Mixed Mimetic Spectral Elements for Atmospheric Simulation

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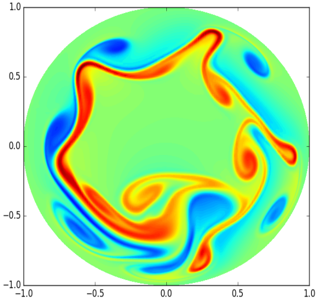


Motivation

Improved representation of dynamical processes and long term statistics

- Preservation of leading order balance relations
  - Geostrophic balance (horizontal), $\mathbf{f}\hat{\mathbf{z}} \times \mathbf{u} = -\nabla h$
  - Hydrostatic balance (vertical), $\theta \frac{\partial \Pi}{\partial z} = -g$
- Preservation of conservation laws
  - Mass
  - Vorticity
  - Energy (balanced exchanges)
  - Potential enstrophy (for exact integration only)

Structure preserving formulations

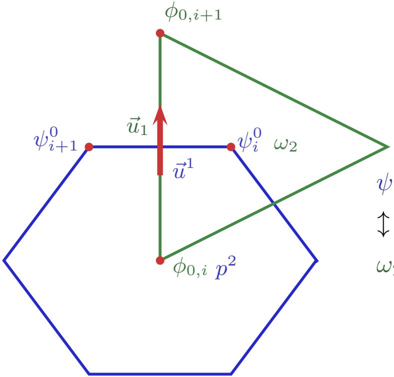
$$\Downarrow$$

Solving PDEs (for geophysical flows)

$$\Uparrow$$

Mimetic discretisations

Mimetic discretisations, a topological perspective



$$\psi^0 \rightarrow \nabla^\perp \rightarrow \vec{u}^1 \rightarrow \nabla\cdot \rightarrow p^2$$

$$\updownarrow\star \qquad \updownarrow\star \qquad \updownarrow\star$$

$$\omega_2 \leftarrow \nabla\times \leftarrow \vec{u}_1 \leftarrow \nabla \leftarrow \phi_0$$

$\nabla \times \nabla\phi = 0$ $\nabla \cdot \nabla^\perp\psi = 0$	$\mathrm{d}\mathrm{d}u = 0$
$\int \nabla \times \mathbf{u} da = \oint \mathbf{u} \cdot d\mathbf{l}$ $\int \nabla \cdot \mathbf{u} dv = \int \mathbf{u} \cdot da$	$\int_{\partial\Omega} u = \int_\Omega \mathrm{d}u$

Mixed mimetic spectral elements (1D)

- The standard (nodal) spectral element basis is given as $f^0(\xi) = a_i^0 l_i(\xi)$
- Introduce a secondary *edge* basis [Gerritsma, 2011] as $g^1(\xi) = b_i^1 e_i(\xi)$
- Integration of the edge functions between GLL nodes are orthogonal such that

$$\int_{\xi_i}^{\xi_{i+1}} e_j(\xi) = \delta_{i,j}$$

- Start from the premise that in 1D we wish to exactly satisfy the fundamental theorem of calculus between nodes ξ_i and ξ_{i+1}:

$$\int_{\xi_i}^{\xi_{i+1}} \frac{df^0(\xi)}{d\xi} = f^0(\xi_{i+1}) - f^0(\xi_i) = a_{i+1}^0 - a_i^0 = b_i^1 = \int_{\xi_i}^{\xi_{i+1}} g^1(\xi)$$

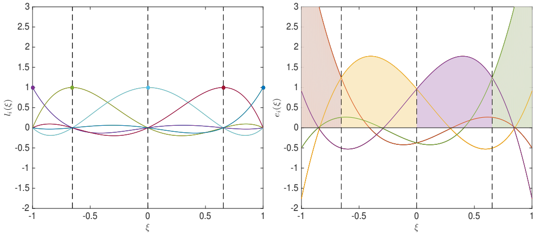


Figure: Nodal (left) and edge (right) functions for a 4^{th} order spectral element

Mixed mimetic spectral elements (2D)

Via tensor product combinations of $l_i(\xi)$ and $e_i(\xi)$ we define the function spaces:

