Inviscid instability of a unidirectional flow sheared in two transverse directions

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Shear flow stability

•Navier-Stokes equations: nonlinear PDEs having a parameter called Reynolds number

•In stability analysis consider base flow + small perturbation

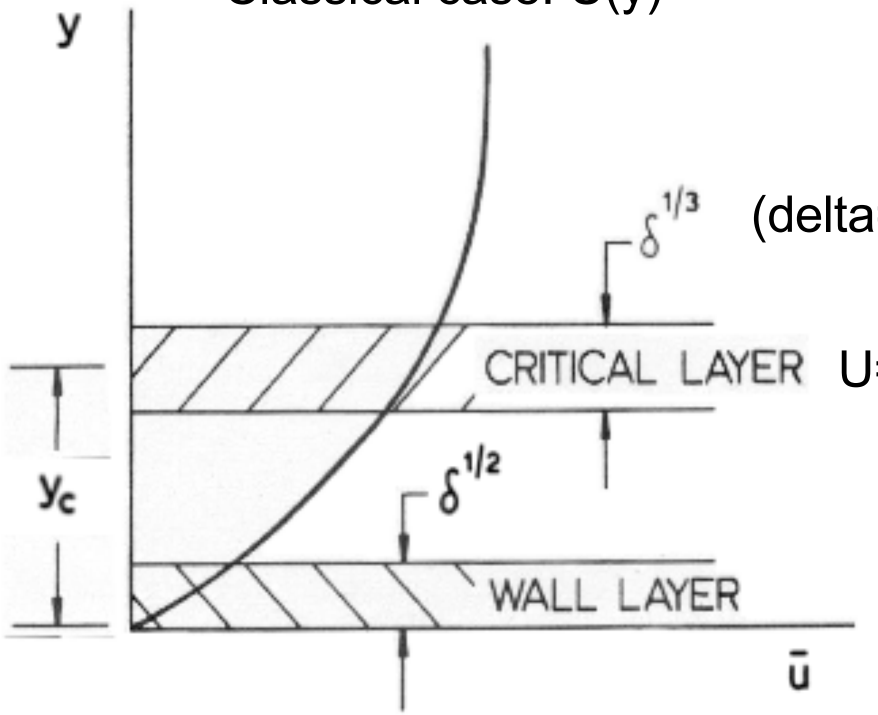
•Linearised NS can be solved by using normal mode (eigenvalue problem)

•Given wavenumber *alpha*, the complex wave speed *c* is obtained as eigenvalue

•Imaginary part of c is the growth rate of the perturbation (Im c positive is unstable case, i.e. the perturbation grows exponentially)

Inviscid stability of shear flows

Classical case: U(y)



(delta=1/R)

U=c: wave speed

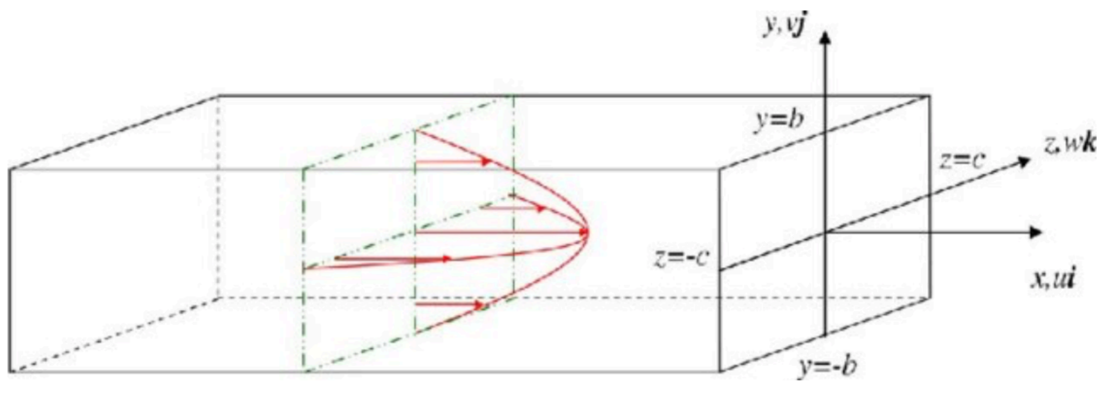
- Rayleigh's equation (viscosity is ignored)
- Singular at the critical level: viscosity is needed in the critical layer
- Matched asymptotic expansion must be used to analyse the critical layer

Picture taken from Maslowe (2009)

