An h-adaptive unfitted finite element method for interface elliptic boundary value problems

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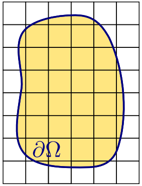
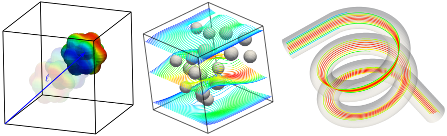
Previous talk

Unfitted *boundaries*:

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = u^{\mathrm{D}} \quad \text{on } \partial\Omega$$

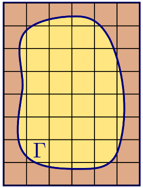




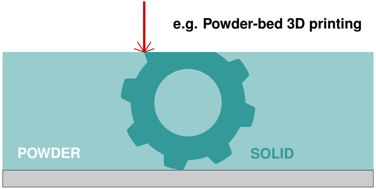
In this talk

Unfitted *interfaces*:

$$-\nabla \cdot (\alpha \nabla u) = f \quad \text{in } \Omega^{\mathrm{in}} \cup \Omega^{\mathrm{out}}$$
$$u = 0 \quad \text{on } \partial\Omega$$
$$[\![u]\!] = 0 \quad \text{on } \Gamma$$
$$\boldsymbol{n}^{\mathrm{out}} \cdot [\![\alpha \nabla u]\!] = 0 \quad \text{on } \Gamma$$



e.g. Powder-bed 3D printing



Building platform

An overview of unfitted FE methods for interface problems

Interface Poisson problem

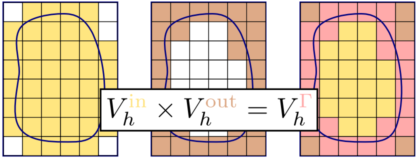
$$-\nabla \cdot (\alpha \nabla u) = f \quad \text{in } \Omega^{\text{in}} \cup \Omega^{\text{out}}$$

$$u = 0 \quad \text{on } \partial\Omega$$

$$[\![u]\!] = 0 \quad \text{on } \Gamma$$

$$\boldsymbol{n}^{\text{out}} \cdot [\![\alpha \nabla u]\!] = 0 \quad \text{on } \Gamma$$

Structure of approximation



Approximation method

$$\begin{aligned} A_h(u_h, v_h) \doteq (\alpha \nabla u_h, \nabla v_h)_{\Omega^{\text{in}} \cup \Omega^{\text{out}}} & - \langle \{\!\{\alpha \nabla u_h\}\!\} \cdot \boldsymbol{n}^+, [\![v_h]\!] \rangle_\Gamma \\ & - \langle [\![u_h]\!], \{\!\{\alpha \nabla v_h\}\!\} \cdot \boldsymbol{n}^+ \rangle_\Gamma \\ & + \langle \tau [\![u_h]\!], [\![v_h]\!] \rangle_\Gamma \end{aligned}$$

SIPM or other Nitsche formulations

Known results

- Naive SIPM+FEM: τ unbounded for arbitrarily small cut size [HH02]
- Robustness to cut location via, e.g., stabilization (CutFEM) [Bur+15]
- Robustness to material contrast via harmonic $\{\!\{\cdot\}\!\}$ and diag. prec. [Bur+15]

Is AgFEM suitable for interface elliptic BVPs?

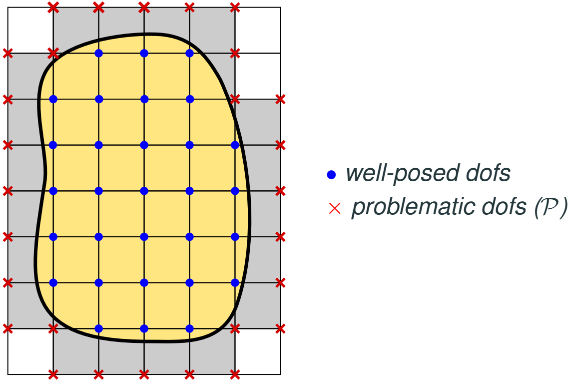
Contents:

