Transition in subcritical shear flows –
Invariant solutions and the edge of chaos.

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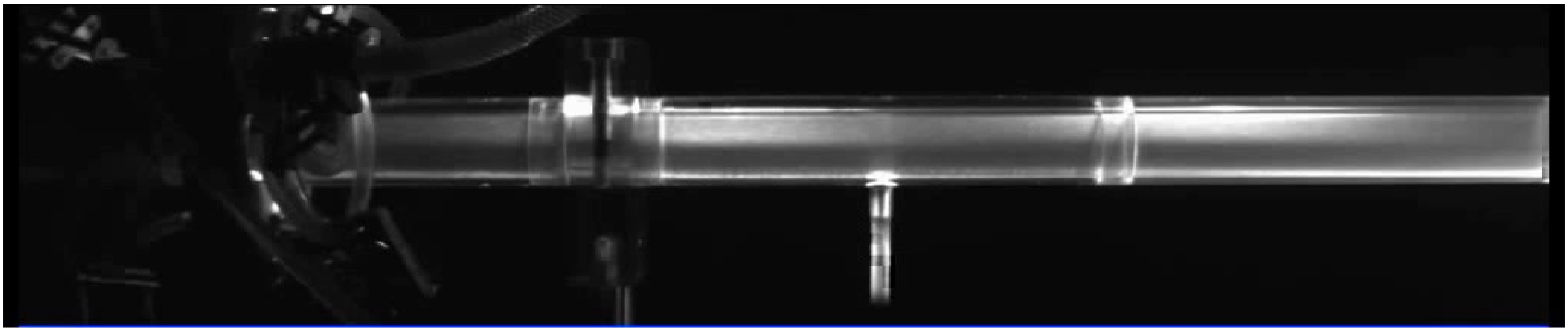
[3] School of Physics, Georgia Tech. (chaosbook.org)

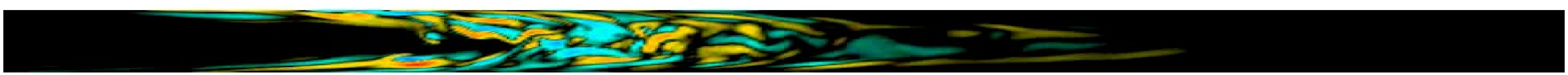
[4] LIMSI-CNRS, Orsay, France.

Localised turbulence in a pipe



Peixinho & Mullin, PRL

Illuminated flakes

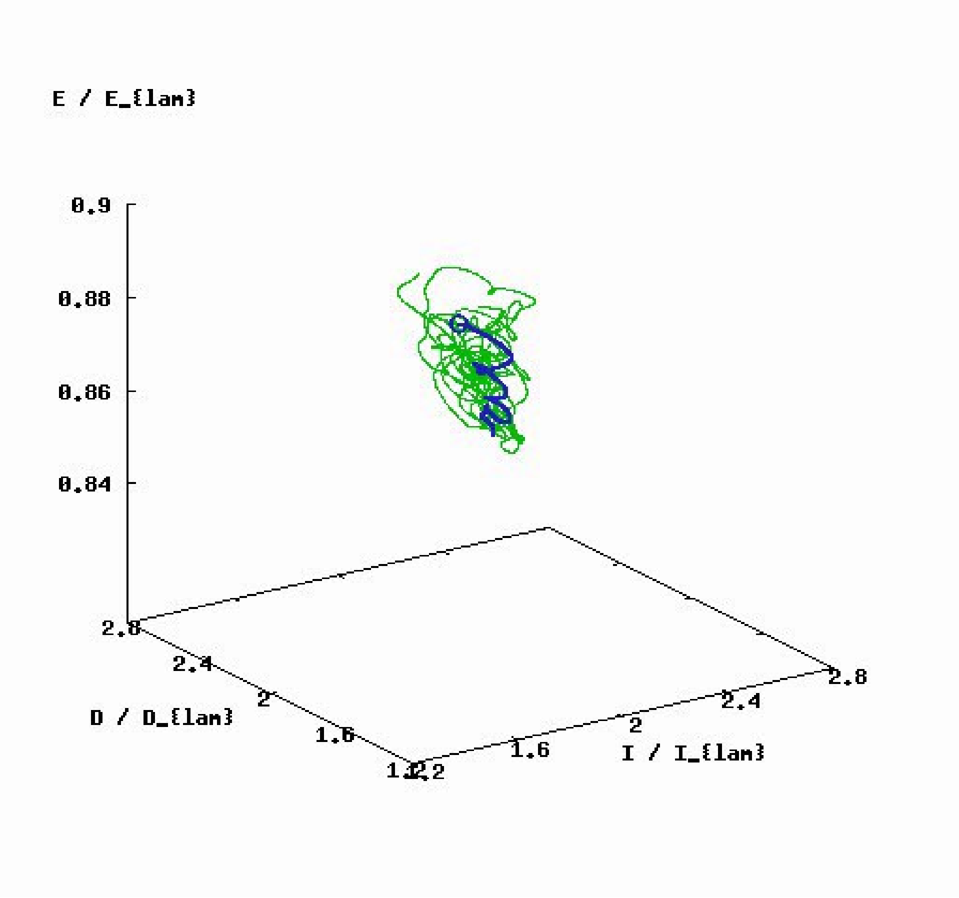


Simulation

Axial vorticity

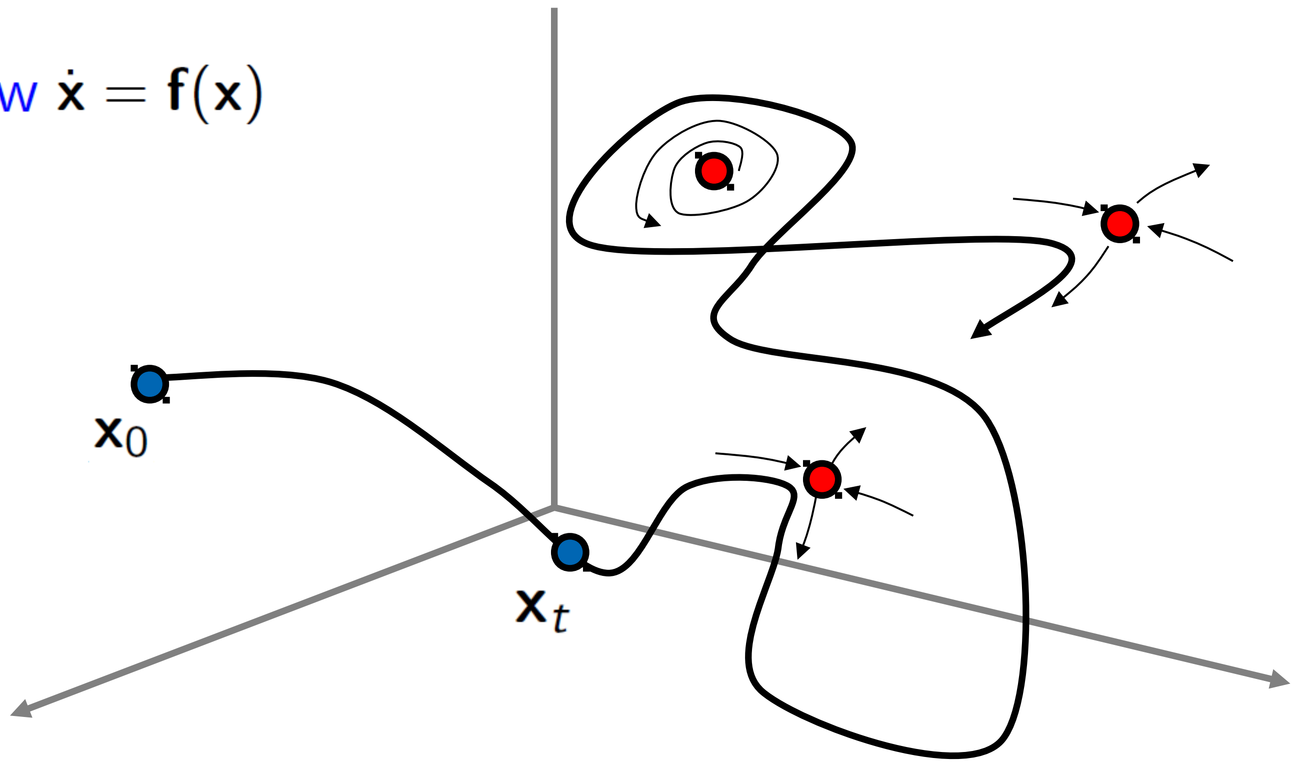
APPROACH: TURBULENCE AS A CHAOTIC DYNAMICAL SYTSTEM

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$



Trajectory in phase space, structured by stable/unstable manifolds of the equilibrium points.

flow $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$



flow-map $\boldsymbol{\Phi}^t$

$$\mathbf{x}_t = \boldsymbol{\Phi}^t(\mathbf{x}_0)$$

$$\mathbf{x}_0 = \boldsymbol{\Phi}^t(\mathbf{x}_0)\,, \qquad \text{[equilibrium = fixed point]}$$

$$\mathbf{x}_p = \boldsymbol{\Phi}^T(\mathbf{x}_p)\,, \qquad \text{[periodic orbit (PO)]}$$

Jacobian-free Newton-Krylov

Knoll and Keyes (2004)
Viswanath (2007)

want roots $\mathbf{x}_p$ such that

$$\mathbf{F}(\mathbf{x}_p) = \mathbf{0} \quad \text{where} \quad \mathbf{F}(\mathbf{x}) = \mathbf{\Phi}(\mathbf{x}) - \mathbf{x}.$$

Newton iteration, given guess $\mathbf{x}_0$:

$$(a) \;\; \mathbf{x}_{i+1} = \mathbf{x}_i + \delta\mathbf{x}_i \quad \text{where} \quad (b) \;\; \left.\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right|_{\mathbf{x}_i} \delta\mathbf{x}_i = -\mathbf{F}(\mathbf{x}_i)\,.$$

(b) is in form

$$A\,\delta\mathbf{x} = \mathbf{b}\,. \qquad \rightarrow \text{GMRES}$$

for given $\delta\mathbf{x}$,

$$\left.\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right|_{\mathbf{x}_i} \delta\mathbf{x} \;\approx\; \frac{1}{\epsilon}\left(\mathbf{F}(\mathbf{x}_i + \epsilon\,\delta\mathbf{x}) - \mathbf{F}(\mathbf{x}_i)\right).$$

- Only involves evaluations of **F**(**x**).
- No preconditioner necessary!

Jacobian-free Newton-Krylov

Knoll and Keyes (2004)
Viswanath (2007)

- Need to solve

$$\mathbf{F}(\mathbf{x}, T) = \boldsymbol{\Phi}^T(\mathbf{x}) - \mathbf{x} = \mathbf{0}\,,$$

for $(\mathbf{x}, T)$.

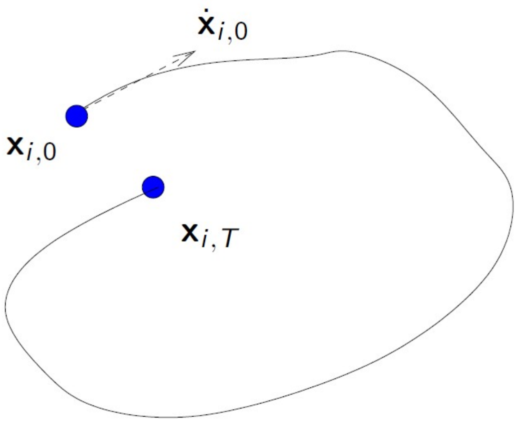
- Augment whole system: Put

$$\tilde{\mathbf{x}}_i = (\mathbf{x}_i,\ T_i) \quad \text{and} \quad \tilde{\mathbf{b}} = (-\mathbf{F}(\tilde{\mathbf{x}}_i),\, 0)\,.$$

- Now want to solve a system of the form

$$A\,\tilde{\delta\mathbf{x}}_i = \tilde{\mathbf{b}}\,,$$

$$\langle \dot{\mathbf{x}}_i | \delta\mathbf{x}_i \rangle = 0\,.$$



- Let

$$A\,\tilde{\delta\mathbf{x}} = \left(\left.\frac{\partial \mathbf{F}}{\partial \tilde{\mathbf{x}}}\right|_{\tilde{\mathbf{x}}_i} \tilde{\delta\mathbf{x}},\ \langle \dot{\mathbf{x}}_i | \delta\mathbf{x} \rangle \right)\,.$$

Jacobian-free Newton-Krylov - Hookstep

Knoll and Keyes (2004)
Viswanath (2007)

$$A\ \delta\mathbf{x} = \mathbf{b}\,.$$

GMRES:

Look for solution $\delta\mathbf{x}$ in $\mathrm{span}\{\mathbf{K}_1,\ \mathbf{K}_2, \ldots,\ \mathbf{K}_m\}$.
i.e.

$$\delta\mathbf{x} = c_1\,\mathbf{K}_1 + c_2\,\mathbf{K}_2 + \ldots + c_m\,\mathbf{K}_m$$

- Put $\mathbf{K}_1 = \mathbf{b}/||\mathbf{b}||$.
- Evaluate $\tilde{\mathbf{K}}_{i+1} = A\,\mathbf{K}_i$.
- Orthonormalise $\tilde{\mathbf{K}}_{i+1}$ against $\mathbf{K}_j$, $j < i$ by Gram-Schmidt $\to\ \mathbf{K}_{i+1}$.
- Minimise

$$\mathrm{err} = ||A\ \delta\mathbf{x} - \mathbf{b}\,|| \qquad \text{over coefficients } c_j,\ j \le i+1$$

Hookstep:

- Minimise

$$\mathrm{err} = ||A\ \delta\mathbf{x} - \mathbf{b}\,|| \qquad \text{over coefficients } c_j,\ j \le i+1$$

subject to constraint

$$||\delta\mathbf{x}|| < \delta\,.$$

- δ is size of trust region.

