$ is the projective limit of the corresponding affine group schemes (it is an exercise to show that the functor Rep_k takes projective limits to direct limits), i.e. the monodromy groups of the various $M_{r'}$. But all the arrows in this projective limit are closed embeddings and all the groups are reduced linear algebraic groups. It follows that if we pick some $\rho \in |k^*|$ with $|\rho| < |r|$ minimizing

$$\dim G(M_{r'}, \mathbf{str}(\mathbb{D}(r'))) \quad \text{and} \quad \#\pi_0 G(M_{r'}, \mathbf{str}(\mathbb{D}(r'))),$$

then $G(M_\rho, \mathbf{str}(\mathbb{D}(\rho))) = G(M, \mathscr{T})$. $\square$

Given a reduced algebraic group G over k, following [25], we introduce the groups $p(G)$ and $G^{(p)}$. $p(G)$ is the smallest closed subgroup of G containing all the elements of order a power of p. Obviously it is a normal subgroup.

Lemma 80 ([25], Claim on p. 28)**.** *Notation as above.*

i) $p(G)$ is an algebraic subgroup of G.

ii) The connected component of the quotient $G^{(p)} := G/p(G)$ is either trivial or a torus and $\pi_0 G^{(p)}$ is a finite group of order prime to p. $\square$

Lemma 81. *Let G be an algebraic quotient of $\Pi^{\mathbf{loc}}$. Then*

i) G is reduced.

ii) If G is finite it is trivial.

iii) G is connected and equals $p(G)$.

Proof: The proof of *i*) follows from Corollary 79 and the fact (which was used to prove the Corollary) that all the monodromy groups of $\mathbf{str}(\mathbb{D}(\rho))$ are reduced. The validity of *ii*) is a consequence of *i*) and the fact that the local ring $\varinjlim_\rho \mathscr{O}(\mathbb{D}(\rho)) = k\{x\}$ is strict henselian ([32], 45.5, p. 193). To prove *iii*), we start by observing that the connected component $(G^{(p)})^\circ$ is trivial or a torus and that $\pi_0(G^{(p)})$ is of order prime to p. By item *ii*), $\Pi^{\mathbf{loc}}$ has no non-trivial finite etale quotients, so $\pi_0(G^{(p)}) = \{1\}$. By Lemma 76, every diagonal quotient of $\Pi^{\mathbf{loc}}$ is also trivial. Hence, $G^{(p)} = \{1\}$. $\square$

All monodromy groups in $\mathscr{T}$ are generated by the elements whose order is a power of p. The converse is the following theorem, whose proof is given below in Theorem 88: given a connected reduced algebraic group $G = p(G)$, we construct a stratified module in $\mathscr{T}$ whose monodromy is G.

Theorem 82. *Any reduced connected algebraic group G which is generated as an algebraic group by its elements of order a power of p is a quotient of $\Pi^{\mathbf{loc}}$.*

5.6 Didatic: Theorem 82 for $\mathbb{SL}(2)$

This is a purely didactic digression (inspired by the didatic digression of [25]).

We will keep the notations from sections 5.2 and 5.3. Here we construct a stratified module over $\mathbb{D}(\rho)$ which has monodromy group in the category $\mathscr{T}$ isomorphic to $\mathbb{SL}(2)$ (one should bear Lemma 78 in mind). We will take matrices $\varphi_n \in SL_2(R_n)$ and form $M := M(\varphi_\bullet)$. The monodromy group G of M will then be a closed subgroup of $\mathbb{SL}(2)$ and our task is to show it actually equals $\mathbb{SL}(2)$.

If $G \neq \mathbb{SL}(2)$, there would exist, by Chevalley's theory of reconstruction, a symmetric power W of the canonical representation $k^{\oplus 2}$ such that G *fixes a line* in W. In fact, because G is reduced and connected, it has dimension ≤ 2 and hence must be solvable. By Borel's Fixed Point Theorem, the fixed line occurs already in $k^{\oplus 2}$. Since in the general case we know almost nothing about G except that it has no one-dimensional non-trivial representations (Lemma 76), we will carry out the computations with a general symmetric power.

The representation W of G corresponds (via Tannakian duality) to some symmetric power of M and the line fixed corresponds to a sub-object of M in $\mathscr{T}$. Since one dimensional objects are trivial, there is a $\rho' < \rho$ and an $s \in M|\mathbb{D}(\rho')$ which is a k-linear combination of the columns of a fundamental matrix. Hence, to obtain a contradiction, we should take φ_n in such a way that the fundamental matrix of its symmetric powers satisfy:

1. The coefficients of the entries grow very fast.

2. The entries of the first *row* have infinitely many distinct non-zero coefficients.

Condition 1. above is very natural. Condition 2. is to avoid the following fact: $\xi, \eta \in k[[x]]$ might have convergence radius zero but some non-trivial linear combination $a\xi + b\eta$ has positive convergence radius.

The idea is to take matrices $\varphi_n \in SL_2(R_n)$ of the form

$$\begin{pmatrix} 1 & a_n x^{p^n} \\ 0 & 1 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 1 & 0 \\ a_n x^{p^n} & 1 \end{pmatrix}.$$

Take an increasing sequence of non-negative integers $\{n_i\}_{i\in\mathbb{N}}$ such that the differences $n_{i+1} - n_i$ tend to infinity as i tends to infinity. Let $a_n \in k^*$ be a sequence such that $\lim_n |a_n| r^{p^n} = \infty$ for any $r > 0$. Define φ_n by

$$\varphi_n = \begin{cases} \begin{pmatrix} 1 & a_n x^{p^n} \\ 0 & 1 \end{pmatrix}, \text{ if } n = n_i \text{ with } i \text{ odd} \\ \\ \begin{pmatrix} 1 & 0 \\ a_n x^{p^n} & 1 \end{pmatrix}, \text{ if } n = n_i \text{ with } i \text{ even} \\ \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ otherwise.} \end{cases} \tag{5.6}$$

Note that for any infinite subset $N \subseteq \{n_i\}$, $\sum_{n \in N} a_n x^{p^n}$ has convergence radius 0. Let $M := M(\varphi_\bullet)$ be the stratified module over $\mathbb{D}(\rho)$ obtained via the φ_n as in Definition 73.

Lemma 83. *The monodromy group G of M is $\mathbb{SL}(2)$.*

Proof: We know that $G \subseteq \mathbb{SL}(2)$. If $G \neq \mathbb{SL}(2)$, there would be an integer $\delta \geq 1$ such that G fixes a line in the representation $\mathscr{S}^\delta(V)$ of $\mathbb{SL}(2)$ (here $V = k^{\oplus 2}$ is the canonical representation and $\mathscr{S}^?$ is the ?th symmetric power). But fixing a line means that there is a rank one sub-object of $L \subset \mathscr{S}^\delta(M)$ in $\mathscr{T}$. By Lemma 76, such an object will be trivial. Translating this in terms of a fundamental matrix Φ for $\mathscr{S}^\delta(M)$, there would be a non-zero vector $\alpha := (\alpha_1, \ldots, \alpha_r)^t \in k^{\oplus r}$ such that $\Phi \cdot \alpha \in (k[[x]])^{\oplus r}$ has positive convergence radius; here $r := \dim_k \mathscr{S}^\delta(V) = \delta + 1$. We will show that this is impossible. In order to do computations, we (1) note that $\mathscr{S}^\delta(M)$ is $M(\mathscr{S}^\delta(\varphi_\bullet))$ (2) take a basis for $\mathscr{S}^\delta(M)$ obtained from the basis of M using the lexicographic order and (3) rename φ_n to mean its symmetric δ-power. Example with $\delta = 2$:

$$\varphi_{n_i} = \begin{pmatrix} 1 & a_{n_i} x^{p^{n_i}} & a_{n_i}^2 x^{2p^{n_i}} \\ 0 & 1 & 2a_{n_i} x^{p_i^n} \\ 0 & 0 & 1 \end{pmatrix} \; i \text{ odd}; \quad \varphi_{n_i} = \begin{pmatrix} 1 & 0 & 0 \\ 2a_{n_i} x^{p^{n_i}} & 1 & 0 \\ a_{n_i}^2 x^{2p^{n_i}} & a_{n_i} x^{p^{n_i}} & 1 \end{pmatrix} \; i \text{ even.}$$

