Stochastic Landau–Lifshitz Equation on Real Line

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Deterministic Landau-Lifshitz equation (LL)

- Magnetic domain: $\mathbb{D} \subseteq \mathbb{R}^d, d \geq 1$.
- Magnetisation: $\mathbf{u} : \mathbb{R}^+ \times \mathbb{D} \to \mathbb{R}^3$.

Landau-Lifshitz equation:

$$\frac{d\mathbf{u}(t)}{dt} = \lambda_1 \mathbf{u}(t) \times H_{eff} - \lambda_2 \mathbf{u}(t) \times (\mathbf{u}(t) \times H_{eff})$$

where $\times$ is the cross product in $\mathbb{R}^3$, $\lambda_1, \lambda_2 > 0$ and H_{eff} is the effective field such that

$$H_{eff} = -\nabla E_{total}$$

Deterministic Landau-Lifshitz equation (LL)

For $H_{eff} = -\nabla E_{exch} = \Delta \mathbf{u}$, where $\Delta \mathbf{u} = \sum_{i=1}^{d} \frac{\partial^2 \mathbf{u}}{\partial x_i^2}$.

Landau-Lifshitz equation:

$$\frac{d\mathbf{u}(t)}{dt} = \lambda_1 \mathbf{u}(t) \times \Delta \mathbf{u}(t) - \lambda_2 \mathbf{u}(t) \times (\mathbf{u}(t) \times \Delta \mathbf{u}(t)).$$

Initial conditions:

$$\mathbf{u}(0, x) = \mathbf{u_0}(x),$$
$$|\mathbf{u_0}(x)| = 1.$$

Property:

$$|\mathbf{u}(t, x)| = 1 \quad \forall x \in \mathbb{D}, \forall t > 0.$$

Previous work

Bounded domain:

- A. Visintin. On Landau-Lifshitz' equations for ferromagnetism. *Japan J. Appl. Math.*, 2(1):69–84, 1985
- F. Alouges and A. Soyeur. On global weak solutions for Landau-Lifshitz equations: existence and nonuniqueness. *Nonlinear Anal.*, 18(11):1071–1084, 1992

Previous work

Unbounded domain:

- F. Alouges and A. Soyeur. On global weak solutions for Landau-Lifshitz equations: existence and nonuniqueness. *Nonlinear Anal.*, 18(11):1071–1084, 1992
- A. Fuwa and M. Tsutsumi. Local well posedness of the Cauchy problem for the Landau-Lifshitz equations. *Differential Integral Equations*, 18(4):379–404, 2005

Stochastic Landau-Lifshitz equation (SLL)

- ζ: white noise.
- $H_{eff} = \Delta \mathbf{u} + \zeta$.
- Physical problems: λ_2 is small.

Stochastic Landau-Lifshitz equation:

$$\frac{d\mathbf{u}(t)}{dt} = \lambda_1 \mathbf{u}(t) \times (\Delta \mathbf{u}(t) + \zeta) - \lambda_2 \mathbf{u}(t) \times (\mathbf{u}(t) \times (\Delta \mathbf{u}(t))).$$

Stochastic Landau-Lifshitz equation (SLL)

- $\zeta = \dot{W}$ with W Wiener process.

Stochastic Landau-Lifshitz equation:

$$d\mathbf{u}(t) = (\lambda_1 \mathbf{u}(t) \times \Delta \mathbf{u}(t) - \lambda_2 \mathbf{u}(t) \times (\mathbf{u}(t) \times \Delta \mathbf{u}(t))dt + \lambda_1 \mathbf{u}(t) \times \circ dW(t)$$

with

$$W(t) = \sum_{i=1}^{\infty} W_i(t)\mathbf{g}_i$$

where W_i sequence of independent one dimensional Brownian motion defined on a common probability space and $\mathbf{g}_i : \mathbb{D} \to \mathbb{R}^3$ are given functions such that the sequence $\mathbf{g}_i \subset H^1$ and $\sum_{i=1}^{\infty} |\mathbf{g}_i|_{H^1}^2 < \infty$.
Initial conditions:

$$\mathbf{u}(0,x) = \mathbf{u_0}(x),$$
$$|\mathbf{u_0}(x)| = 1.$$

Stochastic Landau-Lifshitz equation (SLL)