Code at openpipeflow.org (non-problem specific: black box) arxiv:1908.06730

www.openpipeflow.org/index.php?title=Newton-Krylov_method

openpipeflow

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Newton-Krylov method

Free to download:

CODE (MATLAB / GNU Octave): Template-Example: File:Newton Lorenz m.zip / File:Newton Lorenz m.tgz

CODE (FORTRAN): Template-Example: File:Newton Lorenz.f90.

Info on main subroutine: File:NewtonHook.f90 (essentially same for FORTRAN/MATLAB versions)

— —

(An Extended overview of the Newton-Krylov method is here (pdf,arxiv): [1] See section 4 on using the code.)

— —

The codes above implement the **Jacobian-free Newton-Krylov (JFNK) method** for solving

$$\mathbf{F}(\mathbf{x}) = \mathbf{0},$$

where $\mathbf{x}$ and $\mathbf{F}$ are n-vectors, **supplemented with a Hookstep--Trust-region approach**.

This is a powerful method that can solve for $\mathbf{x}$ for a complicated nonlinear $\mathbf{F}(\mathbf{x})$. For example, to find an *equilibrium* solution or a *periodic orbit*, let $\mathbf{F}(\mathbf{x})$ =
condition $\mathbf{x}$.

Newton-Raphson method [edit]

To find the roots x of a function $f(x)$ in one dimension, given an initial guess x_0, the Newton-Raphson method generates improvements using the iteration
The iteration can be re-expressed as

$$x_{i+1} = x_i + \delta x_i \quad \text{where} \quad f'(x_i)\,\delta x_i = -f(x_i)\,.$$

The extension of Newton's method to an n-dimensional system is then

$$(a) \quad \mathbf{x}_{i+1} = \mathbf{x}_i + \delta\mathbf{x}_i \quad \text{where} \quad (b) \quad \left.\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right|_{\mathbf{x}_i} \delta\mathbf{x}_i = -\mathbf{F}(\mathbf{x}_i)\,.$$

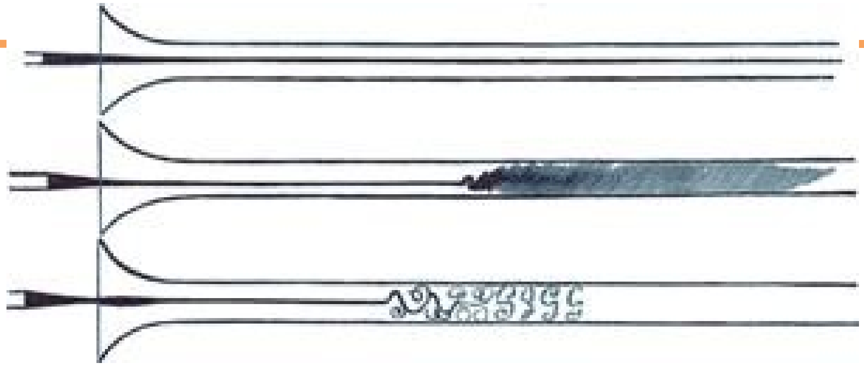
PROBLEMS

1. Where to get starting guess x_0 !?

2. Look for recurrences: ‘small’ $||\mathbf{x}_t - \mathbf{x}_{t-\Delta t}||$

how small? what norm!?

Osborne Reynolds' Experiments, 1883



Observed importance of combination

$Re = LU / \nu$

L, diameter

U, mean axial flow

ν, kinematic viscosity

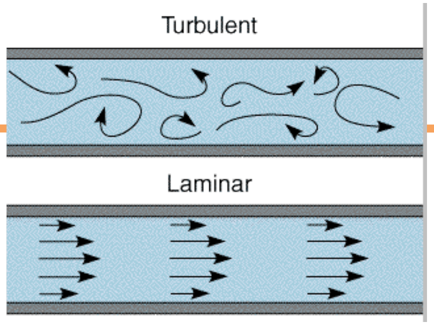
"The only idea I had formed before commencing the experiments, was that at some critical velocity the motion must become unstable, so that any disturbance from perfectly steady motion would result in eddies."

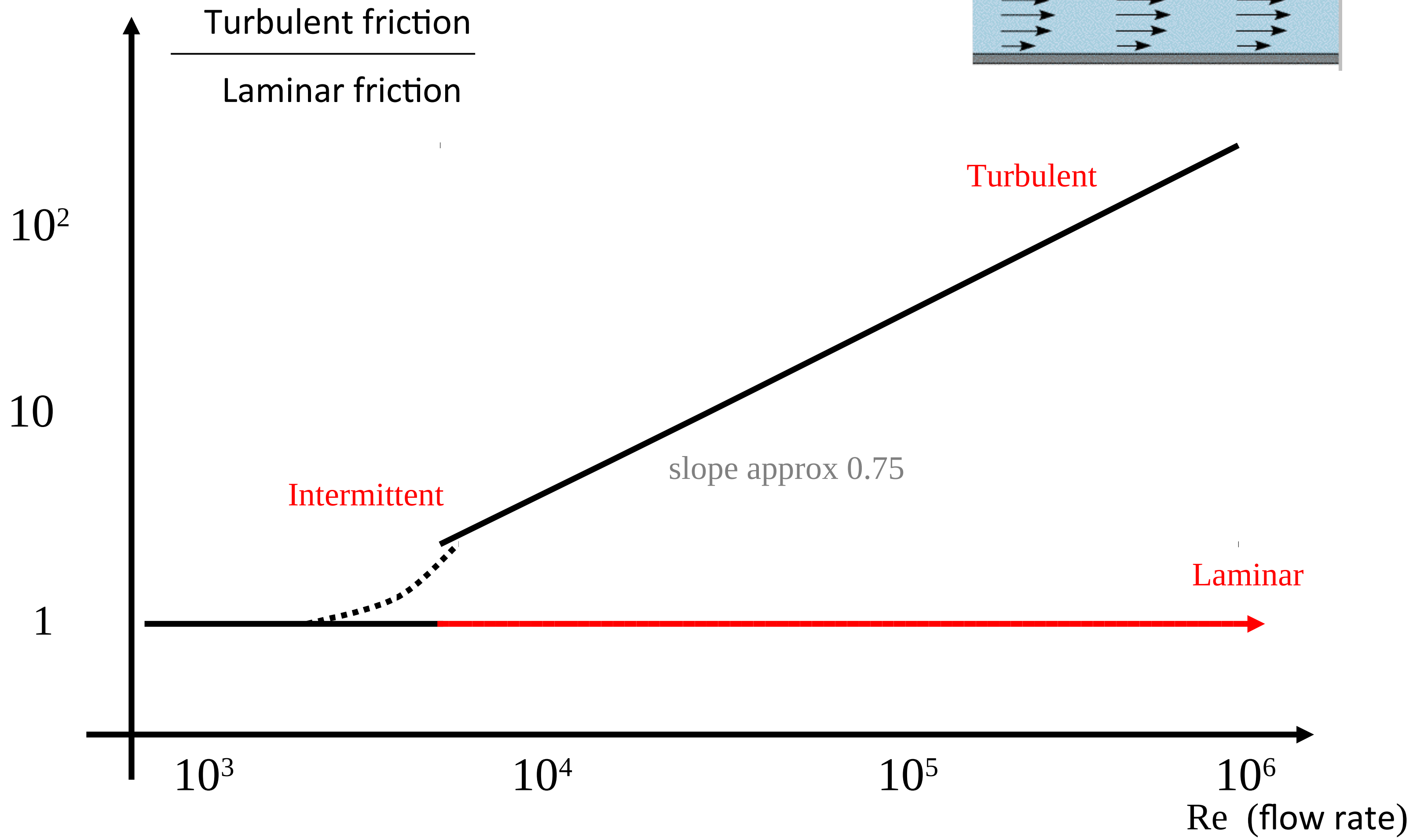
i.e. surprised to not find critical flow rate for linear instability

"…the steady motion breaks down suddenly… for disturbances of the magnitude that cause it to break down… while it is stable for a smaller disturbance…"

i.e. finite amplitude disturbance required to trigger turbulence

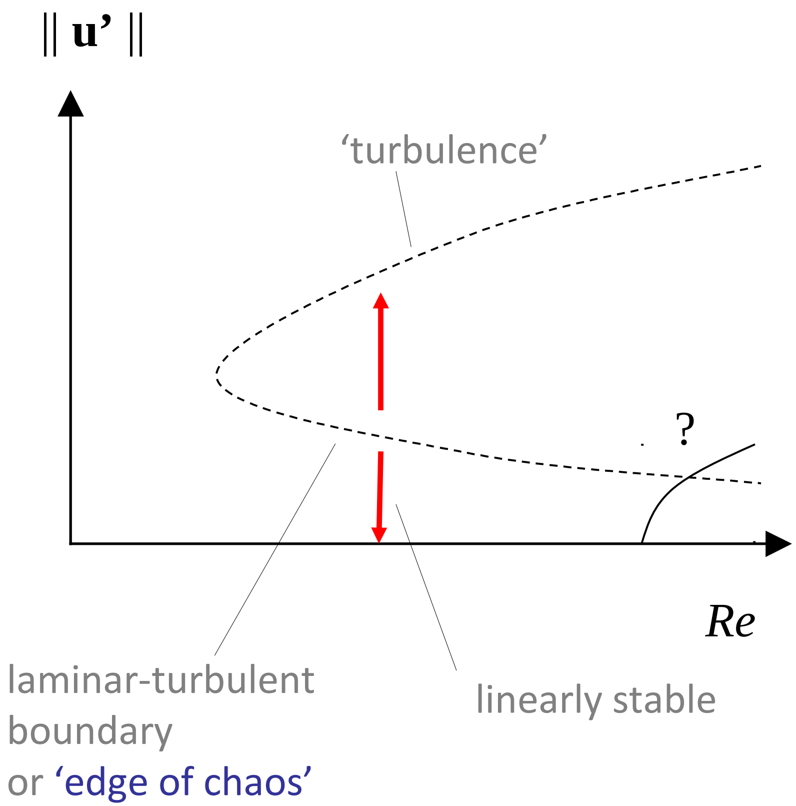
Turbulent friction / Laminar friction (drag).

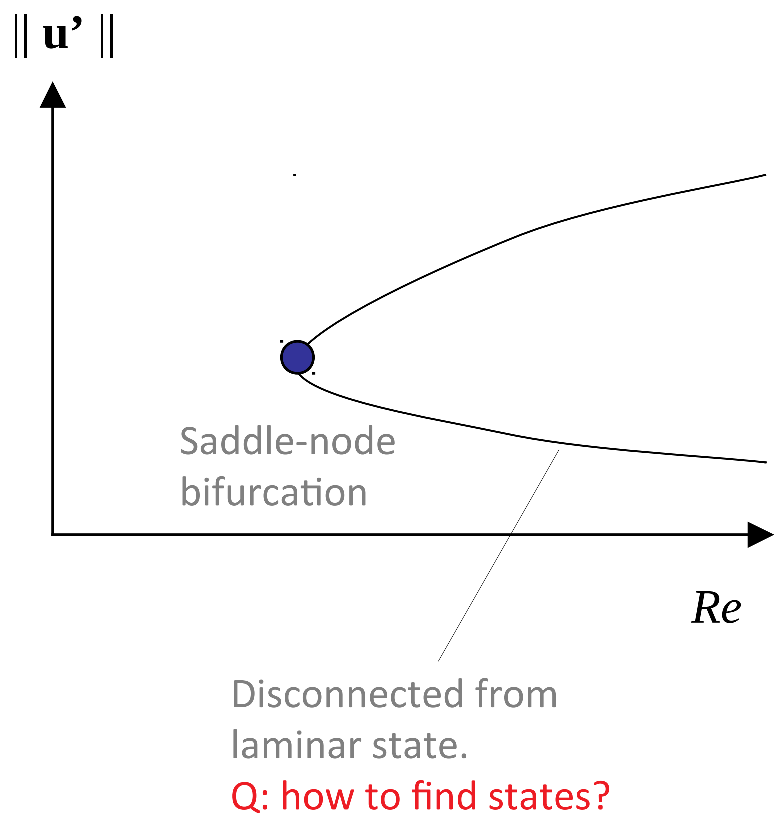




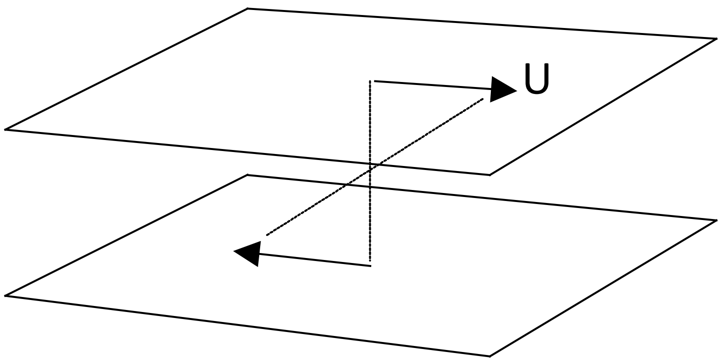
Adapted from Nikuradse (1950) / Blasius (1913)

