Weak Galerkin Finite Element Methods for Elliptic and Parabolic Problems on Polygonal Meshes

MWNDEA 2020

Naresh Kumar

Department of Mathematics
Indian Institute of Technology Guwahati,
Guwahati, India

Outline

- Weak Galerkin Finite Element Methods.
  - Motivation
  - Implementation
- WG-FEM for Model PDEs.
  - Second Order Elliptic Problems.
  - Second Order Parabolic Problems.
- Numerical Results.

Joint Work With: Prof. Bhupen Deka.

Basic Notation

- We denote by $H^m(J;\mathcal{B})$, $1 \leq m < \infty$, the space of all measurable functions $\phi : J \to \mathcal{B}$ for which

$$\|u\|_{H^m(J;\mathcal{B})} = \left(\sum_{j=0}^{m} \int_0^T \left\| \frac{\partial^j u(t)}{\partial t^j} \right\|_{\mathcal{B}}^2 dt \right)^{\frac{1}{2}} < \infty.$$

- The minimal regularity space

$$X = L^2(0, T; H_0^{m+1}(\Omega)) \cap H^1(0, T; H^{m-1}(\Omega)),$$

equipped with the norm

$$\|v\|_X^2 = \|v\|^2_{L^2(0,T;H^{m+1}(\Omega))} + \|\partial_t v\|^2_{L^2(0,T;H^{m-1}(\Omega))}.$$

- We will use $\|\cdot\|$ for L^2-norm.

Second Order Elliptic Problems

Find $u \in H_0^1(\Omega)$ such that

$$(a\nabla u, \nabla v) = (f, v) \;\; \forall v \in H_0^1(\Omega).$$

Procedures in the standard Galerkin finite element method:

- Partition Ω into triangles or tetrahedra.
- Construct a subspace, denoted by $S_h \subset H_0^1(\Omega)$, using piecewise polynomials.
- Seek for a finite element solution u_h from S_h such that

$$(a\nabla u_h, \nabla v) = (f, v) \;\; \forall v \in S_h.$$

The classical gradient ∇u for $u \in C^1(\Omega)$ can be computed as:

$$\int_K \nabla u.\phi = -\int_K u\nabla.\phi + \int_{\partial K} u(\phi.n) \;\; \forall \phi \in [C^1(\Omega)]^2$$

Thus, u can be extended to $\{u_0, u_b\}$ with ∇u being extended to $\nabla_w u$.

Motivation: Weak Function

- Let K be any polygonal or polyhedral domain with interior K^0 and boundary ∂K.
- A weak function on the region K refers to a pair of scalar-valued functions $v = \{v_0, v_b\}$ such that $v_0 \in L^2(K)$ and $v_b \in H^{\frac{1}{2}}(\partial K)$.
- Denote by $\mathcal{V}(K)$ the space of weak functions on K; *i. e.*,

$$\mathcal{V}(K) = \{v = \{v_0, v_b\} : v_0 \in L^2(K), v_b \in H^{\frac{1}{2}}(\partial K)\}.$$

- For any weak function $v = \{v_0, v_b\}$, its weak gradient $\nabla_w v$ is defined (interpreted) as a linear functional on $H(\text{div}, K)$ whose action on each $q \in H(\text{div}, K)$ is given by

$$\int_K \nabla_w v . q dK = -\int_K v_0 \nabla \cdot q dK + \int_{\partial K} v_b q \cdot \mathbf{n} ds, \tag{1}$$

where $\mathbf{n}$ is the outward normal to ∂K.

