Stochastic partial differential equations on the sphere.

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Monash Workshop on Numerical Differential Equations and Applications
February 13, 2020

The talk is based on joint results with

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TALK OUTLINE

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1. Introduction to CMB data
2. Spherical random fields
3. Evolution of CMB: SPDEs

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Introduction to CMB data

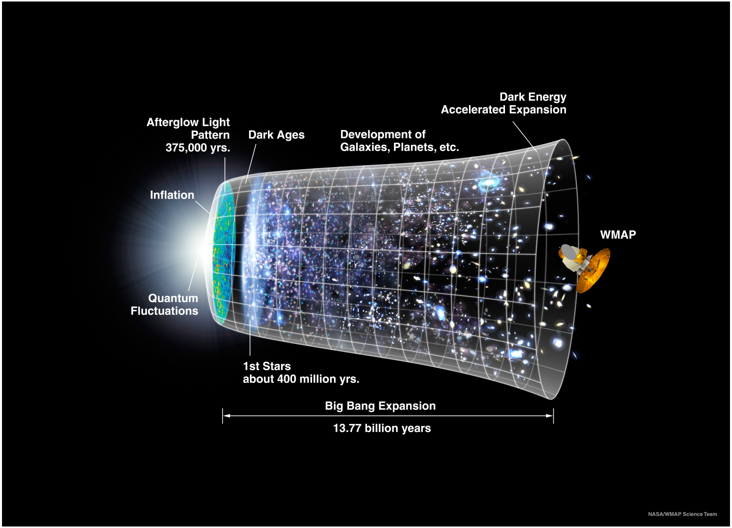


Image credit: NASA / WMAP Science Team

- CMB is the remnant heat left over from the Big Bang;
- Predicted by Ralph Alpher and Robert Herman in 1948;
- Observed by Arno Penzias and Robert Wilson in 1965;
- Hundreds of cosmic microwave background experiments have been conducted to measure CMB;
- Most detailed space mission to date was conducted by the European Space Agency, via the Planck Surveyor satellite (in the range of frequencies from 30 to 857 GHz).

Missions

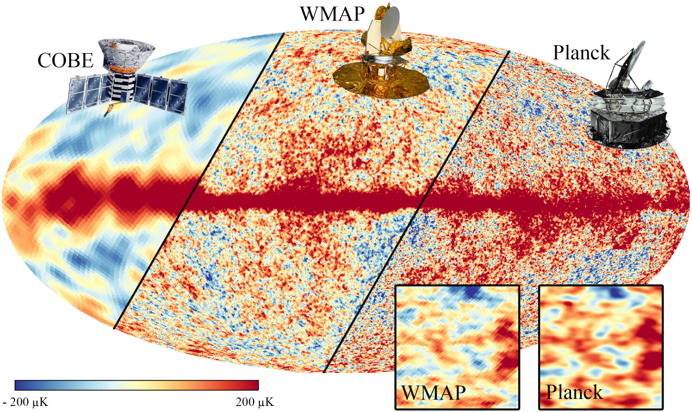
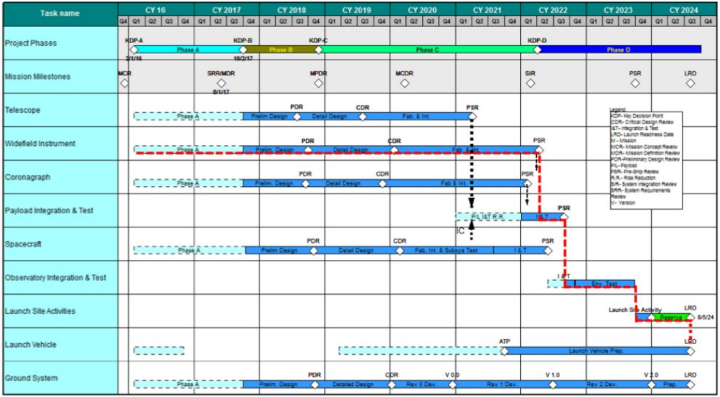


Image credit: https://jgudmunds.wordpress.com

Next Generation Missions

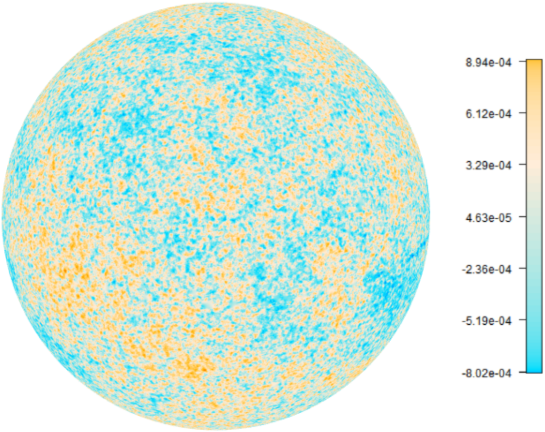


Next Generation Explorer: CMB-S4

(Sponsored by Simons Foundations, NSF and US Department of Energy)

What does CMB data look like?