Assume by absurd that there was an α as above. Let i_0 be an integer such that for all $i > i_0$, $p^{n_{i+1} - n_i} > \delta + 1$. Because the convergence radius of

$$\varphi_{n_{i_0}}^{-1} \cdots \varphi_0^{-1} \cdot \Phi \cdot \alpha$$

is positive if and only of the convergence radius of $\Phi \cdot \alpha$ is, we can assume without loss of generality that $p^{n_{i+1} - n_i} > \delta$ for all i. Denote the (i,j)-th coefficient of the truncation $\varphi_0 \cdots \varphi_n$ by $y_{ij}^{(n)}$ and let $d_{ij}^{(n)} := \deg(y_{ij}^{(n)})$. To simplify notation, let also $d_j^{(i)}$ denote $d_{1j}^{(n_i)}$. For convenience of the reader, we make explicit the formula to obtain $y_{1j}^{(n_{i+1})}$ from $y_{1j}^{(n_i)}$. The matrices φ_n are written $(\varphi_{ij}^{(n)})$.

$$
\begin{aligned}
y_{1j}^{(n_{i+1})} &= y_{1j}^{(n_i)} + \textstyle\sum_{l<j} y_{1l}^{(n_i)} \varphi_{lj}^{(n_{i+1})}, \text{ if } i \text{ is even,} \\
y_{1j}^{(n_{i+1})} &= y_{1j}^{(n_i)} + \textstyle\sum_{l>j} y_{1l}^{(n_i)} \varphi_{lj}^{(n_{i+1})}, \text{ if } i \text{ is odd.}
\end{aligned}
\tag{5.7}
$$

The lemma will be a consequence of the direct **computation:**

1. $\prod_{j=1}^{r} y_{1j}^{(n_i)} \neq 0$ for $i > 0$.
2. Let i be *even*. Then the leading term of $y_{1j}^{(n_{i+1})}$ will be lead.term$(y_{11}^{(n_i)}) \times$ lead.term$(\varphi_{1j}^{(n_{i+1})})$. In particular, $d_j^{(i+1)} = d_1^{(i)} + (j-1)p^{n_{i+1}}$.
3. Let i be *odd*. The leading term of $y_{1j}^{(n_{i+1})}$ is lead.term$(y_{1r}^{(n_i)}) \times$ lead.term$(\varphi_{rj}^{(n_{i+1})})$. In particular, $d_j^{(i+1)} = d_r^{(i)} + (r-j)p^{n_{i+1}}$.
4. $d_{j+1}^{(i)} - d_j^{(i)} = (-1)^{i+1}p^{n_i}$ for $i > 0$.

To see how the above formulae prove the lemma, let y_{ij} be the (i,j)-th coefficient of Φ, so that $\xi := \alpha_1 y_{11} + \ldots + \alpha_r y_{1r}$ has convergence radius greater than 0. If ξ_n denotes the coefficient of x^n in ξ then, except for finitely many, $|\xi_n| \leq c^n$ (some $c > 0$). Let

$$\alpha = (\alpha_1, \ldots, \alpha_J, 0, \ldots, 0) \qquad \alpha_J \neq 0.$$

From the equations (5.7), one deduces that $y_{1j} \equiv y_{1j}^{(n_i)} \mod x^{p^{n_{i+1}}}$. As $d_j^{(i)} < p^{n_{i+1}}$,

> *the coefficient of degree $d_j^{(i)}$ in y_{1j} is the coefficient of the leading term of $y_{1j}^{(n_i)}$.*

For i odd, formula (4) above shows that the coefficient of degree $d_J^{(i)}$ in ξ is the coefficient of the leading term of $y_{1J}^{(n_i)}$ multiplied by α_J. Using the formulae (2) – (4) one more time, it follows that for i odd the coefficient of the leading term of $y_{1j}^{(n_i)}$ is

$$a_{n_i}^{j-1} \cdot (a_{n_{i-1}} \cdots a_{n_0})^{\delta}$$

and

$$d_j^{(i)} = (j-1)p^{n_i} + \delta(p^{n_{i-1}} + \ldots + p^{n_0}).$$

The convergence condition on ξ implies that $\alpha_J = 0$.

Proof of the computation: By the formulae in (5.7) and the inequality $p^{n_{i+1}-n_i} > \delta + 1$, it is easy to prove by induction that $d_j^{(i)} < p^{n_{i+1}}$.

To prove formula (1), we again use induction. For $i = 1$ this is immediate. From the expression in (5.7), we see that all non-zero terms in the summation have no monomials of degree $\leq \deg y_{1j}^{(n_i)}$. Hence $y_{1j}^{(n_{i+1})} \neq 0$.

We prove (2) – (4) simultaneously by induction that is, assume that (2) – (4) hold up to i. First take $i > 0$ even so that $\varphi_{n_{i+1}}$ is upper triangular. Since $\varphi_{lj}^{(n_{i+1})}$ is either zero or has degree $(j - l)p^{n_{i+1}}$ and $d_1^{(i)} > d_2^{(i)} \cdots > d_r^{(i)}$, the term of highest degree in the summation in (5.7) is $y_{11}^{(n_i)} \varphi_{1j}^{(n_{i+1})}$. Also, its degree is bigger than $d_j^{(i)}$. It follows that the leading term of $y_{1j}^{(n_{i+1})}$ is the product of the leading terms of $y_{11}^{(n_i)}$ and $\varphi_{1j}^{(n_{i+1})}$. This proves (2) and (4) The case $i > 0$ odd is entirely analogous and proves (3) and (4) This finishes the computation and consequently the proof of the lemma. □

5.7 Differential equations with no non-trivial convergent solutions and proof of main theorem

In this section we will give a proof of the main theorem. The method is similar to the method used in section 5.6. The idea of proof is as follows. Let $G \subseteq \mathbb{GL}(m)$ be as in the statement of Theorem 82. We want to find matrices $\varphi_n \in G(R_n)$ such that the canonical embedding of the monodromy group H of $M(\varphi_\bullet)$ in G (Lemma 78) is in fact an *isomorphism*. Chevalley taught us how to deal with the construction of quotients G/H by finding lines in some representation W of G which are fixed by H but not by G (by general Tannakian theory, such a representation is related to the standard representation $G \subseteq \mathbb{GL}(m)$ by linear algebra). We follow his wise idea (as did [25]) with the constant support of Lemma 76, which states that such a line will in fact correspond to a *fixed element* (for the H-action). The proof of Theorem 82 will follow from the non-existence of such fixed vectors (if we chose the φ_n carefully!) Section 5.7.1 below shows what sort of constraint will appear if there exists such a fixed vector and section 5.7.2 shows how to pick the matrices φ_n as to violate these constraints. The proof was inspired by the proof of the main result of [25].

5.7.1 Some properties impeding convergent solutions

In this section we study conditions on the matrices φ_n so that $M = M(\varphi_\bullet)$ has no convergent solution (Lemma 84). Convergent means that there exists a horizontal morphism from the trivial object of $\mathbf{str}(\mathbb{D}(\rho))$ to M, for some small ρ. Of course, this is equivalent to finding an $\alpha \in V = 0^*M$ with $\Phi \cdot \alpha \in V \otimes \mathscr{O}(\rho)$, as any element of $\hat{M} = M \otimes k[[x]]$ killed by all differential operators of positive order is a k-linear combination of the columns of Φ.

In order to see what kind of constraint the convergence of $\Phi \cdot \alpha$ will impose on the norms $\|\varphi_n \cdot \alpha\|_\rho$, we want, firstly, to take φ_n polynomial. This will allow us to use the degree to compare the coefficients of $\Phi \cdot \alpha$ with those of $\varphi_n \cdot \alpha$. Secondly, we will want the degrees to grow fast so that $\varphi_0 \cdots \varphi_{n+1}$ is the sum of $\varphi_0 \cdots \varphi_n$ with terms of higher degree (provided by φ_{n+1}). As this is not really feasible (for the pair $n, n+1$), we follows [25] and introduce very large "gaps" in the sequence φ_n:

$$\varphi_0 \cdots \varphi_n = \varphi_0 \cdots \varphi_{n+1} = \ldots = \varphi_0 \cdots \varphi_{n'-1}$$

for some $n' \gg n$ (see property 2 below).

And finally, in order to avoid that the degree of $\varphi_0 \cdots \varphi_n \alpha$ stays bounded, we will impose that the group generated by the various $\varphi_n(k)$ with $n \geq N$ generate a subgroup of $\mathbb{GL}(V)$ which does not fix α (Property 4).

Let $A_n := k[x^{p^n}]$ and let V be a k-vector space with a fixed basis: $k^m \cong V$. Take matrices $\varphi_n \in \mathrm{GL}(V \otimes_k A_n)$ with the following properties:

Property 1. $\varphi_n(0) = \mathrm{I}$.