- $(\Omega, \mathcal{F}, \mathbb{P})$ a filtration probability space
- $\mathbf{u} : \Omega \times \mathbb{R}^+ \times \mathbb{D} \to \mathbb{R}^3$
- $W : \Omega \times \mathbb{R}^+ \to \mathbb{R}$
- $\mathbf{g} : \mathbb{D} \to \mathbb{R}^3$

Stochastic Landau-Lifshitz equation:

$$d\mathbf{u}(t) = (\mathbf{u}(t) \times \Delta\mathbf{u}(t) - \lambda\mathbf{u}(t) \times (\mathbf{u}(t) \times \Delta\mathbf{u}(t)) + \frac{1}{2}((\mathbf{u}(t) \times \mathbf{g}) \times \mathbf{g}))dt$$
$$+ (\mathbf{u}(t) \times \mathbf{g})dW(t)$$

Initial conditions:

$$\mathbf{u}(0, x) = \mathbf{u_0}(x),$$
$$|\mathbf{u_0}(x)| = 1.$$

Previous work

Bounded domain:

- Z. Brzeźniak, B. Goldys, and T. Jegaraj. Weak solutions of a stochastic Landau-Lifshitz-Gilbert equation.
  Appl. Math. Res. Express. AMRX, (1):1–33, 2013
- Z. a. Brzeźniak, B. Goldys, and T. Jegaraj. Large deviations and transitions between equilibria for stochastic Landau-Lifshitz-Gilbert equation.
  Arch. Ration. Mech. Anal., 226(2):497–558, 2017
- B. Goldys, K.-N. Le, and T. Tran. A finite element approximation for the stochastic Landau-Lifshitz-Gilbert equation.
  J. Differential Equations, 260(2):937–970, 2016
- F. Alouges, A. de Bouard, and A. Hocquet. A semi-discrete scheme for the stochastic Landau-Lifshitz equation.
  Stoch. Partial Differ. Equ. Anal. Comput., 2(3):281–315, 2014

Difference method

We work on SLL with $\mathbb{D} = \mathbb{R}$. Given $h > 0$, we consider $\{x_i\}_{i \in \mathbb{Z}}$ where $x_i = ih$. We denote by $\mathbb{Z}_h := \{x_i\}_{i \in \mathbb{Z}}$.
Let $\mathbf{u}^h : \Omega \times \mathbb{R}^+ \times \mathbb{Z}_h \to \mathbb{R}^3$ and $\mathbf{g} : \mathbb{Z}_h \to \mathbb{R}^3$.
We define

$$D^+\mathbf{u}^h(x) := \frac{\mathbf{u}^h(x+h) - \mathbf{u}^h(x)}{h},$$

$$D^-\mathbf{u}^h(x) := \frac{\mathbf{u}^h(x) - \mathbf{u}^h(x-h)}{h},$$

$$\begin{aligned}\tilde{\Delta}\mathbf{u}^h(x) &:= D^+D^-\mathbf{u}^h(x) = D^-D^+\mathbf{u}^h(x) \\ &= \frac{\mathbf{u}^h(x+h) - 2\mathbf{u}^h(x) + \mathbf{u}^h(x-h)}{h^2}.\end{aligned}$$

We define

$$|\mathbf{u}^h|_{L_h^\infty} = \sup_{x \in \mathbb{Z}_h} |\mathbf{u}^h(x)|, \qquad |\mathbf{u}^h|_{L_h^2} = \left(h \sum_{x \in \mathbb{Z}_h} |\mathbf{u}^h(x)|^2 \right)^{\frac{1}{2}}.$$