- $\alpha^0_{i,j}(\boldsymbol{\xi}) = l_i(\xi) \otimes l_j(\eta) \in V^0$; C^0 continuous across elements
- $\boldsymbol{\beta}^1_{i,j}(\boldsymbol{\xi}) = \{l_i(\xi) \otimes e_j(\eta), e_i(\xi) \otimes l_j(\eta)\} \in V^1$; C^0 normal components
- $\gamma^2_{i.j}(\boldsymbol{\xi}) = e_i(\xi) \otimes e_j(\eta) \in V^2$ discontinuous across elements

And basis function expansions:

- $\psi^0_{i,j} = \hat{\psi}_{i,j}\alpha^0_{i,j}$
- $\mathbf{u}^1_{i,j} = \{\hat{u}_{i,j}\beta^{1,\xi}_{i,j}, \hat{v}_{i,j}\beta^{1,\eta}_{i,j}\}$
- $p^2_{i,j} = \hat{p}_{i,j}\gamma^1_{i,j}$

In the strong form:

$$\mathbf{u}^1 = \nabla^\perp \psi^0 \implies \mathbf{u}^1_h = \{(\hat{\psi}_{i,j-1} - \hat{\psi}_{i,j})\beta^{1,\xi}_{i,j}, (\hat{\psi}_{i,j} - \hat{\psi}_{i-1,j})\beta^{1,\eta}_{i,j}\} = \boldsymbol{\beta}^1_h \mathsf{E}^{1,0}\hat{\psi}^0_h$$

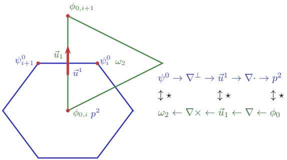
$$p^2 = \nabla \cdot \mathbf{u}^1 \implies p^2_h = (\hat{u}_{i,j} - \hat{u}_{i-1,j} + \hat{v}_{i,j} - \hat{v}_{i,j-1})\gamma^0_{i,j} = \gamma^2_h \mathsf{E}^{2,1}\hat{u}^1_h$$

Weak form adjoint relations:

$$\left\langle \alpha^0, \nabla \times \mathbf{u}^1 \right\rangle_\Omega = -\left\langle \nabla^\perp \alpha^0, \mathbf{u}^1 \right\rangle_\Omega \implies \left\langle \alpha^0_i, \alpha^0_j \right\rangle_{\Omega_a} \hat{\omega}^0_j = -(\mathsf{E}^{1,0}_{k,i})^\top \left\langle \boldsymbol{\beta}^1_k, \boldsymbol{\beta}^1_l \right\rangle_{\Omega_a} \hat{u}^1_l$$

$$\left\langle \beta^1, \nabla \phi^2 \right\rangle_\Omega = -\left\langle \nabla \cdot \beta^1, \phi^2 \right\rangle_\Omega \implies \left\langle \boldsymbol{\beta}^1_i, \boldsymbol{\beta}^1_j \right\rangle_{\Omega_a} \hat{u}^1_j = -(\mathsf{E}^{2,1}_{k,i})^\top \left\langle \gamma^2_k, \gamma^2_l \right\rangle_{\Omega_a} \hat{\phi}^2_l$$