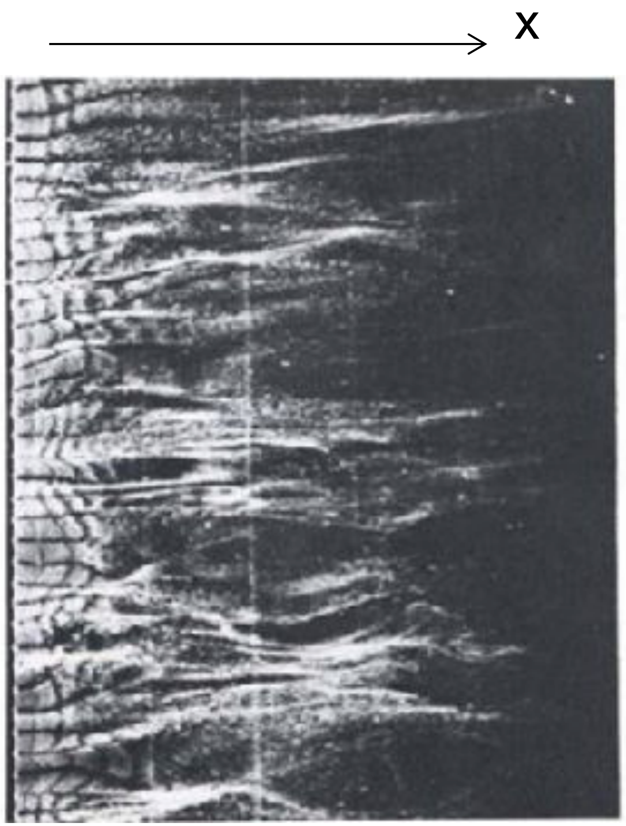
Inviscid stability of shear flows

•However, most physically relevant unidirectional flows vary in two transverse directions, so more general base flow U(y,z) must be considered!

•E.g. stability of flows over corrugated walls, or through non-circular pipe



Streaks



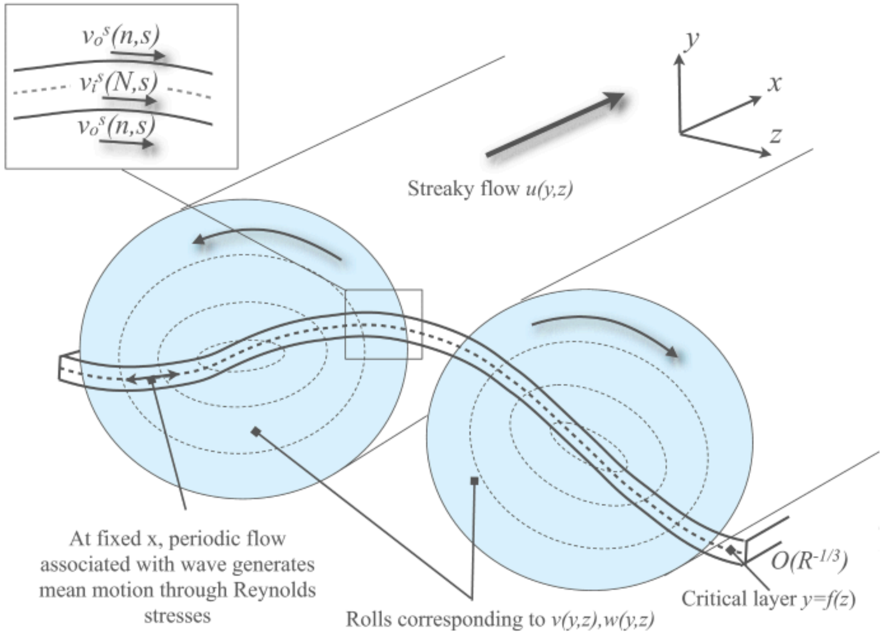
Streaks in a boundary layer flow over flat plate

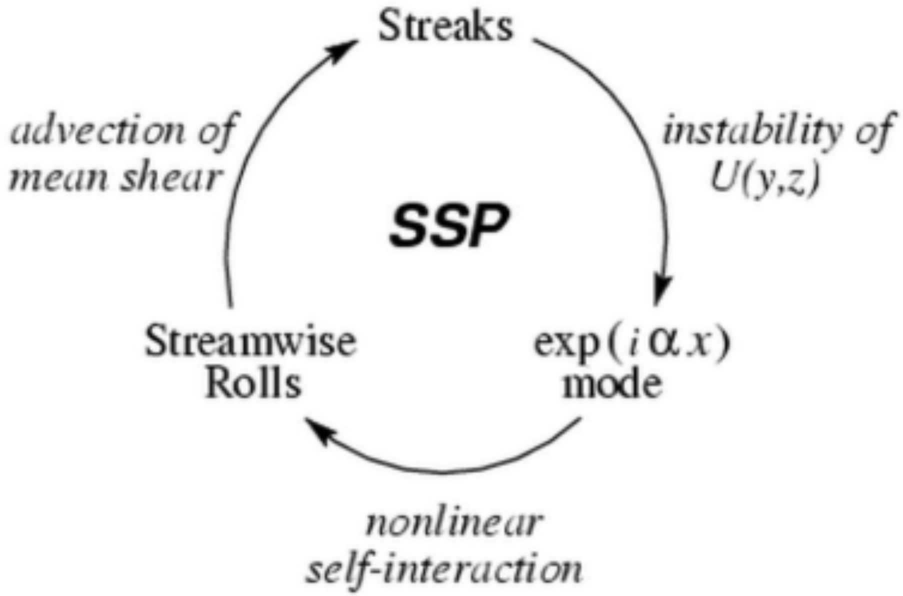
•Streaks can be visualized as thread-like structures