Inclusion Result

- The Sobolev space $H^1(K)$ can be embedded into the space $\mathcal{V}(K)$ by an inclusion map $i_{\mathcal{V}} : H^1(K) \mapsto \mathcal{V}(K)$ defined as follows

$$i_{\mathcal{V}}(\phi) = \{\phi|_K, \phi|_{\partial K}\}, \ \ \phi \in H^1(K).$$

- With the help of the inclusion map $i_{\mathcal{V}}$, the Sobolev space $H^1(K)$ can be viewed as a subspace of $\mathcal{V}(K)$ by identifying each $\phi \in H^1(K)$ with $i_{\mathcal{V}}(\phi)$.
- Analogously, a weak function $v = \{v_0, v_b\} \in \mathcal{V}(K)$ is said to be in $H^1(K)$ if it can be identified with a function $\phi \in H^1(K)$ through the above inclusion map.
- For $u \in H^1(K)$, we have

$$i_{\mathcal{V}}(u) = \{u|_K, u|_{\partial K}\}.$$

It is not hard to see that the weak gradient is identical with the strong/classical gradient.

Discrete Weak Gradient

Discrete Weak Gradient Operator: A discrete weak gradient operator, denoted by $\nabla_{w,m}$, is defined as the unique polynomial $(\nabla_{w,m} v) \in [\mathcal{P}_m(K)]^2$ that satisfies the following equation

$$\int_K \nabla_{w,m} v.\phi dK = -\int_K v_0(\nabla \cdot \phi) dK + \int_{\partial K} v_b(\phi \cdot \mathbf{n}) ds \ \ \forall \phi \in [\mathcal{P}_m(K)]^2, \qquad (2)$$

where $v = \{v_0, v_b\}$ such that $v_0 \in L^2(K)$ and $v_b \in H^{\frac{1}{2}}(\partial K)$.

Weak Galerkin Space

Weak Galerkin space is defined as: $(\mathcal{P}_k(K),\ \mathcal{P}_j(\partial K),\ \left[\mathcal{P}_l(K)\right]^2)$

- $k \geq 1$ is the degree of polynomials in the interior of the element K,
- $j \geq 0$ is the degree of polynomials on the boundary of K and
- $l \geq 0$ is the degree of polynomials employed in the computation of weak gradients or weak first order partial derivatives.
- $k,\ j,\ l$ are selected in such a way that minimize the number of unknowns in the numerical scheme without compromising the accuracy of the numerical approximation.

Lowest Order Weak Galerkin Space

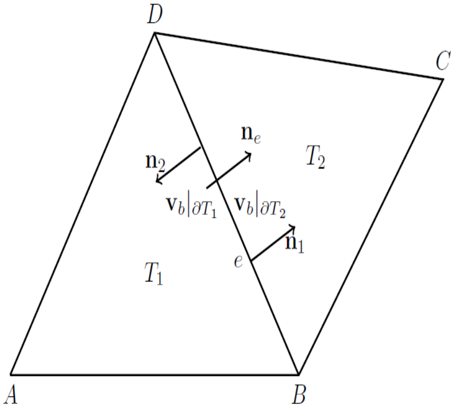
- A lowest order WG-FEM space is $(\mathcal{P}_1(K),\ \mathcal{P}_0(\partial K),\ \left[\mathcal{P}_0(K)\right]^2)$.

Weak Galerkin Approximation

We choose weak Galerkin space is $(\mathcal{P}_k(K),\ \mathcal{P}_k(\partial K),\ \left[\mathcal{P}_{k-1}(K)\right]^2)$.

For $k \geq 1$, let V_h be WG FE space associated with $\mathcal{T}_h$ & defined as:

$$V_h = \{v = \{v_0, v_b\} :\ v_0|_{K^0} \in \mathcal{P}_k(K),\ v_b|_e \in \mathcal{P}_k(e),\ e \in \partial K,\ K \in \mathcal{T}_h\}.$$



Weak Galerkin Approximation Cont...

- Note that functions in V_h are defined on each element, and there are two-sided values of v_b on each interior edge/face e, depicted as $v_b|_{\partial T_1}$ and $v_b|_{\partial T_2}$ in above Figure.