Property 2. *There is an increasing sequence of positive integers* $\{n_i\}_{i \in \mathbb{N}}$ *such that* $\varphi_n = \mathrm{I}$ *if* $n \notin \{n_i\}$. *Also,* $\{n_{i+1} - n_i\}_{i \in \mathbb{N}}$ *is increasing (and hence tends to infinity).*

Property 3. *If we write* $\varphi_n = \mathrm{I} + \Gamma_n$, *then the degree of* Γ_n *in* x *is bounded by* bp^n, $b > 0$.

Property 4. *Let* G *be the subgroup of* $\mathbb{GL}(V)$ *generated by the subset* $\cup_{n \geq 0} \varphi_n(k)$. *Then, for any* $N \in \mathbb{N}$, G *is also generated by* $\cup_{n \geq N} \varphi_n(k)$.

We note that the degree of a matrix in $\mathrm{GL}(V \otimes_k k[x]) = \mathrm{GL}_m(k[x])$ is well defined as is the degree of a vector $f \in V \otimes_k k[x]$. Also, given an element

$$\xi = \sum_{i=0}^{\infty} v_i \otimes x^i \in V \otimes_k k[[x]], \tag{5.8}$$

we define, for some $\rho \in |k^*|$,

$$\|\xi\|_\rho = \sup_i \{|v_i| \rho^i\},$$

where $|\cdot| : V \longrightarrow \mathbb{R}_{\geq 0}$ is the maximum norm with respect to the basis giving $k^m \cong V$. Under these conventions, $V \otimes_k \mathscr{O}(\rho)$ is the subspace of all ξ with $\lim_i |v_i| \rho^i = 0$.

Note that picking a different basis for V gives a different $\|\cdot\|_\rho : V \otimes_k \mathscr{O}(\rho) \longrightarrow \mathbb{R}_{\geq 0}$, but the topology is the same. In particular, the concept of a sequence in $V \otimes_k \mathscr{O}(\rho)$ having bounded norm is well defined.

We will also find convenient to call the monomials $v_i \otimes x^i$ of ξ in (5.8) the *terms* of ξ.

The matrices φ_n define a stratified module $M(\varphi_\bullet)$ over $\mathbb{D}(\rho)$, $\rho \in |k^*|$, and we let Φ be the fundamental matrix (in the notation of Definition 73). Recall that we are denoting by G the subgroup of $\mathbb{GL}(V)(k)$ generated by $\cup_{n\geq 0}\varphi_n(k)$.

Lemma 84. *Notations as above. Assume that for some $\alpha \in V - \{0\}$, $\Phi \cdot \alpha \in V \otimes_k k[[x]]$ is actually in $V \otimes_k \mathscr{O}(\rho)$. If α is **not fixed** by G then there exists an infinite subsequence $S \subseteq \{n_i\}_{i\in\mathbb{N}}$ such that $\Gamma_s \cdot \alpha \neq 0$ and $\|\Gamma_s \cdot \alpha\|_\rho$ is bounded for all $s \in S$.*

Proof: First of all, $\Phi \cdot \alpha \in V \otimes_k \mathscr{O}(\rho)$ if and only if $\varphi_n^{-1} \cdots \varphi_0^{-1}\Phi \cdot \alpha \in V \otimes_k \mathscr{O}(\rho)$. Thus, we can assume that $p^{n_{i+1}} > bp^{n_i}$ for all $i \geq 0$. Let Φ_n denote $\varphi_0 \cdots \varphi_n$. It is easy to see that the degree of $\Phi_{n_i} \cdot \alpha$ is less than $p^{n_{i+1}}$. These normalizations are made to study the terms of $\Phi \cdot \alpha$ and show that they are related to the $\Gamma_* \cdot \alpha$ in a way that the condition $\Phi \cdot \alpha \in V \otimes_k \mathscr{O}(\rho)$ gives the desired bound.

More precisely, take an arbitrary $i \in \mathbb{N}$ and let us study the next step $\Phi_{n_{i+1}} \cdot \alpha$ in the sequence. We have

$$\Phi_{n_{i+1}} \cdot \alpha = \Phi_{n_i} \cdot \alpha + (\Phi_{n_{i+1}} - \Phi_{n_i}) \cdot \alpha =$$

$$= \Phi_{n_i} \cdot \alpha + \Phi_{n_i} \cdot \Gamma_{n_{i+1}} \cdot \alpha.$$

Of course, we might have $\varphi_{n_{i+1}} \cdot \alpha = \alpha$ and the second term above is zero. But, since $\alpha \notin V^G$, Property 4 above guarantees that there exists $j > i$ such that $\varphi_{n_\nu} \cdot \alpha = \alpha$ for all $\nu \in \{i+1, \ldots, j-1\}$, but $\Gamma_{n_j} \cdot \alpha \neq 0$. Hence $\Phi_{n_i} \cdot \Gamma_{n_j} \cdot \alpha \neq 0$ (Φ_n is invertible) and

$$\Phi_{n_j} \cdot \alpha = \Phi_{n_i} \cdot \alpha + \Phi_{n_{j-1}} \cdot \Gamma_{n_j} \cdot \alpha \quad \text{and} \quad \Phi_{n_{j-1}} \cdot \Gamma_{n_j} \cdot \alpha \neq 0.$$

Now, the condition on the degrees shows that the degree of $\Phi_{n_i} \cdot \alpha$ is less than the term of least degree in $\Phi_{n_{j-1}} \cdot \Gamma_{n_j} \cdot \alpha$ since the degree of $\Phi_{n_i} \cdot \alpha$ is at most bp^{n_i} and non-zero terms of $\Gamma_{n_j} \cdot \alpha \neq 0$ have degree greater or equal p^{n_j}. This has the important consequence that all terms $? \otimes x^d$ appearing in $\Phi_{n_i} \cdot \alpha$ are also appearing in $\Phi \cdot \alpha$. If we follow the same reasoning with j in the place of i, follows that the same terms $? \otimes x^d$ appearing in $\Phi_{n_{j-1}} \cdot \Gamma_{n_j} \cdot \alpha$ appear in $\Phi \cdot \alpha$. Because $\Phi \cdot \alpha$ belongs to $V \otimes_k \mathscr{O}(\rho)$, there exists a constant $c > 0$ such that $\|\Phi_{n_{j-1}} \cdot \Gamma_{n_j} \cdot \alpha\|_\rho \leq c$.

All terms in $\Phi_{n_{j-1}} \cdot \Gamma_{n_j} \cdot \alpha$ are of the form $? \otimes x^d$, with

$$d = \varepsilon_0 r_0 p^{n_0} + \ldots + \varepsilon_{j-1} r_{j-1} p^{n_{j-1}} + r_j p^{n_j}, \quad \varepsilon \in \{0,1\}, \quad r \in \{1, \ldots, b\}.$$

By the (easy) Lemma 85 below, the terms of $\Phi_{n_{j-1}}\Gamma_{n_j} \cdot \alpha$ whose degree is between p^{n_j} and bp^{n_j} will be the terms of corresponding degree in $\Gamma_{n_j} \cdot \alpha$. Hence, $\|\Gamma_{n_j} \cdot \alpha\|_\rho \leq c$, as we wanted. $\square$

Lemma 85. *Let $\{0,\ldots,b\}^{\oplus\mathbb{N}}$ be the restricted product of the set $\{0,\ldots,b\}$ with respect to the subset $\{0\}$. Assume that $p^{n_{i+1}} > bp^{n_i}$ for all $i \geq 0$, then the map $\{0,\ldots,b\}^{\oplus\mathbb{N}} \longrightarrow \mathbb{N}$ given by*

$$(m_0, m_1, \ldots) \mapsto \sum_{i=0}^{\infty} m_i p^{n_i}$$

is injective.

Proof: Very easy. $\square$

5.7.2 Proof of Theorem 82

We now proceed as in section 7 of [25].

Take any reduced connected algebraic group G such with $p(G) = G$. Note that any unipotent subgroup of G is contained in $p(G)$ and hence the closed normal subgroup $\mathbb{U}(G)$ generated by all the unipotent and connected closed subgroups of G is contained in $p(G)$. The quotient $G/\mathbb{U}(G)$ is a torus or trivial and hence contains no elements of order a power of p, it follows that $\mathbb{U}(G) = p(G)$.