Subcritical instability, nonlinearity important

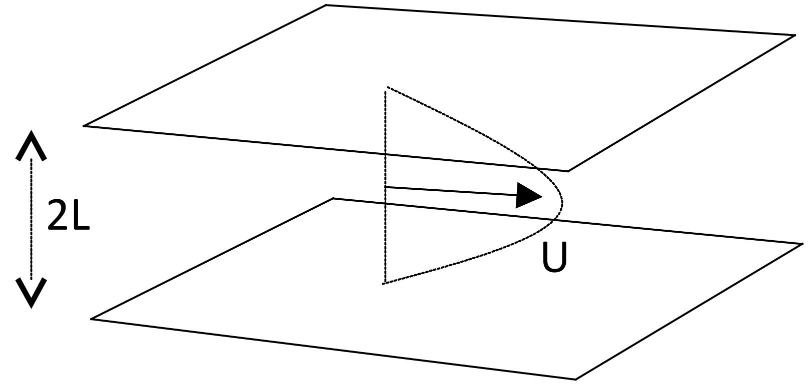




Shear Flows



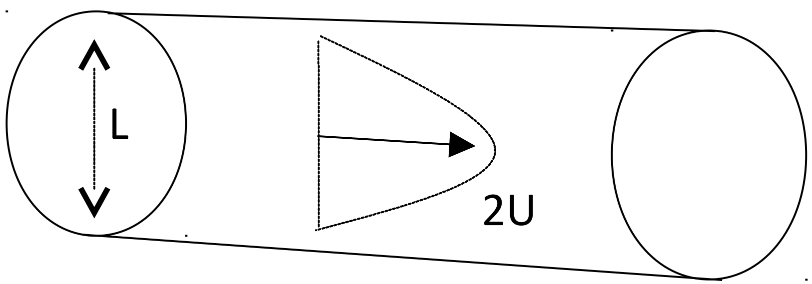
Couette flow



channel flow

Reynolds number Re = LU / ν

Kinematic viscosity ν

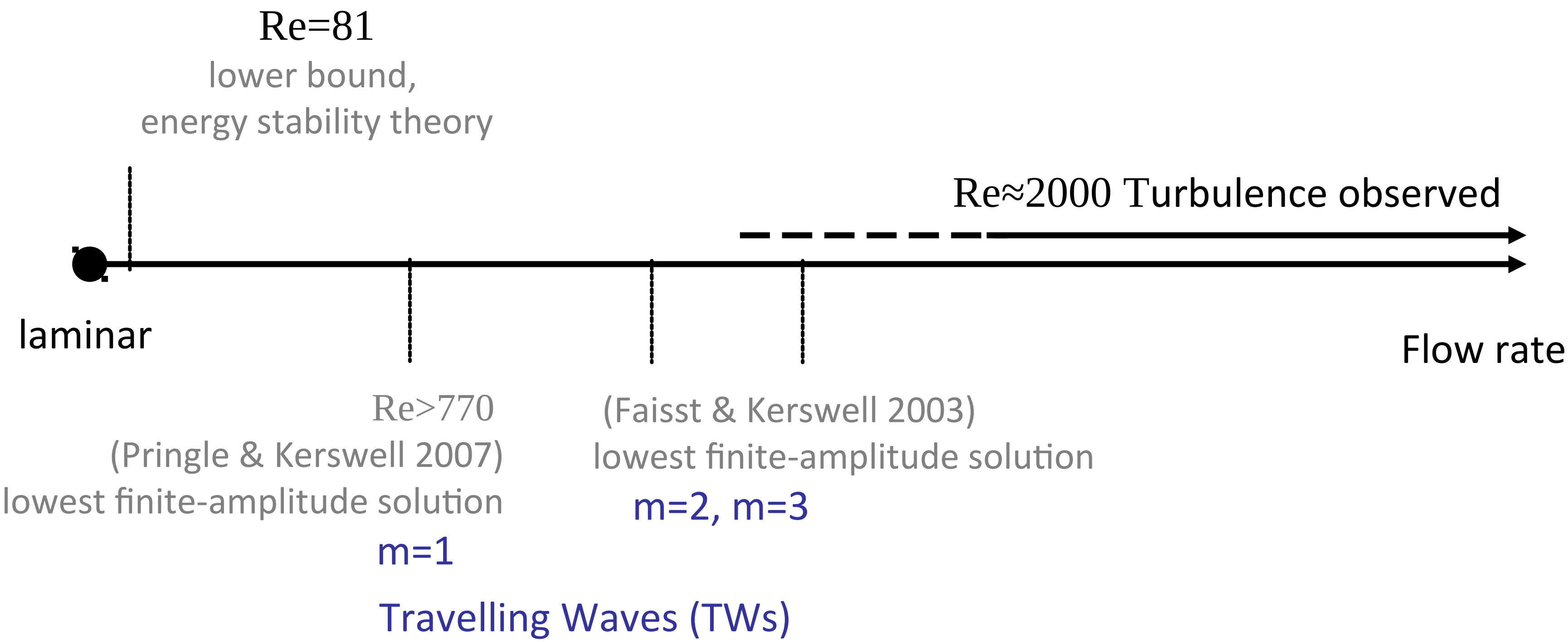


pipe flow

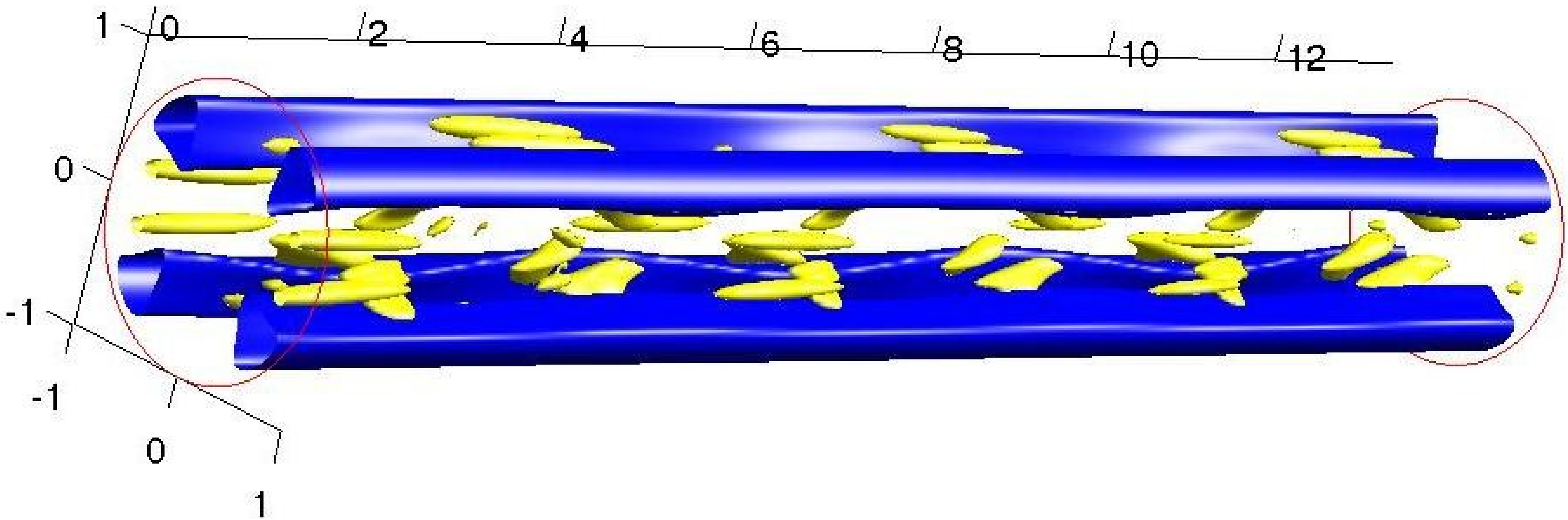
Stability of shear-flows

	Re turbulence observed	linear instability
Pipe flow	1720	inf.?
Channel flow	950	5772
ASBL	367	54370
Couette flow	312	inf.

Pipe flow


Re=81
lower bound,
energy stability theory
Re≈2000 Turbulence observed
laminar
Flow rate
Re>770
(Pringle & Kerswell 2007)
lowest finite-amplitude solution
m=1
(Faisst & Kerswell 2003)
lowest finite-amplitude solution
m=2, m=3
Travelling Waves (TWs)

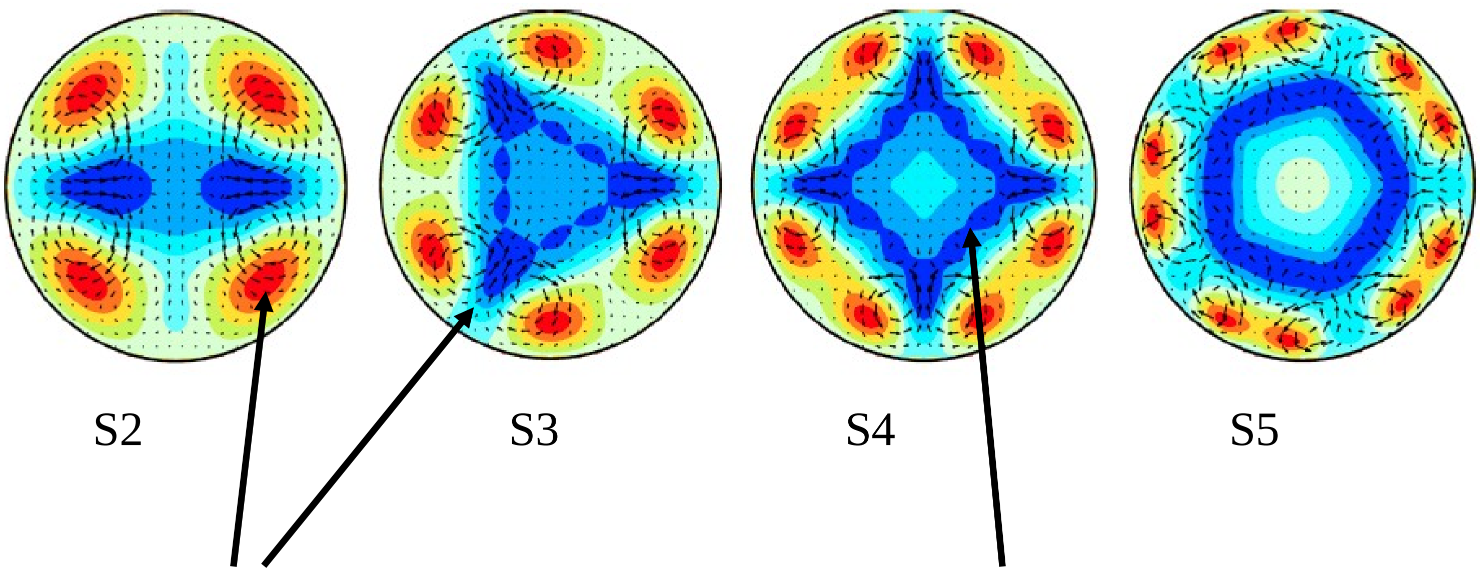
Travelling Waves (TWs) / Vortex-Wave Interaction (VWI) state / ‘Exact’ Coherent Structures (ECS) / Invariant Solutions



[Boundary-layer:] Hall & Smith (1991), via asymptotic theory.

