A fictitious domain approach for the finite element discretization of FSI

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Outline

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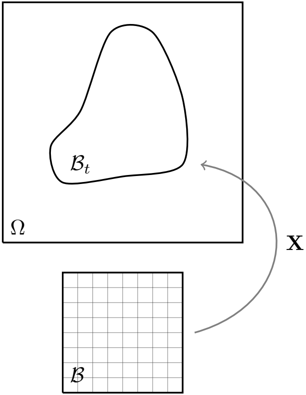
Outline

Fluid-structure interaction

$\Omega \subset \mathbb{R}^d$, $d = 2, 3$
$\mathbf{x}$ Eulerian variable in Ω

$\mathcal{B}_t$ deformable structure domain
$\mathcal{B}_t \subset \mathbb{R}^m$, $m = d, d-1$
s Lagrangian variable in $\mathcal{B}$
$\mathbf{X}(\cdot, t) : \mathcal{B} \to \mathcal{B}_t$ position of the solid
$\mathbb{F} = \dfrac{\partial \mathbf{X}}{\partial s}$ deformation gradient



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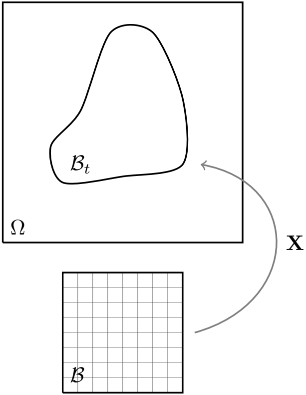

$\mathbf{u}(\mathbf{x}, t)$ material velocity

$$\mathbf{u}(\mathbf{x}, t) = \frac{\partial \mathbf{X}}{\partial t}(s, t) \quad \text{where} \quad \mathbf{x} = \mathbf{X}(s, t)$$

Numerical approaches to FSI

Boundary fitted approaches The fluid problem is solved on a mesh that deforms around a Lagrangian structure mesh, using *arbitary Lagrangian–Eulerian* (ALE) coordinate system.
In case of large deformation the boundary fitted fluid mesh can become severely distorted.

Non boundary fitted approaches A separate structural discretization is superimposed onto a background fluid mesh

- fictitious domain <Glowinski-Pan-Périaux '94, Yu '05>
- level set method <Chang-Hou-Merriman-Osher '96>
- immersed boundary method (IBM) <Peskin '02>
- Nitsche-XFEM method <Burman-Fernández '14, Alauzet-Fabrèges-Fernández-Landajuela '16>
- immersogeometric FSI (thin structures) <Kamensky-Hsu-Schillinger-Evans-Aggarwal-Bazilevs-Sacks-Hughes '15>
- divergence conforming B-splines <Casquero-Zhang-Bona-Casas-Dalcin-Gomez '18>

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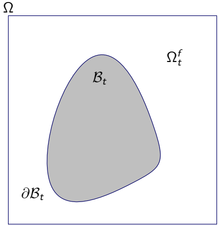
Our approach originates from the *immersed boundary method* IBM and moved towards a *fictitious domain method* FDM.

FSI problem (thick incompressible solid)



$$\rho_f\left(\frac{\partial \mathbf{u}_f}{\partial t}+\mathbf{u}_f\cdot\boldsymbol{\nabla}\,\mathbf{u}_f\right)=\operatorname{div}\boldsymbol{\sigma}_f \qquad \text{in } \Omega\setminus\mathcal{B}_t$$

$$\operatorname{div}\mathbf{u}_f=0 \qquad \text{in } \Omega\setminus\mathcal{B}_t$$

$$\rho_s\frac{\partial^2\mathbf{X}}{\partial t^2}=\operatorname{div}_s(|\mathbb{F}|\boldsymbol{\sigma}_s^f\mathbb{F}^{-\top}+\mathbb{P}(\mathbb{F})) \qquad \text{in } \mathcal{B}$$

$$\operatorname{div}_s\mathbf{u}_s=0 \qquad \text{in } \mathcal{B}$$

$$\mathbf{u}_f=\frac{\partial\mathbf{X}}{\partial t} \qquad \text{on } \partial\mathcal{B}_t$$

$$\boldsymbol{\sigma}_f\mathbf{n}_f=-(\boldsymbol{\sigma}_s^f+|\mathbb{F}|^{-1}\mathbb{P}\mathbb{F}^\top)\mathbf{n}_s \qquad \text{on } \partial\mathcal{B}_t$$

$$\boldsymbol{\sigma}_f=-p_f\mathbb{I}+\nu_f\,\boldsymbol{\nabla}_{sym}\,\mathbf{u}_f \qquad \boldsymbol{\sigma}_s^f=-p_s\mathbb{I}+\nu_s\,\boldsymbol{\nabla}_{sym}\,\mathbf{u}_s \qquad \mathbf{u}_s=\frac{\partial\mathbf{X}}{\partial t}$$

$\mathbb{P}(\mathbb{F})$ Piola–Kirchhoff stress tensor such that $\mathbb{P}=|\mathbb{F}|\boldsymbol{\sigma}_s^e\mathbb{F}^{-\top}$ and $\mathbb{P}(\mathbb{F})=\frac{\partial W}{\partial\mathbb{F}}$
where W is the potential energy density

$+$ initial and boundary conditions

Outline

Fictitious domain approach

<Boffi–Cavallini–G. '15>

- Fluid velocity and pressure are extended into the solid domain

$$\mathbf{u} = \begin{cases} \mathbf{u}_f & \text{in } \Omega \setminus \mathcal{B}_t \\ \mathbf{u}_s & \text{in } \mathcal{B}_t \end{cases} \qquad p = \begin{cases} p_f & \text{in } \Omega \setminus \mathcal{B}_t \\ p_s & \text{in } \mathcal{B}_t \end{cases}$$

Fictitious domain approach

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- Body motion $\mathbf{u}(\mathbf{x}, t) = \dfrac{\partial \mathbf{X}}{\partial t}(s, t)$ for $\mathbf{x} = \mathbf{X}(s, t)$

Fictitious domain approach

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- Body motion $\mathbf{u}(\mathbf{x}, t) = \dfrac{\partial \mathbf{X}}{\partial t}(s, t)$ for $\mathbf{x} = \mathbf{X}(s, t)$
- We introduce two functional spaces Λ and $\mathcal{Z}$ and a bilinear form $\mathbf{c} : \Lambda \times \mathcal{Z} \to \mathbb{R}$ such that

$$\mathbf{c}(\mu, \mathbf{z}) = 0 \quad \forall \mu \in \Lambda \qquad \Rightarrow \qquad \mathbf{z} = 0$$

Notation:

$$a(\mathbf{u},\mathbf{v}) = (\nu\,\boldsymbol{\nabla}_{sym}\,\mathbf{u}, \boldsymbol{\nabla}_{sym}\,\mathbf{v}) \quad \text{with } \nu = \begin{cases} \nu_f & \text{in } \Omega \setminus \mathcal{B}_t \\ \nu_s & \text{in } \mathcal{B}_t \end{cases}$$

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{\rho_f}{2}\left((\mathbf{u}\cdot\boldsymbol{\nabla}\,\mathbf{v},\mathbf{w}) - (\mathbf{u}\cdot\boldsymbol{\nabla}\,\mathbf{w},\mathbf{v})\right)$$

$$(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u}\mathbf{v}d\mathbf{x}, \quad (\mathbf{X},\mathbf{z})_{\mathcal{B}} = \int_{\mathcal{B}} \mathbf{X}\mathbf{z}d\mathbf{s}$$

$$\delta_\rho = \rho_s - \rho_f$$

Variational form with Lagrange multiplier

Problem

For $t \in [0, T]$, find $\mathbf{u}(t) \in H^1_0(\Omega)^d$, $p(t) \in L^2_0(\Omega)$, $\mathbf{X}(t) \in W^{1,\infty}(\mathcal{B})^d$, and $\boldsymbol{\lambda}(t) \in \mathbf{\Lambda}$ such that

$$\rho \frac{d}{dt}(\mathbf{u}(t), \mathbf{v}) + a(\mathbf{u}(t), \mathbf{v}) + b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) - (\operatorname{div} \mathbf{v}, p(t)) + \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{v}(\mathbf{X}(\cdot, t))) = 0 \qquad \forall \mathbf{v} \in H^1_0(\Omega)^d$$

$$(\operatorname{div} \mathbf{u}(t), q) = 0 \qquad \forall q \in L^2_0(\Omega)$$

$$\delta_\rho \left(\frac{\partial^2 \mathbf{X}}{\partial t^2}(t), \mathbf{z} \right)_{\mathcal{B}} + (\mathbb{P}(\mathbb{F}(t)), \boldsymbol{\nabla}_s \mathbf{z})_B - \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{z}) = 0 \quad \forall \mathbf{z} \in H^1(\mathcal{B})^d$$

$$\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}(\mathbf{X}(\cdot, t), t) - \frac{\partial \mathbf{X}(t)}{\partial t} \right) = 0 \qquad \forall \boldsymbol{\mu} \in \mathbf{\Lambda}$$

Definition of $\mathbf{c}$

The fact that $\overline{\mathbf{X}} \in W^{1,\infty}(\mathcal{B})^d$ implies $\mathbf{v}(\overline{\mathbf{X}}(\cdot)) \in H^1(\mathcal{B})^d$

Case 1

$\mathcal{Z} = H^1(\mathcal{B})^d$, Λ dual space of $H^1(\mathcal{B})^d$, $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ duality pairing

$$\mathbf{c}(\boldsymbol{\lambda}, \mathbf{z}) = \langle \boldsymbol{\lambda}, \mathbf{z} \rangle_{\mathcal{B}} \quad \boldsymbol{\lambda} \in \Lambda = (H^1(\mathcal{B})^d)', \ \mathbf{z} \in H^1(\mathcal{B})^d$$

Case 2

$\mathcal{Z} = H^1(\mathcal{B})^d$, $\Lambda = H^1(\mathcal{B})^d$

$$\mathbf{c}(\boldsymbol{\lambda}, \mathbf{z}) = \int_{\mathcal{B}} (\boldsymbol{\nabla}_s \boldsymbol{\lambda} \cdot \boldsymbol{\nabla}_s \mathbf{z} + \boldsymbol{\lambda} \cdot \mathbf{z}) \, ds \quad \boldsymbol{\lambda} \in \Lambda, \ \mathbf{z} \in H^1(\mathcal{B})^d$$

Energy estimate

Stability estimate

If $\rho_s > \rho_f$, then the following bound holds true

$$\frac{\rho_f}{2}\frac{d}{dt}||\mathbf{u}(t)||_0^2 + \mu||\boldsymbol{\nabla}\,\mathbf{u}(t)||_0^2 + \frac{d}{dt}E(\mathbf{X}(t)) + \frac{1}{2}\delta_\rho\frac{d}{dt}\left\|\frac{\partial\mathbf{X}}{\partial t}\right\|_B^2 = 0$$

where $E\left(\mathbf{X}(t)\right) = \int_{\mathcal{B}} W(\mathbb{F}(s,t))\,ds$

Remark Similar bound holds true if the condition $\rho_s > \rho_f$ is not satisfied.