- We assume that v_b has unique value on each interior edge/face e that is

$$[v]_e = 0 \quad \forall e \in \mathcal{E}_h^0,$$

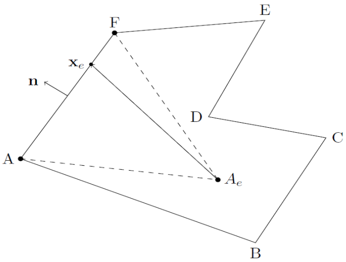
  $[v]_e$ denotes the jump of $v \in V_h$ across an interior edge $e \in \mathcal{E}_h^0$.

- We write $\quad V_h^0 = \{v = \{v_0, v_b\} \in V_h : \; v_b = 0 \; \text{ on } \partial\Omega\}.$

- For $v \in V_h$, the discrete weak gradient of it is defined as the unique polynomial $(\nabla_w v) \in [\mathcal{P}_{k-1}(K)]^2$ that satisfies the following equation

$$\int_K \nabla_{w,m} v.\phi dK = -\int_K v_0(\nabla \cdot \phi) dK + \int_{\partial K} v_b(\phi \cdot \mathbf{n}) ds \;\; \forall \phi \in [\mathcal{P}_{k-1}(K)]^2. \quad (3)$$

Shape Regularity for Polytopal Elements



Why Shape Regularity?
The shape regularity is needed for

1. trace inequality.
2. inverse inequality.
3. domain inverse inequality.

- (*A weak Galerkin mixed finite element method for second order elliptic problems*, Math. Comp., 83 (2014), 2101-2126, by J. Wang and X. Ye for more details)

Elliptic Boundary Value Problem

Consider following BVP

$$-\nabla \cdot \big(\nabla u\big) = f \quad \text{in } \Omega, \tag{4}$$

with boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{5}$$

Weak Formulation: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx. \tag{6}$$

Elliptic Boundary Value Problem Cont...

Weak Galerkin Approximation : Find $u_h = \{u_0, u_b\} \in V_h^0$ such that

$$a_s(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h^0, \tag{7}$$

with

$$a_s(u_h, v_h) = a(u_h, v_h) + s(u_h, v_h), \tag{8}$$

where

- $a(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ is a bilinear map given by

$$a(u_h, v_h) = (\nabla_w u_h, \nabla_w v_h) = \sum_{K \in \mathcal{T}_h} (\nabla_w u_h, \nabla_w v_h)_K, \tag{9}$$

- with a stabilizer $s(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ defined by

$$s(u_h, v_h) = \sum_{K \in \mathcal{T}_h} h_K^{-1} \langle u_0 - u_b, v_0 - v_b \rangle_{\partial K}. \tag{10}$$

- It is important to check that $a_s(\cdot,\cdot)$ is positive so that WG approximation (7) has a unique solution. In fact bilinear map $a_s(\cdot,\cdot)$ induces a norm.

Discrete Norms

- We consider following norm associated with the bilinear map $a_s(\cdot,\cdot)$

$$|||u_h||| = \sqrt{a_s(u_h, u_h)}. \tag{11}$$

- For simplicity, we shall only verify the positive length property for $|||\cdot|||$.
- Assume that $|||w||| = 0$ for some $w = \{w_0, w_b\} \in V_h^0$.
- It follows that $\nabla_w w = 0$ on each element $K \in \mathcal{T}_h$ and $w_0 = w_b$ on ∂K.
- Thus, we have from the definition of weak gradient that for any $\phi \in [\mathcal{P}_{k-1}(K)]^2$.

$$\begin{aligned} 0 &= (\nabla_w w, \phi)_K \\ &= -(w_0, \nabla\cdot\phi)_K + \langle w_b, \phi\cdot\mathbf{n}\rangle_{\partial K} \\ &= (\nabla w_0, \phi) + \langle w_b - w_0, \phi\cdot\mathbf{n}\rangle_{\partial K} \\ &= (\nabla w_0, \phi)_K. \end{aligned}$$

- Letting $\phi = \nabla w_0$ in the above equation yields $\nabla w_0 = 0$ on $K \in \mathcal{T}_h$.