Lemma 86 ([25], 7.6)**.** *Let G equal $p(G)$ as above. Then there are morphisms $u_1, \ldots, u_h : \mathbb{G}_a \longrightarrow G$ taking the 0 to the identity such that G is generated by $\cup_i u_i(k)$.* $\square$

The reader might profit from knowing (using basic group theory) how to prove this Lemma in the case G is reductive. In fact, if G is reductive, then G it is immediately semisimple; G is the semi-direct product $\mathscr{R}(G) \cdot (G, G)$, $\mathscr{R}(G)$ the radical; by rigidity of tori, $\mathscr{R}$ is a central torus and under the assumption $p(G) = G$, it has to be trivial. By Theorem 9.4.1 of [41], it is generated by the root subgroups U_α. The proof of Lemma 86 is more complicated since some care has to be taken for non-reductive groups.

Now, let $u_1, \ldots, u_h$ be as in Lemma 86 above. Choose a sequence $\{a_n\}_{n=1}^{\infty}$ in k^* such that $|a_n|$ grows very fast:

$$\lim_{n\to+\infty} \frac{\log|a_n|}{p^n} = +\infty; \tag{5.9}$$

in particular the inequality

$$|a_n| \leq c r^{p^n}, \quad c, r > 0 \tag{5.10}$$

is only possible for finitely many n.

Pick a sequence of non-negative integers $\{n_i\}_{i\in\mathbb{N}}$ as in Property 3 and define elements $\varphi_n \in G(A_n)$ by

$$\varphi_n := \begin{cases} \mathrm{I}, & \text{if } n \notin \{n_i\}_{i\in\mathbb{N}} \\ u_l(a_{n_i}x^{p^{n_i}}), & \text{if } n = n_i \text{ and } i \equiv l \mod h. \end{cases} \tag{5.11}$$

Let $\pi : G \longrightarrow \mathbb{GL}(V)$ be a representation. Writing

$$\pi \circ u_l(x) = \mathrm{I} + \sum_{r=1}^{b} \gamma_{lr}^{(\pi)} \otimes x^r, \quad \gamma_{lr}^{(\pi)} \in \mathrm{End}_k(V),$$

follows that

$$\pi\varphi_{n_i} = \mathrm{I} + \sum_{r=1}^{b} a_{n_i}^r \cdot \gamma_{lr}^{(\pi)} \otimes x^{rp^{n_i}}, \qquad \text{if } i \equiv l \mod h. \tag{5.12}$$

So, if $\alpha \in V$ is such that

$$\|\pi\varphi_{n_i} \cdot \alpha - \alpha\|_\rho \le c, \quad \text{for some } c > 0$$

we have for $i \equiv l \mod h$

$$|\gamma_{lr}^{(\pi)} \cdot \alpha| \cdot |a_{n_i}|^r \le c\rho^{-rp^{n_i}}. \tag{5.13}$$

In the presence of growth condition (5.9), inequality (5.13) is possible for infinitely many values of i if and only if $\gamma_{lr}^{(\pi)} \cdot \alpha = 0$.

Let Φ be the fundamental matrix for the module $M(\pi\varphi_\bullet)$. If for some $\alpha \in V - \{0\}$ $\Phi \cdot \alpha$ actually belongs to $V \otimes_k \mathscr{O}(\rho)$ $(\rho > 0)$, by Lemma 84, α is fixed by G. In a nutshell:

Lemma 87. *Let $G = p(G)$ and let $u_1, \dots, u_h : \mathbb{G}_a \longrightarrow G$ morphisms whose images generate G. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in k^* satisfying eq. (5.9) and let $\{n_i\}_{i\in\mathbb{N}}$ be a sequence of integers as in Property 2 above. Take $\{\varphi_n\}_{n\in\mathbb{N}}$ as in eq. (5.11) and for some representation $\pi : G \longrightarrow \mathbb{GL}(V)$, let Φ be the fundamental matrix for $M(\pi\varphi_\bullet)$. If $\alpha \in V$ is such that $\Phi \cdot \alpha \in V \otimes_k \mathscr{O}(\rho)$, then $\alpha \in V^G$.* □

Theorem 88. *Keep the notations of Lemma 87 and assume that π embeds G as a closed subgroup of $\mathbb{GL}(V)$. Then the canonical inclusion of the monodromy group $G_{\mathrm{mono}} := G(M(\pi\varphi_\bullet), \mathscr{T}) \subseteq G$ given in Lemma 78 is in fact an equality.*

Proof: Assume that $G_{\mathrm{mono}} \neq G$ and use Lemma 90 below to obtain a representation $\theta : G \longrightarrow \mathbb{GL}(W)$ with

$$\theta = \wedge^r(\pi_{b_1}^{a_1} \oplus \cdots \oplus \pi_{b_s}^{a_s}),$$

such that G_{mono} fixes a line $L \subset W$ which is not fixed by G. Since G_{mono} acts trivially on L (by Lemma 76), there exists $\alpha \in W^{G_{\mathrm{mono}}}$ which is not in W^G. Now, the representation of $\Pi^{\mathbf{loc}}$ obtained by the composition $\Pi^{\mathbf{loc}} \longrightarrow G_{\mathrm{mono}} \longrightarrow G \longrightarrow \mathbb{GL}(W)$ induces, by Tannakian duality, an $N \in \mathscr{T}$ which is just $M(\theta(\varphi_\bullet))$, since

$$\bigwedge^r \left(M(\pi\varphi_\bullet)^{(a_1,\dots,a_s)}_{(b_1,\dots,b_s)} \right) \cong M\left(\wedge^r \left((\pi\varphi_\bullet)^{(a_1,\dots,a_s)}_{(b_1,\dots,b_s)} \right) \right).$$

Let Ψ be the fundamental matrix of the stratified module $M(\theta\varphi_\bullet)$. By Lemma 89 below, follows that $\Psi \cdot \alpha$ is in $W \otimes_k \mathscr{O}(\rho)$ for some $\rho \in |k^*|$. By Lemma 87, $\alpha \in W^G$, which is a contradiction. This shows that the representation W cannot exist and hence that $G_{\mathrm{mono}} = G$. $\square$

Lemma 89. *Let V be a k-vector space and let $\varphi_n \in \mathbb{GL}(V)(A_n)$ be such that $\varphi_n(0) = \mathrm{I}$. Let Φ denote the fundamental matrix for $M(\varphi_\bullet)$. If $\alpha \in V$ is fixed by $\Pi^{\mathbf{loc}}$, then $\Phi \cdot \alpha$ belongs to $V \otimes_k \mathscr{O}(\rho)$ for some $\rho \in |k^*|$.*

Proof: Let $a : \mathbb{1} \longrightarrow V$ be the $\Pi^{\mathbf{loc}}$-equivariant map that takes 1 to α. This corresponds to an arrow

$$A : \mathbb{1} \longrightarrow M(\varphi_\bullet)$$

in some $\mathbf{str}(\mathbb{D}(\rho))$, with $\rho \in |k^*|$, such that taking the fibre at 0 gives a back. But $A(1)$ is of the form $\Phi \cdot \beta$ for some $\beta \in V$. Since $\Phi(0) = \mathrm{I}$, we have $\alpha = (\Phi \cdot \beta)(0) = \beta$ and because A is defined over $\mathbb{D}(\rho)$ we have $\Phi \cdot \alpha = \Phi \cdot \beta = A(1) \in V \otimes_k \mathscr{O}(\rho)$. $\square$

Lemma 90. *Let $H \subseteq G \subseteq \mathbb{GL}(V)$ and assume that $H \neq G$. Then there exist $r, a_1, \dots, a_s, b_1, \dots, b_s \in \mathbb{N}$ such that H is the stabilizer of a line in*

$$\bigwedge^r \left(V^{(a_1,\dots,a_s)}_{(b_1,\dots,b_s)} \right).$$

Proof: From [42], Cor. 1.16, p. 122, there exists a representation U of G which has H as the stability group of a line L. Such a representation U is of the form U'/U'' with subrepresentations $U'' \subset U' \subseteq V^{(a_1,\dots,a_s)}_{(b_1,\dots,b_s)}$. It follows easily that H is the stability group of a subspace $U^{(3)} \subset V^{(a_1,\dots,a_s)}_{(b_1,\dots,b_s)}$. The rest is linear algebra. $\square$

5.8 Appendix

5.8.1 Cartier-Katz

Our goal here is to prove (in an ad-hoc manner) the Cartier-Katz Theorem as stated in section 5.2. Let R be the ring of analytic functions on the disk $\mathbb{D}(\rho)$ ($\rho \in |k^*|$).