Discretized problem

We define

$$E_h = \left\{ \mathbf{v} : \mathbb{Z}_h \to \mathbb{R}^3 : \, |\mathbf{v}|_{E_h} < \infty \right\}$$

with $|\mathbf{v}|^2_{E_h} = |D^+\mathbf{v}|^2_{L^2_h} + |\mathbf{v}|^2_{L^\infty_h}$, and $\mathscr{E}_h$ the space of E_h-valued processes $\mathbf{v}$ endowed with the norm $|\mathbf{v}|^2_{\mathscr{E}_h} = \sup_{t \leq T} \mathbb{E}|\mathbf{v}(t)|^2_{E_h}$.
We get

$$\begin{cases} d\mathbf{u}^h(t,x_i) = (\mathbf{u}^h(t,x_i) \times \tilde{\Delta}\mathbf{u}^h(t,x_i) - \lambda \mathbf{u}^h(t,x_i) \times (\mathbf{u}^h(t,x_i) \times \tilde{\Delta}\mathbf{u}^h(t,x_i)) \\ \qquad\qquad + \frac{1}{2}((\mathbf{u}^h(t,x_i) \times \mathbf{g}(x_i)) \times \mathbf{g}(x_i)))dt + (\mathbf{u}^h(t,x_i) \times \mathbf{g}(x_i))dW(t), \\ \mathbf{u}^h(0,x_i) = \mathbf{u_0}(x_i), \\ |\mathbf{u_0}(x_i)| = 1. \end{cases}$$

Result: This problem involves an SDE which has a unique strong global solution $\mathbf{u}^h(t), t > 0$ on $\mathscr{E}_h$.

Energy estimates

Lemma

Assume that $\mathbf{g} \in H^1$, $T \in (0, \infty)$ *and* $|\mathbf{u_0}(x)| = 1$. *For all* $x_i \in \mathbb{Z}_h$ *and every* $t \in [0, T]$, *we have*

$$|\mathbf{u}^h(t, x_i)| = 1. \tag{0.1}$$

Energy estimates

Lemma

Assume that $\mathbf{g} \in H^1$, $T \in (0, \infty)$ *and* $|\mathbf{u_0}(x)| = 1$. *For all* $x_i \in \mathbb{Z}_h$ *and every* $t \in [0, T]$, *we have*

$$|\mathbf{u}^h(t, x_i)| = 1. \tag{0.1}$$

Moreover, given $1 \leq p < \infty$, *there exists a constant* C *which does not depend on* h *but which may depend on* $\mathbf{g}$ *and* T *such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |D^+\mathbf{u}^h(t)|^{2p}_{L^2_h}\Big] \leq C, \tag{0.2}$$

$$\mathbb{E}\Big[\Big(\int_0^T |\tilde{\Delta}\mathbf{u}^h(t)|^2_{L^2_h} dt\Big)^p\Big] \leq C. \tag{0.3}$$

Piecewise linear Interpolation

We define r_h the interpolation operator by

$$r_h\mathbf{u}^h(t,x) = \mathbf{u}^h(t,x_i), \quad \forall x \in [x_i; x_{i+1}).$$

Then, we get

$$\begin{cases} dr_h\mathbf{u}^h(t,x) = (r_h\mathbf{u}^h(t,x) \times r_h\tilde{\Delta}\mathbf{u}^h(t,x) \\ \qquad\qquad -\lambda r_h\mathbf{u}^h(t,x) \times (r_h\mathbf{u}^h(t,x) \times r_h\tilde{\Delta}\mathbf{u}^h(t,x)) \\ \qquad\qquad +\frac{1}{2}((r_h\mathbf{u}^h(t,x) \times r_h\mathbf{g}(x)) \times r_h\mathbf{g}(x)))dt \\ \qquad\qquad +(r_h\mathbf{u}^h(t,x) \times r_h\mathbf{g}(x))dW(t), \\ r_h\mathbf{u}^h(0,x) = \mathbf{u_0}(x), \\ |\mathbf{u_0}(x)| = 1. \end{cases}$$

Convergence

We define

$$L_m^2 = \{\mathbf{u} : \mathbb{R} \to \mathbb{R}^3 | \int_{\mathbb{R}} |\mathbf{u}(x)|^2 \rho_m(x) dx < \infty\},$$

where $\rho_m(x) = \mathrm{e}^{-\frac{|x|}{m}}$, $m > 0$, and for any n

$$\tau_n^h = \inf\{t > 0| \quad max\left(|r_h D^+ \mathbf{u}^h(t)|_{L^2}, \int_0^t |r_h \tilde{\Delta} \mathbf{u}^h(s)|_{L^2}^2 ds\right) > n\}$$

with

$$\tau_n = \lim_{h \to 0} \tau_n^h \quad a.s.$$

Lemma

For any n sufficiently large, we have

$$\mathbb{P}(\tau_n > 0) = 1.$$

Convergence

Lemma

Assume $\mathbf{g} \in H^1$. *For any n sufficiently large, there exists* $\mathbf{u}^n \in L^2(\Omega, L^\infty((0, T \wedge \frac{\tau_n}{2}), L^2_m(\mathbb{R})))$ *such that as* $h \to 0$, *the following limits exist*