2D Hydrodynamics in $H(div)$



Strong form (pointwise)	Weak form (Galerkin proj.)
$\nabla^\perp,\ \mathsf{E}^{1,0}_{i,j}:\ V^0 \ \to\ V^1$	$\nabla\times,\ -(\mathsf{E}^{1,0}_{j,i})^\top:\ V^1 \ \to\ V^0$
$\mathbf{u}^1 \ = \ \nabla^\perp\psi^0$	$\left\langle\alpha^0, \nabla\times\mathbf{u}^1\right\rangle_\Omega \ = \ -\left\langle\nabla^\perp\alpha^0, u^1\right\rangle_\Omega$
$\mathbf{u}^1_i \ = \ \mathsf{E}^{1,0}_{i,j}\psi^0_j$	$\left\langle\alpha^0_i, \alpha^0_j\right\rangle_{\Omega_a}\hat{\omega}^0_j \ = \ -(\mathsf{E}^{1,0}_{k,i})^\top\left\langle\boldsymbol{\beta}^1_k, \boldsymbol{\beta}^1_l\right\rangle_{\Omega_a}\hat{u}^1_l$
$\nabla\cdot,\ \mathsf{E}^{2,1}:\ V^1 \ \to\ V^2$	$\nabla,\ -(\mathsf{E}^{2,1})^\top:\ V^2 \ \to\ V^1$
$p^2 \ = \ \nabla\cdot\mathbf{u}^1$	$\left\langle\boldsymbol{\beta}^1, \nabla\phi^2\right\rangle_\Omega \ = \ -\left\langle\nabla\cdot\boldsymbol{\beta}^1, \mathbf{u}^1\right\rangle_\Omega$
$p^2_i \ = \ \mathsf{E}^{2,1}_{i,j}\mathbf{u}^1_j$	$\left\langle\boldsymbol{\beta}^1_i, \boldsymbol{\beta}^1_j\right\rangle_{\Omega_a}\hat{u}^1_j \ = \ -(\mathsf{E}^{2,1}_{k,i})^\top\left\langle\gamma^2_k, \gamma^2_l\right\rangle_{\Omega_a}\hat{\phi}^2_l$
$\nabla\cdot\nabla^\perp \ := \ 0,\ \mathsf{E}^{2,1}_{i,j}\mathsf{E}^{1,0}_{j,k} \ = \ 0$	$\nabla\times\nabla \ := \ 0,\ (\mathsf{E}^{1,0}_{j,i})^\top(\mathsf{E}^{2,1}_{k,i})^\top \ = \ 0$

- Let $\mathcal{H}$ be an invariant of the PDE with dependent variables $\mathbf{a}$
- Write the PDE as
$$\frac{\partial \mathbf{a}}{\partial t} = \mathbf{S}\frac{\delta \mathcal{H}}{\delta \mathbf{a}}$$
where $\mathbf{S}$ is a skew-symmetric operator
- $\mathcal{H}$ is conserved as
$$\frac{\partial \mathcal{H}}{\partial t} = \frac{\delta \mathcal{H}}{\delta \mathbf{a}} \cdot \frac{\partial \mathbf{a}}{\partial t} = \frac{\delta \mathcal{H}}{\delta \mathbf{a}} \cdot \mathbf{S}\frac{\delta \mathcal{H}}{\delta \mathbf{a}} = \mathbf{0}$$
- Corresponds to the anti-symmetry of the Poisson bracket

$$\langle A, \mathbf{S}B \rangle = \{A, B\} = -\{B, A\} \Rightarrow \{\mathcal{H}, \mathcal{H}\} = -\{\mathcal{H}, \mathcal{H}\} = 0$$

Example: Rotating Shallow Water Equations (continuous form)

$$\mathcal{H} = \int \frac{1}{2} h\mathbf{u}\cdot\mathbf{u} + \frac{g}{2} h^2 \mathrm{d}\Omega$$

$$\frac{\delta\mathcal{H}}{\delta\mathbf{u}} = h\mathbf{u} = \mathbf{F}$$

$$\frac{\delta\mathcal{H}}{\delta h} = \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + gh = \Phi$$

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{u}\\ h\end{bmatrix} = -\begin{bmatrix}(\omega+f)/h\times & \nabla\\ \nabla\cdot & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{F}\\ \Phi\end{bmatrix}$$

$$\frac{\partial\mathcal{H}}{\partial t} = [\mathbf{F},\Phi]\cdot\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{u}\\ h\end{bmatrix} = -[\mathbf{F},\Phi]\begin{bmatrix}(\omega+f)/h\times & \nabla\\ \nabla\cdot & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{F}\\ \Phi\end{bmatrix} = 0$$