- 13.77 billion year old temperature fluctuations that correspond to the seeds that grew to become the galaxies
- Current CMB data are at 5 arcminutes resolution on the sphere.
- Contains 50,331,648 data collected by Planck mission.



Research directions

- **Direction 1: Stochastic modelling and developing new spherical inference tools: high frequency asymptotics, Minkowski functionals, Rényi functions**

- **Direction 2: Evolution of CMB: SPDEs**

- **Direction 3: Practical statistical analysis of CMB data: R package rcosmo**

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Spherical random fields

The standard statistical model for CMB is an isotropic random field on the sphere $\mathbb{S}^2$

$$T = \{ T(\theta, \varphi) = T_\omega(\theta, \varphi) : 0 \leq \theta < \pi, \quad 0 \leq \varphi < 2\pi, \ \omega \in \Omega \} .$$

CMB can be viewed as a single realization of this random field.

We consider a real-valued second-order spherical random field T that is continuous in the mean-square sense. The field T can be expanded in the mean-square sense as a Laplace series

$$T(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \varphi),$$

where $\{ Y_{lm}(\theta, \varphi)\}$ represents the complex spherical harmonics.

The random coefficients a_{lm} in the Laplace series can be obtained through inversion arguments in the form of mean-square stochastic integrals

$$a_{lm} = \int_0^{\pi} \int_0^{2\pi} T(\theta,\varphi) Y_{lm}^*(\theta,\varphi) \sin\theta d\theta d\varphi.$$

The field is isotropic if

$$\mathbf{E} a_{lm} a_{l'm'}^* = \delta_l^{l'} \delta_m^{m'} C_l, \quad -l \le m \le l, \quad -l' \le m' \le l'.$$

Thus, $\mathbf{E}|a_{lm}|^2 = C_l,\ m = 0, \pm 1, ..., \pm l.$

The series $\{C_1, C_2, ..., C_l, ...\}$ is called the angular power spectrum of the isotropic random field $T(\theta,\varphi)$.

Random spherical hyperbolic diffusion

The papers

- Anh, V., Broadbridge, P., Olenko, A., Wang, Y. (2018) On approximation for fractional stochastic partial differential equations on the sphere. Stoch. Environ. Res. Risk Assess. **32**, 2585-2603.
- Broadbridge, P., Kolesnik, A.D., Leonenko, N., Olenko, A. (2019) Random spherical hyperbolic diffusion. Journal of Statistical Physics. **177**, 889-916.
- Broadbridge, P., Kolesnik, A.D., Leonenko, N., Olenko, A., Omari, D. (2020) Analysis of spherically restricted random hyperbolic diffusion, arXiv:1912.08378, will appear in Entropy.

investigated three SPDEs with random initial condition given by CMB.

Model 1:

The fractional SPDE on $\mathbb{S}^2$

$$\mathrm{d}X(t,\mathbf{x}) + \psi(-\Delta_{\mathbb{S}^2})X(t,\mathbf{x}) = \mathrm{dB}^H(t,\mathbf{x}), \quad t \geq 0, \ \mathbf{x} \in \mathbb{S}^2,$$

where the *fractional diffusion operator*

$$\psi(-\Delta_{\mathbb{S}^2}) := (-\Delta_{\mathbb{S}^2})^{\alpha/2}(I - \Delta_{\mathbb{S}^2})^{\gamma/2}$$

is given in terms of Laplace-Beltrami operator $\Delta_{\mathbb{S}^2}$ on $\mathbb{S}^2$.
$\mathrm{B}^H(t,\mathbf{x})$ is a fractional Brownian motion on $\mathbb{S}^2$ with Hurst index $H \in [1/2, 1)$.