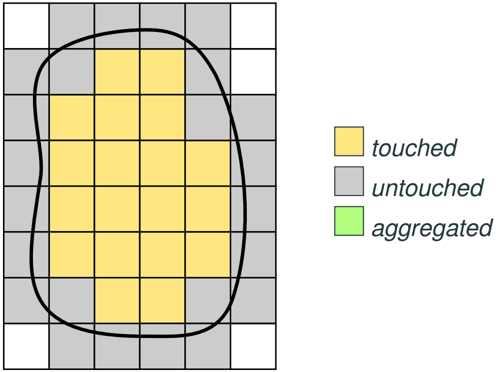
- Construction of aggregated FE spaces
- Approximation of the interface problem
- Numerical analysis
- Numerical experiments
  - Verification in uniform meshes
  - Robustness w.r.t. cut location
  - Robustness w.r.t. material contrast
  - Robustness and optimality in tree-based meshes

Construction of AgFE spaces in interface problems

Recalling the rationale: improve conditioning by removing problematic DOFs

$$V_h^{\text{agg}} := \left\{ u \in V_h : \quad u_{\times} = \sum_{\bullet \in \text{masters}(\times)} C_{\times \bullet} u_{\bullet} \quad \forall \times \in \mathcal{P} \right\}$$





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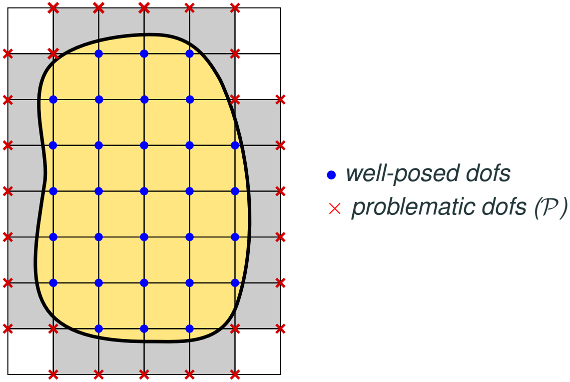

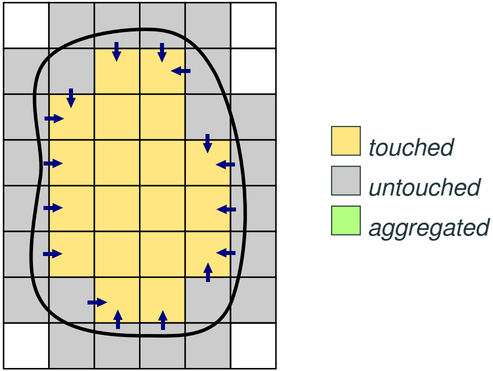

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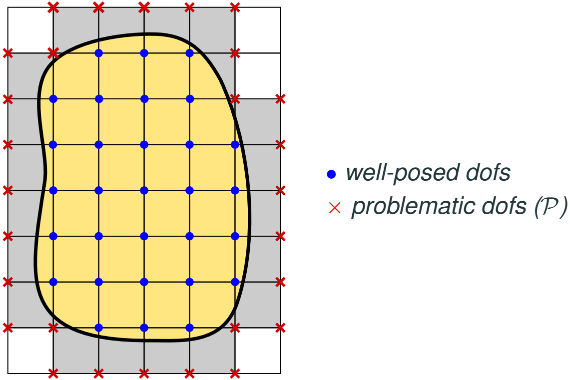

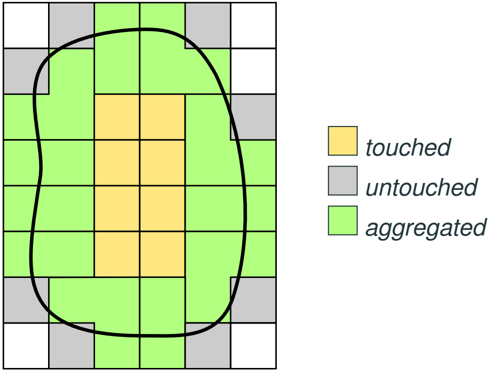

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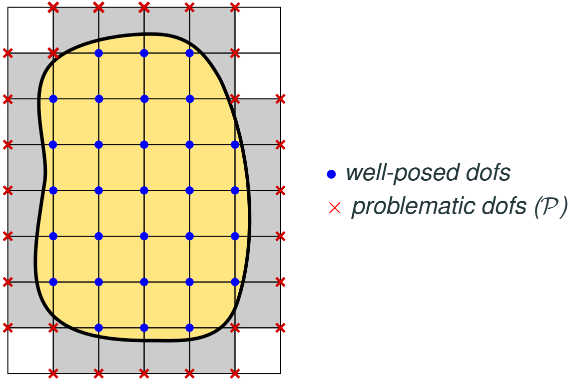