•Streamwise velocity naturally creates inhomogeneity in transverse direction

VWI / SSP

Nonlinear theory for shear flows





Vortex-wave interaction
(Hall & Smith 1991,
Hall & Sherwin 2010)

Self-sustaining process
(Waleffe 1997,
Wang, Gibson & Waleffe 2007)

Derivation of the generalized problem

Our starting point is the Navier–Stokes equations linearised around the background flow $U(y, z)$ at the Reynolds number $R > 0$:

$$\mathrm{i}\alpha u + v_y + w_z = 0, \quad \mathrm{i}\alpha(U - c)u + U_y v + U_z w = -\mathrm{i}\alpha p + R^{-1}(\Delta - \alpha^2)u, \qquad (2.1a,b)$$

$$\mathrm{i}\alpha(U - c)v = -p_y + R^{-1}(\Delta - \alpha^2)v, \quad \mathrm{i}\alpha(U - c)w = -p_z + R^{-1}(\Delta - \alpha^2)w, \quad (2.1c,d)$$

where $\Delta = \partial_y^2 + \partial_z^2$.

Neglecting the viscous terms,

$$\left(\frac{p_y}{(U-c)^2}\right)_y + \left(\frac{p_z}{(U-c)^2}\right)_z - \alpha^2 \frac{p}{(U-c)^2} = 0.$$

Hocking (1968), Goldstein (1976), Benney (1984), Henningson (1987), Hall & Horseman (1991)

Classical stability problem for U(y)

Here, before we begin the analysis of (1.2), we recall some properties of the classical Rayleigh equation in the pressure form to highlight our main idea (Tollmien 1935; Lin 1945, 1955). For the neutral case, the equation possesses a regular singular point at $y = y_c$, where $U - c$ vanishes. In a new coordinate, $n = y - y_c$, the singular point is simply $n = 0$.

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The method of Frobenius can be used to show that the local expansion of the solution contains the term like $n^3 \ln|n| + \cdots$

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Inner analysis shows that the outer solution must be written in the form

$$p = n^3 f_a(n) + \{n^3 \mathcal{L}(n) f_b(n) + g(n)\}, \tag{2.1}$$

where f_a, f_b, g are continuous regular functions and the jump is absorbed into the modified logarithmic function

$$\mathcal{L}(n) = \begin{cases} \ln n & \text{if} \quad n > 0, \\ \ln|n| + i\theta & \text{if} \quad n < 0, \end{cases} \tag{2.2}$$

where θ is a real number describing the phase shift of the wave across the critical layer. If the critical layer is viscous and linear $\theta = -\pi \operatorname{sgn}(\overline{u}_n|_{n=0})$.

(Hereafter we set c=0)

$$\left(\frac{p'}{U^2}\right)' - \alpha^2 \frac{p}{U^2} = 0. \tag{1.2}$$

$$p = n^3 f_a(n) + \{n^3 \mathcal{L}(n) f_b(n) + g(n)\}, \tag{2.1}$$

Substituting (2.1) to Rayleigh equation (1.2), we find two equations

$$6f + 6nf' + n^2 f'' - \alpha^2 n^2 f - \frac{2U_y}{U}(3nf + n^2 f') = 0, \tag{2.3}$$

$$g'' + 2n^2 f_b' + 5n f_b - \alpha^2 g - \frac{2U_y}{U}(g' + n^2 f_b) = 0. \tag{2.4}$$

Here both f_a, f_b satisfy the first equation

Three 2nd order ODEs: Boundary conditions?

We write the local expansion of U as

$$U = \lambda_1 n + \lambda_2 n^2 + \cdots . \tag{2.5}$$

From (2.3)-(2.4) it is easy to find the functions f, g must have the local expansions

$$f = f_0 + f_1 n + O(n^2), \quad g = g_0 + g_2 n^2 + O(n^4), \tag{2.6}$$

where

$$f_1 = \frac{3\lambda_2}{2\lambda_1} f_0, \tag{2.7}$$

$$f_{b0} = -\frac{2\lambda_2 \alpha^2}{3\lambda_1} g_0 \equiv \mu g_0. \tag{2.8}$$

For given f_0, the value and the first derivative of f is known so we can integrate the first equation to compute f; we write the solution computed in this way as $f(n; f_0)$.