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\frac{\tau_n}{2}]} |r_h\mathbf{u}^h - \mathbf{u}^n|^2_{L^2_m}\Big] \to 0. \tag{0.4}$$

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\frac{\tau_n}{2}]} |r_h D^+\mathbf{u}^h - \nabla\mathbf{u}^n|^2_{L^2}\Big] \to 0. \tag{0.5}$$

$$\mathbb{E}\Big[\int_0^{T\wedge\frac{\tau_n}{2}} |r_h\tilde{\Delta}\mathbf{u}^h - \Delta\mathbf{u}^n|^2_{L^2}\Big] \to 0. \tag{0.6}$$

Existence of local strong solution

Lemma

Assume $\mathbf{u_0}(x) \in \mathbb{S}^2$ *for all* $x \in \mathbb{R}$*, where* $\mathbb{S}^2$ *is the unit sphere in* $\mathbb{R}^3$*,* $\nabla \mathbf{u_0} \in L^2(\mathbb{R})$ *and* $\mathbf{g} \in H^1(\mathbb{R})$*. For* $T > 0$*, there exists a solution* $\mathbf{u}^n \in L^2(\Omega, L^\infty((0, T \wedge \frac{\tau_n}{2}), L^2_m(\mathbb{R})))$ *to SLL problem such that*

1.
$$|\mathbf{u}^n(t,x)| = 1\,,$$

Existence of local strong solution

Lemma

Assume $\mathbf{u_0}(x) \in \mathbb{S}^2$ *for all* $x \in \mathbb{R}$, *where* $\mathbb{S}^2$ *is the unit sphere in* $\mathbb{R}^3$, $\nabla \mathbf{u_0} \in L^2(\mathbb{R})$ *and* $\mathbf{g} \in H^1(\mathbb{R})$. *For* $T > 0$, *there exists a solution* $\mathbf{u}^n \in L^2(\Omega, L^\infty((0, T \wedge \frac{\tau_n}{2}), L^2_m(\mathbb{R})))$ *to SLL problem such that*

1. $$|\mathbf{u}^n(t,x)| = 1\,,$$

2. $$\mathbf{u}^n \in C((0,T), L^2_m)\,, \quad \mathbb{P} - a.s.$$

Existence of local strong solution

Lemma

Assume $\mathbf{u_0}(x) \in \mathbb{S}^2$ *for all* $x \in \mathbb{R}$, *where* $\mathbb{S}^2$ *is the unit sphere in* $\mathbb{R}^3$, $\nabla \mathbf{u_0} \in L^2(\mathbb{R})$ *and* $\mathbf{g} \in H^1(\mathbb{R})$. *For* $T > 0$, *there exists a solution* $\mathbf{u}^n \in L^2(\Omega, L^\infty((0, T \wedge \frac{\tau_n}{2}), L^2_m(\mathbb{R})))$ *to SLL problem such that*

1. $$|\mathbf{u}^n(t,x)| = 1\,,$$

2. $$\mathbf{u}^n \in C((0,T), L^2_m)\,, \quad \mathbb{P} - a.s.$$

3. *for every* $p > 0$

$$\mathbb{E}\left[\sup_{t\in[0,T\wedge\frac{\tau_n}{2}]} |\nabla \mathbf{u}^n(t)|^p_{L^2}\right] < \infty\,,$$

Existence of local strong solution

Lemma

4. *for every $p > 0$*

$$\mathbb{E}\Big[\Big(\int_0^{T\wedge\frac{\tau_n}{2}} |\Delta \mathbf{u}^n(t)|^2_{L^2}\,dt\Big)^p\Big] < \infty\,,$$