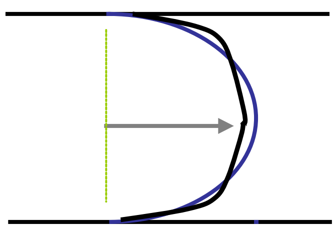
[Plane-couette:] Waleffe (1998), via continuation from Taylor-Couette

Travelling Wave solutions (TWs)



Streaks near walls

Slower core

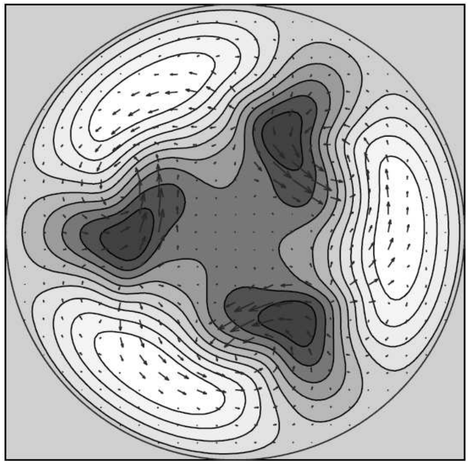


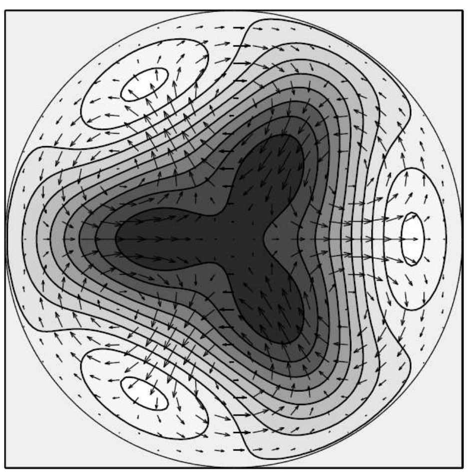
Faisst & Eckhardt (2003)
Wedin & Kerswell (2004)

[Pipe:] via (painful) continuation from system with body force

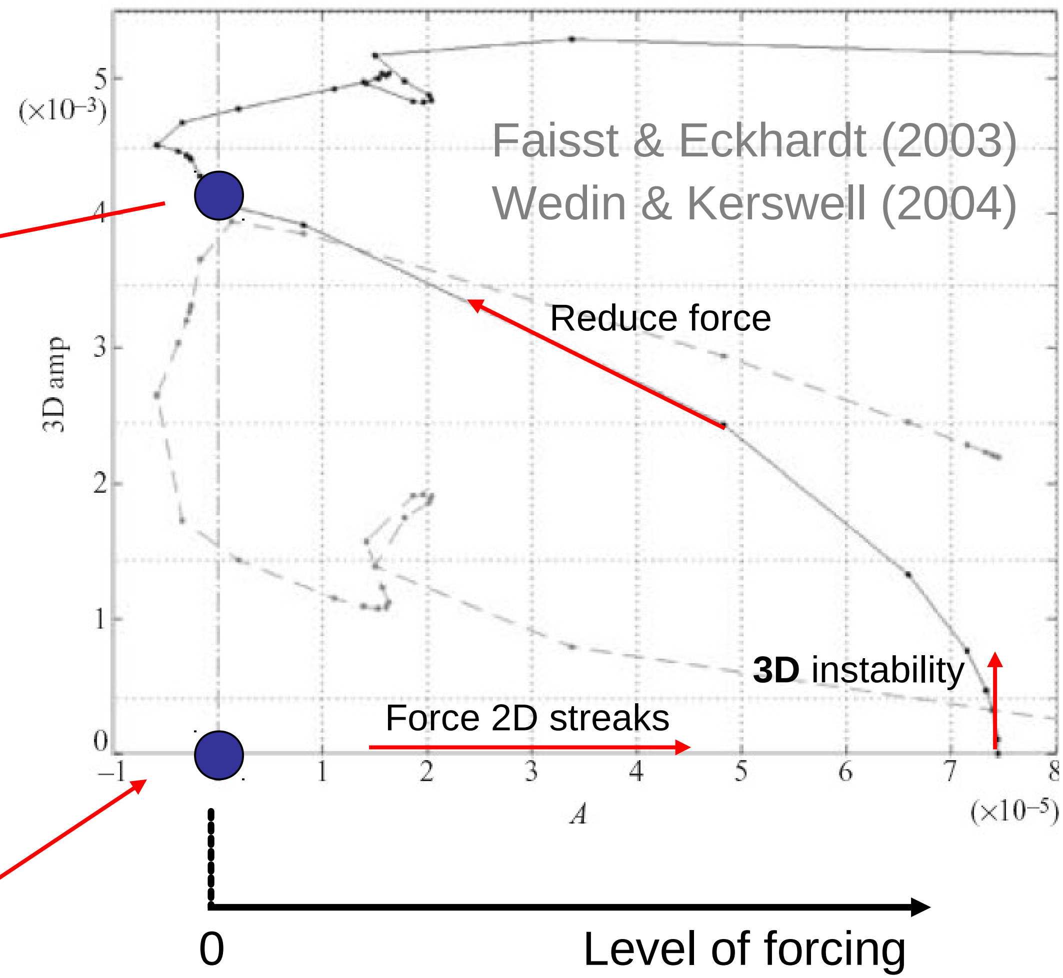
Discovery of TWs. (Self sustaining cycle completed 'by hand'!)

END:
unforced 3D state, self-sustained



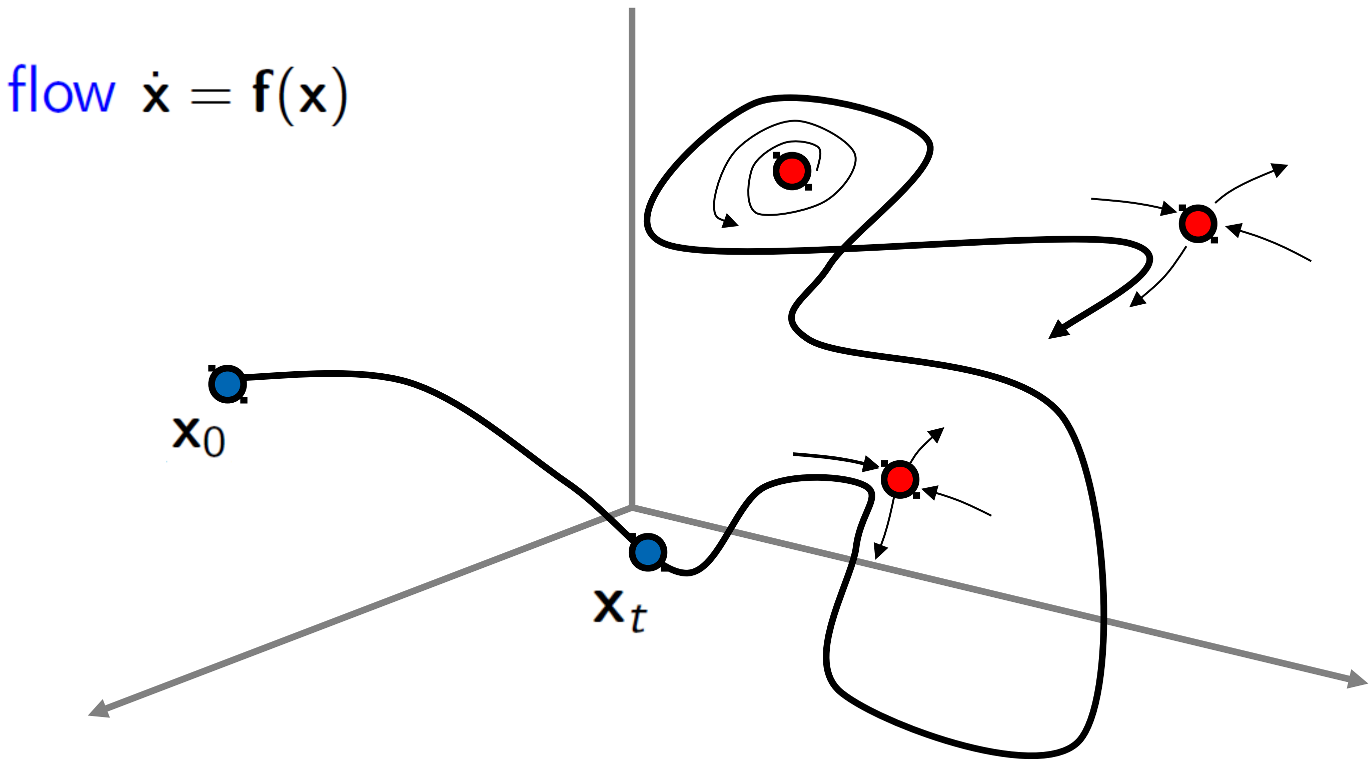


START:
least-stable 2D eigen mode → Force



- A lot can go wrong!
- State really linked to dynamics?

Trajectory in phase space, structured by stable/unstable manifolds of the equilibrium points.

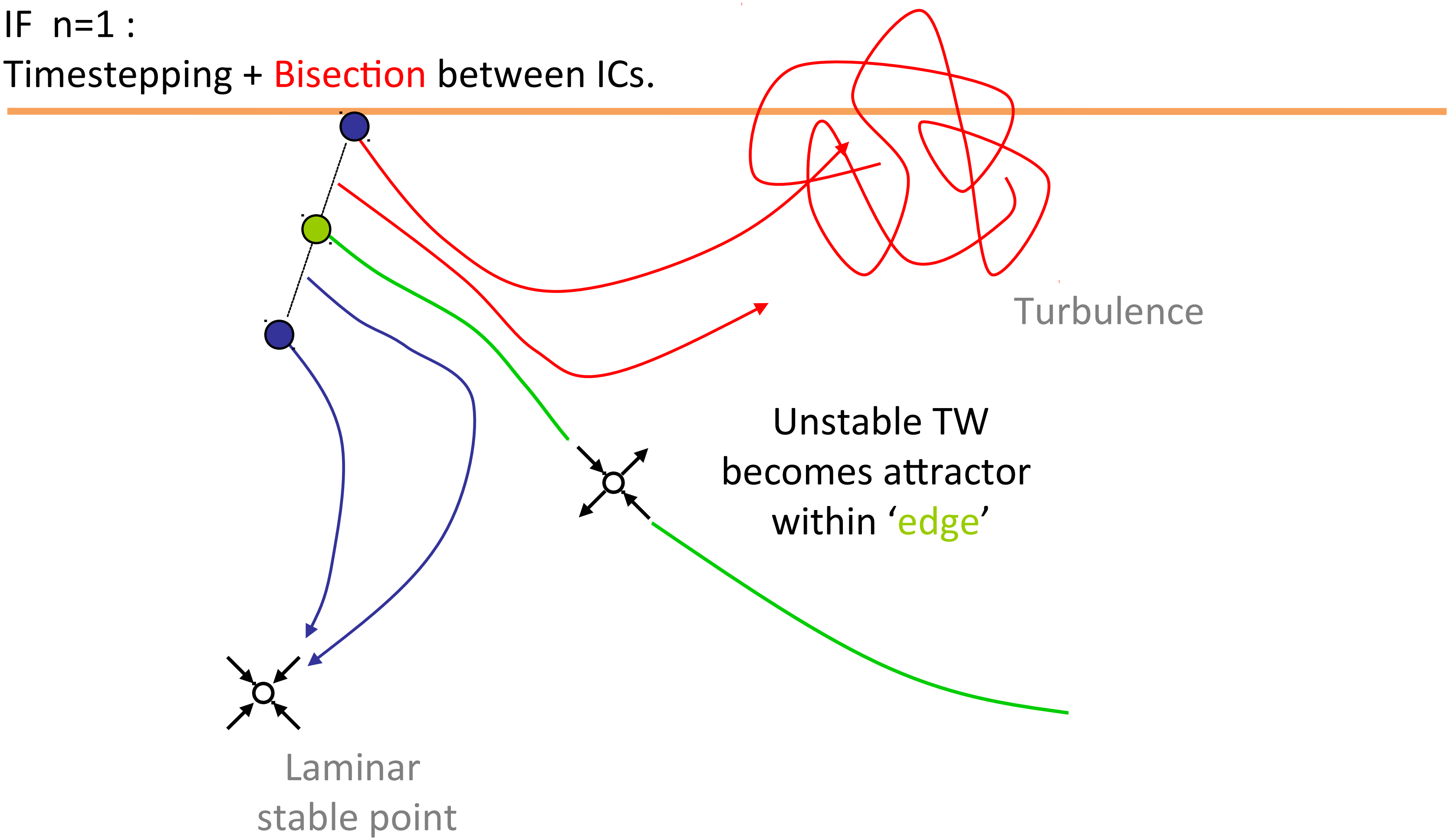


Dimension of the space N → inf.,

In simulations, N = O(10^5-10^6), dimension of unstable manifolds n = O(10)

IF n=1 :

Timestepping + Bisection between ICs.