Semi-discrete:

$$\frac{1}{\Delta t}\begin{bmatrix}\mathbf{u}^{n+1}-\mathbf{u}^{n}\\ h^{n+1}-h^{n}\end{bmatrix} = -\begin{bmatrix}(\omega+f)/h\times & \nabla\\ \nabla\cdot & \mathbf{0}\end{bmatrix}\begin{bmatrix}\bar{\mathbf{F}}\\ \bar{\Phi}\end{bmatrix}$$

$$\bar{\mathbf{F}} = \frac{1}{3}h^n\mathbf{u}^n + \frac{1}{6}h^n\mathbf{u}^{n+1} + \frac{1}{6}h^{n+1}\mathbf{u}^n + \frac{1}{3}h^{n+1}\mathbf{u}^{n+1}$$

$$\bar{\Phi} = \frac{1}{6}\mathbf{u}^n\cdot\mathbf{u}^n + \frac{1}{6}\mathbf{u}^n\cdot\mathbf{u}^{n+1} + \frac{1}{6}\mathbf{u}^{n+1}\cdot\mathbf{u}^{n+1} + \frac{g}{2}h^n + \frac{g}{2}h^{n+1}$$

Example: Rotating Shallow Water Equations (discrete form)

Discrete variational derivatives may be computed as

$$\lim_{\epsilon\to 0}\frac{\mathcal{H}_h(a_h+\epsilon b_h)-\mathcal{H}_h(a_h)}{\epsilon}=\left\langle b_h,\frac{\delta\mathcal{H}_h}{\delta a_h}\right\rangle \qquad \forall b_h\in V^0/V^1/V^2$$

Such that

$$\left\langle \boldsymbol{\beta}_h,\frac{\delta\mathcal{H}_h}{\delta\mathbf{u}_h}\right\rangle=\langle\boldsymbol{\beta}_h,\boldsymbol{\beta}_h\rangle\hat{F}_h=\langle\boldsymbol{\beta}_h,h_h\mathbf{u}_h\rangle \qquad \forall\boldsymbol{\beta}_h\in V^1$$

$$\left\langle \gamma_h,\frac{\delta\mathcal{H}_h}{\delta h_h}\right\rangle=\langle\gamma_h,\gamma_h\rangle\hat{\Phi}_h=\langle\gamma_h,\frac{1}{2}\mathbf{u}_h\cdot\mathbf{u}_h+gh_h\rangle \qquad \forall\gamma_h\in V^2$$

We also define the potential vorticity, q_h as:

$$\langle\alpha_h,h_hq_h\rangle=-(\mathsf{E}^{1,0})^\top\langle\boldsymbol{\beta}_h,\mathbf{u}_h\rangle+\langle\alpha_h,f_h\rangle \qquad \forall\alpha_h\in V^0$$

$$\frac{1}{\Delta t}\begin{bmatrix}\langle\boldsymbol{\beta}_h,\mathbf{u}_h^{n+1}-\mathbf{u}_h^n\rangle\\ \langle\gamma_h,h_h^{n+1}-h_h^n\rangle\end{bmatrix}=-\begin{bmatrix}\langle\boldsymbol{\beta}_h,q_h\times\boldsymbol{\beta}_h\rangle & -(\mathsf{E}^{2,1})^\top\langle\gamma_h,\gamma_h\rangle\\ \langle\gamma_h,\gamma_h\rangle\mathsf{E}^{2,1} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\hat{F}_h\\ \hat{\Phi}_h\end{bmatrix}$$

Rotating Shallow Water Equations: Other Conservation Properties

- Conservation of mass, $\hat{1}_h^\top \hat{h}_h$: holds pointwise due to strong form divergence

$$\hat{1}_h^\top \mathsf{E}^{2,1} \hat{F}_h = 0$$

- Conservation of vorticity, $\hat{1}_h^\top \langle \alpha_h, \alpha_h \rangle \hat{\omega}_h$: holds in the weak form as

$$\hat{1}_h^\top (\mathsf{E}^{1,0})^\top \langle \boldsymbol{\beta}_h, q_h \times \boldsymbol{\beta}_h \rangle \hat{F}_h = 0$$

$$(\mathsf{E}^{1,0})^\top (\mathsf{E}^{2,1})^\top = 0$$

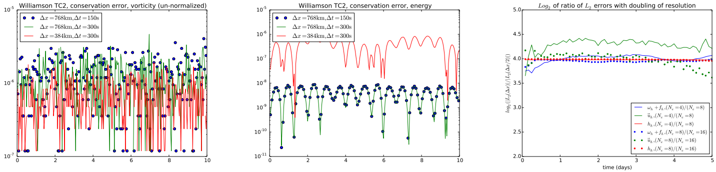
$$\langle \alpha_h, \alpha_h \rangle \hat{\omega}_h = -(\mathsf{E}^{1,0})^\top \langle \boldsymbol{\beta}_h, \boldsymbol{\beta}_h \rangle \hat{u}_h$$