Discrete Norms Contd.

- It follows that $w_0 =$ constant on every $K \in \mathcal{T}_h$.
- This, together with the fact that $w_b = w_0$ on ∂K and $w_b = 0$ on $\partial\Omega$, implies that $w_0 = 0$ and $w_b = 0$.
- We define following discrete H^1-norm

$$\|v\|_{1,h} = \Big(\sum_{K \in \mathcal{T}_h} (\|\nabla v_0\|_K^2 + h_K^{-1} \|v_0 - v_b\|_{\partial K}^2) \Big)^{\frac{1}{2}}, \; v = \{v_0, v_b\} \in V_h^0.$$

- The following lemma indicates that discrete H^1-norm is equivalent to triple bar norm (*c.f.* Lemma 5.3, A weak Galerkin finite element method with polynomial reduction, JCAM, 285 (2015) 4558)

Lemma

There exist two positive constants C_1 and C_2 such that

$$C_1 \|v\|_{1,h} \le |||v||| \le C_2 \|v\|_{2,h} \;\; \forall v_h \in V_h. \qquad \square$$

Some Useful Results

Trace Inequality. (see, Lemma A.3, *A weak Galerkin mixed finite element method for second order elliptic problems*, Math. Comp., 83 (2014), 2101-2126). Let K be an element with e as an edge. For any function $\varphi \in H^1(K)$, the following trace inequality holds true

$$\|\varphi\|^2_{L^2(e)} \leq C(h_K^{-1}\|\varphi\|^2_{L^2(K)} + h_K\|\nabla\varphi\|^2_{L^2(K)}).$$

Inverse Inequality. (see, Lemma A.6, Math. Comp., 83 (2014), 2101-2126). For any piecewise polynomial φ of degree p on $\mathcal{T}_h$, there exists constant $C = C(p)$ such that

$$\|\nabla\varphi\|_{L^2(K)} \leq C(p)h_K^{-1}\|\varphi\|_{L^2(K)}, \ \ \forall K \in \mathcal{T}_h.$$

Poincaré-type Inequality. (see, Lemma 7.1, *Weak Galerkin finite element methods on polytopal meshes*, IJNAM, 12 (2015), 31-53). Assume that the finite element partition $\mathcal{T}_h$ is shape regular. Then,there exists a constant C independent of the mesh size h such that

$$\|\varphi_0\| \leq C|||\varphi|||, \ \ \forall \varphi = \{\varphi_0, \varphi_b\} \in V_h^0.$$

Polynomial Approximation in Weak Galerkin Space

L^2-Projections.

- For each element $K \in \mathcal{T}_h$, denote by Q_0 the usual L^2 projection operator from $L^2(K)$ onto $\mathcal{P}_k(K)$ and by Q_b the L^2 projection from $L^2(e)$ onto $\mathcal{P}_k(e)$ for any $e \in \mathcal{E}_h$. Then
    - We shall combine Q_0 with Q_b by writing $Q_h = \{Q_0, Q_b\}$. More precisely, for $\phi \in H^1(K)$, we have $Q_h\phi = \{Q_0\phi, Q_b\phi\}$.
- In addition to Q_h, let $\mathbb{Q}_h$ be an another local L^2 projection from $[L^2(K)]^2$ onto $[\mathcal{P}_{k-1}(K)]^2$.