Time advancing scheme - Backward Euler BE

Problem

Given $\mathbf{u}_0 \in H_0^1(\Omega)^d$ and $\mathbf{X}_0 \in W^{1,\infty}(\mathcal{B})^d$, for $n = 1, \ldots, N$, find $(\mathbf{u}^n, p^n) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$, $\mathbf{X}^n \in W^{1,\infty}(\mathcal{B})^d$, and $\boldsymbol{\lambda}^n \in \boldsymbol{\Lambda}$, such that

$$\rho_f \left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t}, \mathbf{v} \right) + a(\mathbf{u}^{n+1}, \mathbf{v}) + b(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p^{n+1}) + \mathbf{c}(\boldsymbol{\lambda}^{n+1}, \mathbf{v}(\mathbf{X}^{n+1}(\cdot))) = 0 \quad \forall \mathbf{v} \in H_0^1(\Omega)^d$$

$$(\operatorname{div} \mathbf{u}^{n+1}, q) = 0 \quad \forall q \in L_0^2(\Omega)$$

$$\delta_\rho \left(\frac{\mathbf{X}^{n+1} - 2\mathbf{X}^n + \mathbf{X}^{n-1}}{\Delta t^2}, \mathbf{z} \right)_{\mathcal{B}} + (\mathbb{P}(\mathbb{F}^{n+1}), \nabla_s \mathbf{z})_{\mathcal{B}} - \mathbf{c}(\boldsymbol{\lambda}^{n+1}, \mathbf{z}) = 0 \quad \forall \mathbf{z} \in H^1(\mathcal{B})^d$$

$$\mathbf{c} \left(\boldsymbol{\mu}, \mathbf{u}^{n+1}(\mathbf{X}^{n+1}(\cdot)) - \frac{\mathbf{X}^{n+1} - \mathbf{X}^n}{\Delta t} \right) = 0 \quad \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}$$

Time advancing scheme - Mofified backward Euler MBE

Problem

Given $\mathbf{u}_0 \in H^1_0(\Omega)^d$ and $\mathbf{X}_0 \in W^{1,\infty}(\mathcal{B})^d$, for $n = 1, \ldots, N$, find $(\mathbf{u}^n, p^n) \in H^1_0(\Omega)^d \times L^2_0(\Omega)$, $\mathbf{X}^n \in W^{1,\infty}(\mathcal{B})^d$, and $\boldsymbol{\lambda}^n \in \boldsymbol{\Lambda}$, such that

$$\rho_f\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t},\mathbf{v}\right) + a(\mathbf{u}^{n+1},\mathbf{v}) + b(\mathbf{u}^n,\mathbf{u}^{n+1},\mathbf{v})$$

$$- (\operatorname{div}\mathbf{v}, p^{n+1}) + \mathbf{c}(\boldsymbol{\lambda}^{n+1}, \mathbf{v}(\mathbf{X}^n(\cdot))) = 0 \qquad \forall \mathbf{v} \in H^1_0(\Omega)^d$$

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$$\delta_\rho\left(\frac{\mathbf{X}^{n+1}-2\mathbf{X}^n+\mathbf{X}^{n-1}}{\Delta t^2},\mathbf{z}\right)_{\mathcal{B}} + (\mathbb{P}(\mathbb{F}^{n+1}), \nabla_s\mathbf{z})_{\mathcal{B}}$$

$$- \mathbf{c}(\boldsymbol{\lambda}^{n+1}, \mathbf{z}) = 0 \qquad \forall \mathbf{z} \in H^1(\mathcal{B})^d$$

$$\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}^{n+1}(\mathbf{X}^n(\cdot)) - \frac{\mathbf{X}^{n+1}-\mathbf{X}^n}{\Delta t}\right) = 0 \qquad \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}$$

Energy estimate for the time discrete problem

Proposition (Unconditional stability)

Assume that W is convex and $\delta_\rho = \rho_s - \rho_f > 0$
For both BE and MBE schemes, the following estimate holds true for all $n = 1, \dots, N$

$$
\begin{aligned}
&\frac{\rho_f}{2\Delta t}\left(\|u^{n+1}\|_0^2 - \|u^n\|_0^2\right) + \nu\|\nabla \mathbf{u}^{n+1}\|_0^2 \\
&+ \frac{\delta_\rho}{2\Delta t}\left(\left\|\frac{\mathbf{X}^{n+1}-\mathbf{X}^n}{\Delta t}\right\|_{0,\mathcal{B}}^2 - \left\|\frac{\mathbf{X}^{n}-\mathbf{X}^{n-1}}{\Delta t}\right\|_{0,\mathcal{B}}^2\right) \\
&+ \frac{1}{\Delta t}(E(\mathbf{X}^{n+1}) - E(\mathbf{X}^n)) \le 0
\end{aligned}
$$

where $E(\mathbf{X})$ is the elastic potential energy given by

$$E(\mathbf{X}) = \int_{\mathcal{B}} W(\mathbb{F}(\mathbf{s},t))\,ds$$

Operator matrix form of time advancing schemes

BE

$$\begin{bmatrix} A_f(\mathbf{u}^{n+1}) & B_f^\top & 0 & C_f^\top(\mathbf{X}^{n+1}) \\ B_f & 0 & 0 & 0 \\ 0 & 0 & A_s & -C_s^\top \\ C_f(\mathbf{X}^{n+1}) & 0 & -C_s & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ p^{n+1} \\ \mathbf{X}^{n+1} \\ \lambda^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \\ \mathbf{g} \\ \mathbf{d} \end{bmatrix}$$

MBE

$$\begin{bmatrix} A_f(\mathbf{u}^{n}) & B_f^\top & 0 & C_f^\top(\mathbf{X}^{n}) \\ B_f & 0 & 0 & 0 \\ 0 & 0 & A_s & -C_s^\top \\ C_f(\mathbf{X}^{n}) & 0 & -C_s & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ p^{n+1} \\ \mathbf{X}^{n+1} \\ \lambda^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \\ \mathbf{g} \\ \mathbf{d} \end{bmatrix}$$

Analysis of the saddle point problem (MBE)

For simplicity, we take $\mathbb{P}(\mathbb{F}) = \kappa\mathbb{F} = \kappa\,\boldsymbol{\nabla}_s\,\mathbf{X}$.

Problem

Let $\overline{\mathbf{X}} \in W^{1,\infty}(\mathcal{B})^d$ be invertible with Lipschitz inverse and $\overline{\mathbf{u}} \in L^\infty(\Omega)$. Given $\mathbf{f} \in L^2(\Omega)^d$, $\mathbf{g} \in L^2(\mathcal{B})^d$, and $\mathbf{d} \in L^2(\mathcal{B})^d$, find $\mathbf{u} \in H^1_0(\Omega)^d$, $p \in L^2_0(\Omega)$, $\mathbf{X} \in H^1(\mathcal{B})^d$, and $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ such that

$$
\begin{aligned}
&\mathbf{a}_f(\mathbf{u},\mathbf{v}) - (\operatorname{div}\mathbf{v}, p) + \mathbf{c}(\boldsymbol{\lambda}, \mathbf{v}(\overline{\mathbf{X}})) = (\mathbf{f},\mathbf{v}) && \forall \mathbf{v} \in H^1_0(\Omega)^d\\
&(\operatorname{div}\mathbf{u}, q) = 0 && \forall q \in L^2_0(\Omega)\\
&\mathbf{a}_s(\mathbf{X},\mathbf{z}) - \mathbf{c}(\boldsymbol{\lambda},\mathbf{z}) = (\mathbf{g},\mathbf{z})_{\mathcal{B}} && \forall \mathbf{z} \in H^1(\mathcal{B})^d\\
&\mathbf{c}(\boldsymbol{\mu}, \mathbf{u}(\overline{\mathbf{X}}) - \mathbf{X}) = \mathbf{c}(\boldsymbol{\mu},\mathbf{d}) && \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathbf{a}_f(\mathbf{u},\mathbf{v}) = \alpha(\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + b(\overline{\mathbf{u}},\mathbf{u},\mathbf{v}) && \forall \mathbf{u},\mathbf{v} \in H^1_0(\Omega)^d\\
&\mathbf{a}_s(\mathbf{X},\mathbf{z}) = \beta(\mathbf{X},\mathbf{z})_{\mathcal{B}} + \gamma(\boldsymbol{\nabla}_s\,\mathbf{X}, \boldsymbol{\nabla}_s\,\mathbf{z})_{\mathcal{B}} && \forall \mathbf{X},\mathbf{z} \in H^1(\mathcal{B})^d
\end{aligned}
$$

Finite element discretization

We consider

- Background grid $\mathcal{T}_h$ for the domain Ω (meshsize h_x)
- $(V_h, Q_h) \subseteq H^1_0(\Omega)^d \times L^2_0(\Omega)$ stable pair for the Stokes equations

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- Grid $\mathcal{S}_h$ for $\mathcal{B}$ (meshsize h_s)
- $S_h \subseteq H^1(\mathcal{B})^d$ continuous Lagrange elements

$$S_h = \{\mathbf{Y} \in C^0(\mathcal{B}; \Omega) : \mathbf{Y} \in P^1\}$$

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Remark

- If $\mathbf{c}$ is a duality pairing, we represent it by the scalar product in $L^2(\mathcal{B})$.
- Stabilized $P1 - P1$ elements for Stokes could also be used