- Conservation of potential enstrophy, $\hat{q}_h \langle \alpha_h, h_h \alpha_h \rangle \hat{q}_h$: holds in the weak form *for exact integration only*: The product rule

$$q \times \nabla^\perp q \neq \frac{1}{2} \nabla q^2$$

holds only for exact integration

Rotating Shallow Water Equations (results)



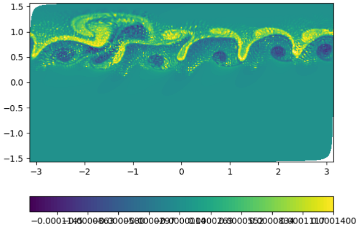


Figure: Vorticity field for the Galewsky test case (day 7), inviscid solution with exact energy conservation (left) and with viscosity (right).

The 3D compressible Euler equations (continuous form)

$$\frac{\partial \mathbf{u}}{\partial t} = -(\boldsymbol{\omega} + f e_z) \times \mathbf{u} - \nabla(\frac{1}{2}u^2 + gz) - \theta\nabla\Pi$$
$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\mathbf{u})$$
$$\frac{\partial \Theta}{\partial t} = -\nabla \cdot (\rho\theta\mathbf{u}), \qquad \Theta = \rho\theta$$
$$\Pi = c_p\left(\frac{R\Theta}{p_0}\right)^{R/c_v}, \qquad c_p = R + c_v$$

Energy is defined as

$$\mathcal{H} = \int \frac{1}{2}\rho u^2 + \rho g z + \frac{c_v}{c_p}\Theta\Pi \mathrm{d}\Omega$$

Setting

$$\frac{\delta\mathcal{H}}{\delta\mathbf{u}} = \rho\mathbf{u} = \mathbf{U}, \qquad \frac{\delta\mathcal{H}}{\delta\rho} = \frac{1}{2}u^2 + gz = \Phi, \qquad \frac{\delta\mathcal{H}}{\delta\Theta} = \Pi$$

gives

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{u}\\ \rho\\ \Theta\end{bmatrix} = \begin{bmatrix}-\frac{1}{\rho}(\boldsymbol{\omega} + f e_z)\times & -\nabla & -\theta\nabla\\ -\nabla\cdot & \mathbf{0} & \mathbf{0}\\ -\nabla\cdot(\theta\cdot) & \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{U}\\ \Phi\\ \Pi\end{bmatrix}$$

Discrete variational derivatives may be computed as

$$\lim_{\epsilon\to 0}\frac{\mathcal{H}_h(a_h+\epsilon b_h)-\mathcal{H}_h(a_h)}{\epsilon}=\left\langle b_h,\frac{\delta\mathcal{H}_h}{\delta a_h}\right\rangle \qquad \forall b_h\in V^0/V^1/V^2/V^3$$

Such that

$$\left\langle \boldsymbol{\beta}_h,\frac{\delta\mathcal{H}_h}{\delta\mathbf{u}_h}\right\rangle=\langle\boldsymbol{\beta}_h,\boldsymbol{\beta}_h\rangle\hat{U}_h=\langle\boldsymbol{\beta}_h,\rho_h\mathbf{u}_h\rangle \qquad \forall\boldsymbol{\beta}_h\in V^2$$

$$\left\langle \gamma_h,\frac{\delta\mathcal{H}_h}{\delta\rho_h}\right\rangle=\langle\gamma_h,\gamma_h\rangle\hat{\Phi}_h=\langle\gamma_h,\frac{1}{2}\mathbf{u}_h\cdot\mathbf{u}_h+gz_h\rangle \qquad \forall\gamma_h\in V^3$$