Approximation Results. (See, Lemma 4.1, *A weak Galerkin mixed finite element method for second order elliptic problems*, Math. Comp., 2014).

$$\begin{aligned} \|u - Q_0 u\|^2_{L^2(K)} + h_K^2 \|\nabla(u - Q_0 u)\|^2_{L^2(K)} &\leq Ch_K^{2(k+1)} \|u\|^2_{k+1,K}, \\ \|\nabla u - \mathbb{Q}_h(\nabla u)\|^2_{L^2(K)} + h_K^2 \|\nabla(\nabla u - \mathbb{Q}_h(\nabla u))\|^2_{L^2(K)} &\leq Ch^{2k} \|u\|^2_{k+1,K}. \quad \square \end{aligned}$$

Error Analysis

- As a traditional way, we split our error into two components using an intermediate operator. We write

$$u - u_h = (u - Q_h u) + (Q_h u - u_h).$$

- For simplicity, we introduce the following notation

$$e_h := \{e_0, e_b\} = u_h - Q_h u. \tag{12}$$

Convergence Result for H^1: Let $u_h \in V_h$ be the weak Galerkin finite element solution of the problem (7) .Assume that the exact solution is so regular that $u \in H^{k+1}(\Omega)$.Then, there exist a constant C such that

$$|||e_h||| \leq Ch^k \|u\|_{k+1,\Omega}. \qquad \square$$

(See L. Mu, J. Wang, and X. Ye, *Weak Galerkin Finite Element Methods On Polytopal Meshes* Int. Jour. Numer. Anal. Model., 12(2015) 31-54.)

Error Analysis Contd.

Now, for L^2 norm error estimate, duality argument leads to following convergence result.

- Convergence Results for L^2-norm: Let $u_h \in V_h$ be the weak Galerkin finite element solution of the problem (7). Assume that the exact solution is so regular that $u \in H^{k+1}(\Omega)$. Then, there exist a constant C such that

$$\|e_0\| \leq Ch^{k+1}\|u\|_{k+1,\Omega}. \quad \square$$

(See L. Mu, J. Wang, and X. Ye, *Weak Galerkin Finite Element Methods On Polytopal Meshes* Int. Jour. Numer. Anal. Model., 12(2015) 31-54.)

Weak Galerkin Method for Parabolic Problems

Consider the following second order parabolic problem

$$u_t - \nabla \cdot (\mathcal{A}\nabla u) \;=\; f \text{ in } \Omega,\; t \in J, \tag{13}$$

$$u \;=\; 0 \text{ on } \partial\Omega,\; t \in J, \tag{14}$$

$$u(\cdot, 0) \;=\; \psi \text{ in } \Omega. \tag{15}$$

Where Ω is a polygonal domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, $J = (0, T],\ T < \infty$ and $\mathcal{A}$ is a symmetric positive definite matrix.

- Weak Formulation: Find $u(\cdot, t) \in H_0^1(\Omega)$ such that

$$(u_t, v) + (\mathcal{A}\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega),\ t \in J,$$
$$u(\cdot, 0) = \psi.$$

- Semi-discrete weak Galerkin finite element approximation: Find $u_h(t) = \{u_0(\cdot, t), u_b(\cdot, t)\} \in V_h^0$ such that

$$(u_{ht}, v_0) + a_s(u_h, v) = (f, v_0) \ \ \forall v = \{v_0, v_b\} \in V_h^0, \quad t > 0, \tag{16}$$

- with

$$u_h(\cdot, 0) = Q_h\psi \quad in \quad \Omega.$$

- Where $a_s(\cdot, \cdot)$ is same as in (8) with stabilizer term (10).

Semi-discrete error estimates for parabolic problems:

Theorem

Assume that $u \in H^{r+1}(\Omega)$. Then there exists a positive constant $C > 0$ independent of the mesh size h such that

$$\|e_h(\cdot,t)\|^2 \leq \|e_h(\cdot,0)\|^2 + Ch^{2r}\int_0^t \|u\|_{r+1}^2 ds,$$

and

$$\begin{aligned} |\!|\!| e_h(\cdot,t) |\!|\!|^2 \leq \; & \|e_h(\cdot,0)\|^2 + Ch^{2r}\Big(\|\psi\|_{r+1}^2 \\ & + \|u(\cdot,t)\|_{r+1}^2 + \int_0^t \|u\|_{r+1}^2 ds + \int_0^t \|u_t\|_{r+1}^2 ds\Big). \qquad \square \end{aligned}$$

(See Theorem (4.2), Hongoin Zhang et al, Weak Galerkin Finite Element Method For Second Order Parabolic Equations), International J. Numer. Anal. and Modeling 13 (2016), 525-544.