<Annese, Phd Thesis '17>

Discrete saddle point problem

Problem

Find $\mathbf{u}_h \in V_h$, $p_h \in Q_h$, $\mathbf{X}_h \in S_h$ and $\lambda_h \in \Lambda_h$ such that

$$
\begin{aligned}
& a_f(\mathbf{u}_h, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p_h) + \mathbf{c}(\boldsymbol{\lambda}_h, \mathbf{v}(\overline{\mathbf{X}}(\cdot))) = (\mathbf{f}, \mathbf{v}) && \forall \mathbf{v} \in V_h \\
& (\operatorname{div} \mathbf{u}_h, q) = 0 && \forall q \in Q_h \\
& a_s(\mathbf{X}_h, \mathbf{z}) - \mathbf{c}(\boldsymbol{\lambda}_h, \mathbf{z}) = (\mathbf{g}, \mathbf{z})_{\mathcal{B}} && \forall \mathbf{z} \in S_h \\
& \mathbf{c}(\boldsymbol{\mu}, \mathbf{u}_h(\overline{\mathbf{X}}(\cdot)) - \mathbf{X}_h) = \mathbf{c}(\boldsymbol{\mu}, \mathbf{d}) && \forall \boldsymbol{\mu} \in \Lambda_h.
\end{aligned}
$$

Alternative (equivalent) matrix form

$$\left[\begin{array}{ccc|c} A_f & B_f^\top & 0 & C_f^\top \\ B_f & 0 & 0 & 0 \\ 0 & 0 & A_s & -C_s^\top \\ \hline C_f & 0 & -C_s & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u} \\ p \\ \mathbf{X} \\ \hline \boldsymbol{\lambda} \end{array}\right] = \left[\begin{array}{c} \mathbf{f} \\ 0 \\ \mathbf{g} \\ \hline \mathbf{d} \end{array}\right]$$

or

$$\left[\begin{array}{ccc|c} A_f & 0 & C_f^\top & B_f^\top \\ 0 & A_s & -C_s^\top & 0 \\ C_f & -C_s & 0 & 0 \\ \hline B_f & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u} \\ \mathbf{X} \\ \boldsymbol{\lambda} \\ \hline p \end{array}\right] = \left[\begin{array}{c} \mathbf{f} \\ \mathbf{g} \\ \mathbf{d} \\ \hline 0 \end{array}\right].$$

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Theoretical results <B.–Gastaldi '17>

This problem has been rigorously analyzed both at continuous and discrete level (existence, uniqueness, stability, and convergence)

Abstract saddle point formulation

Set: $\mathbb{V} = H^1_0(\Omega)^d \times H^1(\mathcal{B})^d \times \mathbf{\Lambda}$ and $\mathbf{V} = (\mathbf{v}, \mathbf{z}, \boldsymbol{\lambda}) \in \mathbb{V}$

$$\mathbb{A}(\mathbf{U}, \mathbf{V}) = \mathbf{a}_f(\mathbf{u}, \mathbf{v}) + \mathbf{a}_s(\mathbf{X}, \mathbf{z}) + \mathbf{c}(\boldsymbol{\lambda}, \mathbf{v}(\overline{\mathbf{X}}) - \mathbf{z}) - \mathbf{c}(\boldsymbol{\mu}, \mathbf{u}(\overline{\mathbf{X}}) - \mathbf{X})$$

$$\mathbb{B}(\mathbf{V}, q) = (\operatorname{div} \mathbf{v}, q)$$

Problem (continuous)

Find $(\mathbf{U}, p) \in \mathbb{V} \times L^2_0(\Omega)$ *such that*

$$\mathbb{A}(\mathbf{U}, \mathbf{V}) + \mathbb{B}(\mathbf{V}, p) = (\mathbf{f}, \mathbf{v}) + (\mathbf{g}, \mathbf{z})_{\mathcal{B}} + \mathbf{c}(\boldsymbol{\mu}, \mathbf{d}) \quad \forall \mathbf{V} \in \mathbb{V}$$

$$\mathbb{B}(\mathbf{U}, q) = 0 \quad \forall q \in L^2_0(\Omega).$$

Set: $\mathbb{V}_h = V_h \times S_h \times \mathbf{\Lambda}_h$

Problem (discrete)

Find $(\mathbf{U}_h, \boldsymbol{\lambda}_h) \in \mathbb{V}_h \times \mathbf{\Lambda}_h$ *such that*

$$\mathbb{A}(\mathbf{U}_h, \mathbf{V}) + \mathbb{B}(\mathbf{V}, p_h) = (\mathbf{f}, \mathbf{v}) + (\mathbf{g}, \mathbf{z})_{\mathcal{B}} + \mathbf{c}(\boldsymbol{\mu}, \mathbf{d}) \quad \forall \mathbf{V} \in \mathbb{V}_h$$

$$\mathbb{B}(\mathbf{U}_h, q) = 0 \quad \forall q \in Q_h.$$

Main steps of the proof

Discrete case

Discrete inf-sup condition for $\mathbb{B}$

Since $V_h \times Q_h$ is stable for the Stokes equation, there exists a positive constant $\overline{\beta}_{\mathrm{div}}$ such that for all $q_h \in Q_h$

$$\sup_{\mathbf{V}_h \in \mathbb{V}_h} \frac{\mathbb{B}(\mathbf{V}_h, q_h)}{|||\mathbf{V}_h|||_{\mathbb{V}}} = \sup_{\mathbf{v}_h \in V_h} \frac{(\mathrm{div}\, \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1} \geq \overline{\beta}_{\mathrm{div}} \|q_h\|_0$$

Main steps of the proof

Discrete case

Discrete inf-sup condition for $\mathbb{B}$

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$$\sup_{\mathbf{V}_h \in \mathbb{V}_h} \frac{\mathbb{B}(\mathbf{V}_h, q_h)}{|||\mathbf{V}_h|||_{\mathbb{V}}} = \sup_{\mathbf{v}_h \in V_h} \frac{(\mathrm{div}\,\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1} \geq \overline{\beta}_{\mathrm{div}} \|q_h\|_0$$

The main issue is to show the invertibility of the operator matrix

$$\left[\begin{array}{cc|c} A_f & 0 & C_f^\top \\ 0 & A_s & -C_s^\top \\ \hline C_f & -C_s & 0 \end{array}\right]$$

on the discrete kernel of $\mathbb{B}$:

$$\mathbb{K}_{\mathbb{B},h} = \{\mathbf{V} \in \mathbb{V}_h : \mathbb{B}(\mathbf{V}, q) = 0 \ \forall q \in Q_h\}.$$

Main steps of the proof (cont'ed)

Discrete inf-sup for $\mathbb{A}$

There exists $\kappa_0 > 0$, independent of h_x and h_s, such that

$$\inf_{\mathbf{U}\in\mathbb{K}_{\mathbb{B},h}} \sup_{\mathbf{V}\in\mathbb{K}_{\mathbb{B},h}} \frac{\mathbb{A}(\mathbf{U},\mathbf{V})}{|||\mathbf{U}|||_{\mathbb{V}}|||\mathbf{V}|||_{\mathbb{V}}} \ge \kappa_0.$$

Proposition

There exists $\alpha_1 > 0$ independent of h_x and h_s such that

$$\mathbf{a}_f(\mathbf{u}_h,\mathbf{u}_h) + \mathbf{a}_s(\mathbf{X}_h,\mathbf{X}_h) \ge \alpha_1(\|\mathbf{u}_h\|_1^2 + \|\mathbf{X}_h\|_{1,\mathcal{B}}^2) \quad \forall(\mathbf{u}_h,\mathbf{X}_h)\in\mathbb{K}_h$$

where

$$\mathbb{K}_h = \left\{(\mathbf{v}_h,\mathbf{z}_h)\in V_{0,h}\times S_h:\ \mathbf{c}(\boldsymbol{\mu}_h,\mathbf{v}_h(\overline{\mathbf{X}})-\mathbf{z}_h)=0\ \forall\boldsymbol{\mu}_h\in\boldsymbol{\Lambda}_h\right\}$$
$$V_{0,h} = \{\mathbf{v}_h\in V_h : (\operatorname{div}\mathbf{v}_h, q_h)=0\ \forall q_h\in Q_h\}$$

Proposition

There exists a constant $\beta_1 > 0$ independent of h_x and h_s such that for all $\boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h$ it holds true

$$\sup_{(\mathbf{v}_h,\mathbf{z}_h)\in V_{0,h}\times S_h} \frac{\mathbf{c}(\boldsymbol{\mu}_h, \mathbf{v}_h(\overline{\mathbf{X}}) - \mathbf{z}_h)}{(\|\mathbf{v}_h\|_1^1 + \|\mathbf{z}_h\|_{1,\mathcal{B}}^2)^{1/2}} \geq \beta_1 \|\boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}}.$$

The proof depends on the choice of $\mathbf{c}$.

Case 1 $\mathbf{c}(\boldsymbol{\mu}, \mathbf{z}) = \langle \boldsymbol{\mu}, \mathbf{z} \rangle$ for $\boldsymbol{\mu} \in \boldsymbol{\Lambda}_h$ $\mathbf{z} \in S_h$

The above inf-sup condition holds true if the L^2-projection onto S_h is bounded in $H^1(\mathcal{B})^d$.

This can be proved by assuming that the mesh in $\mathcal{B}$ is quasi-uniform or satisfies weaker assumptions as in <Bramble–Pasciak–Steinbach '02> <Crouzeix–Thomée '87>

Case 2 $\mathbf{c}(\boldsymbol{\mu}, \mathbf{z}) = \int_{\mathcal{B}} (\boldsymbol{\nabla}_s \boldsymbol{\mu} \boldsymbol{\nabla}_s \mathbf{z} + \boldsymbol{\mu}\mathbf{z}) ds$ for $\boldsymbol{\mu} \in \boldsymbol{\Lambda}_h$ $\mathbf{z} \in S_h$

The result follows directly from the continuous inf-sup condititon.