Thus $f_a = f(n; f_{a0}) = f_{a0} f(n; 1), \qquad f_b = f(n; f_{b0}) = f_{b0} f(n; 1)$

$$g(n; g_0) = g_0 g(n, 1).$$

So if we rewrite the two parameters in the solution as $a = f_{a0}, b = g_0$, the general solution can be written as

$$p = a\{n^3 f(n;1)\} + b\{\mu n^3 \mathcal{L}(n) f(n;1) + g(n;1)\}. \tag{2.9}$$

The two unknown constants a, b are fixed by the boundary conditions on the walls, namely $p_n = 0$ there. Those two conditions can be written in the matrix form

$$M \begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{0} \tag{2.10}$$

with 2 by 2 matrix M computed by $f(n;1), g(n;1)$. In order to have non-trivial solution

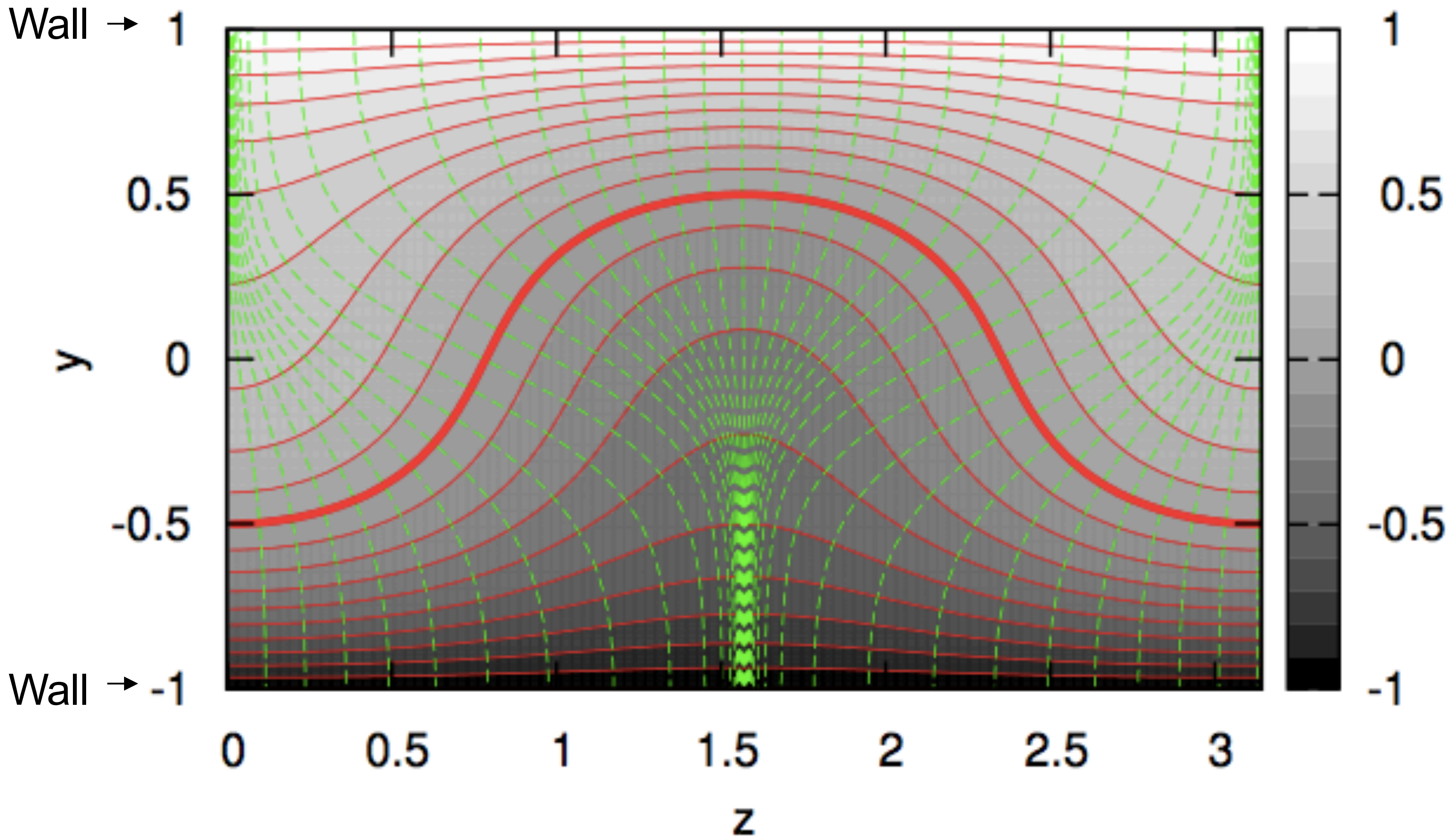
$$\det M(\alpha, c) = 0 \tag{2.11}$$

must be satisfied. We must vary α, c to ensure the real and imaginary parts of the determinant to vanish - this gives the wavelength and phase speed of the neutral wave.