This equation is solved under the initial condition $X(0,\mathbf{x}) = \mathrm{u}(t_0,\mathbf{x})$, where $\mathrm{u}(t_0,\mathbf{x})$, $t_0 \geq 0$, is the solution of the fractional stochastic Cauchy problem at time t_0:

$$\frac{\partial \mathrm{u}(t,\mathbf{x})}{\partial t} + \psi(-\Delta_{\mathbb{S}^2})\mathrm{u}(t,\mathbf{x}) = 0$$

$$\mathrm{u}(0,\mathbf{x}) = T_0(\mathbf{x}).$$

Model 2:

The hyperbolic diffusion equation

$$\frac{1}{c^2}\frac{\partial^2 q(\mathbf{x},t)}{\partial t^2} + \frac{1}{D}\frac{\partial q(\mathbf{x},t)}{\partial t} = \Delta q(\mathbf{x},t), \quad \mathbf{x} \in \mathbb{R}^3,\ t \geq 0,\ D > 0,\ c > 0,$$

subject to the random initial conditions:

$$q(\mathbf{x},t)|_{t=0} = \eta(\mathbf{x}), \qquad \left.\frac{\partial q(\mathbf{x},t)}{\partial t}\right|_{t=0} = 0,$$

where Δ is the Laplacian in $\mathbb{R}^3$ and $\eta(\mathbf{x}) = \eta(\mathbf{x},\omega),\ \mathbf{x} \in \mathbb{R}^3,\ \omega \in \Omega$ is the random field.

We investigated $T_H(\mathbf{x},t),\ \mathbf{x} \in S^2,\ t > 0$, which is a restriction of the spatial-temporal hyperbolic diffusion field $q(\mathbf{x},t)$ to the sphere $\mathbb{S}^2$.

Model 3:

Hyperbolic diffusion equation on the sphere

$$\frac{1}{c^2}\frac{\partial^2 u(\theta,\varphi,t)}{\partial t^2}+\frac{1}{D}\frac{\partial u(\theta,\varphi,t)}{\partial t}=k^2\Delta_{(\theta,\varphi)}\,u(\theta,\varphi,t),$$

$$\theta\in[0,\pi),\ \varphi\in[0,2\pi),\ t>0,$$

where $\Delta_{(\theta,\varphi)}$ is the Laplace-Beltrami operator on the sphere

$$\Delta_{(\theta,\varphi)}=\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\varphi^2}$$

The random initial conditions are determined by the isotropic random field on the sphere

$$u(\theta,\varphi,t)\big|_{t=0}=T(\theta,\varphi),$$

$$\left.\frac{\partial u(\theta,\varphi,t)}{\partial t}\right|_{t=0}=0.$$

Theorem 1

The random solution $u(\theta, \varphi, t)$ of the initial value problem is

$$u(\theta, \varphi, t) = \exp\left(-\frac{c^2 t}{2D}\right) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta, \varphi) \xi_{lm}(t), \qquad t \geq 0,$$

where

$$\xi_{lm}(t) = \sqrt{\frac{4\pi}{2l+1}} a_{lm} Y_{l0}^{*}(\mathbf{0})[A_l(t) + B_l(t)]$$

are stochastic processes with

$$A_l(t) = \left[\cosh\left(tK_l\right) + \frac{c^2}{2DK_l} \sinh\left(tK_l\right)\right] \mathbf{1}_{\left\{l \leq \frac{\sqrt{D^2k^2+c^2}-Dk}{2Dk}\right\}}$$

and

$$B_l(t) = \left[\cos\left(tK_l'\right) + \frac{c^2}{2DK_l'} \sin\left(tK_l'\right)\right] \mathbf{1}_{\left\{l > \frac{\sqrt{D^2k^2+c^2}-Dk}{2Dk}\right\}}.$$

The covariance function of the random solution $u(\theta, \varphi, t)$ is given by

$$R(\cos\Theta, t, t') = \mathbf{Cov}(u(\theta, \varphi, t), u(\theta', \varphi', t')) = \exp\left(-\frac{c^2}{2D}(t + t')\right)$$

$$\times (4\pi)^{-1} \sum_{l=0}^{\infty} (2l + 1) C_l P_l(\cos\Theta)[A_l(t)A_l(t') + B_l(t)B_l(t')],$$

where $\Theta = \Theta_{PQ}$ is the angular distance between the points (θ, φ) and (θ', φ') and $P_l(\cdot)$ is the l-th Legendre polynomial, i.e.