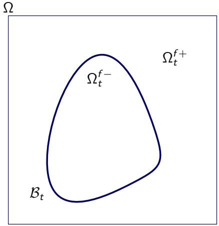
Error estimates

Theorem

The following error estimates hold true

$$
\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_h\|_{H^1_0(\Omega)^d}+\|p-p_h\|_{L^2(\Omega)}+\|\mathbf{X}-\mathbf{X}_h\|_{H^1(\mathcal{B})^d}+\|\lambda-\lambda_h\|_\Lambda \\
&\qquad\leq C \inf_{(\mathbf{v},q,\mathbf{z},\mu)\in V_h\times Q_h\times S_h\times S_h}\Big(\|\mathbf{u}-\mathbf{v}\|_{H^1_0(\Omega)^d}+\|p-q\|_{L^2(\Omega)} \\
&\qquad\qquad+\|\mathbf{X}-\mathbf{z}\|_{H^1(\mathcal{B})^d}+\|\lambda-\mu\|_\Lambda\Big)
\end{aligned}
$$

FSI problem (thin solid)



$$\rho_f\left(\frac{\partial \mathbf{u}_f}{\partial t}+\mathbf{u}_f\cdot\nabla\,\mathbf{u}_f\right)=\operatorname{div}\boldsymbol{\sigma}_f \quad \text{in } \Omega\setminus\mathcal{B}_t$$

$$\operatorname{div}\mathbf{u}_f=0 \quad \text{in } \Omega\setminus\mathcal{B}_t$$

$$\rho_s\frac{\partial \mathbf{u}_s}{\partial t}=\operatorname{div}_s(\mathbb{P}(\mathbb{F}))+\mathbf{f}_{\mathsf{FSI}} \quad \text{in } \mathcal{B}$$

$$\mathbf{u}_f=\mathbf{u}_s \quad \text{on } \mathcal{B}_t$$

$$\boldsymbol{\sigma}_f^+\mathbf{n}^+ + \boldsymbol{\sigma}_f^-\mathbf{n}^- = -\mathbf{f}_{\mathsf{FSI}} \quad \text{on } \mathcal{B}_t$$

$$\boldsymbol{\sigma}_f=-p_f\mathbb{I}+\nu_f\,\nabla_{sym}\,\mathbf{u}_f \qquad \mathbf{u}_s=\frac{\partial\mathbf{X}}{\partial t}$$

+ initial and boundary conditions

Variational form with Lagrange multiplier (thin solid)

- integrate by parts
- use f_{FSI} as Lagrange multiplier
- set $\mathcal{Z} = H^{1/2}(\mathcal{B})^d$, Λ dual space of $H^{1/2}(\mathcal{B})^d$, $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ duality pairing

$$\mathbf{c}(\boldsymbol{\lambda},\mathbf{z}) = \langle\boldsymbol{\lambda},\mathbf{z}\rangle_{\mathcal{B}} \quad \boldsymbol{\lambda}\in\Lambda = (H^{1/2}(\mathcal{B})^d)',\ \mathbf{z}\in H^{1/2}(\mathcal{B})^d$$

- obtain the same variational form as before.

Variational form

Given $\mathbf{u}_0 \in H^1_0(\Omega)^d$ and $\mathbf{X}_0 \in W^{1,\infty}(\mathcal{B})^d$, for $t \in [0, T]$, find $\mathbf{u}(t) \in H^1_0(\Omega)^d$, $p(t) \in L^2_0(\Omega)$, $\mathbf{X}(t) \in W^{1,\infty}(\mathcal{B})^d$, and $\boldsymbol{\lambda}(t) \in \mathbf{\Lambda}$ such that

$$
\begin{aligned}
&\rho \frac{d}{dt}(\mathbf{u}(t), \mathbf{v}) + a(\mathbf{u}(t), \mathbf{v}) + b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) \\
&\qquad - (\operatorname{div} \mathbf{v}, p(t)) + \mathbf{c}(\boldsymbol{\lambda}, \mathbf{v}(\mathbf{X}(\cdot, t))) = 0 && \forall \mathbf{v} \in H^1_0(\Omega)^d \\
&(\operatorname{div} \mathbf{u}(t), q) = 0 && \forall q \in L^2_0(\Omega) \\
&\delta_\rho \left(\frac{\partial^2 \mathbf{X}}{\partial t^2}, \mathbf{z} \right)_{\mathcal{B}} + (\mathbb{P}(\mathbb{F}(t)), \boldsymbol{\nabla}_s \mathbf{z})_{\mathcal{B}} - \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{z}) = 0 && \forall \mathbf{z} \in H^1(\mathcal{B})^d \\
&\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}(\mathbf{X}(\cdot, t), t) - \frac{\partial \mathbf{X}(t)}{\partial t} \right) = 0 && \forall \boldsymbol{\mu} \in \mathbf{\Lambda} \\
&\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega, \qquad \mathbf{X}(0) = \mathbf{X}_0 \quad \text{in } \mathcal{B}.
\end{aligned}
$$

The analysis can be performed as in the thick solid case, but the inf-sup for $\mathbf{c}$ requires a different approach

Inf-sup condition for **c**

There exists a constant $\beta_0 > 0$ such that for all $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ it holds true

$$\sup_{(\mathbf{v},\mathbf{z})\in V_0\times H^1(\mathcal{B})^d} \frac{\mathbf{c}(\boldsymbol{\mu}, \mathbf{v}(\overline{\mathbf{X}}) - \mathbf{z})}{(\|\mathbf{v}\|_1^2 + \|\mathbf{z}\|_{1,\mathcal{B}}^2)^{1/2}} \geq \beta_0 \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}$$

where V_0 is the space of free divergence velocities.

Inf-sup condition for $\mathbf{c}$

There exists a constant $\beta_0 > 0$ such that for all $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ it holds true

$$\sup_{(\mathbf{v},\mathbf{z})\in V_0\times H^1(\mathcal{B})^d} \frac{\mathbf{c}(\boldsymbol{\mu}, \mathbf{v}(\overline{\mathbf{X}}) - \mathbf{z})}{(\|\mathbf{v}\|_1^2 + \|\mathbf{z}\|_{1,\mathcal{B}}^2)^{1/2}} \geq \beta_0 \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}$$

where V_0 is the space of free divergence velocities.

Proof By definition

$$\|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} = \sup_{\mathbf{z}\in H^{1/2}(\mathcal{B})^d} \frac{\langle \boldsymbol{\mu}, \mathbf{z}\rangle}{\|\mathbf{z}\|_{H^{1/2}(\mathcal{B})^d}} = \sup_{\mathbf{z}\in H^{1/2}(\mathcal{B})^d} \frac{\mathbf{c}(\boldsymbol{\mu}, \mathbf{z})}{\|\mathbf{z}\|_{H^{1/2}(\mathcal{B})^d}}$$

We construct a maximizing sequence $\mathbf{z}_n \in H^{1/2}(\mathcal{B})^d$ and functions $\mathbf{v}_n \in V_0$ such $\mathbf{v}_n(\overline{\mathbf{X}}(\cdot)) = z_n$ with $\|\mathbf{v}_n\|_1 \leq c\|z_n\|_{H^{1/2}(\mathcal{B})^d}$. Then

$$\sup_{(\mathbf{v},\mathbf{z})\in V_0\times H^1(\mathcal{B})^d} \frac{\mathbf{c}(\boldsymbol{\mu}, \mathbf{v}(\overline{\mathbf{X}}) - \mathbf{z})}{\|\mathbf{V}\|_{\mathbb{V}}} \geq \sup_{\mathbf{v}\in V_0} \frac{\mathbf{c}(\boldsymbol{\mu}, \mathbf{v}(\overline{\mathbf{X}}))}{\|\mathbf{v}\|_1}$$
$$\geq \frac{\mathbf{c}(\boldsymbol{\mu}, \mathbf{v}_n(\overline{\mathbf{X}}))}{\|\mathbf{v}_n\|_1} \geq \frac{1}{c}\frac{\mathbf{c}(\boldsymbol{\mu}, \mathbf{z}_n)}{\|\mathbf{z}_n\|_{H^{1/2}(\mathcal{B})^d}} \geq \frac{1}{2c}\|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}$$

Discrete inf-sup condition for $\mathbf{c}$

We assume that the domain Ω is convex. If h_x/h_s is sufficiently small and the mesh $\mathcal{S}_h$ is quasi-uniform, then there exists a constant $\beta_1 > 0$ independent of h_x and h_s such that for all $\boldsymbol{\mu}_h \in \Lambda_h$ it holds true

$$\sup_{(\mathbf{v}_h,\mathbf{z}_h)\in V_{0,h}\times S_h} \frac{\mathbf{c}(\boldsymbol{\mu}_h, \mathbf{v}_h(\overline{\mathbf{X}}) - \mathbf{z}_h)}{(\|\mathbf{v}_h\|_1^2 + \|\mathbf{z}_h\|_{1,\mathcal{B}}^2)^{1/2}} \geq \beta_1 \|\boldsymbol{\mu}_h\|_{\mathbf{\Lambda}}.$$

Proof Let $\bar{\mathbf{u}} \in V_0$ be the element where the supremum of the continuous inf-sup condition is attained and $\bar{\mathbf{u}}_h \in V_{0,h}$ be the approximation of $\bar{\mathbf{u}}$. Then

$$\mathbf{c}(\boldsymbol{\mu}_h, \bar{\mathbf{u}}_h(\overline{\mathbf{X}})) = \mathbf{c}(\boldsymbol{\mu}_h, \bar{\mathbf{u}}(\overline{\mathbf{X}})) + \mathbf{c}(\boldsymbol{\mu}_h, \bar{\mathbf{u}}_h(\overline{\mathbf{X}}) - \bar{\mathbf{u}}(\overline{\mathbf{X}})).$$

By trace theorem and inverse inequality $\|\bar{\mathbf{u}}_h(\overline{\mathbf{X}}) - \bar{\mathbf{u}}(\overline{\mathbf{X}})\|_{0,\mathcal{B}} \leq C h_x^{1/2} \|\bar{\mathbf{u}}\|_1$ and $\|\boldsymbol{\mu}_h\|_{0,\mathcal{B}} \leq C h_s^{-1/2} \|\boldsymbol{\mu}_h\|_{\mathbf{\Lambda}}$. Hence

$$\begin{aligned}
\mathbf{c}(\boldsymbol{\mu}_h, \bar{\mathbf{u}}_h(\overline{\mathbf{X}})) &\geq \frac{1}{2c} \|\boldsymbol{\mu}\|_{\mathbf{\Lambda}} \|\bar{\mathbf{u}}\|_1 - C \|\boldsymbol{\mu}_h\|_{0,\mathcal{B}} h_x^{1/2} \|\bar{\mathbf{u}}\|_1 \\
&\geq \|\boldsymbol{\mu}\|_{\mathbf{\Lambda}} \|\bar{\mathbf{u}}\|_1 \left(\frac{1}{2c} - C \left(\frac{h_x}{h_s} \right)^{1/2} \right)
\end{aligned}$$

Error estimate for the monolithic scheme

For simplicity

- we take $\mathbb{P} = \kappa\mathbb{F} = \kappa \nabla_s \mathbf{X}$
- we consider small displacements from the reference/initial configuration, hence the current configuration is identified with the reference configuration $\mathcal{B} = \Omega_0^s$ and $\mathbf{v}|_{\mathcal{B}} = \mathbf{v}(\mathbf{X}(\mathbf{s}, 0))$ for all $\mathbf{v} \in H_0^1(\Omega)^d$.