As usual, $F : R \longrightarrow R$ denotes the Frobenius $f \mapsto f^p$. Given an R-module M and a subgroup of the additive group $(M, +)$, N, which is stable by multiplication by elements of the form f^{p^n}, we consider it as an R-module via p^n-powers. Naturally there is a canonical R-linear homomorphism $N \otimes_{R,F^n} R \longrightarrow M$.

Modules with connections, Cartier's Theorem

Let M be an R-module with a connection $D : M \longrightarrow M$, i.e. D is additive and $D(f \cdot s) = \frac{df}{dx} \cdot s + f \cdot D(s)$ for all $f \in R$. Assume now that D has p-curvature 0: $D^p = 0$. Define M^D to be the subgroup of M consisting of all $s \in M$ killed by D. It is an R-module via p-powers and it is a classical result of Cartier that *the natural homomorphism of R-modules $\iota : M^D \otimes_{R,F} R \longrightarrow M$ is an isomorphism.*

We give a proof of this fact following [19], Theorem 5.1 (where the finite type hypothesis is assumed). First, we show that ι is injective. Let $s \in \ker \iota$ and write it as $\sum_0^m s_i \otimes x^i \in M^D \otimes R$, with $s_m \neq 0$ and $m < p$. Then $0 = D^m(\sum x^i s_i) = c s_m$ and this is impossible because $c \in k^*$. So ι is injective. To prove surjectivity, we let

$$P : M \longrightarrow M, \qquad s \mapsto \sum_{r=0}^{p-1} \frac{(-1)^r}{r!} x^r D^r(s).$$

It is easy to verify that (i) $\mathrm{im}(P) \subseteq M^D$. Let

$$T : M \longrightarrow M^D \otimes_{R,F} R, \qquad s \mapsto \sum_{r=0}^{p-1} P \circ D^r(s) \otimes \frac{x^r}{r!}.$$

By expanding $P(D^r(s))$ using the definition of P, we see that $\iota \circ T = \mathrm{id}$, and hence ι is also surjective. $\square$

The Cartier-Katz Theorem

The goal here is to prove that the category of F-divided modules **Fdiv** is equivalent to the category of stratified modules $\mathbf{str} = \mathbf{str}(\mathbb{D}(\rho))$. Recall that all F-divided modules over R are free.

Now let M be an R-module with a stratification ∇. Denote the maps $\nabla(\partial_n)$ by the usual D_n and let

$$M_n = \{s \in M;\ D_j(s) = 0, \forall 0 < j < p^n\}.$$

It is an R-module via p^n-powers and $\delta_n := D_{p^n}$ will map M_ν into itself for each $\nu \geq n$, since $\delta_n D_\nu = D_\nu \delta_n$. Using that

$$\partial_{p^n}(f^{p^n}) = \left(\frac{df}{dx}\right)^{p^n},$$

we see that δ_n defines a connection on the R-module M_n and this connection is of p-curvature zero. Moreover, $M_n^{\delta_n} = M_{n+1}$ and follows from the above result of Cartier that $M_{n+1} \otimes_{R,F} R \cong M_n$. Note that if M_0 is finite over R, then so is M_n (seen as a module via p^n-powers). So we have obtained a functor from **str** to **Fdiv**; call it U. Note that, in particular, stratified modules will be locally free if we know (as we do) that F-divided modules are locally free.

We now construct an equivalence inverse to this functor. Let $\{M_i\}$ be in **Fdiv**, we want to find a stratification on M_0. For each ∂_n, we define $\nabla(\partial_n)(s)$ as follows. Take $e_1, \ldots, e_r$ freely generating M_ν ($p^\nu > n$) and write $s = \sum f_i \otimes e_i$. Then

$$\nabla(\partial_n)(s) = \sum_i \partial_n(f_i) \otimes e_i.$$

It is immediate to see that this is well defined (i.e. is independent of the basis and of ν) and that ∇ gives a stratification on M_0 such that

$$\bigcap_{n=1}^{p^\nu - 1} \ker \nabla(\partial_n) = \operatorname{im}(M_\nu \longrightarrow M_0).$$

Call this functor V, which is certainly a *tensor functor*.

Then the above equality states that there is a natural equivalence $\mathrm{id} \Rightarrow UV$, the other natural equivalence $\mathrm{id} \Rightarrow VU$ is also trivial. This proves the Cartier-Katz Theorem.

5.8.2 Differential operators in rigid geometry of positive characteristic

This discussion of differential operators is based on the exposition of W. Traves in his PhD thesis (Toronto, 1998). Most importantly, the proof of Theorem 98 is just an adaptation of the classical result presented there (strangely enough, this very natural result is overlooked in EGAIV$_4$, while the corresponding one for Ω^1 is carefully analyzed, loc.cit. Cor. 17.2.4). The necessity for such an appendix is to clarify the definition of stratification given in the main text (section 5.2). This requires a study of how differential operators behave with respect to the opens of the rigid site over an affinoid; more precisely, given an affinoid X, we want to obtain the *coherence* of the presheaf $\mathscr{D}_X^n$ which, to each affinoid domain $U \subseteq X$, associates the $\mathscr{O}_X(U)$-module of differential operators $D^n(\mathscr{O}_X(U))$. Theorem 98 fills this requirement.

Let Affd_k denote the category of k-affinoid algebras and let A be an object of this category. We remind the reader that k is still algebraically closed; this will allow us to make various simplifications on the nature of differential operators.

Definition 91. *The algebra of principal parts of order n over A, $P^n_{A/k}$, is the commutative k-algebra $A \otimes_k A/\mathfrak{I}^{n+1}$, where $\mathfrak{I}$ is the kernel of multiplication $a \otimes b \mapsto ab$. The modified algebra of principal parts of order n, ${}^f P^n_{A/k}$, is the algebra $A \widehat{\otimes}_k A/\mathfrak{I}^{n+1}$, where $\mathfrak{I}$ and the structural morphism are defined analogously. For notational convenience we also define $P^\infty_A := A \otimes_k A$ and $A \widehat{\otimes}_k A := {}^f P^\infty_A$. We let $d_0, d_1 : A \longrightarrow P^n_A$ denote the natural k-algebra homomorphism $a \mapsto a \otimes 1$ and $a \mapsto 1 \otimes a$. When referring to P^n_A as an A-module we will always mean the structure obtained via d_0. Analogous definitions for ${}^f P^n_A$ are in force (also for $n = \infty$).*

Lemma 92. *Under the hypothesis that $k = k^p$, the natural homomorphism $P^n_A \longrightarrow {}^f P^n_A$ is an isomorphism for any $n \geq 0$ and ${}^f P^n_A$ is a finite A-algebra.*

Proof: We note that there is a commutative diagram of k-algebras

$$\begin{tikzcd} P^n_A \arrow[r] & {}^f P^n_A \\ (A \otimes_{F, A^{(n)}, F} A)/\mathfrak{I}^{n+1} \arrow[u] \arrow[r] & (A \widehat{\otimes}_{A^{(n)}} A)/\mathfrak{I}^{n+1} \arrow[u] \end{tikzcd}$$

and that the vertical arrows are clearly isomorphisms. The bottom arrow is also an isomorphism by the fact that $F : A^{(n)} \longrightarrow A$ is finite plus Prop. 6 of 3.7.3, p. 165 of [4]. Hence the top arrow is also an isomorphism. $\square$

Now we recall the concept of differential operator. Given A-modules M and N, and a k-linear map $\partial : M \longrightarrow N$ we can consider the A-linearization

$$\partial' : P^\infty_A \otimes_{d_1, A} M \longrightarrow N, \qquad (a_0 \otimes a_1) \otimes s \mapsto a_0 \partial(a_1 m),$$

where we see $P^\infty_A \otimes M$ as an A-module via multiplication on the left of P^∞_A. The k-linear map ∂ is also A-linear if and only if ∂' kills all elements of the form $(1 \otimes a - a \otimes 1) \otimes s \in P^\infty_A \otimes M$.

Definition 93. *The k-linear map ∂ between A-modules M and N is a differential operator of order $\leq n$ if the linearization $\partial' : P^\infty_A \otimes M \longrightarrow N$ factors through $P^n_A \otimes_{d_1, A} M$.*

If we treat elements of A as k-linear homomorphisms of A-modules $a : M \longrightarrow N$, then it is immediate to see that for a differential operator ∂ of order $\leq n$, $a \circ \partial$ is also a differential operator of order $\leq n$. The A-module of all differential operators of order $\leq n$ (the action of A being the composition on the left) is denoted by $D^n(M, N)$. The next lemma is tautological.