Fully Discrete Weak Galerkin approximation

- Let $k > 0$ be a time step-size. At the time level $t = t_n = nk$, with integer $0 \leq n \leq N; Nk = T$, and denote by $U^n = U_h^n \in V_h$ the approximation of $u(t_n)$ to be determined.
- Weak Formulation: Find $U^n = U_h^n \in V_h$ such that

$$(\bar{\partial} U^n, v_0) + a(U^n, v) = (f(t_n), v_0) \quad \forall\ v = \{v_0, v_b\} \in V_h^0,\ \ n \geq 1 \tag{17}$$

- with $\quad \bar{\partial} U^n = \frac{U^n - U^{n-1}}{k}$.

Fully -discrete error estimates:

Theorem

Assume that $u \in C^2([0,T]; H^{r+1}(\Omega))$. *Then there exists a positive constant* $C > 0$ *independent of the mesh size* h *such that for* $0 < n \leq N$

$$\|e^n\|^2 \leq \|e^0\|^2 + C\big(h^{2r}\|u\|_{r+1,\infty}^2 + k^2 \int_0^{t_n} \|u_{tt}\|^2 ds\big),$$

and

$$\begin{aligned} |||e^n|||^2 \quad &\leq \quad C\{\|e^0\|^2 + h^{2r}\Big(\|\psi\|_{r+1}^2 + \|u(\cdot,t)\|_{r+1,\infty}^2 + \|u_t\|_{r+1,\infty}^2 \\ &+k^2 \int_0^t \|u_{tt}\|_{r+1}^2 ds\Big) + k^2 \int_0^t \|u_{tt}\|^2 ds\}, \quad \square \end{aligned}$$

where $\|u\|_{r+1,\infty} = \max_{0 \leq t \leq T}\{\|u(t)\|_{r+1}\}$.

(See Theorem (4.2), H. Zhang et al, Weak Galerkin Finite Element Method For Second Order Parabolic Equations), International J. Numer. Anal. and Modeling 13 (2016), 525-544.

Weak Galerkin Method Under Minimal Regularity

Theorem

Let $u(x,t)$ and $u_h(x,t)$ be the solution to the problem (13)-(15) and the semi-discrete WG scheme (16) respectively.Assume that the exact solution $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$ and $u_t \in H_0^1(\Omega) \cap H^{k-1}(\Omega)$ then there exist a constant C such that

$$\|e\|_{L^2(0,T;L^2)} \leq Ch^{k+1}\|u\|_{L^2(0,T;H^{k+1})}$$

Theorem

Let u and U be the solution of (13)-(16) and (17) respectively.then for $u_0 \in H^2 \cap H_0^1(\Omega)$, $f \in H^1(0,T;L^2(\Omega))$, there exist a constant C independent of h and k such that

$$\|u - U\|_{L^2(0,T;L^2(\Omega))} \leq C(k + h^2)\{\|u\|_{L^2(0,T;H^2)} + \|u_t\|_{L^2(0,T;L^2)}\}$$

Joint Work With Dr Bhupen Deka

Numerical Result

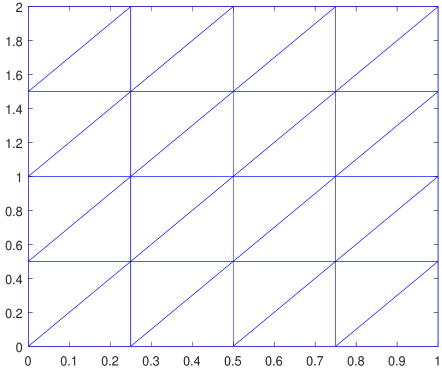


Figure: Triangulation

Numerical Result Cont...