Regularity assumptions

$$\mathbf{u}(t) \in H^{1+l}(\Omega), \quad p(t) \in H^l(\Omega),$$
$$\mathbf{X}(t) \in H^{1+m}(\mathcal{B}), \quad \boldsymbol{\lambda}(t) \in H^{-1/2+l}(\mathcal{B})$$

- Thick solid Depending of the elastic response of the solid material, we can have a continuous pressure. Hence $0 < l \leq 1/2$ and $0 < m \leq 1$.
- Thin solid The pressure is discontinuous across the structure, hence we assume that $0 < l < 1/2$ and $0 < m \leq 1$

Space-time error estimates for negligible displacements

<Annese PhD Thesis '17>

Theorem

In the case of thick solid, we assume that $\rho_s > \rho_f$.

- $$\frac{\rho_f}{2}\|\mathbf{u}(t^n)-\mathbf{u}_h^n\|^2_{0,\Omega}+\frac{1}{2}\|\mathbf{X}(t^n)-\mathbf{X}_h^n\|^2_{1,\mathcal{B}} + \frac{\delta_\rho}{2}\left\|\frac{\partial \mathbf{X}}{\partial t}(t^n)-\frac{\mathbf{X}_h^n-\mathbf{X}_h^{n-1}}{\Delta t}\right\|^2_{0,\mathcal{B}} \le C\left(h_f^{2l}+h_s^{2m}+h_s^{2l}+\Delta t^2\right)$$
- $\Delta t\sum_{k=1}^n \|\nabla_{sym}(\mathbf{u}(t^k)-\mathbf{u}_h^k)\|^2_{0,\Omega} \le C\left(h_f^{2l}+h_s^{2m}+h_s^{2l}+\Delta t^2\right)$
- $\Delta t\sum_{k=1}^n \|\boldsymbol{\lambda}(t^k)-\boldsymbol{\lambda}_h^k\|^2_{\boldsymbol{\Lambda}} \le C\left(h_f^{2l}+h_s^{2m}+h_s^{2l}+\Delta t^2\right)$

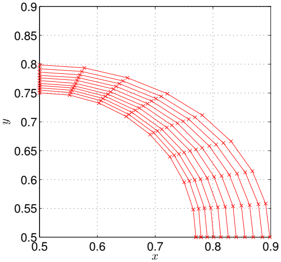
Ellipse immersed in a static fluid

$\mathbb{P} = \kappa\mathbb{F}$ **c** scalar product in L^2

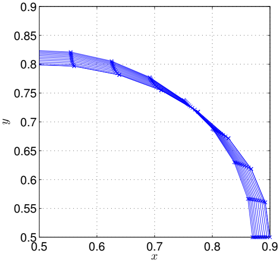
Fluid initially at rest: $\mathbf{u}_{0h} = 0$

$$\mathbf{X}_0(s) = \begin{pmatrix} 0.2\cos(2\pi s) + 0.45 \\ 0.1\sin(2\pi s) + 0.45 \end{pmatrix} \quad s \in [0,1],$$

$h_x = 1/32$, $h_s = 1/32$, $\Delta t = 10^{-2}$, $\mu = 1$, $\kappa = 5$



Standard IBM with PW update of the immersed boundary



IBM with DLM

Error analysis

Codimension 1

h_x	$\|\|p - p_h\|\|_{L^2}$	L^2-rate	$\|\|\mathbf{u} - \mathbf{u}_h\|\|_{L^2}$	L^2-rate
1/4	2.9606	-	0.0223	-
1/8	2.1027	0.49	0.0102	1.12
1/16	1.4349	0.55	0.0039	1.38
1/24	1.1572	0.53	0.0021	1.52
1/32	0.9750	0.60	0.0013	1.60
1/40	0.8874	0.42	0.0010	1.22

Outline

Computational aspects

Recall that we have to solve at each time step the linear system

$$\begin{bmatrix} A_f & B_f^\top & 0 & C_f(\mathbf{X}_h^n)^\top \\ B_f & 0 & 0 & 0 \\ 0 & 0 & A_s & -C_s^\top \\ C_f(\mathbf{X}_h^n) & 0 & -C_s & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_h^{n+1} \\ p_h^{n+1} \\ \mathbf{X}_h^{n+1} \\ \boldsymbol{\lambda}_h^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \\ \mathbf{g} \\ \mathbf{d} \end{bmatrix}$$

The matrix $C_f(\mathbf{X}_h^n)$ takes into account the relation between fluid and solid mesh.

Computational aspects

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The matrix $C_f(\mathbf{X}_h^n)$ takes into account the relation between fluid and solid mesh.
Let φ_j and χ_i be basis functions for $\mathbf{V}_h$ and $\boldsymbol{\Lambda}_h$, respectively, then

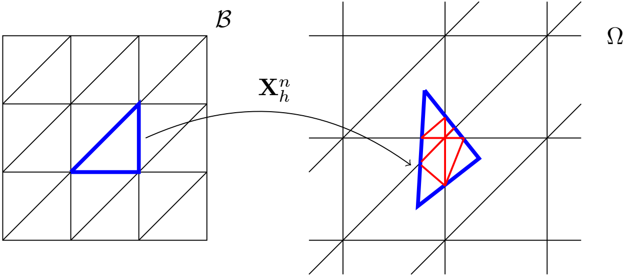
$$C_f(\mathbf{X}_h^n)_{ij} = \mathbf{c}(\chi_i, \varphi_j(\mathbf{X}_h^n)) = \int_{\mathcal{B}} \chi_i(\mathbf{s})\varphi_j(\mathbf{X}_h^n(\mathbf{s}))d\mathbf{s}$$

$$C_f(\mathbf{X}_h^n)_{ij} = \int_{\mathcal{B}} \chi_i(\mathbf{s})\varphi_j(\mathbf{X}_h^n(\mathbf{s}))d\mathbf{s}$$

We construct the matrix element by element in the solid mesh.

$$C_f(\mathbf{X}_h^n)_{ij} = \int_{\mathcal{B}} \chi_i(\mathbf{s})\varphi_j(\mathbf{X}_h^n(\mathbf{s}))d\mathbf{s}$$

We construct the matrix element by element in the solid mesh.



In order to evaluate $\varphi_j(\mathbf{X}_h^n(\mathbf{s}))$ we need to find the intersection of the fluid mesh with the mapping of the solid mesh and to triangulate it.

A simpler example

Interface problem

$$
\begin{aligned}
&-\operatorname{div}(\beta_1 \nabla u_1) = f_1 && \text{in } \Omega_1 \\
&-\operatorname{div}(\beta_2 \nabla u_2) = f_2 && \text{in } \Omega_2 \\
&u_1 = 0 && \text{on } \partial\Omega_1 \setminus \Gamma \\
&u_2 = 0 && \text{on } \partial\Omega_2 \setminus \Gamma \\
&u_1 = u_2 && \text{on } \Gamma \\
&\beta_1 \nabla u_1 \cdot \mathbf{n} = \beta_2 \nabla u_2 \cdot \mathbf{n} && \text{on } \Gamma
\end{aligned}
$$

with interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$

Equivalent formulation with Lagrange multiplier

- $\Omega = \Omega_1 \cup \Omega_2$
- $f \in L^2(\Omega)$ such that $f|_{\Omega_1} = f_1$
- $\beta \in W^{1,\infty}(\Omega)$ such that $\beta|_{\Omega_1} = \beta_1$

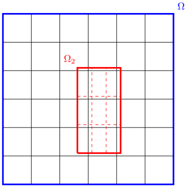
Equivalent formulation (DLM): look for $u \in H_0^1(\Omega)$, $u_2 \in H^1(\Omega_2)$, and $\lambda \in \Lambda = [H^1(\Omega_2)]'$ such that

$$\int_\Omega \beta \nabla u \nabla v \, dx + \langle \lambda, v|_{\Omega_2} \rangle = \int_\Omega f v \, dx \qquad \forall v \in H_0^1(\Omega)$$

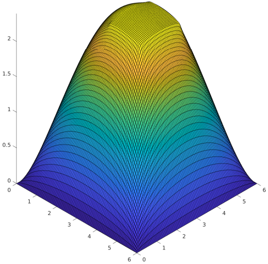
$$\int_{\Omega_2} (\beta_2 - \beta) \nabla u_2 \nabla v_2 \, dx - \langle \lambda, v_2 \rangle = \int_{\Omega_2} (f_2 - f) v_2 \, dx \qquad \forall v_2 \in H^1(\Omega_2)$$

$$\langle \mu, u|_{\Omega_2} - u_2 \rangle = 0 \qquad \forall \mu \in \Lambda$$

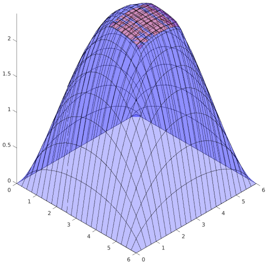
Dependence on the alignment of the meshes



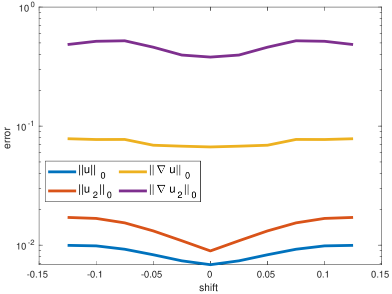
$\Omega = [0,6]^2$, $\Omega_2 = [e-0.1, 1+\pi] \times [2+s, 4+s]$
$\beta_1 = 1$, $\beta_2 = 10$, $f_1 = f_2 = 1$
$N = 24$, $N_2 = 10$
shift $s = -0.125 : 0.025 : 0.125$