$$\left\langle \gamma_h,\frac{\delta\mathcal{H}_h}{\delta\Theta_h}\right\rangle=\langle\gamma_h,\gamma_h\rangle\hat{\Pi}_h=c_p\left(\frac{R}{p_0}\right)^{R/c_v}\langle\gamma_h,(\Theta_h)^{R/c_v}\rangle \qquad \forall\gamma_h\in V^3$$

$$\langle\alpha_h,\rho_h q_h\rangle=-(\mathrm{E}^{2,1})^\top\langle\boldsymbol{\beta}_h,\mathbf{u}_h\rangle+\langle\alpha_h,f_h\rangle \qquad \forall\alpha_h\in V^1$$

The discrete Euler equations are then given as

$$\frac{\partial}{\partial t}\begin{bmatrix}\langle\boldsymbol{\beta}_h,\mathbf{u}_h\rangle\\ \langle\gamma_h,\rho_h\rangle\\ \langle\gamma_h,\Theta_h\rangle\end{bmatrix}=$$

$$\begin{bmatrix}-\langle\boldsymbol{\beta}_h,q_h\times\boldsymbol{\beta}_h\rangle & (\mathrm{E}^{3,2})^\top\langle\gamma_h,\gamma_h\rangle & \langle\boldsymbol{\beta}_h,\theta_h\boldsymbol{\beta}_h\rangle\langle\boldsymbol{\beta}_h,\boldsymbol{\beta}_h\rangle^{-1}(\mathrm{E}^{3,2})^\top\langle\gamma_h,\gamma_h\rangle\\ -\langle\gamma_h,\gamma_h\rangle\mathrm{E}^{3,2} & \mathbf{0} & \mathbf{0}\\ -\langle\gamma_h,\gamma_h\rangle\mathrm{E}^{3,2}\langle\boldsymbol{\beta}_h,\boldsymbol{\beta}_h\rangle^{-1}\langle\boldsymbol{\beta}_h,\theta_h\boldsymbol{\beta}_h\rangle & \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\hat{U}_h\\ \hat{\Phi}_h\\ \hat{\Pi}_h\end{bmatrix}$$

Discrete form (cont.)

Multiplying both sides by $[\hat{U}_h^\top, \hat{\Phi}_h^\top, \hat{\Pi}_h^\top]$:

$$\left\langle \mathbf{U}_h, \frac{\partial \mathbf{u}_h}{\partial t} \right\rangle + \left\langle \frac{1}{2}\mathbf{u}_h \cdot \mathbf{u}_h, \frac{\partial \rho_h}{\partial t} \right\rangle + \left\langle gz_h, \frac{\partial \rho_h}{\partial t} \right\rangle + \left\langle \Pi_h, \frac{\partial \Theta_h}{\partial t} \right\rangle = \frac{\partial K_h}{\partial t} + \frac{\partial P_h}{\partial t} + \frac{\partial I_h}{\partial t} = 0$$

where

$$K_h = \frac{1}{2}\langle \mathbf{U}_h, \mathbf{u}_h \rangle, \qquad P_h = \langle \rho_h, gz_h \rangle, \qquad I_h = \frac{c_v}{c_p}\langle \Pi_h, \Theta_h \rangle = c_v \left(\frac{R}{p_0}\right)^{R/c_v} \int (\Theta_h)^{c_p/c_v} \mathrm{d}\Omega$$

Energetic exchanges given as

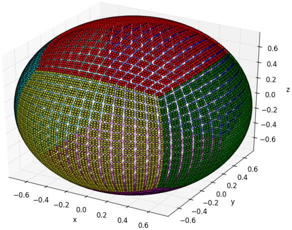
$$\frac{\partial K_h}{\partial t} = \hat{U}_h^\top (\mathsf{E}^{3,2})^\top \langle \gamma_h, gz_h \rangle + \langle \mathbf{U}_h, \theta_h \boldsymbol{\beta}_h \rangle \langle \boldsymbol{\beta}_h, \boldsymbol{\beta}_h \rangle^{-1} (\mathsf{E}^{3,2})^\top \langle \gamma_h, \Pi_h \rangle$$

$$\frac{\partial P_h}{\partial t} = -\langle gz_h, \gamma_h \rangle \mathsf{E}^{3,2} \hat{U}_h$$

$$\frac{\partial I_h}{\partial t} = -\langle \Pi_h, \gamma_h \rangle \mathsf{E}^{3,2} \hat{U}_h \langle \boldsymbol{\beta}_h, \boldsymbol{\beta}_h \rangle^{-1} \langle \boldsymbol{\beta}_h, \theta_h \mathbf{U}_h \rangle$$