Example

Let $\Omega = [0,1] \times [0,1]$, and the exact solution is

$$u = exp(-t)sin(\pi x)sin(\pi y)sin(\pi x + \pi y - t).$$

The right-hand sides f in (13) are determined from the choice for u and $\mathcal{A}(x) = \begin{bmatrix} 1 & xy \\ xy & x^2y^2 - 1 \end{bmatrix}$.

WG based on $\{P_1, P_1, P_0\}$ *space with* $k = 10^{-4}$

h	$\|\|e\|\|$	Order	$\|\|\|e\|\|\|$	Order
$1/4$	0.3432696306	–	1.411571357	–
$1/8$	0.02851317328	1.91	0.41909813963	0.87
$1/16$	0.007142629039	1.97	0.208296331455	1.00
$1/32$	0.000985569	1.98	0.0705478	1.01

Numerical Result Cont...

WG based on $\{P_2, P_2, P_1\}$ *space with* $k = 10^{-4}$

h	$\|e\|$	Order	$\|\|\|e\|\|\|$	Order
$1/4$	2.949236×10^{-2}	–	3.363017×10^{-1}	–
$1/8$	3.794609×10^{-3}	2.95	8.970083×10^{-2}	1.90
$1/16$	4.814101×10^{-4}	2.97	2.406542×10^{-2}	1.89
$1/32$	6.317516×10^{-5}	2.92	7.25737×10^{-3}	1.80

WG based on $\{P_3, P_2, P_2\}$ *space with* $k = 10^{-4}$

h	$\|e\|$	Order	$\|\|\|e\|\|\|$	Order
$1/4$	0.213523	–	1.11226	–
$1/8$	0.0131348	4.02	0.134914	3.04
$1/16$	0.000633517	4.37	0.0125416	3.42
$1/32$	3.60792×10^{-5}	4.13	0.00142561	3.13

Numerical Result Cont...

Example

Let $\Omega = [0,2] \times [0,2]$, and the exact solution is

$$u = \frac{200}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big(\frac{(1+(-1)^{m+1})(1-cos\frac{n\pi}{2})}{mn} \Big) sin(\frac{m\pi x}{2}) \\ sin(\frac{m\pi x}{2}) exp(-\pi^2(m^2+n^2)t/36)$$

with initial condition

$$u_0 = \begin{cases} 50 \;\; if \;\; y \leq 1 \\ 0 \;\; otherwise \end{cases}$$

The right-hand sides f in (13) are determined from the choice for u and $\mathcal{A}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Numerical Result Cont...

WG based on $\{P_1, P_1, P_0\}$ *space with* $k = 10^{-4}$

h	$\|e\|$	Order	$\|\|\|e\|\|\|$	Order
$1/4$	8.0587×10^{-3}	–	2.198428	–
$1/8$	2.00095×10^{-3}	2.00	1.152289	0.93
$1/16$	4.97292×10^{-4}	2.01	5.909163×10^{-1}	0.96
$1/32$	1.22191×10^{-4}	2.02	2.996170×10^{-1}	0.97

References

[1] H. Zhang, Y. Zou, Y. Xu, Q. Zhai, H. Yue, *A Weak Galerkin Finite Element Method for Second Order Parabolic Equations*, Jour. Numer. Analysis Modeling , 2016.

[2] J. Wang, X. Ye, *A Weak Galerkin Mixed Finite Element Method for Second Order Elliptic Problems*, Math. Comp., 2014.

[3] L.Mu, J. Wang, X. Ye, *A Weak Galerkin Finite Element Method on Polytopal Meshes*, Jour. Numer. Analysis Modeling , 2015.

[4] L.Mu, J. Wang, X. Ye, *A weak Galerkin finite element method with polynomial reduction*, Jour. Comp. Applied Math., 2015.

[5] N. Kumar, B. Deka, *A Weak Galerkin Finite Element Method for Second Order Parabolic Problems Under Minimal Regularity*, Under Preparations.

[6] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North Holland , 1978.

Thank you