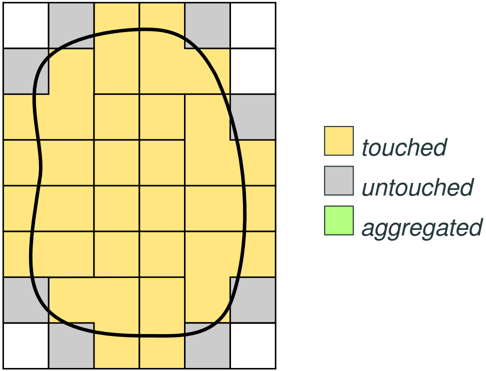

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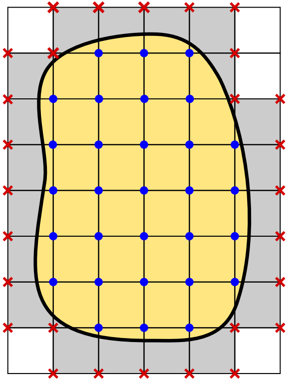
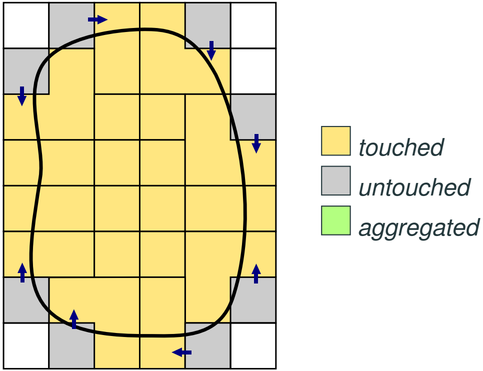


- *well-posed dofs* (●)
- *problematic dofs ($\mathcal{P}$)* (×)



Construction of AgFE spaces in interface problems

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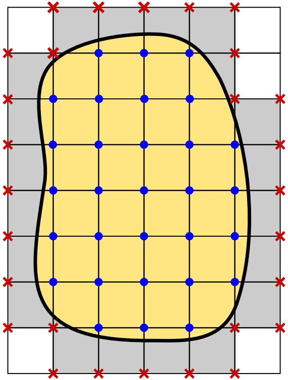
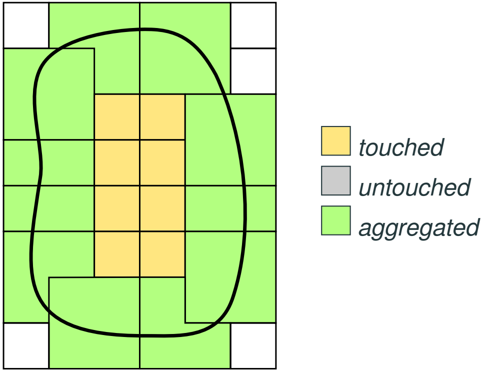
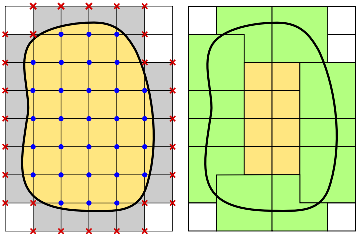


● *well-posed dofs*

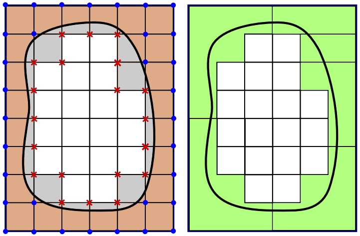
× *problematic dofs ($\mathcal{P}$)*



In our context: constrain by cell aggregation at both subregions



$\longrightarrow V_h^{\mathrm{in},\mathrm{agg}}$



$\longrightarrow V_h^{\mathrm{out},\mathrm{agg}}$

$$V_h^{\mathrm{agg}} = V_h^{\mathrm{in},\mathrm{agg}} \times V_h^{\mathrm{out},\mathrm{agg}}$$

Analogously, given n different materials, $V_h^{\mathrm{agg}} = V_h^{1,\mathrm{agg}} \times \ldots \times V_h^{\mathrm{n},\mathrm{agg}}$.