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2 - 1)^l.$$

Convergence study of approximate solutions

The approximation of truncation degree $L \in \mathbb{N}$ to the solution is

$$u_L(\theta,\varphi,t) = \exp\left(-\frac{c^2 t}{2D}\right) \sum_{l=0}^{L-1} \sum_{m=-l}^{l} Y_{lm}(\theta,\varphi)\xi_{lm}(t).$$

Theorem 2

For $t > 0$ the truncation error is bounded by

$$\|u(\theta,\varphi,t) - u_L(\theta,\varphi,t)\|_{L_2(\Omega\times\mathbb{S}^2)} \le C\left(\sum_{l=L}^{\infty}(2l+1)C_l\right)^{1/2}.$$

Moreover, for $L > \frac{\sqrt{D^2k^2+c^2}-Dk}{2Dk}$ it holds

$$\|u(\theta,\varphi,t) - u_L(\theta,\varphi,t)\|_{L_2(\Omega\times\mathbb{S}^2)} \le C\exp\left(-\frac{c^2 t}{2D}\right)\left(\sum_{l=L}^{\infty}(2l+1)C_l\right)^{1/2}.$$

Corollary 3

Let the angular power spectrum $\{C_l, l = 0, 1, 2, ...\}$ of the random field $T(\theta, \varphi)$ from the initial condition decay algebraically with order $\alpha > 2$, i.e. $C_l \leq C \cdot l^{-\alpha}$ for all $l \geq l_0$. Then,

(i) *for $L > \max(l_0, \frac{\sqrt{D^2k^2+c^2}-Dk}{2Dk})$ the truncation error is bounded by*

$$\|u(\theta, \varphi, t) - u_L(\theta, \varphi, t)\|_{L_2(\Omega\times\mathbb{S}^2)} \leq C \exp\left(-\frac{c^2 t}{2D}\right) L^{-\frac{\alpha-2}{2}},$$

(ii) *for any $\varepsilon > 0$ it holds*

$$\mathbf{P}\Big(|u(\theta, \varphi, t) - u_L(\theta, \varphi, t)| \geq \varepsilon\Big) \leq \frac{C \exp\left(-c^2 t/D\right)}{L^{\alpha-2}\varepsilon^2},$$

(iii) *for all $\theta \in [0, \pi)$, $\varphi \in [0, 2\pi)$ and $t > 0$ it holds*

$$|u(\theta, \varphi, t) - u_L(\theta, \varphi, t)| \leq L^{-\beta} \quad \mathbf{P} - a.s.,$$

where $\beta \in \left(0, \frac{\alpha-3}{2}\right)$ and $\alpha > 3$.

Hölder continuity of solutions and their approximations

Theorem 4

Let $u(\theta, \varphi, t)$ be the solution to the initial value problem and the angular power spectrum $\{C_l, l = 0, 1, 2, \ldots\}$ of the random field from the initial condition satisfies

$$\sum_{l=0}^{\infty} (2l+1)^3 C_l < \infty.$$

Then there exists a constant C such that for all $t > 0$ it holds

$$\|u(\theta, \varphi, t+h) - u(\theta, \varphi, t)\|_{L_2(\Omega \times \mathbb{S}^2)} \leq Ch, \quad \text{when} \quad h \to 0+,$$

where the constant C depends only on the parameters c, D and k.

Corollary 5

If the assumptions of Theorem 4 hold true, then there exists a constant C_L *such that for all* $t > 0$ *it holds*

$$\|u_L(\theta, \varphi, t+h) - u_L(\theta, \varphi, t)\|_{L_2(\Omega\times\mathbb{S}^2)} \leq C_L h, \quad when \quad h \to 0+,$$

where the constant C_L *depends only on the parameters* c, D *and* k.

Theorem 6

Let $u(\theta, \varphi, t)$ be the solution to the initial value problem and the angular power spectrum $\{C_l, l = 0, 1, 2, ...\}$ of the random field from the initial condition satisfies

$$\sum_{l=0}^{\infty}(2l+1)^{1+2\gamma}C_l < \infty, \quad \gamma \in [0,1].$$

Then, there exists a constant C such that for all $t > 0$ it holds

$$\mathbf{MSE}(u(\theta, \varphi, t) - u(\theta', \varphi', t)) \leq C\sum_{l=0}^{\infty} C_l\,(2l+1)^{1+2\gamma}\,(1-\cos\Theta)^{\gamma},$$

where Θ is the angular distance between (θ, φ) and (θ', φ') and the constant C depends only on the parameters c, D and k.

Short and long memory

The random field $u(\theta, \varphi, t)$ is short memory if

$$\int_0^{+\infty} |R(\cos\Theta, t+h, t)| dh < +\infty$$

for all t and $\Theta \in [0, \pi]$.
If the integral is divergent, the field has long memory.