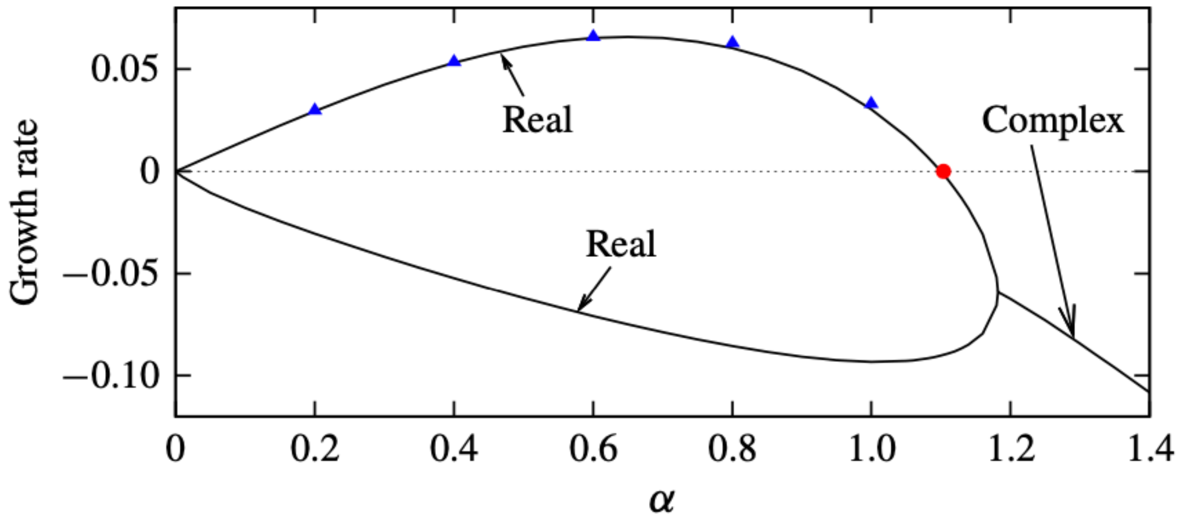
Generalised problem for U(y,z)

$$\left(\frac{p_y}{(U-c)^2}\right)_y + \left(\frac{p_z}{(U-c)^2}\right)_z - \alpha^2 \frac{p}{(U-c)^2} = 0.$$

Streak-like model flow profile



$$U\left(y,z\right) = y + \left(1 - y^2\right)\frac{\cos(2z) - 2y}{3}.$$



Line: NS result, R=10000

Blue triangle: Rayleigh, usual method

Red circle: Rayleigh, new method

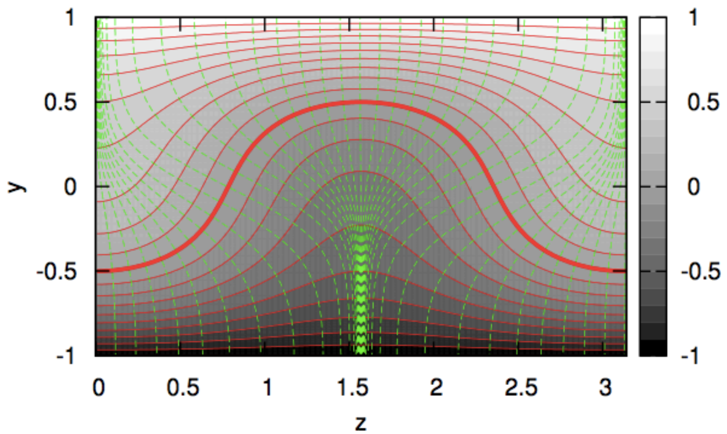
Thus we consider the new transformation from (y, z) to the coordinate (η, ζ), which we now refer to the base flow fitted coordinate. Here $\eta(y, z)$ is some known function, and later it is related to the base flow (e.g. $\eta = U$). The critical layer and the walls are specified by $\eta = 0$ and some constants. For the other coordinate $\zeta(y, z)$ we require the orthogonality condition

$$\eta_y \zeta_y + \eta_z \zeta_z = 0. \tag{3.14}$$

We need initial condition of this differential equation to fix ζ: here we require that ζ coincides with the arc length of the critical layer at $\eta = 0$.