Lemma 94 (Universal Property). *Let M and N be a A-modules and consider the A-module $\mathrm{Hom}_A(P_A^n \otimes_{d_1,A} M, N)$ (via the A-module structure of N). The natural A-linear homomorphism*

$$\mathrm{Hom}_A(P_A^n \otimes_{d_1,A} M, N) \longrightarrow D^n(M,N), \quad \varphi \mapsto (M \longrightarrow P_A^n \otimes M \longrightarrow N)$$

is an isomorphism. □

We now give a more computational way of verifying that a k-linear map between A-modules $\partial : M \longrightarrow N$ is a differential operator. Introduce the k-linear map

$$\mathrm{ad}_a(\partial)(s) := \partial \circ a - a \circ \partial : M \longrightarrow N.$$

Obviously ∂ is A-linear (differential operator of order ≤ 0) if and only if $\mathrm{ad}_a(\partial) = 0$ for all $a \in A$. Using that for $\xi = \sum_i x_i \otimes y_i \in \mathfrak{I}$ we have

$$\xi = \sum (x_i \otimes 1) \cdot (1 \otimes y_i - y_i \otimes 1),$$

it follows that ∂ is a differential operator of order $\leq n$ if and only if for every sequence $a_0, \ldots, a_n$ of elements in A,

$$\mathrm{ad}_{a_0} \circ \cdots \circ \mathrm{ad}_{a_n} \circ \partial = 0.$$

5.8.3 Behaviour of differential operators with respect to etale extensions

Definition 95. *An arrow $f : A \longrightarrow B$ in Affd_k is formally smooth (resp. formally etale) if given a commutative diagram in Affd_k*

$$\begin{tikzcd} B \arrow[r, "\psi"] & C/\mathfrak{N} \\ A \arrow[u, "f"] \arrow[r, "\varphi"'] & C \arrow[u, "\pi"'] \end{tikzcd} \tag{5.14}$$

with $\mathfrak{N}^2 = 0$, there exists an arrow (resp. unique) $\psi' : B \longrightarrow C$ making

$$\begin{tikzcd} B \arrow[r, "\psi"] \arrow[dr, "\psi'"] & C/\mathfrak{N} \\ A \arrow[u, "f"] \arrow[r, "\varphi"'] & C \arrow[u, "\pi"'] \end{tikzcd} \tag{5.15}$$

commute.

Definition 96 (Huber; [15], Prop. 8.1.1, p. 240)**.** *A morphism in* Affd_k $f : A \longrightarrow B$ *is etale if there exists an isomorphism of A-algebras* $B \cong A\langle x_1, \ldots, x_n\rangle/(f_1, \ldots, f_n)$ *with*

$$\det\left(\frac{\partial f_j}{\partial x_i}\right) \in B^\times.$$

We note that an etale morphism is flat. As usual, we have

Lemma 97. *Let* $B := A\langle x_1, \ldots, x_n\rangle/(f_1, \ldots, f_r)$ *and let*

$$J = \left(\frac{\partial f_j}{\partial x_i}\right)$$

be the Jacobian $n \times r$-matrix. Then $A \longrightarrow B$ is formally smooth if some $r \times r$ minor of J is invertible in B. If in addition $r = n$, then B is formally etale.

Proof: We use the following facts about affinoid algebras:

i) All affinoid algebras are k-Banach algebras and any two Banach algebras norms are equivalent ([15], 3.2.1, p. 48).

ii) On an affinoid algebra T with Banach algebra norm $|\cdot|$, the subring

$$T^\circ := \{a \in T;\ \|a\|_{\mathrm{spec}} \le 1\}$$

equals the subring

$$\{a \in T;\ \sup_n |a^n| < \infty\}$$

(loc.cit., 3.4.5, p. 56).

iii) If $\mathfrak{N} \subset C$ is a nilpotent ideal of an affinoid algebra, then the natural projection $C \longrightarrow C/\mathfrak{N}$ preserves the spectral semi-norm.

iv) Given an affinoid algebra C and a power series $F \in C\langle x_1, \ldots, x_n\rangle$, for any two $a, y \in (C^\circ)^n$ we have

$$F(a+y) = F(a) + \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(a) y_j + \sum_{i,j} F_{ij} y_i y_j, \quad F_{ij} \in C.$$

Now to the proof of the Lemma. Consider a diagram as (5.14) and keep the notation. We seek to find ψ' as in diagram (5.15). Because $\|\psi(x_i)\|_{\mathrm{spec}} \le 1$ (use the maximum modulus principle), follows that any $c_i \in C$ above $\psi(x_i)$ has spectral semi-norm bounded by 1. Let $c = (c_1, \ldots, c_n) \in (C^\circ)^n$ reduce to $(\psi(x_1), \ldots, \psi(x_n))$. Using φ, we regard each f_i as an element of $C\langle x\rangle$. Note that some $r \times r$ minor of the corresponding Jacobian matrix – the first one say – is invertible in $C/\mathfrak{N}$ and consequently in C. Then, the system of $r \times r$ equations

$$-f_i(c) = \sum_{j=1}^{r} \frac{\partial f_i}{\partial x_j}(c) Y_j, \quad i = 1, \ldots, r \tag{5.16}$$

has a unique solution $y_1, \ldots, y_r \in C$; reducing the above system to $C/\mathfrak{N}$, uniqueness tells us that $y_i \in \mathfrak{N}$. Now define $\Psi : A\langle x_1, \ldots, x_n\rangle \longrightarrow C$ by $x_i \mapsto c_i + y_i$ for $1 \leq i \leq r$ and $x_i \mapsto c_i$ for $j > r$. Since $\mathfrak{N}$ is nilpotent, the formula in *iv*) above shows that Ψ factors through B and gives the desired ψ'. The case $r = n$ clearly implies that ψ' is uniquely defined by the solutions of the system (5.16). □

We now want to show that the algebras $P^n_A = P^n_{A/k}$ have a local nature for the etale topology on Affd_k. That is, let $f : A \longrightarrow B$ be an arrow in Affd_k and consider the natural homomorphism of k-algebras

$$P^n(f) : P^n_A \otimes_{d_0,A} B \longrightarrow P^n_B, \quad (a_0 \otimes a_1 \mod \mathfrak{I}^{n+1}_A) \otimes b \mapsto b \cdot (f(a_0) \otimes f(a_1)) \mod \mathfrak{I}^{n+1}_B \tag{5.17}$$

Theorem 98. *If f above is etale, then $P^n(f)$ is an isomorphism.*

Proof: First we need a candidate for the inverse of $P^n(f)$. We begin by noting that the k-algebra $P^n_A \otimes_{d_0,A} B$ is in fact an affinoid algebra; P^n_A is affinoid and since $d_0 : A \longrightarrow P^n_A = (A \otimes_{A^{(n)}} A)/\mathfrak{I}^{n+1}$ is finite, $P^n_A \otimes_{d_0,A} B$ is $P^n_A \widehat{\otimes}_{d_0,A} B$ ([4], Prop. 6 of 3.7.3, p. 165) and the complete tensor product is affinoid (loc.cit., Prop. 4 of 7.1.4, p. 268). Now we will use Lemma 97 in order to copy the procedure of proof of the same result in the classical case. Start with the diagram in Affd_k

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}_B"] & B \\ A \arrow[u, "f"] \arrow[r, "d_1 \otimes 1_B"'] & P^n_A \otimes_A B \arrow[u, "\varepsilon_A \otimes_A \mathrm{id}_B = \psi"'] \end{tikzcd} \tag{5.18}$$

where ε_A is the augmentation $P^n_A \longrightarrow A$. Because f is flat, follows that the $(\ker \psi)^{n+1} = 0$ and because f is formally etale, there exists a unique homomorphism $\delta : B \longrightarrow P^n \otimes_A B$ making

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}_B"] \arrow[dr, "\delta"] & B \\ A \arrow[u, "f"] \arrow[r, "d_1 \otimes 1_B"'] & P^n_A \otimes_A B \arrow[u, "\psi"'] \end{tikzcd} \tag{5.19}$$

commute. We want to show that the k-algebra homomorphism δ is a differential operator of order $\leq n$ from B to the B-module $P_A^n \otimes_A B$. Given $b \in B$, it is easy to see that

$$\mathrm{ad}_b(\delta) = \delta \circ (1 \otimes b) - (1 \otimes b) \circ \delta = (\delta(b) - 1 \otimes b) \circ \delta$$

and proceeding inductively

$$\mathrm{ad}_{b_n} \circ \cdots \circ \mathrm{ad}_{b_0} \circ \delta = \left(\prod_{j=0}^{n} (\delta(b_j) - 1 \otimes b_j) \right) \circ \delta. \tag{5.20}$$

By the commutativity of the upper triangle in diagram (5.19), follows that $\psi(\delta(b) - 1 \otimes b) = 0$ and hence the left hand side in equation (5.20) is zero. This means that δ is a differential operator of order $\leq n$. By the universality of P_B^n, there must be a factorization

$$\begin{tikzcd} & P_B^n \arrow[d, "\varphi"] \\ B \arrow[ur, "d_1"] \arrow[r, "\delta"'] & P_A^n \otimes_A B \end{tikzcd} \tag{5.21}$$

with φ B-linear. This is our candidate for the inverse of $P^n(f)$. To prove this, we first prove

Claim: $P^n(f) \circ \delta = d_{1,B}$.