Implementation details

- Cubed sphere discretisation



- Piola transform mappings from computational to physical space for fields in $V^0/V^1/V^2/V^3$
- Spectral element in horizontal, lowest order ($p = 1$) mixed formulation in the vertical
- Horizontally explicit/vertically implicit time integration [Ullrich and Jablonowski, MWR, 2012]
  - Stiffly stable RK3 in horizontal
  - Strang carryover in the vertical
  - Vertical solve breaks energy conservation!!!
  - TODO: fully implicit time integration to recover energy conservation

Implicit vertical solve

- Vertical resolution too small to explicitly resolve the sound waves
- Pressure gradient term must be solved implicitly
- Natural logarithm of the equation of state gives:

$$\ln(\Pi) = \frac{R}{c_v}\left(\ln(\Theta) + \ln\left(\frac{R}{p_0}\right)\right) + \ln(c_p)$$

- Second order (in time) expansion:

$$\frac{\Pi^{n+1} - \Pi^n}{\Delta t \Pi^n} \approx \frac{R}{c_v}\frac{\Theta^{n+1} - \Theta^n}{\Delta t \Theta^n}$$

- Substituting in the temperature evolution equation [Gassmann, 2013]:

$$\Theta^n \Pi^{n+1} \approx \Theta^n \Pi^n - \frac{\Delta t R}{c_v}\Pi^n \nabla \cdot (\rho\theta\mathbf{u})$$

- And then into the vertical momentum equation:

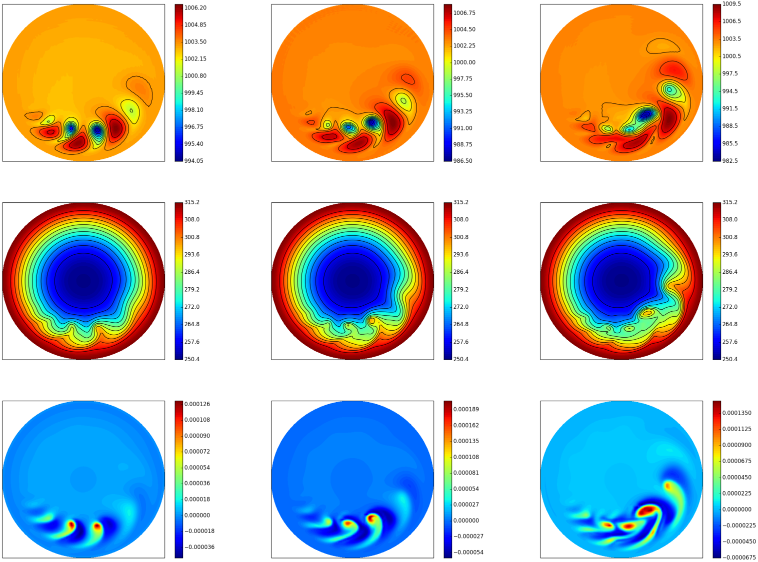
$$w^{n+1} + \frac{\Delta t}{2}\frac{\partial (w^{n+1})^2}{\partial z} - \frac{\Delta t^2 R}{c_v}\theta\frac{\partial}{\partial z}\left((\Theta^n)^{-1}\Pi^n\frac{\partial \Theta^{n+1} w^{n+1}}{\partial z}\right) = w^n - \Delta t\theta\frac{\partial \Pi^n}{\partial z} - \Delta t\frac{\partial gz}{\partial z}$$

- NOTE: this pressure gradient formulation is NOT energetically consistent!!!