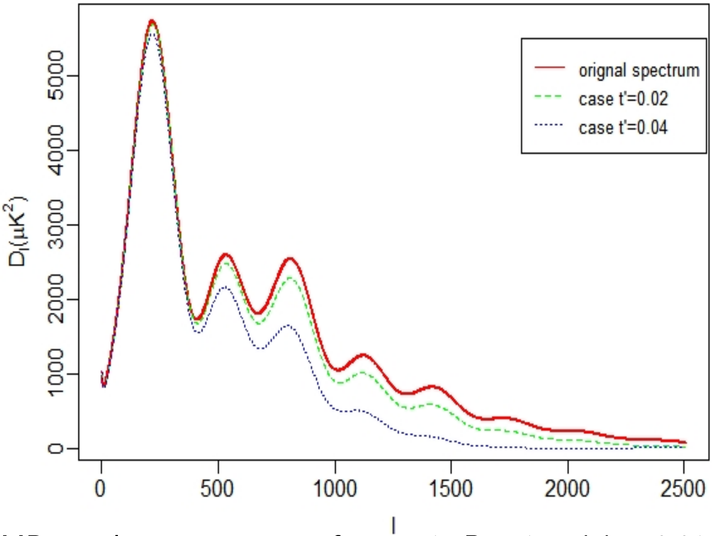
Let $G(\cdot)$ be a spectral measure of the initial condition field, i.e. its covariance function has the form

$$\mathrm{Cov}(T(\theta, \varphi), T(\theta, \varphi)) = R(\cos\Theta) = \int_0^{\infty} \frac{\sin(2\mu \sin\frac{\Theta}{2})}{2\mu \sin\frac{\Theta}{2}} \, G(d\mu).$$

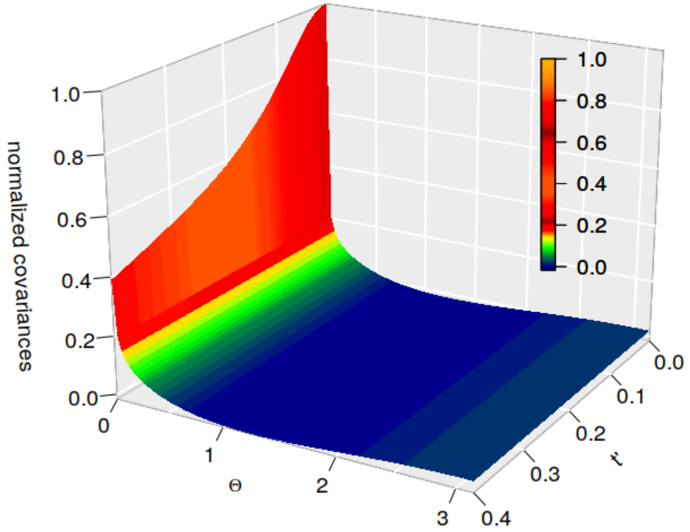
Theorem 7

The random field $u(\theta, \varphi, t)$ has short-memory if and only if $\mu^{-2} G(d\mu)$ is integrable in a neighbourhood of the origin.

Numerical studies



Scaled CMB angular power spectra for $c = 1$, $D = 1$ and $k = 0.01$ at time $t' = 0$, 0.02 and 0.04.



Covariance for $c = 1$, $D = 1$ and $k = 0.01$ at time lag t' and angular distance Θ.

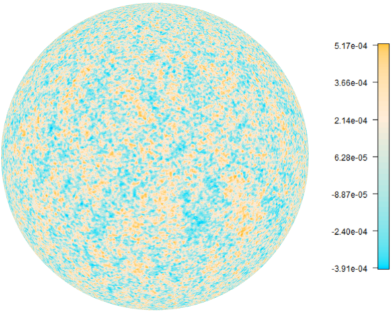


Figure: Error field of the approximation with $L = 200$ to the solution with $c = 1$, $D = 1$ and $k = 0.01$ at time $t' = 0.04$.