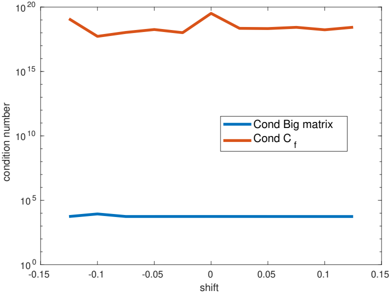
exact



DLM solution



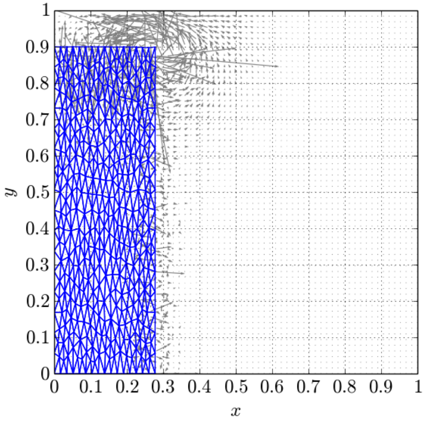
Errors for the DLM solution



Condition numbers

Stretched rectangular solid

Enhanced Bercovier-Pironneau element: $P_1 \text{iso} P_2 \setminus P_1 + P_0$
Solid element: P_1
Viscosity $\nu_f = \nu_s = 0.01$, structure elastic constant $\kappa = 100$
$h_x = 1/32$, $h_s = 1/16$



Parallel computing

<Boffi–G.–Scacchi work in progress>

Fluid element: $Q_2 \setminus P_1$, Solid element: Q_1, Time step: 0.01

Linear elastic solid

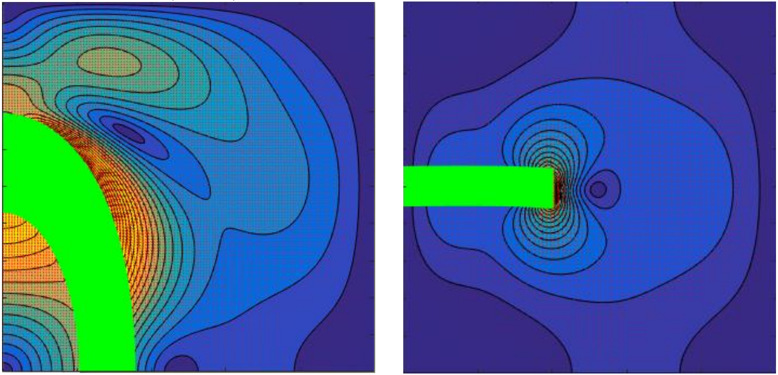
$\mathbb{P} = \kappa \mathbb{F}$ $\kappa = 10$

$\nu_f = \nu_s = 0.1$, $\rho_f = \rho_s = 1$

Nonlinear elastic solid

$W = \frac{a}{2b} exp(b \mathrm{tr}(\mathbb{F}^\top \mathbb{F}) - 2)$

$\nu_f = \nu_s = 0.2$, $\rho_f = \rho_s = 1$



Linear solid model

procs= 32, $T = 20$

dofs	vol. loss (%)	its	$T_{sol}(s)$	$T_{ass}(s)$	$T_{coup}(s)$
47190	0.16	9	$1.28\,10^{-1}$	$1.18\,10^{-2}$	$1.24\,10^{-1}$
83398	0.13	9	$2.01\,10^{-1}$	$3.98\,10^{-2}$	$9.48\,10^{-1}$
129846	0.12	9	$2.54\,10^{-1}$	$3.11\,10^{-2}$	$9.61\,10^{-1}$
186534	$9.92\,10^{-2}$	9	$4.90\,10^{-1}$	$4.45\,10^{-2}$	3.12

dofs= 83398, $T = 10$

procs	its	$T_{sol}(s)$	$T_{ass}(s)$	$T_{coup}(s)$
4	9	$3.84\,10^{-1}$	$1.43\,10^{-1}$	10.05
8	9	$2.40\,10^{-1}$	$9.09\,10^{-2}$	2.96
16	9	$1.38\,10^{-1}$	$3.75\,10^{-2}$	$7.71\,10^{-1}$
32	9	$1.09\,10^{-1}$	$2.68\,10^{-2}$	$3.25\,10^{-1}$
64	9	$1.11\,10^{-1}$	$1.60\,10^{-2}$	$1.34\,10^{-1}$

Nonlinear solid model

procs= 32, $T = 20$

dofs	vol. loss (%)	its	$T_{sol}(s)$	$T_{ass}(s)$	$T_{coup}(s)$
47190	0.63	2 (147)	4.35 (1.69)	$1.13\,10^{-2}$	$8.58\,10^{-2}$
83398	0.39	2 (145)	7.44 (2.73)	$1.90\,10^{-2}$	$1.94\,10^{-1}$
129846	0.35	2 (225)	20.84 (7.07)	$2.96\,10^{-2}$	$4.10\,10^{-1}$
186534	0.30	2 (179)	22.87 (6.82)	$4.23\,10^{-2}$	$8.33\,10^{-1}$

dofs= 83398, $T = 2$

procs	its (lits)	$T_{sol}(s)$	$T_{ass}(s)$	$T_{coup}(s)$
4	3 (331)	48.70 (12.60)	$1.49\,10^{-1}$	1.07
8	3 (323)	40.64 (11.93)	$9.00\,10^{-2}$	$7.18\,10^{-1}$
16	3 (319)	28.34 (8.69)	$4.60\,10^{-2}$	$3.83\,10^{-1}$
32	3 (312)	12.55 (3.73)	$2.55\,10^{-2}$	$3.16\,10^{-1}$
64	3 (310)	15.13 (4.78)	$9.05\,10^{-3}$	$1.48\,10^{-1}$

Outline

Second order time schemes

<Boffi-G.-Wolf '19>

We consider three second order schemes:

- **Backward Differentiation Formula** BDF2
- **Crank-Nicolson** using either midpoint CNm or trapezoidal CNt rule for the integration of nonlinear terms

We set:

$$\partial_{\Delta t} y^{n+1} = \begin{cases} \dfrac{3y^{n+1} - 4y^n + y^{n-1}}{2\Delta t} & \text{for BDF2} \\ \dfrac{y^{n+1} - y^n}{\Delta t} & \text{for Crank-Nicolson} \end{cases}$$

BDF2 scheme

Problem

Given $\mathbf{u}_{0h} \in V_h$ and $\mathbf{X}_{0h} \in S_h$, for $n = 0, \dots, N-1$ find $(\mathbf{u}_h^n, p_h^n) \in V_h \times Q_h$, $\mathbf{X}_h^n \in S_h$, and $\boldsymbol{\lambda}_h^n \in \boldsymbol{\Lambda}_h$, such that

$$
\begin{aligned}
&\rho_f \left(\partial_{\Delta t}\mathbf{u}_h^{n+1}, \mathbf{v}_h\right)_\Omega + b\left(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h\right) + a\left(\mathbf{u}_h^{n+1}, \mathbf{v}_h\right) \\
&\qquad - \left(\operatorname{div}\mathbf{v}_h, p_h^{n+1}\right)_\Omega + \mathbf{c}\left(\lambda_h^{n+1}, \mathbf{v}_h(\mathbf{X}_h^{n+1})\right) = 0 && \forall \mathbf{v}_h \in \mathbf{V}_h \\
&\left(\operatorname{div}\mathbf{u}_h^{n+1}, q_h\right)_\Omega = 0 && \forall q_h \in Q_h \\
&(\dot{\mathbf{X}}_h^{n+1}, \mathbf{w}_h)_{\mathcal{B}} = \left(\partial_{\Delta t}\mathbf{X}_h^{n+1}, \mathbf{w}_h\right)_{\mathcal{B}} && \forall \mathbf{w}_h \in \S_h \\
&\delta\rho\left(\partial_{\Delta t}\dot{\mathbf{X}}_h^{n+1}, \mathbf{z}_h\right)_{\mathcal{B}} + \left(\mathbb{P}(\mathbb{F}_h^{n+1}), \nabla_s \mathbf{z}_h\right)_{\mathcal{B}} - \mathbf{c}\left(\lambda_h^{n+1}, \mathbf{z}_h\right) = 0 && \forall \mathbf{z}_h \in S_h \\
&\mathbf{c}\left(\mu_h, \mathbf{u}_h^{n+1}(\mathbf{X}_h^{n+1}) - \partial_{\Delta t}\mathbf{X}_h^{n+1}\right) = 0 && \forall \mu_h \in \boldsymbol{\Lambda}_h \\
&\mathbf{u}_h^0 = \mathbf{u}_{0h}, \quad \mathbf{X}_h^0 = \mathbf{X}_{0h}.
\end{aligned}
$$

The other two schemes have the same structure with due modifications.

Stability estimates

We can show that BDF2 and CNm are stable.

Stability estimate for Crank-Nicolson CNm scheme

Let $\delta\rho \geq 0$ and assume that the energy density $W \in \mathcal{C}^1$ is convex. Then the following estimate holds true:

$$\frac{\rho_f}{2\Delta t}\left(\|\mathbf{u}_h^{n+1}\|_\Omega^2 - \|\mathbf{u}_h^n\|_\Omega^2\right) + \frac{\nu}{4}\|\nabla_{sym}\mathbf{u}_h^{n+1} + \nabla_{sym}\mathbf{u}_h^n\|_\Omega^2$$
$$+ \frac{\delta\rho}{2\Delta t}\left[\left\|\frac{\mathbf{X}_h^{n+1} - \mathbf{X}_h^n}{\Delta t}\right\|_{\mathcal{B}}^2 - \left\|\frac{\mathbf{X}_h^n - \mathbf{X}_h^{n-1}}{\Delta t}\right\|_{\mathcal{B}}^2\right]$$
$$+ \frac{E(\mathbf{X}_h^{n+1}) - E(\mathbf{X}_h^n)}{\Delta t} \leq 0$$

The stability analysis for CNt is not straigtforward (not even for Navier-Stokes equations).