Test case setup

- Baroclinic instability (on z-levels) [Ullrich et. al. (2014) QJRMS]
- 24×24 elements ($p = 3$) on each face $\Delta x \approx 128km$
- Second order ($p = 1$) in the vertical
- 30 vertical levels
- $\Delta t = 120s$
- $w(z = 0) = w(z = 30\mathrm{km}) = 0$
- $\theta(z = 0) = \theta^b(\lambda, \phi)$, $\theta(z = 30\mathrm{km}) = \theta^t(\lambda, \phi)$
- $\partial \Pi / \partial z|_{z=0} = \partial \Pi / \partial z|_{z=30\mathrm{km}} = 0$
- Biharmonic viscosity in horiztonal
- Rayleigh damping in upper layer vertical momentum equation [Klemp et. al., MWR, 2008]
- $6 \times 4^2 = 96$ processors

Results: baroclinic instability test case (days 8–10)



Results: baroclinic instability test case (cont.)

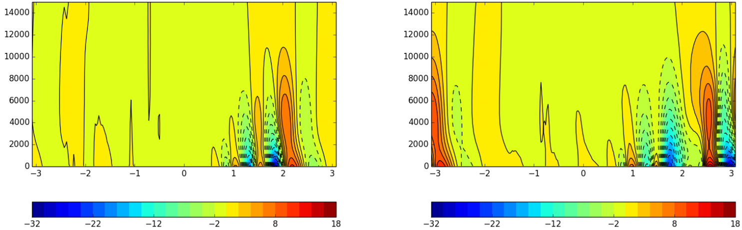


Figure: Pressure perturbation at 50°N, days 8 (left) and 10 (right).

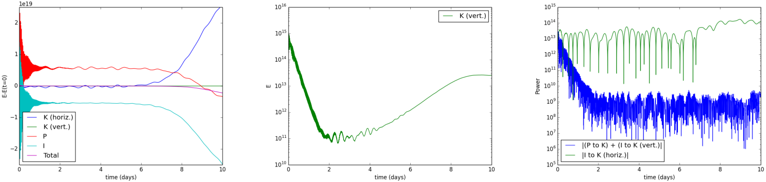


Figure: Kinetic, potential and internal energy evolution (left and center) power exchanges (right).

Results: non-hydrostatic gravity wave

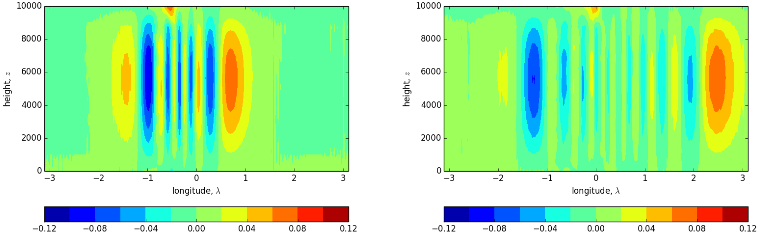


Figure: Potential temperature perturbation, θ', at the equator at times 30 minutes (left) and 60 minutes (right).

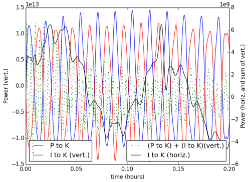


Figure: Power exchanges for the non-hydrostatic gravity wave test.

Thanks for listening!

TODO:

- Implicit energy conserving time integrator
  - preconditioning issues (coupling vertical and horiztonal dynamics)
- Energetically balanced upwinding of temperature fluxes/pressure gradients?

D. Lee and A. Palha "A Mixed Mimetic Spectral Element Model of the 3D Compressible Euler Equations on the Cubed Sphere" *J. Comput. Phys.*, **401** (2020) 108993

D. Lee and A. Palha "A Mixed Mimetic Spectral Element Model of the Rotating Shallow Water Equations on the Cubed Sphere" *J. Comput. Phys.*, **375** (2018) 240–262

D. Lee, A. Palha and M. Gerritsma "Discrete conservation properties for shallow water flows using mixed mimetic spectral elements" *J. Comput. Phys.*, **357** (2018) 282–304

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