Laminar turbulent boundary calculated by bisection:

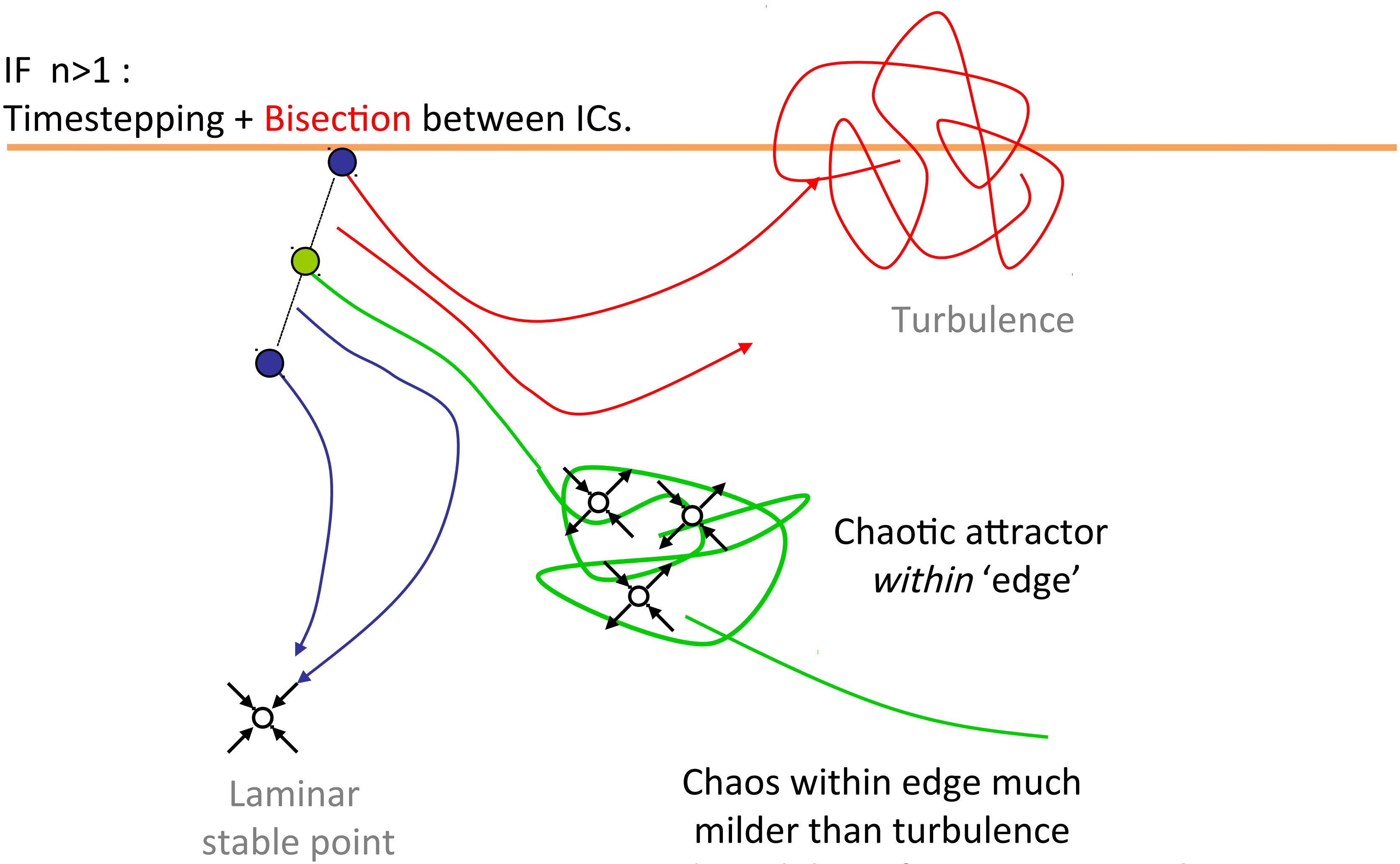
Skufca, Yorke & Eckardt (2006) *for a reduced model of shear flow*

Schneider, Eckhardt & Yorke (2007) *for a short periodic pipe*

Itano & Toh (2000) *for channel flow.*

IF n>1 :

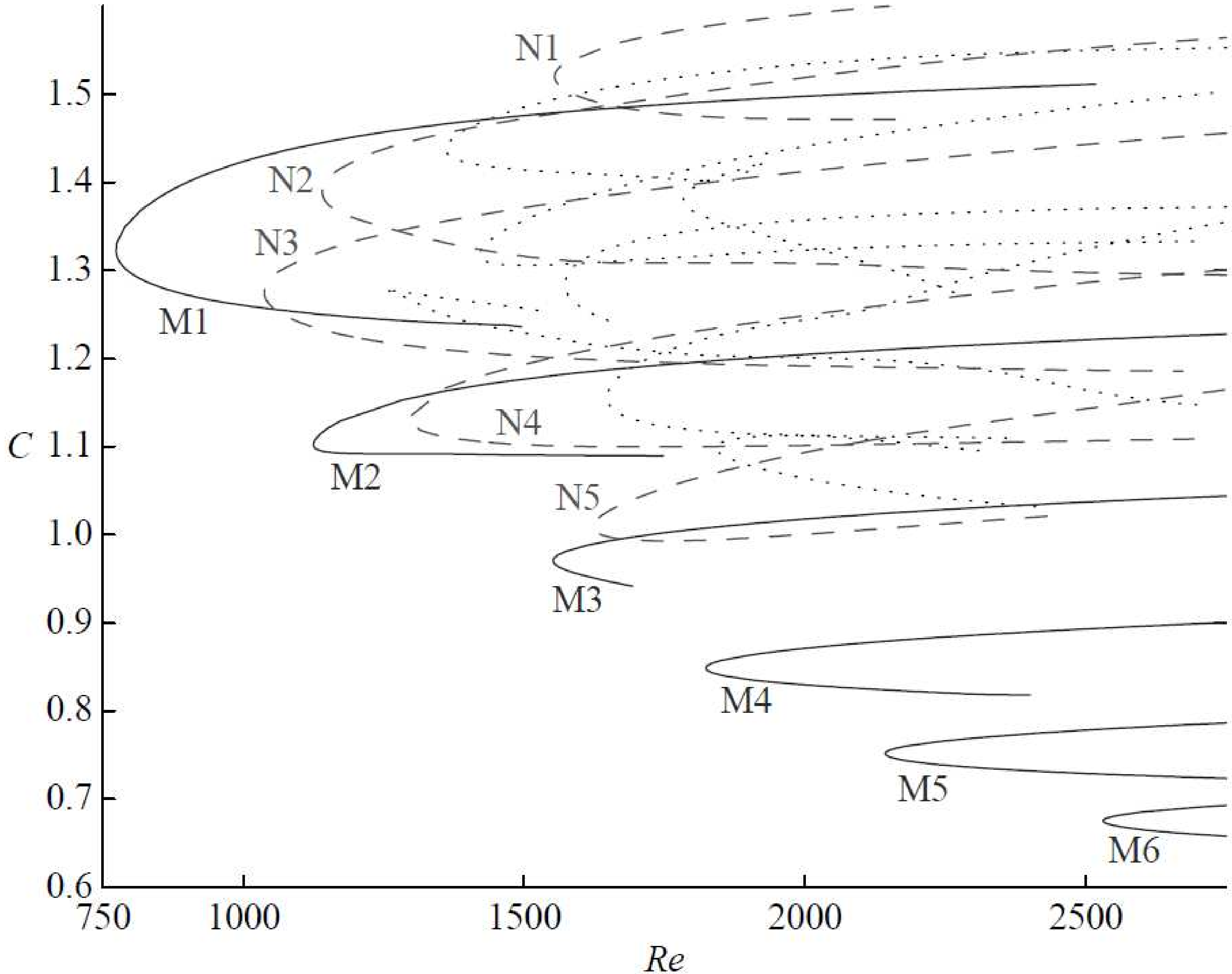
Timestepping + Bisection between ICs.



Chaos within edge much milder than turbulence
→ good candidates for Newton search

Duguet, W. & Kerswell 2008,10 JFM
long pipe, localised coherent structures within laminar-turbulent boundary

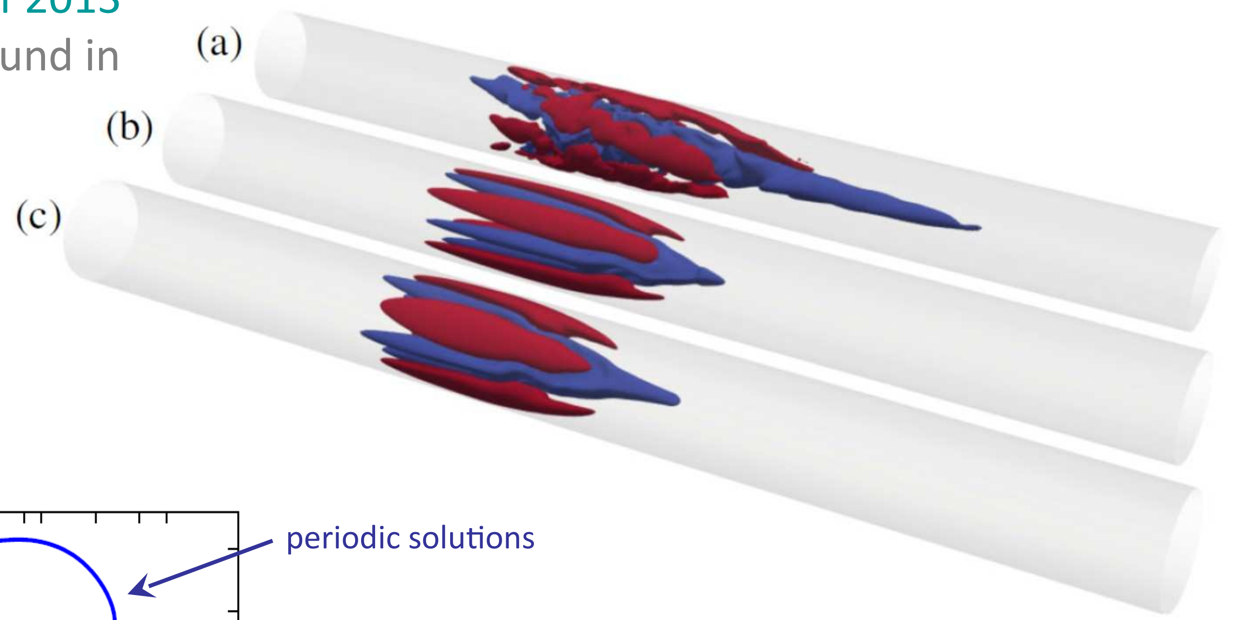
Discovery of many (spatially periodic) TWs solutions for pipe flow