Matrix form

The fully discrete problem requires at each time step the solution of a big linear system

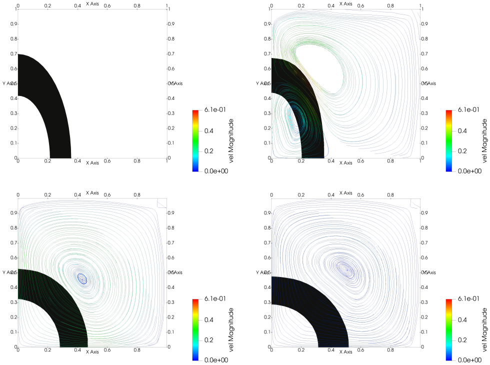
$$\begin{pmatrix} A(u_h^{n+1}) & -B^T & 0 & 0 & C_f(\overline{\mathbf{X}}_h)^T \\ -B & 0 & 0 & 0 & 0 \\ 0 & 0 & M_s & -\frac{3}{2\Delta t} M_s & 0 \\ 0 & 0 & \frac{3\delta_\rho}{2\Delta t} M_s & A_s & -C_s^T \\ C_f(\overline{\mathbf{X}}_h) & 0 & 0 & -\frac{3}{2\Delta t} C_s & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_h^{n+1} \\ p_h^{n+1} \\ \dot{\mathbf{X}}_h^{n+1} \\ \mathbf{X}_h^{n+1} \\ \lambda_h^{n+1} \end{pmatrix} = \begin{pmatrix} g_1 \\ 0 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix}$$

where $\overline{\mathbf{X}}_h$ represents an extrapolated value for $\mathbf{X}_h^{n+1}$.

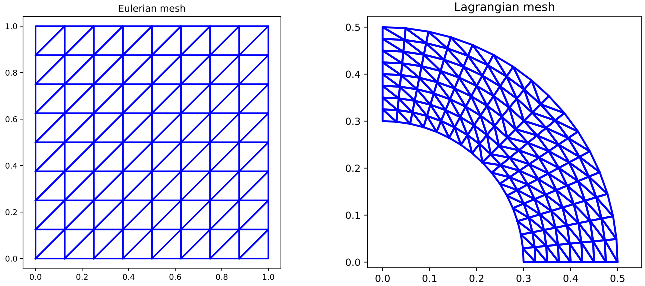
Deformed annulus

Material properties: $\mathbb{P}(\mathbb{F}) = \kappa\mathbb{F}$ with $\kappa = 10$, $\nu = 0.1$, $\rho_f = \rho_s = 1$.
The BDF2 method was used with $\Delta t = 0.05$, $T = 1$.
The snapshots were taken at $t = 0$, $t = 0.1$, $t = 0.5$ and $t = 1$.

$$\mathbf{u}(x,0) = 0, \mathbf{X}(s,0) = \begin{pmatrix} \frac{1}{1.4}s_1 \\ 1.4s_2 \end{pmatrix}.$$



Numerical results



Meshes for the fluid and the structure

Material coefficients: $\rho_f = \rho_s = 1$, $\nu = 1$, $\kappa = 10$.
The time interval considered is $[0, 0.2]$.

	DOFs $\mathbf{u}_h$	DOFs p_h	DOFs $\mathbf{X}_h$	DOFs λ_h
coarse mesh (M $=$ 8)	578	209	306	306
fine mesh (M $=$ 16)	2178	801	1122	1122

Convergence results for the fully implicit scheme

Velocity

	BDF1		BDF2		CNm		CNt	
Δt	L^2 error	rate	L^2 error	rate	L^2 error	rate	L^2 error	rate
0.05	$9.05 \cdot 10^{-2}$		$3.62 \cdot 10^{-2}$		$2.28 \cdot 10^{-1}$		$2.26 \cdot 10^{-1}$	
0.025	$4.87 \cdot 10^{-2}$	0.89	$5.05 \cdot 10^{-3}$	2.84	$6.23 \cdot 10^{-2}$	1.87	$6.04 \cdot 10^{-2}$	1.91
0.0125	$2.54 \cdot 10^{-2}$	0.94	$1.20 \cdot 10^{-3}$	2.07	$2.28 \cdot 10^{-2}$	1.45	$2.07 \cdot 10^{-2}$	1.54
0.00625	$1.29 \cdot 10^{-2}$	0.98	$3.53 \cdot 10^{-4}$	1.77	$5.27 \cdot 10^{-3}$	2.11	$4.03 \cdot 10^{-3}$	2.36

Displacement

	BDF1		BDF2		CNm		CNt	
Δt	L^2 error	rate	L^2 error	rate	L^2 error	rate	L^2 error	rate
0.05	$1.98 \cdot 10^{-3}$		$5.19 \cdot 10^{-4}$		$1.65 \cdot 10^{-3}$		$4.04 \cdot 10^{-4}$	
0.025	$1.05 \cdot 10^{-3}$	0.92	$9.79 \cdot 10^{-5}$	2.41	$9.27 \cdot 10^{-4}$	0.84	$8.48 \cdot 10^{-5}$	2.25
0.0125	$5.31 \cdot 10^{-4}$	0.99	$3.13 \cdot 10^{-5}$	1.64	$4.90 \cdot 10^{-4}$	0.92	$2.47 \cdot 10^{-5}$	1.78
0.00625	$2.70 \cdot 10^{-4}$	0.98	$1.35 \cdot 10^{-5}$	1.22	$2.50 \cdot 10^{-4}$	0.97	$3.47 \cdot 10^{-6}$	2.83

Number of iterates of the nonlinear solver

Δt	BDF1	BDF2	CNm	CNt
0.05	10	5	6	6
0.025	6	5	5	4
0.0125	6	4	4	4
0.00625	4	4	3	3

Convergence results for the semi-implicit scheme

Velocity

Δt	BDF1 L^2 error	BDF1 rate	BDF2 L^2 error	BDF2 rate	CNm L^2 error	CNm rate	CNt L^2 error	CNt rate
0.05	$9.18 \cdot 10^{-2}$		$3.89 \cdot 10^{-2}$		$2.36 \cdot 10^{-1}$		$2.39 \cdot 10^{-1}$	
0.025	$5.05 \cdot 10^{-2}$	0.86	$8.59 \cdot 10^{-3}$	2.18	$7.54 \cdot 10^{-2}$	1.64	$7.06 \cdot 10^{-2}$	1.76
0.0125	$2.63 \cdot 10^{-2}$	0.94	$3.32 \cdot 10^{-3}$	1.37	$4.24 \cdot 10^{-2}$	0.83	$2.22 \cdot 10^{-2}$	1.67
0.00625	$1.33 \cdot 10^{-2}$	0.98	$1.40 \cdot 10^{-3}$	1.24	$2.19 \cdot 10^{-2}$	0.96	$4.19 \cdot 10^{-3}$	2.40

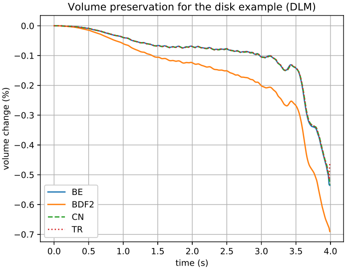
Displacement

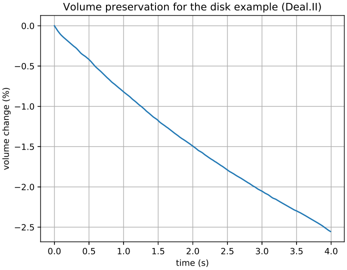
Δt	BDF1 L^2 error	BDF1 rate	BDF2 L^2 error	BDF2 rate	CNm L^2 error	CNm rate	CNt L^2 error	CNt rate
0.05	$2.03 \cdot 10^{-3}$		$7.86 \cdot 10^{-4}$		$1.81 \cdot 10^{-3}$		$6.51 \cdot 10^{-4}$	
0.025	$1.06 \cdot 10^{-3}$	0.93	$3.28 \cdot 10^{-4}$	1.26	$9.75 \cdot 10^{-4}$	0.89	$1.31 \cdot 10^{-4}$	2.31
0.0125	$5.34 \cdot 10^{-4}$	1.00	$1.44 \cdot 10^{-4}$	1.18	$5.10 \cdot 10^{-4}$	0.93	$4.82 \cdot 10^{-5}$	1.44
0.00625	$2.69 \cdot 10^{-4}$	0.99	$6.31 \cdot 10^{-5}$	1.19	$2.55 \cdot 10^{-4}$	1.00	$1.29 \cdot 10^{-5}$	1.90

Volume conservation of the floating disk

A circular disk is placed in a lid-driven cavity.

- $\Omega = (0,1)^2$, disk with diameter of 0.2 initially placed at $(0.6, 0.5)$
- $\rho_f = \rho_s = 1$, $\nu = 0.01$ and $\mathbb{P}(\mathbb{F}) = \kappa\mathbb{F}$ with $\kappa = 0.1$.
- 18818 DOFs for $\mathbf{u}$, 7009 DOFs for p, 4402 DoFs for $\mathbf{X}$ and λ
- $h_f = 0.029$, $h_s = 0.012$, $\Delta t = 0.01$.