Local FE operators: For any $K \in \mathcal{T}_h$, we define

$$A_K(u_h, v_h) \doteq a_K^{\text{bulk}}(u_h, v_h) + a_K^{\Gamma}(u_h, v_h), \qquad l_K(v_h) \doteq l_K^{\text{bulk}}(v_h),$$

with

$$a_K^{\text{bulk}}(u_h, v_h) = \left(\alpha^{\text{in}} \nabla u_h^{\text{in}}, \nabla v_h^{\text{in}}\right)_{\Omega^{\text{in}} \cap K} + \left(\alpha^{\text{out}} \nabla u_h^{\text{out}}, \nabla v_h^{\text{out}}\right)_{\Omega^{\text{out}} \cap K},$$

$$l_K^{\text{bulk}}(v_h) = \left\langle f^{\text{in}}, v_h^{\text{in}} \right\rangle_{\Omega^{\text{in}} \cap K} + \left\langle f^{\text{out}}, v_h^{\text{out}} \right\rangle_{\Omega^{\text{out}} \cap K},$$

and

$$a_K^{\Gamma}(u_h, v_h) = -\langle \boldsymbol{n}^{\text{out}} \cdot \{\!\{\alpha \nabla u_h\}\!\}, [\![v_h]\!]\rangle_{\Gamma \cap K} - \langle [\![u_h]\!], \boldsymbol{n}^{\text{out}} \cdot \{\!\{\alpha \nabla v_h\}\!\}\rangle_{\Gamma \cap K} + \langle \tau_K [\![u_h]\!], [\![v_h]\!]\rangle_{\Gamma \cap K},$$

where $\tau_K = \beta_K h_K^{-1}$, with $\beta_K > 0$ large enough, is a stabilization parameter and weighted average $\{\!\{\cdot\}\!\}$ given by

$$\frac{\alpha^{\text{out}}}{\alpha^{\text{in}} + \alpha^{\text{out}}} (\cdot)^{\text{in}} + \frac{\alpha^{\text{in}}}{\alpha^{\text{in}} + \alpha^{\text{out}}} (\cdot)^{\text{out}}.$$

Approximation of interface problem with Nitsche's method

Local FE operators: For any $K \in \mathcal{T}_h$, we define

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and

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interface-AgFEM inherits key boundary-AgFEM inverse inequalities

Trace inverse inequality for unfitted boundary [BVM18]

For any $v_h \in V_h^{\mathrm{agg}}$ and $K \in \mathcal{T}_h^{\Gamma}$,

$$\|\boldsymbol{n} \cdot \nabla v_h\|_{0,\Gamma_D \cap K}^2 \leqslant C_\partial h_K^{-1} \|\nabla v_h\|_{0,\Omega_K}^2,$$

where Ω_K is the domain of the aggregate where K belongs and C_∂ *independent of mesh size and cut location*.

Trace inverse inequality for unfitted interface

For any $v_h \in V_h^{\mathrm{agg}}$ and $K \in \mathcal{T}_h^{\Gamma}$,

$$\left\|\boldsymbol{n}^{\mathrm{out}} \cdot \{\!\{\alpha \nabla v_h\}\!\}\right\|_{0,\Gamma \cap K}^2 \leqslant C_\partial h_K^{-1} \frac{\alpha^{\mathrm{in}} \alpha^{\mathrm{out}}}{\alpha^{\mathrm{in}} + \alpha^{\mathrm{out}}} \left(\|\nabla v_h\|_{0,\Omega_{K^{\mathrm{in}}}}^2 + \|\nabla v_h\|_{0,\Omega_{K^{\mathrm{out}}}}^2 \right),$$

where $\Omega_K^{\mathrm{in/out}}$ is the aggregate domain at $\Omega^{\mathrm{in/out}}$ and C_∂ *independent of mesh size and cut location*.

interface-AgFEM inherits key boundary-AgFEM inverse inequalities

Trace inverse inequality for unfitted boundary [BVM18]

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For any $v_h \in V_h^{\mathrm{agg}}$ and $K \in \mathcal{T}_h^{\Gamma}$,

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where $\Omega_K^{\mathrm{in/out}}$ is the aggregate domain at $\Omega^{\mathrm{in/out}}$ and C_∂ *independent of mesh size and cut location.*