Setting $\eta = U$, the generalised Rayleigh equation in the base flow fitted coordinate (η, ζ) becomes

$$(\Delta\eta)p_\eta + (\Delta\zeta)p_\zeta - \alpha^2 p + (\eta_y^2 + \eta_z^2)(p_{\eta\eta} - \frac{2p_\eta}{\eta}) + (\zeta_y^2 + \zeta_z^2)p_{\zeta\zeta} = 0. \tag{3.15}$$



Red: eta constant
Green: zeta constant

Critical layer analysis remains similar to the usual case and we can use the Frobenius form $p = \{f_a + f_b \mathcal{L}(\eta)\}\eta^3 + g$. Substituting this to the Rayleigh equation (1.3) we can find the two equations

$$(\triangle\eta)(f'\eta + 3f) + (\eta_y^2 + \eta_z^2)(f''\eta + 4f') \\ + \{(\triangle\zeta)f_\zeta - \alpha^2 f + (\zeta_y^2 + \zeta_z^2)f_{\zeta\zeta}\}\eta = 0. \tag{3.18}$$

$$(\triangle\eta)g' + (\triangle\zeta)g_\zeta - \alpha^2 g + (\eta_y^2 + \eta_z^2)(g'' - \frac{2g'}{\eta}) + (\zeta_y^2 + \zeta_z^2)g_{\zeta\zeta} \\ + (\triangle\eta)f_b\eta^2 + (\eta_y^2 + \eta_z^2)(2f_b'\eta^2 + 3f_b\eta) = 0, \tag{3.19}$$

where both f_a, f_b satisfy the first equation. The small η expansions of f, g are

$$f = f_0(\zeta) + f_1(\zeta)\eta + O(\eta^2), \qquad g = g_0(\zeta) + g_2(\zeta)\eta^2 + O(\eta^4), \tag{3.20}$$

$$f_1 = -\left.\frac{3\triangle\eta}{4(\eta_y^2 + \eta_z^2)}\right|_{U=0} f_0, \tag{3.21}$$

$$f_{b0} = \mu_2 g_0'' + \mu_1 g_0' + \mu_0 g_0, \tag{3.23}$$

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Complicated functions of U!

From f_0 the value and the first η derivative of f is known. This means that the function f can be found by integrating (3.18); we write the solution found in this way as

$$f(n,\zeta;f_0)=f(n,\zeta;\sum_m \widehat{f}_0^{(m)}e^{im\zeta})=\sum_m \widehat{f}_0^{(m)}f(n,\zeta;e^{im\zeta}). \tag{3.24}$$

Here we Fourier expanded the initial condition for the latter convenience. We can compute f_a and f_b using initial conditions f_{a0}, f_{b0}, which can be specified by choosing their Fourier coefficients $\widehat{f}_{a0}^{(m)}, \widehat{f}_{b0}^{(m)}$. Given f_b, the function g can be found by integrating (3.19). The initial conditions $g_0=\sum_m \widehat{g}_0^{(m)}e^{im\zeta}$ can be linked to f_{b0} through (3.23), so

$$\sum_m \widehat{f}_{b0}^{(m)}e^{im\zeta}=\sum_m \mu^{(m)}\widehat{g}_0^{(m)}e^{im\zeta}, \qquad \mu^{(m)}=-m^2\mu_2+im\mu_1+\mu_0. \tag{3.25}$$

Thus the function g can be determined by g_0, so we denote

$$g(n,\zeta;g_0)=g(n,\zeta;\sum_m \widehat{g}_0^{(m)}e^{im\zeta})=\sum_m \widehat{g}_0^{(m)}g(n,\zeta;e^{im\zeta}). \tag{3.26}$$

Finally we write $a_m=\widehat{f}_{a0}^{(m)}$, $b_m=\widehat{g}_0^{(m)}$ to get

$$p=\sum_m a_m\eta^3 f(\eta,\zeta;e^{im\zeta})+b_m\{\eta^3 f(\eta,\zeta;\mu^{(m)}e^{im\zeta})\mathcal{L}(\eta)+g(\eta,\zeta;e^{im\zeta})\}. \tag{3.27}$$

The normal derivative of p, namely p_η must be zero on the walls. This means that all the Fourier coefficients should vanish at two values of η representing the walls, and hence those conditions yield the linear problem

$$M \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \mathbf{0}. \tag{3.28}$$

The determinant is complex, and we adjust α, c to ensure the real and imaginary parts of the determinant to be zero.