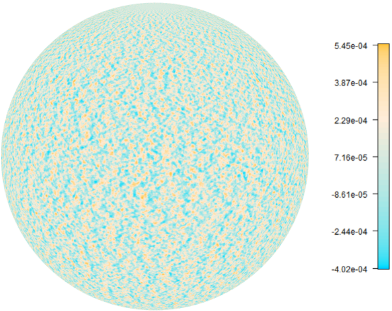
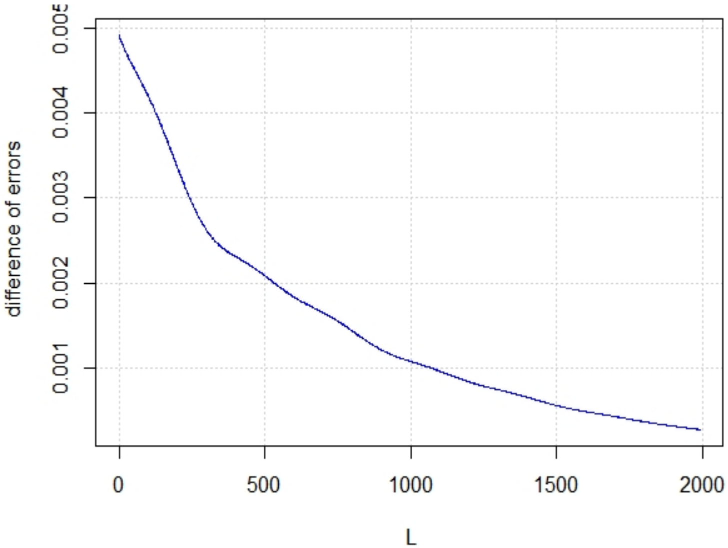


Figure: Error field of the approximation with $L = 400$ to the solution with $c = 1$, $D = 1$ and $k = 0.01$ at time $t' = 0.04$.



Difference of the mean $L_2(\Omega \times \mathbb{S}^2)$-errors and their upper bound for $c = 1$, $D = 1$ and $k = 0.1$ at $t' = 10$.

Sensitivity to parameters

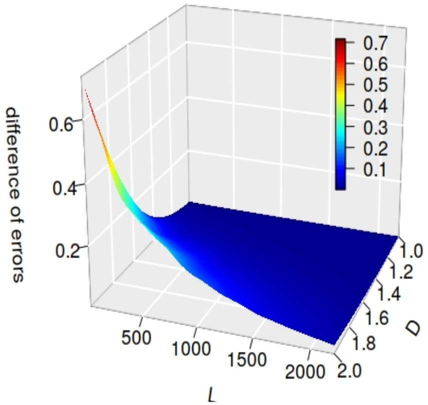


Figure: Difference of the mean $L_2(\Omega \times \mathbb{S}^2)$-errors and their upper bound for $c = 1$ and $k = 0.1$ at $t' = 10$.

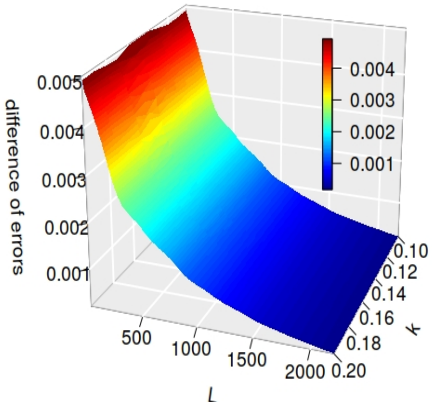
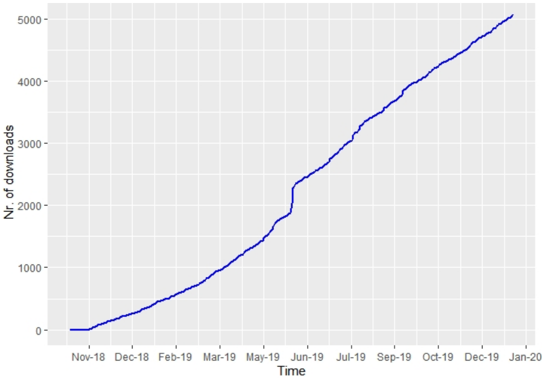


Figure: Difference of the mean $L_2(\Omega \times \mathbb{S}^2)$-errors and their upper bound for $c = 1$ and $D = 1$ at $t' = 10$.

R package rcosmo

The first R package for statistical analysis of CMB and HEALPix data.
The package has more than 100 different functions.

- https://CRAN.R-project.org/package=rcosmo
- D.Fryer, M.Li, A.Olenko. (2019) rcosmo: R Package for Analysis of Spherical, HEALPix and Cosmological Data, arxiv 1907.05648



Directions for future research

- Study sharpness of the obtained upper bounds.
- Develop statistical estimators of equations' parameters and study their properties.
- Extend the methodology to tangent vector fields on the sphere.
- Extend the methodology to tensor random fields on the sphere.
- Add new functions to rcosmo.

References

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This research was supported under ARC Discovery Projects DP160101366.