Proof of the Claim: Because f is formally etale (until now we have only used formal smoothness), we are required to prove that the diagram

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}"] \arrow[dr, "P^n(f)\circ\delta"'] & B \\ A \arrow[u, "f"] \arrow[r, "d_{1,B}\circ f"'] & P_B^n \arrow[u, "\varepsilon_B"'] \end{tikzcd}$$

is commutative since, replacing the diagonal arrow above by $d_{1,B}$, we certainly get a commutative diagram. The bottom triangle is commutative, as $\delta \circ f = d_{1,A} \otimes 1_B$. The upper triangle is commutative because a direct inspection shows that $\psi = \varepsilon_A \otimes \mathrm{id}_B$ (of (5.19)) is none other than $\varepsilon_B \circ P^n(f)$. This proves the claim.

Now we show how the claim proves the Theorem. From the commutative diagram

$$\begin{tikzcd} & B \arrow[dl, "d_{1,B}"'] \arrow[d, "\delta"] \arrow[dr, "d_{1,B}"] & \\ P_B^n \arrow[r, "\varphi"'] & P_A^n \otimes_A B \arrow[r, "P^n(f)"'] & P_B^n \end{tikzcd}$$

and the universal property of P^n_B, it follows that $P^n(f)\circ\varphi = \mathrm{id}$. Analogously, we have the commutative diagram

$$\begin{tikzcd}
 & B \arrow[dl, "\delta"'] \arrow[d, "d_{1,B}"'] \arrow[dr, "\delta"] & \\
P^n\otimes_A B \arrow[r, "P^n(f)"'] & P^n_B \arrow[r, "\varphi"'] & P^n_A\otimes_A B.
\end{tikzcd}$$

It follows that $\varphi\circ P^n(f)$ agrees with the identity on the B-submodule of $P^n_A\otimes_A B$ generated by the image of δ (call this module M). From diagram (5.19), $M\supseteq N\otimes_A B$, where $N\subseteq P^n_A$ is the A-submodule generated by the image of $d_{1,A}$. But it is clear that $N = P^n_A$. It follows that $\varphi\circ P^n(f) = \mathrm{id}$ and we are done. $\square$

Remark: If we keep in mind that the single goal of this discussion is to obtain Theorem 98 above, then we could restrict the hypothesis to the case of *flat* and formally etale A-algebras. It may be the case that all *formally etale* A-algebras are flat, but the author was not able to find a convincing reference for this, even in the classical case of EGAIV_1. Indeed, consulting section 19.7 of loc.cit., we find Thm. 19.7.1 which states that a local, formally smooth homomorphism (of topological rings, the topology being the linear topology obtained from the maximal ideals) $\varphi : (R,\mathfrak{m}) \longrightarrow (S,\mathfrak{n})$ is flat. But in view of the absence of a clear statement other than that of loc.cit. Prop. 22.6.4 concerning the relations between the global and local notions of formal smoothness, it is not safe to assume that all formally etale algebras are flat. We also note that the proof of Theorem 98 would be reduced to referencing, at least in the case of rational domains in the affinoid $\mathrm{Max}(A)$, if we could prove that the ring of analytic functions on such an open is formally etale *for the discrete topologies*; that, of course, does not seem to hold, once more due to the fact that the local notion of formal smoothness is not so well behaved as we are used to in the finite type case.

We are now in a position to define the sheaf of differential operators on the affinoid space $X = \mathrm{Max}(A)$ (with its strong G-topology).

Definition 99. *Given $\mathscr{E}$ a coherent $\mathscr{O}_X$-module, represented by the finite A-module M ([15] Thm. 4.5.2, p. 88), the sheaf of differential operators of order $\leq n$ on $\mathscr{E}$, $\mathscr{D}^n(\mathscr{E})$, is the coherent $\mathscr{O}_X$-module associated to the finite A-module* $\mathrm{Hom}_A(P^n_A\otimes_A M, M) = D^n_A(M)$. *The sheaf of differential operators on $\mathscr{E}$, $\mathscr{D}(\mathscr{E})$, is the direct limit of sheaves $\varinjlim_n \mathscr{D}^n(\mathscr{E})$ of $\mathscr{O}_X$-modules.*

Because affinoids are quasi-compact, it is easy to see that $\mathscr{D}(\mathscr{E})$ is in fact the sheaf associated to the A-algebra $D(M) = \varinjlim_n D^n(M)$.

Using that homomorphism modules $\mathrm{Hom}_A(M, ?)$ behave well under flat base extension ([24], Thm. 7.11, p. 52) and Theorem 98, we have deduced the first part of

Corollary 100. *i) Given a coherent sheaf $\mathscr{E}$ on X, the sheaf $\mathscr{D}^n(\mathscr{E})$ has the following property: for any rational domain $U \subseteq X$, we have $\mathscr{D}^n(\mathscr{E})(U) = D^n_{\mathscr{O}(U)}(\mathscr{E}(U))$.*

ii) If $X = \mathbb{D}(\rho)$ $(\rho \in |k^*|)$, then the following extra data on $\mathscr{E}$ are equivalent:

(a) A stratification on $M := \mathscr{E}(X)$

(b) A homomorphism of $\mathscr{O}_X$-algebras $\nabla : \mathscr{D}(\mathscr{O}_X) \longrightarrow \mathscr{D}(\mathscr{E})$.

Proof: Just *ii*) needs proof. First we note that $D^n(A)$ is the A-module generated by the differential operators ∂_q with $0 \leq q \leq n$; this is an easy consequence of the fact that P^n_A is free on $(1 \otimes x - x \otimes 1)^q$. Having this in mind, it is obvious that the data on (b) induces a stratification on M by using the differential operators $\partial_n : \mathscr{O}(X) \longrightarrow \mathscr{O}(X)$. If we have a homomorphism $\delta : \mathscr{D}(\mathscr{O})(X) \longrightarrow \mathrm{End}_k(M)$ as in (a), it is easy to see, using the ad operators, that δ factors through $D_A(M)$. In fact, δ is given by a compatible set of A-linear homomorphisms

$$\delta_n : D^n(A) \longrightarrow D^n_A(M),$$

which in turn give compatible $\mathscr{O}_X$-linear homomorphisms $\delta_n : \mathscr{D}^n(\mathscr{O}_X) \longrightarrow \mathscr{D}^n(\mathscr{E})$. This is the required data in (b). □

Remark: It follows from Huber's Lemma ([15], 8.1.1, p. 240) that all affinoid domains of an affinoid variety X are etale; hence *i*) of Corollary 100 holds true in this more general setting.

Chapter 6

Natural questions for future work

6.1 Local solutions on proper rigid spaces

We begin by collecting a result of our general theory. Here k is algebraically closed of positive characteristic p and complete with respect to a non-trivial and non-Archimedean absolute value.

Proposition 101. *Let $\mathscr{E}$ be an object of $\mathbf{str}(X)$, $X = \mathbb{G}_m^{\mathrm{an}}/\langle q\rangle$ a Tate elliptic curve. For any point $x \in X$, there exists an admissible U neighbourhood of x in X such that $\mathscr{E}|U$ is trivial.*

Proof: Every Tate elliptic curve is algebraizable; there exists an elliptic curve X_0/k and an isomorphism $X_0^{\mathrm{an}} \cong X$ (of group objects). Hence, using Theorem 55 above, we can assume that $\mathscr{E} = \mathscr{N} \otimes \mathscr{L}$, where $\mathscr{N}$ is nilpotent (in $\mathbf{str}(X)$) and $\mathscr{L}$ has rank one. From Lemma 76, there exists an admissible neighbourhood U of x where $\mathscr{L}$ is trivial (as an object of $\mathbf{str}(U)$). We also know that $\mathscr{N}$ is associated to an etale covering of X and because $\mathscr{O}_{X,x}$ is henselian, it follows that $\mathscr{N}$ will also become trivial if we shrink U (in the rigid topology) even more. $\square$

We are presented with the very same question for other *projective* rigid analytic spaces.