Well-posedness and a priori error estimates

Let $V(h) \doteq V_h^{\mathrm{agg}} + H_0^1(\Omega) \cap H^2(\Omega^{\mathrm{in}} \cup \Omega^{\mathrm{out}})$ and define for any $v \in V_h^{\mathrm{agg}}$ the norms

$$|v|_*^2 \doteq \sum_{K \in \mathcal{T}^\Gamma} \beta_K h_K^{-1} \|[\![v]\!]\|_{0,\Gamma\cap K}^2, \quad \|v\|_{V(h)}^2 \doteq |v|_{1,\Omega^{\mathrm{in}}\cup\Omega^{\mathrm{out}}}^2 + |v|_*^2,$$

$$\text{and } |\!|\!|v|\!|\!|_{V(h)}^2 \doteq \|v\|_{V(h)}^2 + \sum_{K \in \mathcal{T}_{\mathrm{cut}}} h_K^2 \left(\left| v^{\mathrm{in}} \right|_{2,\Omega^+\cap K}^2 + \left| v^{\mathrm{out}} \right|_{2,\Omega^-\cap K}^2 \right).$$

The following results hold:

$A(u_h, u_h) \gtrsim \|u_h\|_{V(h)}^2$	for all $u_h \in V_h^{\mathrm{agg}}$	(stability if $\beta_K \gtrsim \frac{2\alpha^{\mathrm{in}}\alpha^{\mathrm{out}}}{\alpha^{\mathrm{in}}+\alpha^{\mathrm{out}}}$)
$A(u, v) \lesssim |\!|\!|u|\!|\!|_{V(h)}^2 |\!|\!|v|\!|\!|_{V(h)}^2$	for all $u, v \in V(h)$	(continuity)
$\|u - u_h\|_{1,\Omega^{\mathrm{in}}\cup\Omega^{\mathrm{out}}} \lesssim h^p$	for all $u_h \in V_h^{\mathrm{agg}}$ and $u \in V(h)$	(optimal convergence in H^1)
$\|u - u_h\|_{0,\Omega^{\mathrm{in}}\cup\Omega^{\mathrm{out}}} \lesssim h^{p+1}$	for all $u \in V(h)$ and $u_h \in V_h^{\mathrm{agg}}$	(optimal convergence in L^2)

where the constants are **independent of cut location**.

Summary of numerical analysis

Well-posedness and a priori error estimates

Let $V(h) \doteq V_h^{\mathrm{agg}} + H_0^1(\Omega) \cap H^2(\Omega^{\mathrm{in}} \cup \Omega^{\mathrm{out}})$ and define for any $v \in V_h^{\mathrm{agg}}$ the norms

$$|v|_*^2 \doteq \sum_{K \in \mathcal{T}^\Gamma} \beta_K h_K^{-1} \|[\![v]\!]\|_{0,\Gamma\cap K}^2, \quad \|v\|_{V(h)}^2 \doteq |v|_{1,\Omega^{\mathrm{in}} \cup \Omega^{\mathrm{out}}}^2 + |v|_*^2,$$

and $$|\!|\!|v|\!|\!|_{V(h)}^2 \doteq \|v\|_{V(h)}^2 + \sum_{K \in \mathcal{T}_{\mathrm{cut}}} h_K^2 \left(\left|v^{\mathrm{in}}\right|_{2,\Omega^+ \cap K}^2 + \left|v^{\mathrm{out}}\right|_{2,\Omega^- \cap K}^2 \right).$$

The following results hold:

$A(u_h, u_h) \gtrsim \|u_h\|_{V(h)}^2$	for all $u_h \in V_h^{\mathrm{agg}}$	(stability if $\beta_K \gtrsim \frac{2\alpha^{\mathrm{in}}\alpha^{\mathrm{out}}}{\alpha^{\mathrm{in}}+\alpha^{\mathrm{out}}}$)
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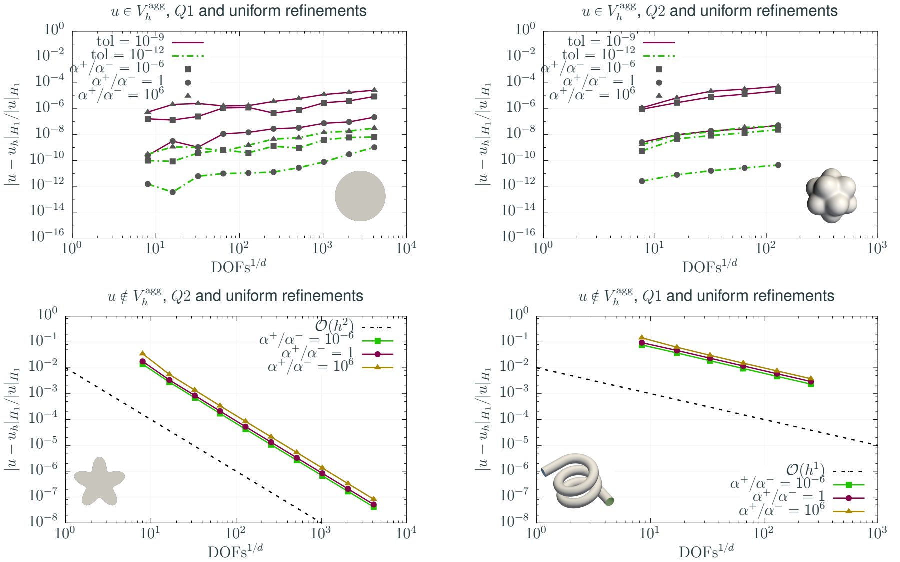
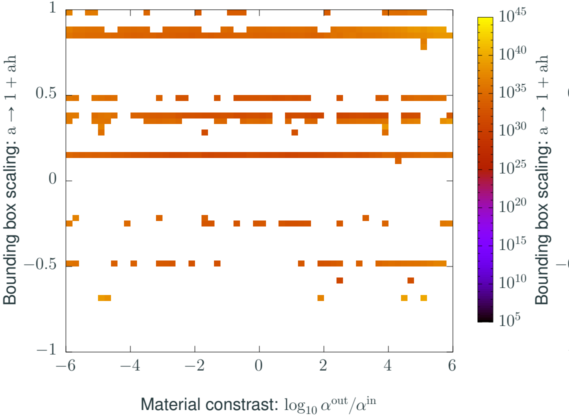
Numerical verification experiments

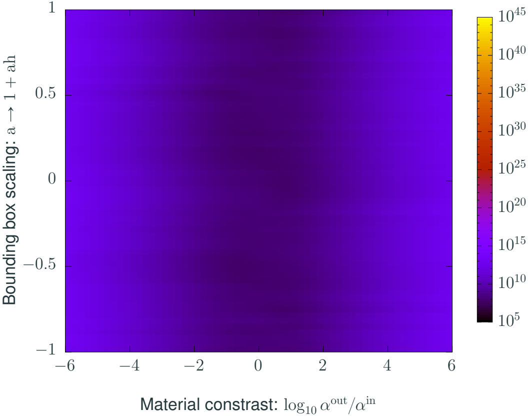


Robustness w.r.t. to cut location in popcorn Q2 with $u \notin V_h^{\mathrm{agg}}$

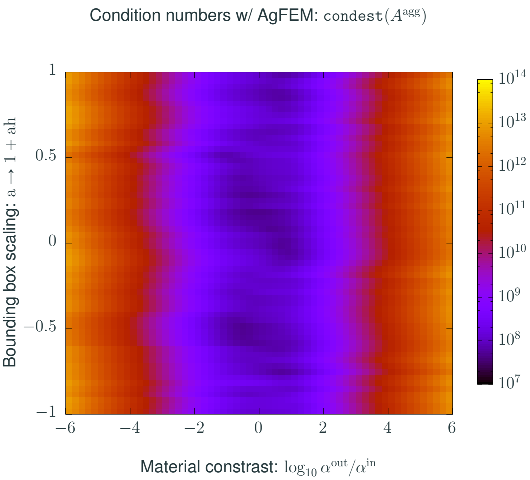
Condition numbers w/ standard FEM: $\texttt{condest}(A^{\mathrm{std}})$