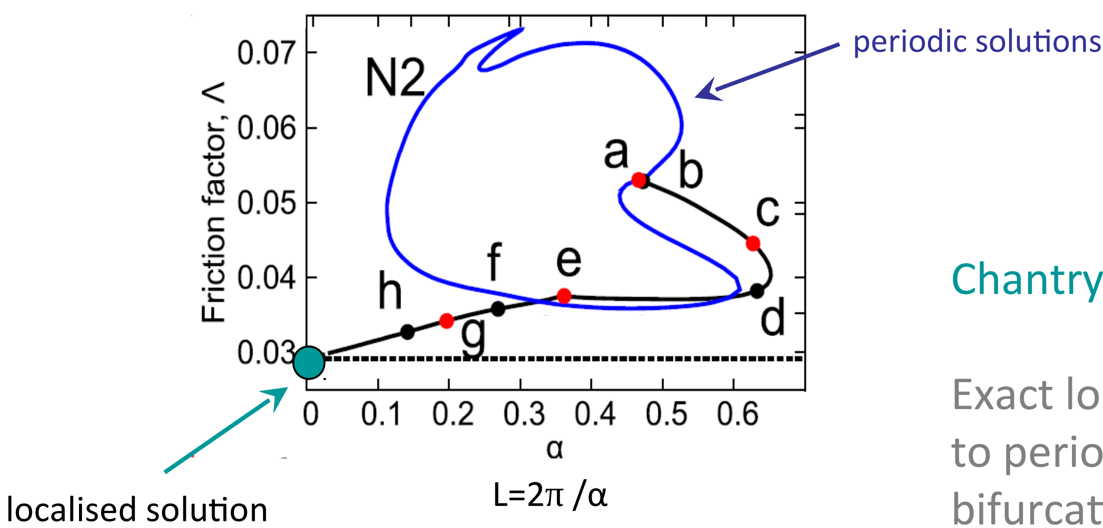


Puff-like invariant solutions

Avila, Mellibovsky, Rolland & Hof 2013

Exact localised periodic orbits found in m=2 + mirror space





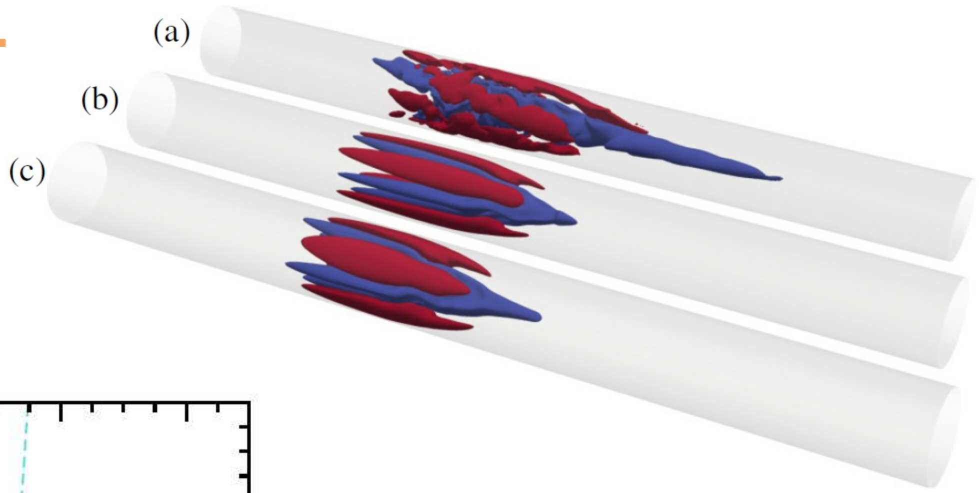
Chantry, Willis & Kerswell 2014

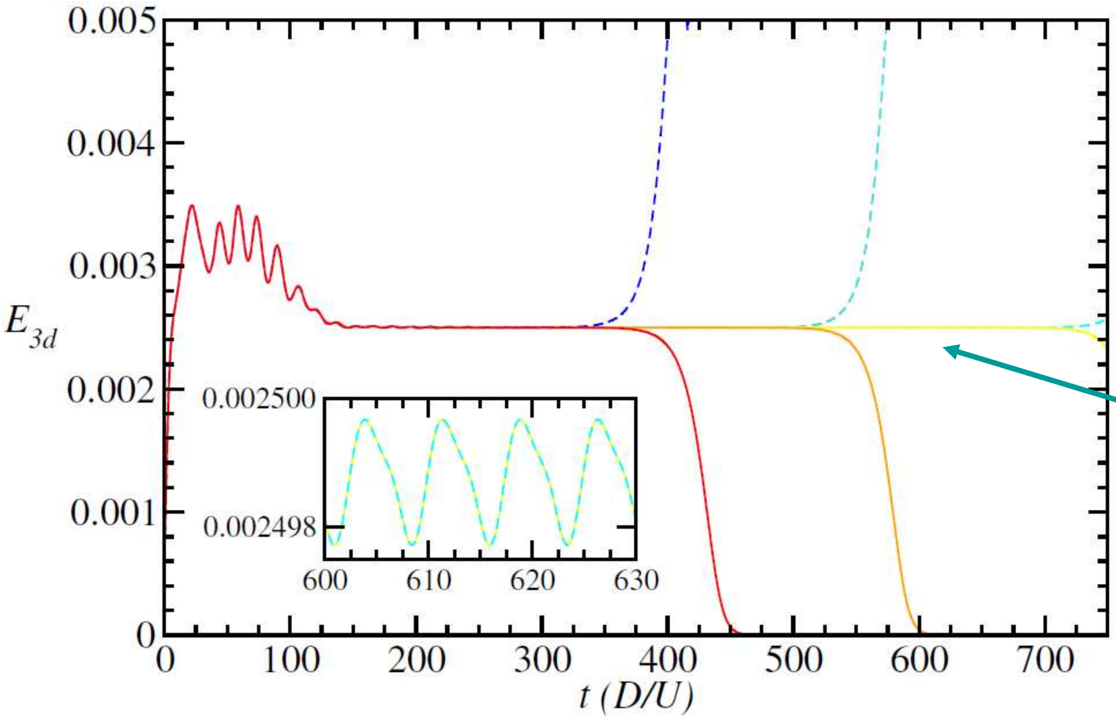
Exact localised periodic orbits connected to periodic TWs via spatial subharmonic bifurcation.

‘Edge tracking’:

Avila, Mellibovsky, Rolland & Hof 2013

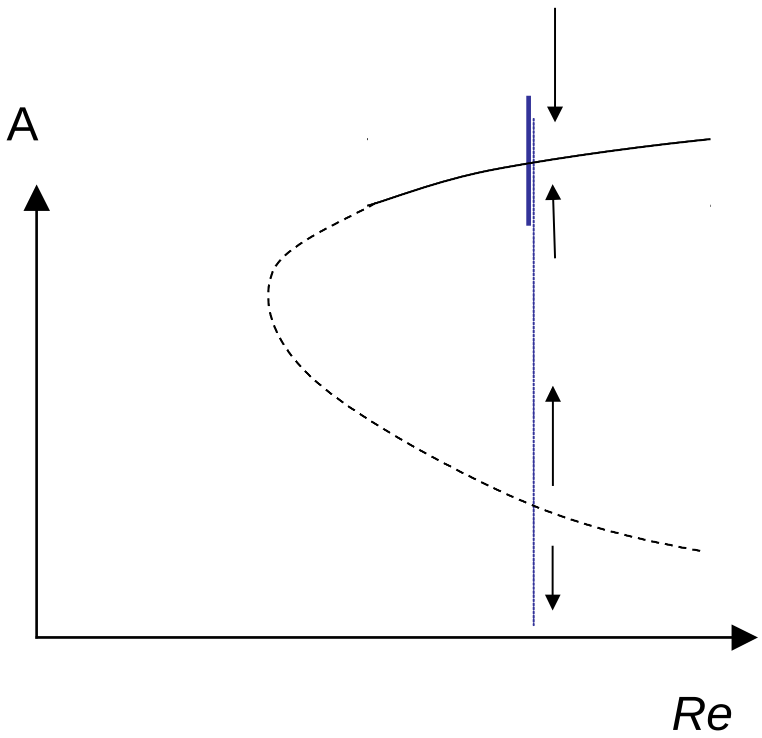
Exact localised solution found in m=2 + mirror space



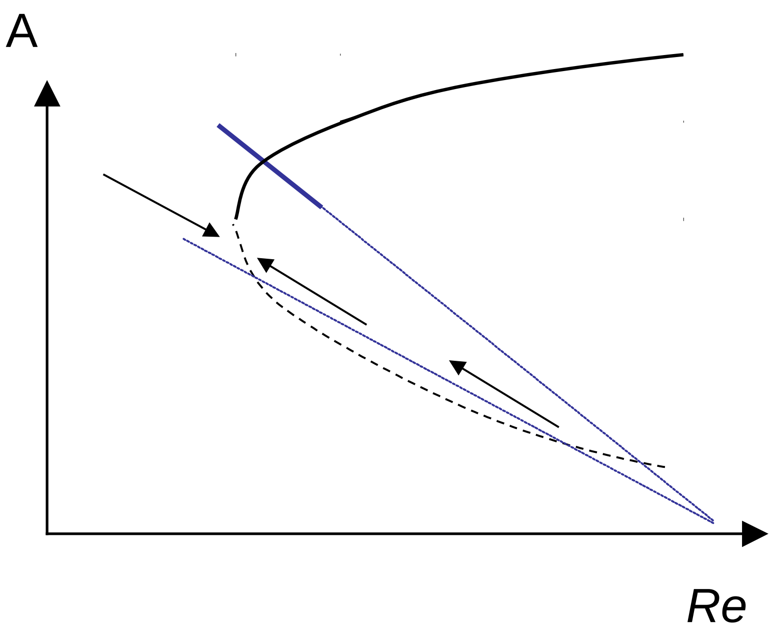


100s of simulations!

TWs disconnected from laminar state. How to find them?