Question 1. *Let X be a connected projective and smooth rigid analytic curve and let $\mathscr{E} \in \mathbf{str}(X)$. Let x be a point of X. Is there an admissible neighbourhood U of x such that $\mathscr{E}|U$ is trivial?*

We explain why this should be true. We have seen above (example in section 5.4) that the F-divided modules which fail to have a complete system of solutions near an arbitrary point come from unipotent matrices. So, in a sense, these should be nilpotent objects of the category $\mathbf{str}(X)$. But nilpotent objects in $\mathbf{str}(X)$ come from etale coverings and hence should be locally trivial.

6.2 Is $\Pi^{\mathbf{str}}$ the algebraic hull of a topological group?

The title of this section says it all. If X/k is as in section 6.1 above, we can ask if there exists a topological group $\Gamma = \varprojlim_i \Gamma_i$, with the Γ_i *discrete*, such that the *continuous*[1] algebraic hull $\varprojlim_i \Gamma_i^{\mathrm{alg}}$ is isomorphic to $\Pi_X^{\mathbf{str}}$. Using the *concrete description* of the stratified fundamental group of an elliptic curve (Tate curve) given in Chapter 3 and section 4.3.2 and some prototypes of fundamental groups in the non-Archimedean setting (see [8] and [1]), we can test which one would be appropriate. To be more specific, we propose to tackle this problem in an *experimental* way: this means that first we should actually find an abstract topological group whose character group (i.e., homomorphisms into $k^\times$) is isomorphic to the character group of the stratified fundamental group in question. In the case of a Tate elliptic curve, this requires that we find some group whose character group is $k^\times \oplus \mathbb{Z}_p/\mathbb{Z}$. Note that a distant analogy is governed by Pontryagin duality (of course, in our case, the group in question is not Hausdorff and we are interested in homomorphisms into $k^\times$, not $\mathbb{C}^\times$).

We now make a brief comment about the aforementioned prototypes of fundamental groups and give indications of why they should not yet give the correct answers.

A. de Jong introduced in [8] a new etale fundamental group for analytic spaces (in the sense of Berkovich) π_1^{et} — this is a topological group which is usually *not* prodiscrete. It is unknown to us how to compute π_1^{et}, even in simple examples; this inability seems justifiable since Y. André underwent the work of defining the *temperate* fundamental groups to fill a lacuna between the rigid analytic fundamental group and (de Jong's) etale fundamental group. Also, the period maps of Gross and Hopkings show that the etale fundamental group of $\mathbb{P}^h_{\mathbb{C}_p}$ is quite big ($\pi_1^{\mathrm{et}}(\mathbb{P}^h_{\mathbb{C}_p})$ surjects onto $\mathrm{SL}_{h+1}(\mathbb{Q}_p)$). So, it seems that π_1^{et} is unproportionally large. Nevertheless it is very useful to define other fundamental groups.

The temperate fundamental group of André is much better behaved. For example

$$\pi_1^{\mathrm{temp}}(X) \cong \mathbb{Z} \times \prod_\ell \mathbb{Z}_\ell,$$

if $X = \mathbb{G}_m^{\mathrm{an}}/\mathbb{Z}$ is a Tate curve over $\mathbb{C}_p$. We *do not* know how to compute π_1^{temp} of a Tate curve over k (positive characteristic), but we believe that it is isomorphic to

[1] The reason we have asked for the topological group in question to be prodiscrete is influenced by the fact that the temperate fundamental group is prodiscrete. Of course, at the moment, we have no indication whatsoever that this is the correct condition to be asked.

$\Gamma = \mathbb{Z} \times \prod_{\ell \neq p} \mathbb{Z}_\ell$ as a prodiscrete topological group (Γ is the projective limit of the projective system

$$\text{id} \times \text{projection} : \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}, \quad m|n, \ (n,p) = 1).$$

It is likely that this group will provide the answer we are looking for, since the group of continuous characters (and here we take k with the *discrete* topology) of Γ is just $k^\times \oplus \mu(k)$, where

$$\mu(k) \cong \bigoplus_{\ell \neq p} (\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

is the group of torsion elements in $k^\times$ (roots of unity).

6.3 Singularities and Riemann's existence theorem

One natural topic of investigation arises from the study of stratified modules in the rigid analytic category. It links to the powerful theory of p-adic differential equations of Dwork, Robba, Christol, Mebkhout and many others. On Chapter 4, we only had to evoke rigid GAGA in order to compare the fundamental group schemes of $\mathbf{str}(X)$ and $\mathbf{str}(X_0)$, where $X = X_0^{\text{an}}$ is the analytification of a projective and smooth algebraic variety defined over k (algebraically closed, of positive characteristic and complete with respect to a non-Archimedean absolute value $|\cdot|$). What about the open case? How to relate $\mathbf{str}(X)$ and $\mathbf{str}(X_0)$?

A substantial step towards understanding this problem in the complex analytic case is the theory of regular singular points of Deligne. The bulk of this theory is the following: let $X_0/\mathbb{C}$ be a smooth algebraic variety and let X be the complex analytic manifold obtained from it. There is a natural faithful tensor functor $(\cdot)^{\text{an}} : \text{DE}(X_0/\mathbb{C}) \longrightarrow \text{DE}(X)$ from the category of algebraic differential equations on X_0 to the category of analytic differential equations on X. Deligne defines a subcategory $\text{RSDE}(X_0/\mathbb{C})$, called the category of differential equations with regular singular points, such that the restriction of $(\cdot)^{\text{an}}$ to $\text{RSDE}(X_0/\mathbb{C})$ induces an equivalence. The objects of $\text{RSDE}(X_0/\mathbb{C})$ are DEs on X_0 which come from DEs with *logarithmic poles* on some smooth compactification $X_0 \subset \overline{X}_0$ (see [9] for more details). The proof of the equivalence $\text{RSDE}(X_0/\mathbb{C}) \cong \text{DE}(X)$ consists of (1) using GAGA and (2) proving that the analogous category of analytic equations, $\text{RSDE}(X)$, is in fact the whole $\text{DE}(X)$. At this point, the problem is entirely analytic and is solved by showing the existence of an extension with logarithmic singularities to the compactification and the meromorphicity of horizontal sections. Let us comment on the *existence of the extension.* One first works locally. Using a simple monodromy argument, it is possible to show that integrable differential equations on a polydisk minus coordinate hyperplanes $\{z \in \mathbb{C}^n;\ |z \leq \delta\} - \{z;\ z_1 \cdots z_m \neq 0\}$ can be extended to differential equations with logarithmic singularities on $\{z_i = 0\}$. If we stop now and translate what we have to the non-Archimedean world (of characteristic zero, for the time being), we obtain a very interesting question.

Question 2. *Let $\mathbb{D}(r)^*$ be the punctured disk of radius r in $\mathbb{C}_p$ and let (M, ∇) be a differential equation over it. Is there an extension (with singularities) of this differential equation to the whole disk $\mathbb{D}(r)$? What kind of poles can one expect at the origin?*

It is our belief that this question already has an answer (known to the experts in p-adic differential equations). So we are also interested in the case where $\mathbb{C}_p$ is replaced by some positive characteristic analogue. But in this direction, any given answer has to be more subtle because of Riemann's existence theorem (RET)[2]. Keeping in mind that finite etale covers "are" differential equations with finite monodromy, the question above is a generalization of RET. Fortunately, RET has already been understood in the non-Archimedean setting (Gabber (unpublished), Lütkebohmert [22], Schmechta [39], Lütkebohmert-Schmechta [23] and Ramero [36]), even in positive characteristic. In fact, in [23] a negative answer to RET is given (Artin-Schreier!) and this is the reason for our belief that the analogue to Question 2 in positive characteristic should be harder. Nevertheless, it is possible that RET is just a consequence, as it is in the complex analytic case ([20], Appendix), of a positive answer to Question 2.

[2]In what follows, we will use the name RET to designate Riemann's existence *problem* (the question) and Riemann's existence theorem (the affirmative answer).

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