Splitting schemes

Thin solid

<Annese-Fernández-G. In preparation>

In this section, we use the stabilized $P1 - P1$ elements for the Stokes equations by adding the Brezzi-Pitkaranta stability term

$$s_h(p,q) = \gamma \sum_{K \in \mathcal{T}_h} h_K^2 (\nabla p, \nabla q).$$

$\mathbf{d}$ is the displacement, so that $\mathbf{X} = \mathbf{X}_0 + \mathbf{d}$, $\dot{\mathbf{d}} = \partial \mathbf{X} / \partial t$

We separate the contribution of the inertial forces, due to the acceleration of the solid mass, and elastic forces, due to the solid deformation.
The *explicit coupling* of the fluid equations with the solid elastic forces, is realized by introducing an extrapolation of the displacement, as follows

$$\mathbf{d}_h^{n*} = \begin{cases} 0 & \text{if } r = 0 \\ \mathbf{d}_h^{n-1} & \text{if } r = 1 \\ \mathbf{d}_h^{n-1} + \tau \dot{\mathbf{d}}_h^{n-1} & \text{if } r = 2. \end{cases}$$

Partitioned scheme

Step 1: find $\mathbf{u}_h^n \in \mathbf{V}_h$, $p_h^n \in Q_h$, $\dot{\mathbf{d}}_h^{n-\frac{1}{2}} \in S_h$, $\boldsymbol{\lambda}_h^n \in \boldsymbol{\Lambda}_h$ such that

$$\rho_f \left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\Delta t}, \mathbf{v} \right) + b(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}) + a(\mathbf{u}_h^n, \mathbf{v})$$

$$- (\operatorname{div} \mathbf{v}, p_h^n) + \mathbf{c}(\boldsymbol{\lambda}_h^n, \mathbf{v}(\mathbf{X}_h^{n-1})) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_h$$

$$(\operatorname{div} \mathbf{u}_h^n, q) + s_h(p_h^n, q) = 0 \qquad \forall q \in Q_h$$

$$\frac{\rho_s}{\Delta t}(\dot{\mathbf{d}}_h^{n-\frac{1}{2}} - \dot{\mathbf{d}}_h^{n-1}), \mathbf{z})_{\mathcal{B}} - \mathbf{c}(\boldsymbol{\lambda}_h^n, \mathbf{z}) = -a_s(\mathbf{d}_h^{n*}, \mathbf{z}) \qquad \forall \mathbf{z} \in S_h$$

$$\mathbf{c}(\boldsymbol{\mu}, \mathbf{u}_h^n(\mathbf{X}_h^{n-1}) - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}) = 0 \qquad \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}_h$$

Step 2: find $\mathbf{d}_h^n \in S_h$, $\dot{\mathbf{d}}_h^n \in S_h$ such that

$$\frac{\rho_s}{\Delta t}(\dot{\mathbf{d}}_h^n - \dot{\mathbf{d}}_h^{n-\frac{1}{2}}, \mathbf{z})_{\mathcal{B}} + a_s(\mathbf{d}_h^n - \mathbf{d}_h^{n*}, \mathbf{z}) = 0 \qquad \forall \mathbf{z} \in S_h$$

$$\frac{\mathbf{d}_h^n - \mathbf{d}_h^{n-1}}{\Delta t} = \dot{\mathbf{d}}_h^n$$

Step 3: update the structure position $\mathbf{X}_h^n$

$$\mathbf{X}_h^n = \mathbf{X}_{0,h} + \mathbf{d}_h^n$$

Energy estimates

- Scheme with $r = 1$, $\mathbf{d}_h^{n*} = \mathbf{d}_h^{n-1}$

$$\rho_f \|\mathbf{u}_h^n\|^2_{0,\Omega} + \rho_s \|\dot{\mathbf{d}}_h^n\|^2_{0,\mathcal{B}} + \|\mathbf{d}_h^n\|^2_{1,\mathcal{B}} \le \rho_f \|\mathbf{u}_{0,h}\|^2_{0,\Omega} + \rho_s \|\mathbf{d}_{1,h}\|^2_{0,\mathcal{B}} + \|\mathbf{d}_{0,h}\|^2_{1,\mathcal{B}} + \Delta t^2 \|\mathbf{d}_{1,h}\|^2_{1,\mathcal{B}} + \frac{\Delta t}{2\rho_s} \|\mathbf{L}_h \mathbf{d}_{0,h}\|^2_{0,\mathcal{B}};$$

- Scheme with $r = 2$, $\mathbf{d}_h^{n*} = \mathbf{d}_h^{n-1} + \Delta t \dot{\mathbf{d}}_h^{n-1}$ let Δt and h_s be such that there exist $\alpha > 0$ such that

$$2 \frac{\Delta t^4 C_I^4}{(\rho_s)^2 h_s^4} \le 1,$$

then for $n \ge 1$

$$\rho_f \|\mathbf{u}_h^n\|^2_{0,\Omega} + \rho_s \|\dot{\mathbf{d}}_h^n\|^2_{0,\mathcal{B}} + \|\mathbf{d}_h^n\|^2_{1,\mathcal{B}} \le \exp\left(\frac{2\gamma t_n}{1 - 2\Delta t \gamma}\right) \left(\rho_f \|\mathbf{u}_{0,h}\|^2_{0,\Omega} + \rho_s \|\mathbf{d}_{1,h}\|^2_{0,\mathcal{B}} + \|\mathbf{d}_{0,h}\|^2_{1,\mathcal{B}}\right)$$

Space time error estimates for negligible displacements

Theorem

Regularity assumptions

$$\mathbf{u}(t) \in H^{1+l}(\Omega), \quad p(t) \in H^{l}(\Omega),$$
$$\mathbf{X}(t) \in H^{1+m}(\mathcal{B}), \quad \boldsymbol{\lambda}(t) \in H^{-1/2+l}(\mathcal{B})$$

Then

$$\frac{\rho_f}{2}\|\mathbf{u}(t^n) - \mathbf{u}_h^n\|_{0,\Omega}^2 + \frac{1}{2}\|\mathbf{X}(t^n) - \mathbf{X}_h^n\|_{1,\mathcal{B}}^2 + \frac{\delta_\rho}{2}\left\|\dot{\mathbf{d}}(t^n) - \dot{\mathbf{d}}_h^n\right\|_{0,\mathcal{B}}^2$$
$$\leq C\left(h_f^{2l} + h_s^{2m} + h_s^{2l} + \Delta t^2\right)$$

Convergence results for the partitioned schemes

Partitioned algorithm - $r = 1$ - Space convergence for $\Delta t = 0.01$

	Fluid velocity		Solid velocity		Displacement	
$h_f = h_s$	L^2 error	rate	L^2 error	rate	L^2 error	rate
1/8	$7.61\,10^{-3}$	-	$5.17\,10^{-4}$	-	$2.99\,10^{-2}$	-
1/16	$5.91\,10^{-3}$	0.37	$4.15\,10^{-4}$	0.32	$1.57\,10^{-2}$	0.93
1/32	$2.28\,10^{-3}$	1.38	$2.19\,10^{-4}$	0.92	$8.28\,10^{-3}$	0.93
1/64	$8.53\,10^{-4}$	1.42	$1.05\,10^{-4}$	1.05	$4.69\,10^{-3}$	0.82
1/128	$2.91\,10^{-4}$	1.55	$5.91\,10^{-5}$	0.83	$2.82\,10^{-3}$	0.73

Partitioned algorithm - $r = 2$ - Space convergence for $\Delta t = 0.01$

	Fluid velocity		Solid velocity		Displacement	
$h_f = h_s$	L^2 error	rate	L^2 error	rate	L^2 error	rate
1/8	$7.60\,10^{-3}$	-	$5.15\,10^{-4}$	-	$2.99\,10^{-2}$	-
1/16	$5.91\,10^{-3}$	0.36	$4.16\,10^{-4}$	0.31	$1.57\,10^{-2}$	0.93
1/32	$2.28\,10^{-3}$	1.38	$2.19\,10^{-4}$	0.93	$8.28\,10^{-3}$	0.93
1/64	$8.53\,10^{-4}$	1.42	$1.06\,10^{-4}$	1.05	$4.69\,10^{-3}$	0.82
1/128	$2.93\,10^{-4}$	1.54	$5.89\,10^{-5}$	0.84	$2.82\,10^{-3}$	0.73

Convergence results for the partitioned schemes (cont'd)

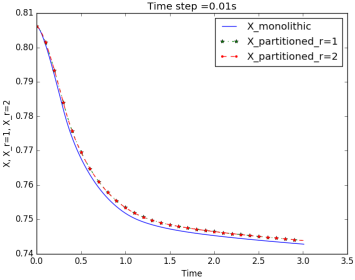
Partitioned algorithm - $r = 1$ - Time convergence for $h_f = h_s = 1/64$

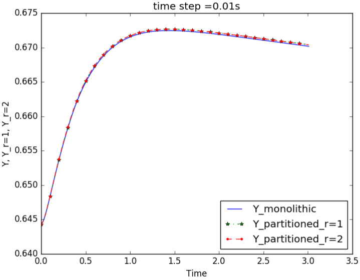
	Fluid velocity		Solid velocity		Displacement	
$h_f = h_s$	L^2 error	rate	L^2 error	rate	L^2 error	rate
$1/16$	$2.40\,10^{-4}$	-	$1.60\,10^{-4}$	-	$1.81\,10^{-3}$	-
$1/32$	$9.90\,10^{-5}$	1.28	$4.36\,10^{-5}$	1.87	$1.08\,10^{-3}$	0.75
$1/64$	$3.08\,10^{-5}$	1.69	$1.29\,10^{-5}$	1.75	$4.37\,10^{-4}$	1.30
$1/128$	$6.86\,10^{-6}$	2.17	$3.63\,10^{-6}$	1.84	$1.05\,10^{-4}$	2.06
$1/256$	$1.57\,10^{-6}$	2.12	$1.11\,10^{-6}$	1.71	$3.33\,10^{-5}$	1.65

Partitioned algorithm - $r = 2$ - Time convergence for $h_f = h_s = 1/64$

	Fluid velocity		Solid velocity		Displacement	
$h_f = h_s$	L^2 error	rate	L^2 error	rate	L^2 error	rate
$1/16$	$2.21\,10^{-4}$	-	$8.32\,10^{-5}$	-	$1.20\,10^{-3}$	-
$1/32$	$6.34\,10^{-5}$	1.81	$6.06\,10^{-5}$	0.46	$6.03\,10^{-4}$	0.98
$1/64$	$4.64\,10^{-6}$	3.77	$6.04\,10^{-6}$	3.33	$1.26\,10^{-4}$	2.25
$1/128$	$6.39\,10^{-7}$	2.86	$1.40\,10^{-6}$	2.11	$5.50\,10^{-5}$	1.20
$1/256$	$3.17\,10^{-7}$	1.01	$6.83\,10^{-7}$	1.03	$2.73\,10^{-5}$	1.01

Partitioned versus monolithic scheme





Conclusions

- The use of the fictitious domain method with Lagrange multiplier can be successfully extended to FSI problems
- The semi-implicit scheme is unconditionally stable in time
- Analysis of stationary problem provides optimal error estimates
- Error estimates in space and time are provided for a simplified situation
- Unconditional stability of high order time advancing schemes and of time splitting schemes has been proved
- Extensions to compressible solids are also available

THANK YOU