Existence of local strong solution

Lemma

4. *for every* $p > 0$

$$\mathbb{E}\Big[\Big(\int_0^{T\wedge\frac{\tau_n}{2}} |\Delta \mathbf{u}^n(t)|_{L^2}^2 dt\Big)^p\Big] < \infty\,,$$

5. *the following equation holds in* $L^\infty((0, T\wedge\frac{\tau_n}{2}), L^2_m)$ $\mathbb{P}$ *a.s., for all* $t \in [0, T\wedge\frac{\tau_n}{2}]$*:*

$$\begin{aligned}\mathbf{u}^n(t) = \mathbf{u}_0^{\mathbf{n}} &+ \int_0^t \mathbf{u}^n(s)\times\Delta\mathbf{u}^n(s)ds \\ &- \lambda\int_0^t \mathbf{u}^n(s)\times(\mathbf{u}^n(s)\times\Delta\mathbf{u}^n(s))ds \\ &+ \frac{1}{2}\int_0^t (\mathbf{u}^n(s)\times\mathbf{g})\times\mathbf{g}ds + \int_0^t \mathbf{u}^n(s)\times\mathbf{g}dW(s).\end{aligned}$$

Uniqueness of solution

Theorem

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions for SLL with the same initial values. Then, we have that $\mathbf{u}_1 = \mathbf{u}_2$ a.s.

Existence of global strong solution

Finally, we can take the limit when $n \to \infty$ and deduce that we have the following theorem

Theorem

Assume $\mathbf{u_0}(x) \in \mathbb{S}^2$ *for all* $x \in \mathbb{R}$*, where* $\mathbb{S}^2$ *is the unit sphere in* $\mathbb{R}^3$*,* $\nabla \mathbf{u_0} \in L^2(\mathbb{R})$ *and* $\mathbf{g} \in H^1(\mathbb{R})$*. For* $T > 0$*, there exists a solution* $\mathbf{u} \in L^2(\Omega, L^\infty((0,T), L^2_m(\mathbb{R})))$ *to SLL problem such that*

1. $$|\mathbf{u}(t,x)| = 1\,,$$
2. $$\mathbf{u} \in C((0,T), L^2_m)\,, \quad \mathbb{P} - a.s.$$
3. *for every* $p > 0$
$$\mathbb{E}\Big[\sup_{t\in[0,T]} |\nabla \mathbf{u}(t)|^p_{L^2}\Big] < \infty\,,$$

Existence of global strong solution

Theorem

4. *for every* $p > 0$

$$\mathbb{E}\left[\left(\int_0^T |\Delta \mathbf{u}(t)|_{L^2}^2\, dt\right)^p\right] < \infty\,,$$

5. *the following equation holds in* $L^\infty((0,T), L^2_m)$ $\mathbb{P}$ *a.s., for all* $t \in [0,T]$*:*

$$\begin{aligned}\mathbf{u}(t) = \mathbf{u_0} &+ \int_0^t \mathbf{u}(s) \times \Delta \mathbf{u}(s) ds \\ &- \lambda \int_0^t \mathbf{u}(s) \times (\mathbf{u}(s) \times \Delta \mathbf{u}(s)) ds \\ &+ \frac{1}{2} \int_0^t (\mathbf{u}(s) \times \mathbf{g}) \times \mathbf{g} ds + \int_0^t \mathbf{u}(s) \times \mathbf{g} dW(s).\end{aligned}$$

Continuous dependence on initial conditions

Theorem

Let $\mathbf{u}_{0i} : \mathbb{R} \to \mathbb{S}^2$ $(i = 1, 2)$ *be such that* $\mathbf{u}_{01} - \mathbf{u}_{02} \in L^2(\mathbb{R})$. *Let* $\mathbf{u_1}$ *and* $\mathbf{u_2}$ *be two solutions for SLL with initial values* $\mathbf{u}_{01}$ *and* $\mathbf{u}_{02}$ *respectively. Then,* $\mathbf{u_1} - \mathbf{u_2} \in L^2(\mathbb{R})$ *and*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |\mathbf{u_1} - \mathbf{u_2}|^2_{L^2}\Big] \leq C|\mathbf{u}_{01} - \mathbf{u}_{02}|^2_{L^2}.$$

THANK YOU