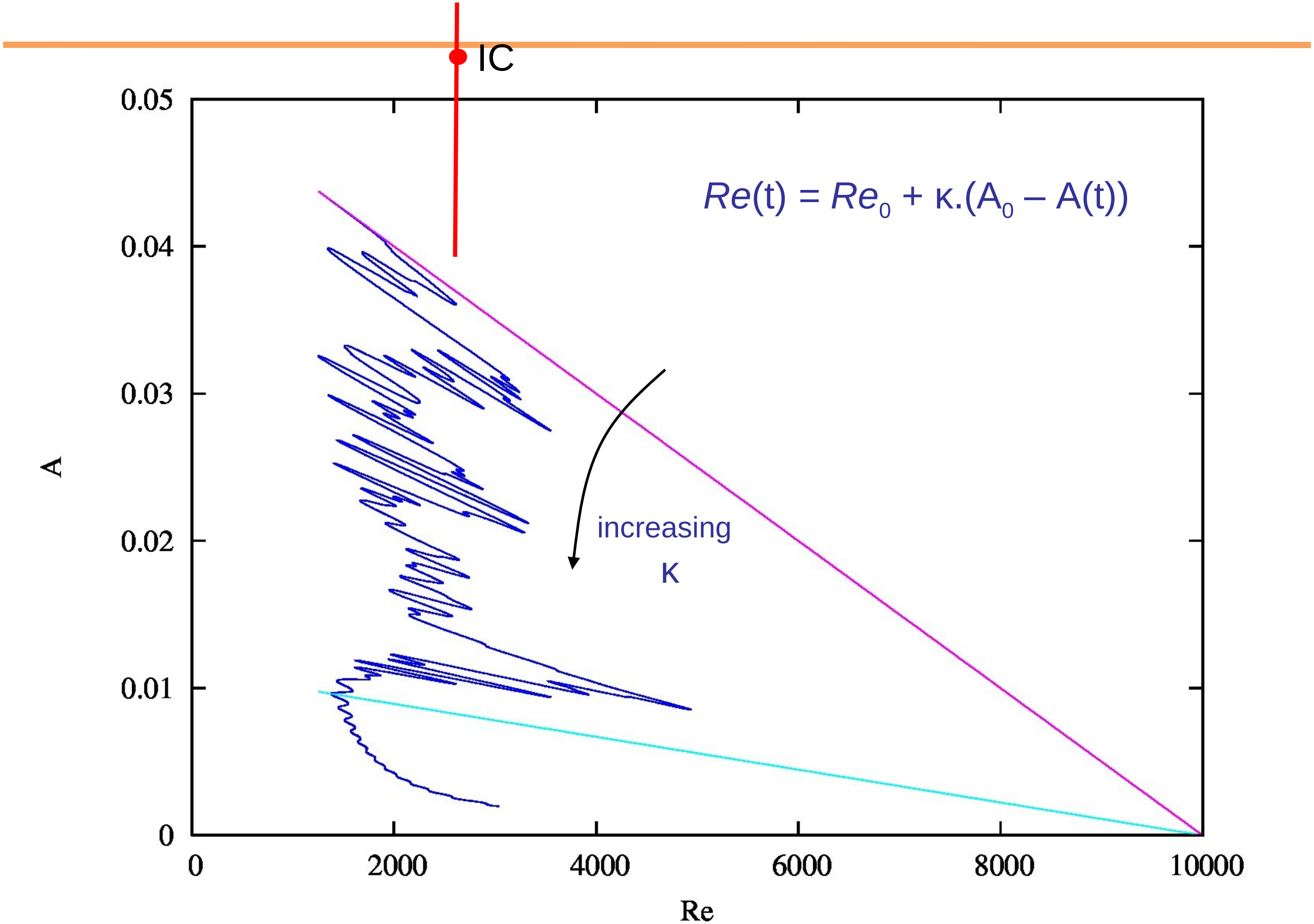
$Re = const.$

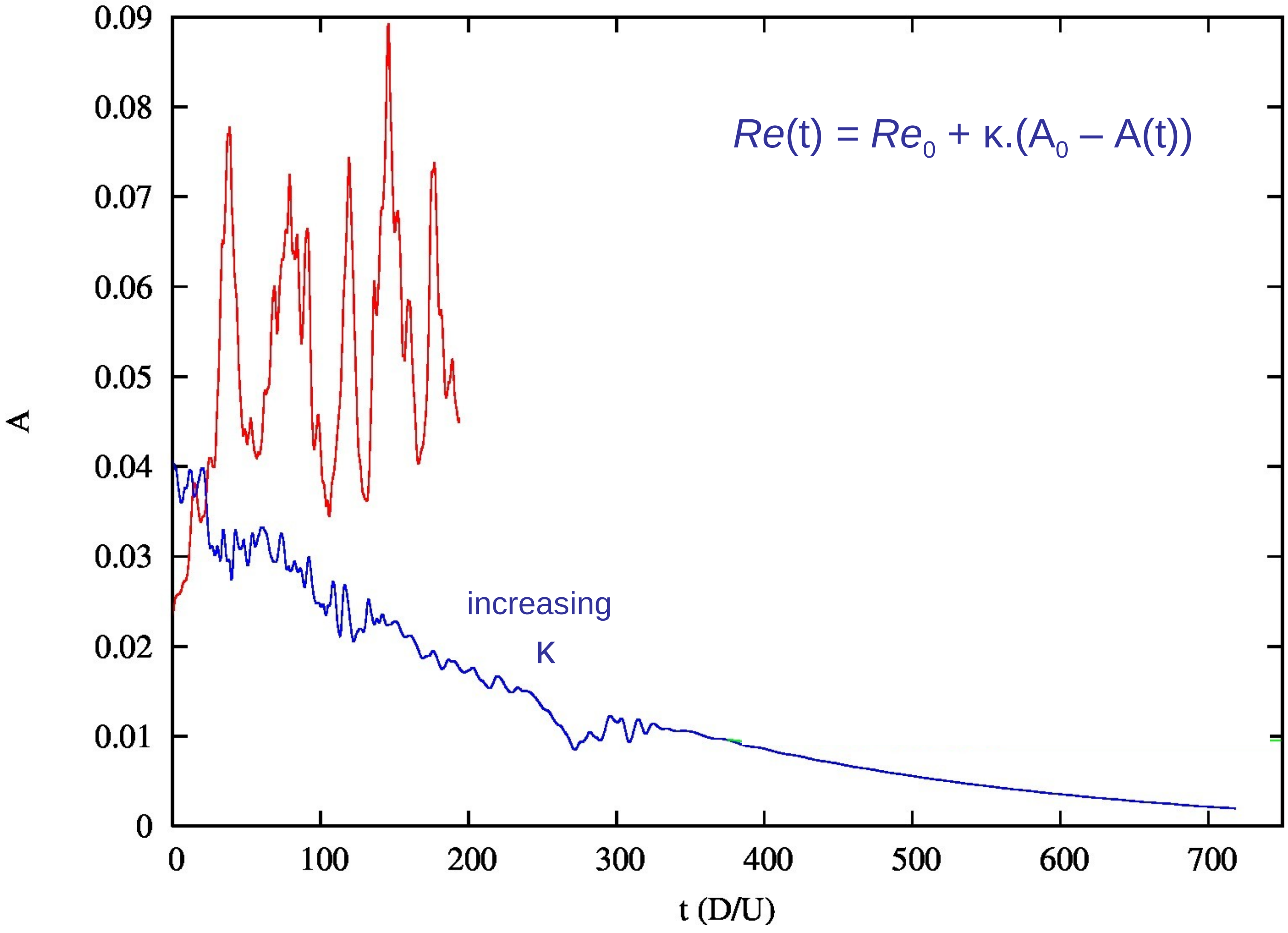


$Re(t) = Re_0 + \kappa.(A_0 - A(t))$

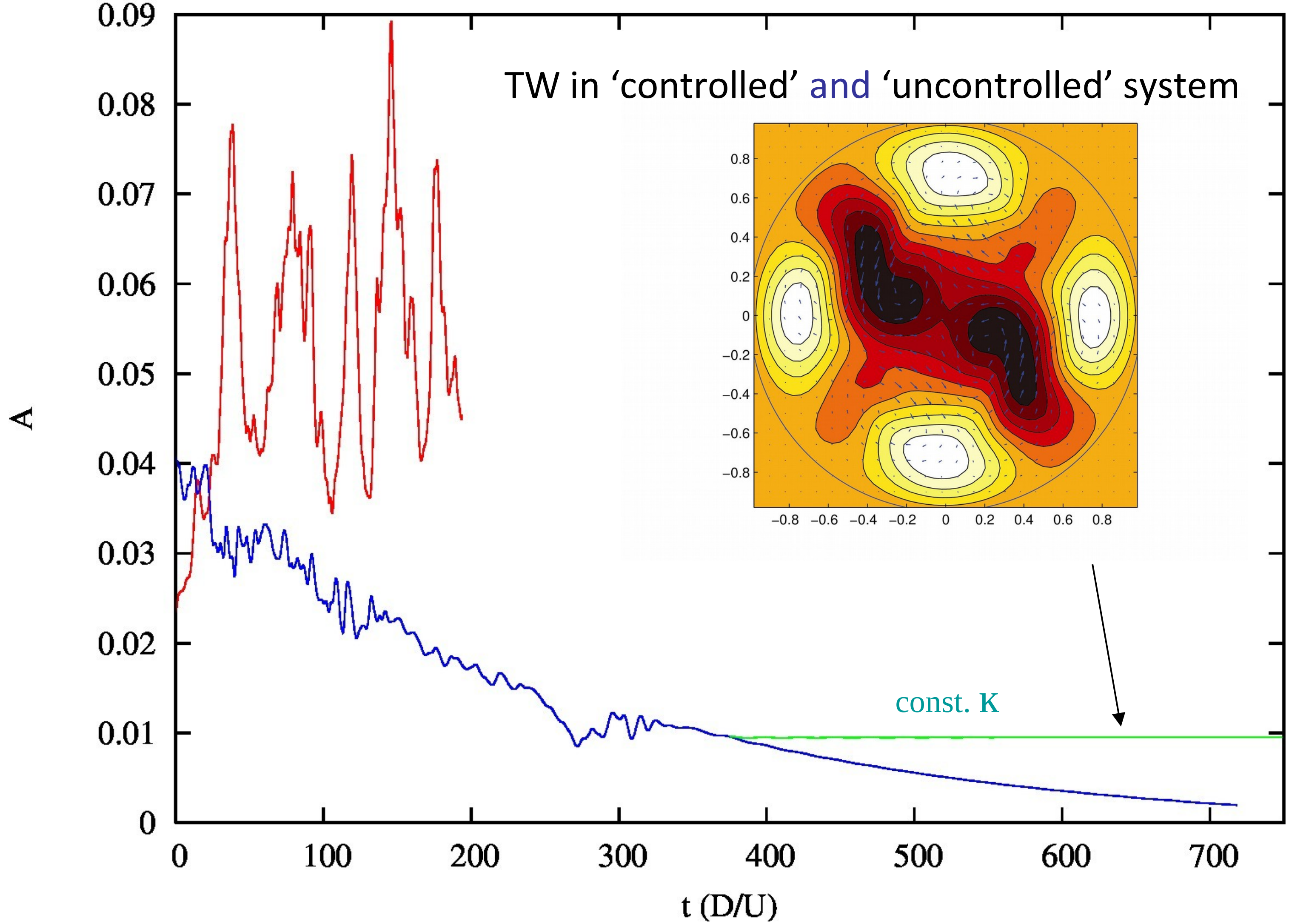
(Willis, Duguet, Omel'Chenko & Wolfrum, 2017, JFM)

Pipe simulation: L = 2π /1.25 R, m = 2, no S&R etc.

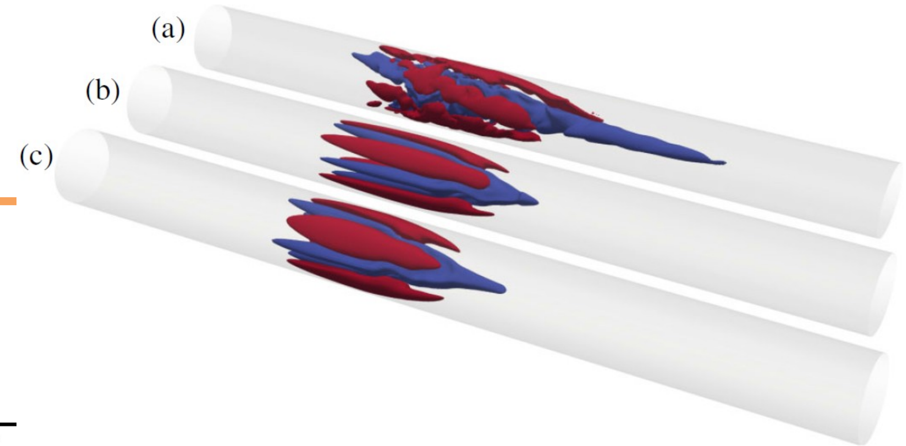


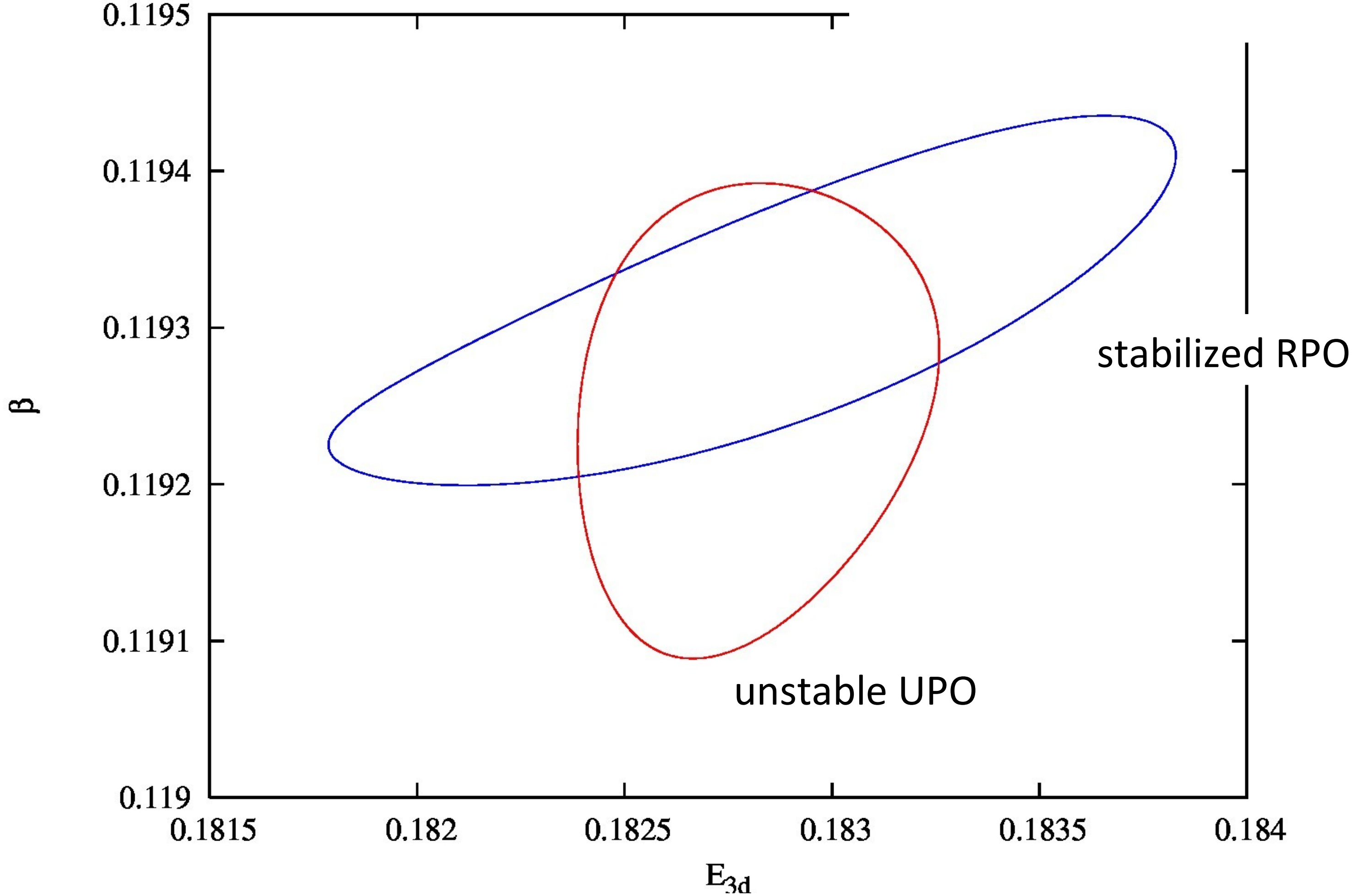


0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
A
0
100
200
300
400
500
600
700
t (D/U)
$Re(t) = Re_0 + \kappa.(A_0 – A(t))$
increasing
κ



TW in 'controlled' and 'uncontrolled' system
const. K
A
t (D/U)
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
100
200
300
400
500
600
700


(a)
(b)
(c)


0.1195
0.1194
0.1193
0.1192
0.1191
0.119
β
0.1815
0.182
0.1825
0.183
0.1835
0.184
E3d
stabilized RPO
unstable UPO

‘Method’:

1. Find a ‘suitable’ amplitude measure A
2. Link control parameter to A(t), e.g. $Re(t) = Re_0 + \kappa\,(A_0 - A(t))$
3. Increase slowly $\kappa = \kappa(t) \rightarrow$ reduction in A(t)
4. Fix κ if hit a stable point / orbit!