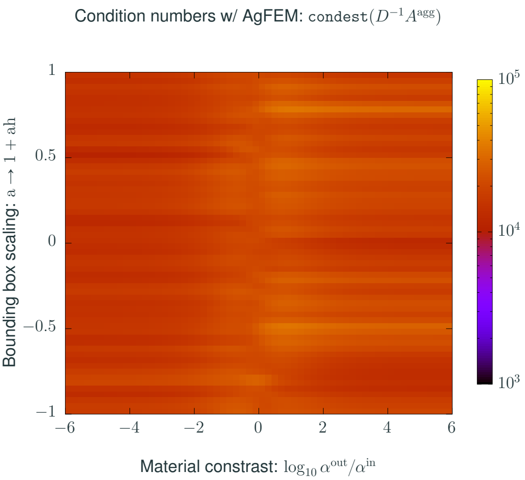


Condition numbers w/ AgFEM: $\texttt{condest}(A^{\mathrm{agg}})$

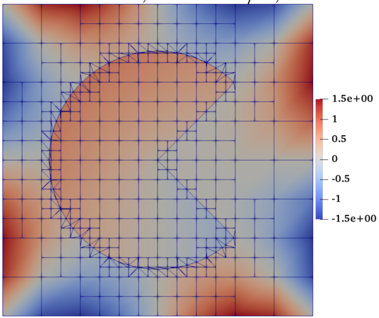


Robustness w.r.t. to material contrast in popcorn Q2 with $u \notin V_h^{\mathrm{agg}}$

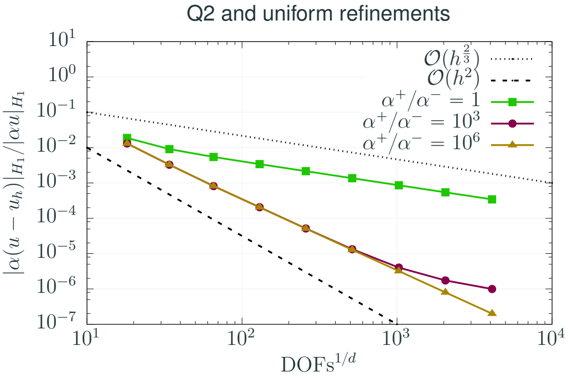




$$v^{\pm} = r^{\alpha^{\pm}} \sin \alpha^{\pm}\theta, \ \alpha^{-} = 2/3, \ \alpha^{+} = 4$$



$v^{-} \in H^{1+\frac{2}{3}}(\Omega)$



Robustness and optimality in h-adaptivity: Fichera-corner

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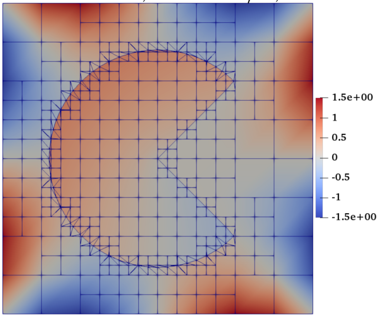

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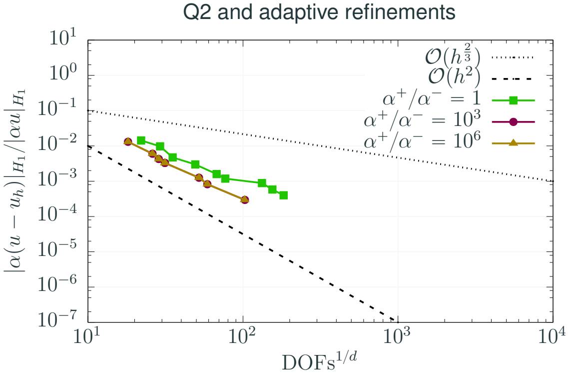

Conclusions and further work

AgFEM shows good properties, in the context of interface elliptic BVPs, but further study is required:

1. more experimentation on tree-based meshes
2. accuracy of the quantities at the interface (traces, fluxes...)
3. comparison against other well-behaved methods
4. multimaterial, elasticity, moving interfaces, scalability...

References

Erik Burman et al. “CutFEM: discretizing geometry and partial differential equations”. In: *International Journal for Numerical Methods in Engineering* 104.7 (2015), pp. 472–501.

Santiago Badia, Francesc Verdugo, and Alberto F Martín. “The aggregated unfitted finite element method for elliptic problems”. In: *Computer Methods in Applied Mechanics and Engineering* 336 (2018), pp. 533–553.

Anita Hansbo and Peter Hansbo. “An unfitted finite element method, based on Nitsche’s method, for elliptic interface problems”. In: *Computer methods in applied mechanics and engineering* 191.47-48 (2002), pp. 5537–5552.

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Thank you!