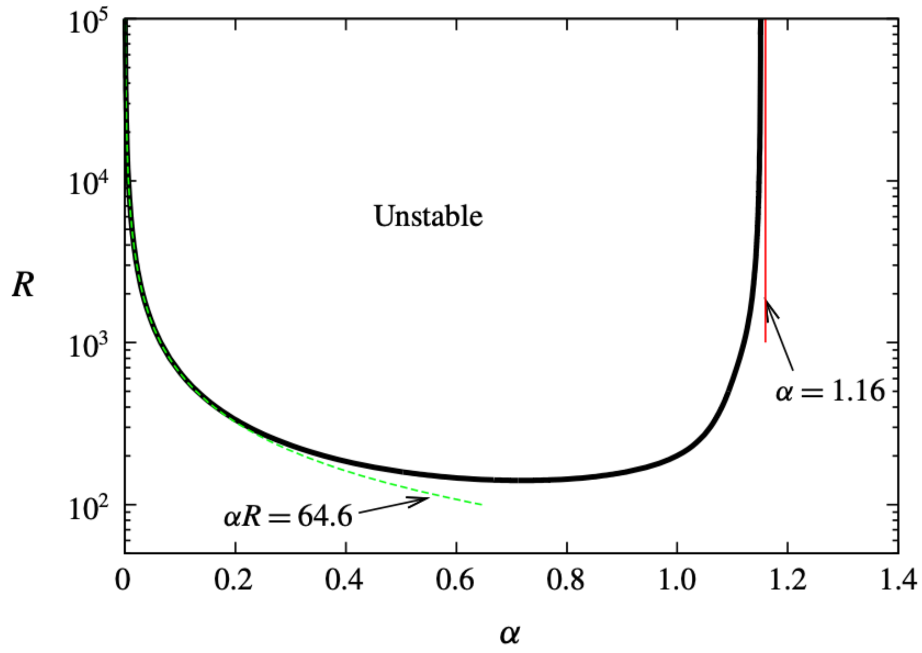


FIGURE 3. (Colour online) The model flow (3.12) becomes unstable above the thick black solid curve according to the linearised Navier–Stokes equations (2.1). The red solid line is the neutral inviscid limit solution found by (1.2). The green dashed curve is the long-wavelength asymptotic limit. Approximately 200 collocation points are used in $\eta \in [-1, 1]$, whilst 35 Fourier harmonics are used for ζ.

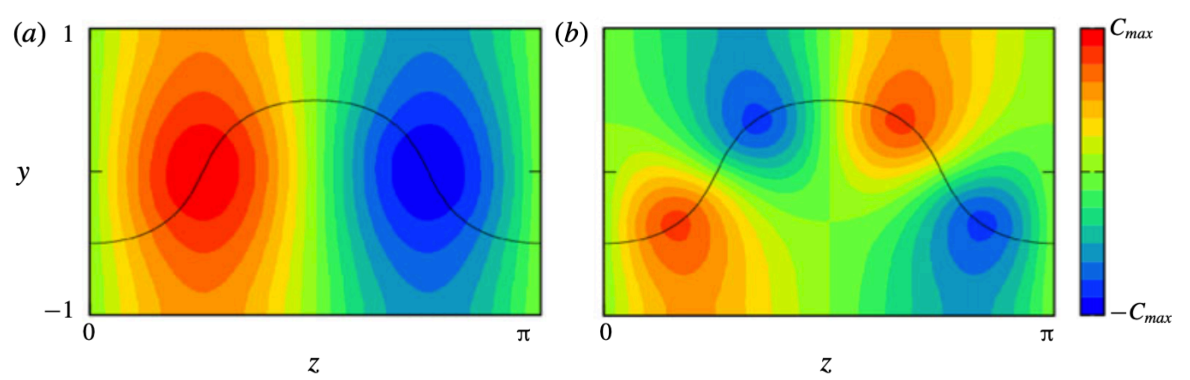
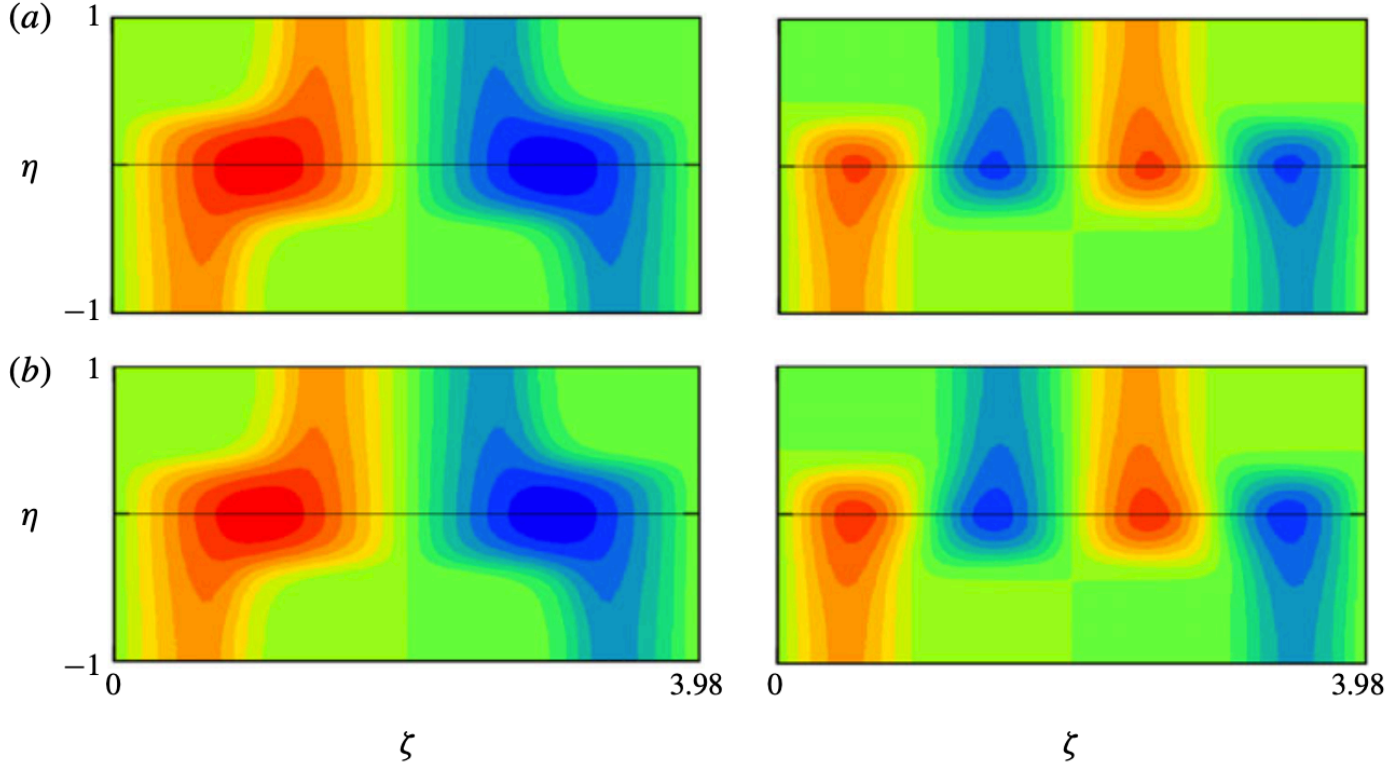


FIGURE 4. (Colour online) The pressure eigensolution on the right-hand branch of the full neutral curve in figure 1, for $R = 10\,000$. Panels (*a*) and (*b*) are the real part ($C_{max} = 1$) and the imaginary part ($C_{max} = 0.2$), respectively. The black solid curve is the critical curve.



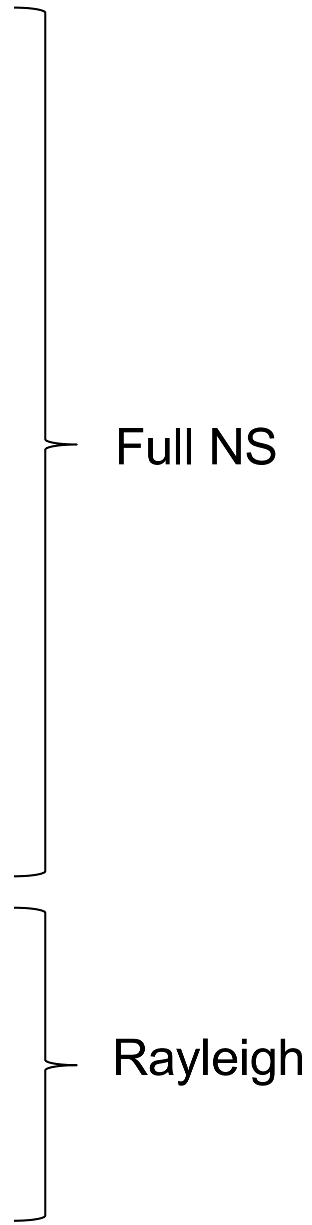


FIGURE 5. (Colour online) (*a*) The same results as figure 4, but plotted in the (η, ζ) coordinates. (*b*) The inviscid solution obtained by applying the new method to (1.2). This result corresponds to the red line in figure 3.

Remark 1: We only need singular basis function near the critical layer

Remark 2: Computationally much cheaper than solving NS

Remark 3: Necessary condition for existence of a neutral mode

For the classical case

$$-\frac{2\lambda_2\alpha^2}{3\lambda_1}$$

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$$\mathcal{L}(n) = \begin{cases} \ln n & \text{if} \quad n > 0, \\ \ln|n| + i\theta & \text{if} \quad n < 0, \end{cases} \tag{2.2}$$

where θ is a real number describing the phase shift of the wave across the critical layer. If the critical layer is viscous and linear $\theta = -\pi \operatorname{sgn}(\overline{u}_n|_{n=0})$.

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If the generalised problem has a neutral mode,

$$(\Delta U)\{4U_{yz}U_yU_z + (U_{yy} - U_{zz})(U_y^2 - U_z^2)\} \leqslant 0$$

somewhere in the flow.

Conclusion

- In order to solve the generalized inviscid stability problem (a singular PDE) the method of Frobenius is used in curved coordinates to construct appropriate basis functions

- The new Rayleigh solver is more efficient than the full NS solver

End