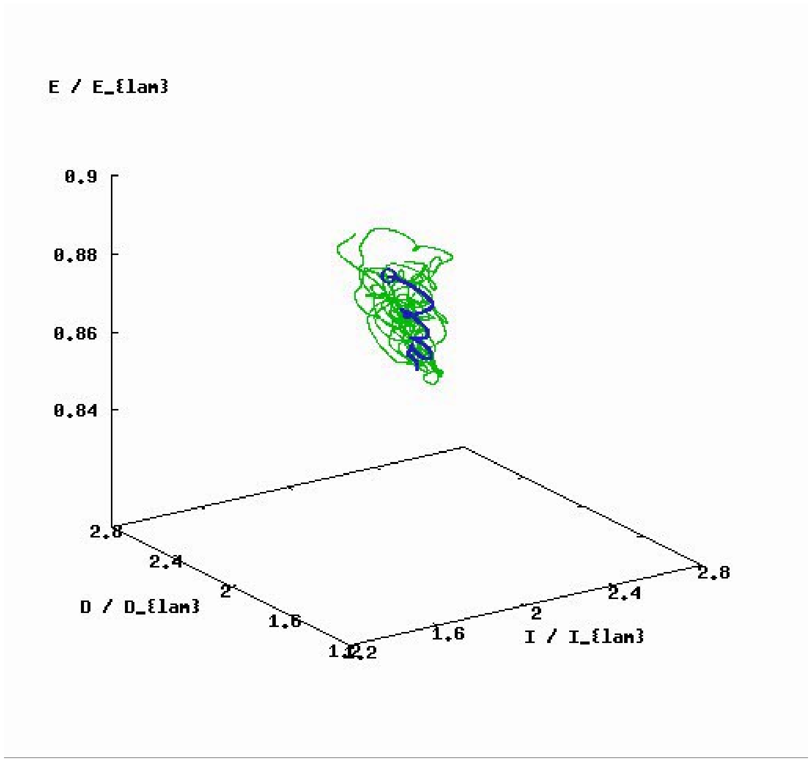
(Willis, Duguet, Omel’Chenko & Wolfrum, 2017, JFM)

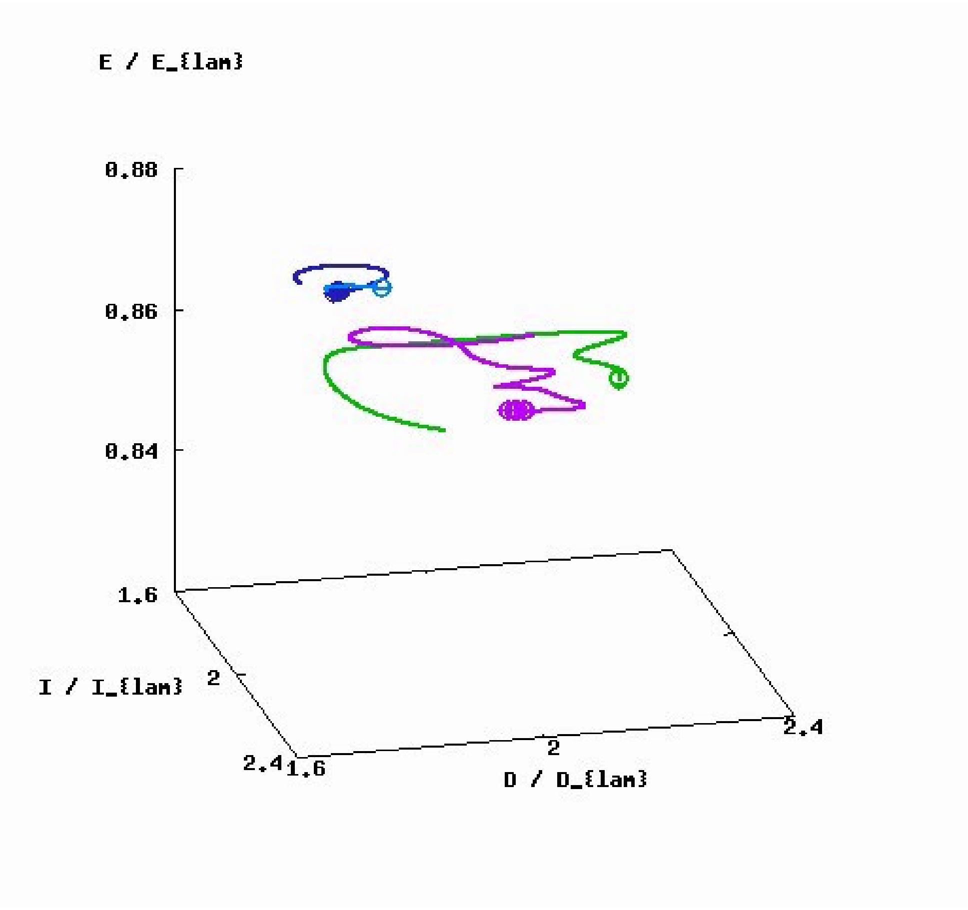
Key points:

- TWs (and POs) are weakly unstable solutions of the N-S equations
- Some are found in the laminar-turbulent boundary → transition
- With hindsight, we could have found them yonks ago!

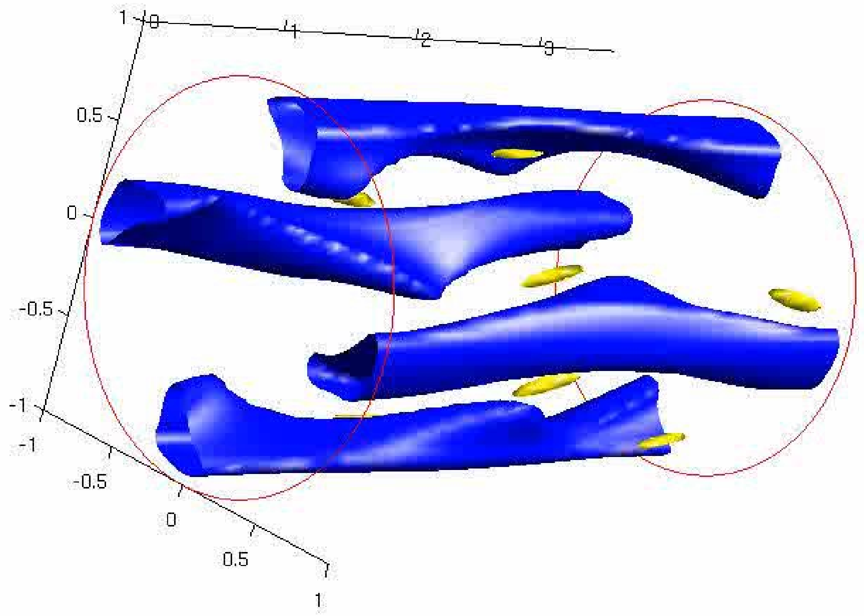
(Willis, Duguet, Omel’Chenko & Wolfrum, 2017, JFM)

Recurrent cycles (periodic orbits, POs) in turbulence



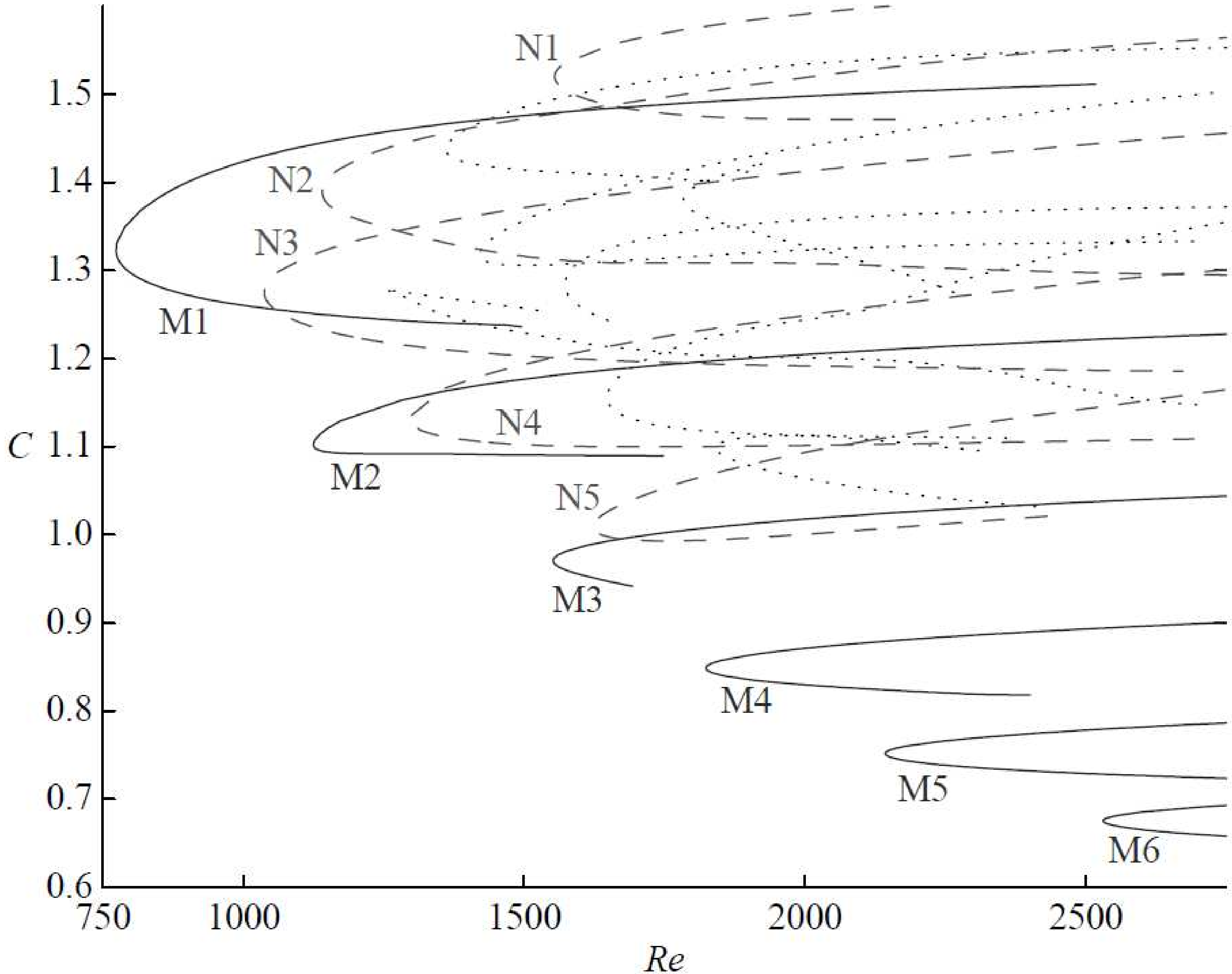


Need to go into moving frame...



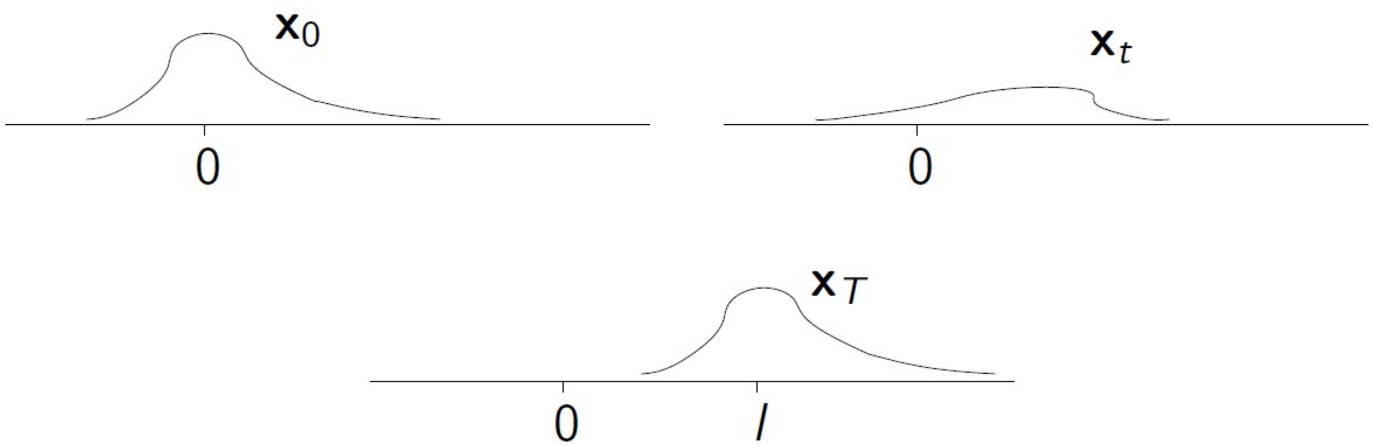
yellow, $\lambda_2 = -0.3$ blue, $u_z = -0.1$

Which phase speed !?

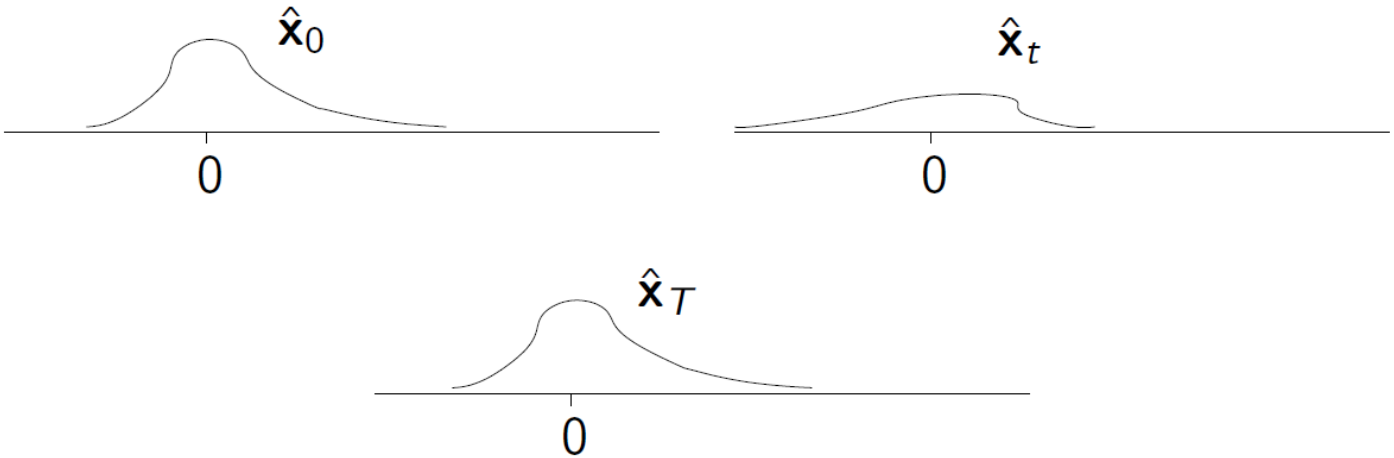


‘Slicing’

Original dynamics of RPO:

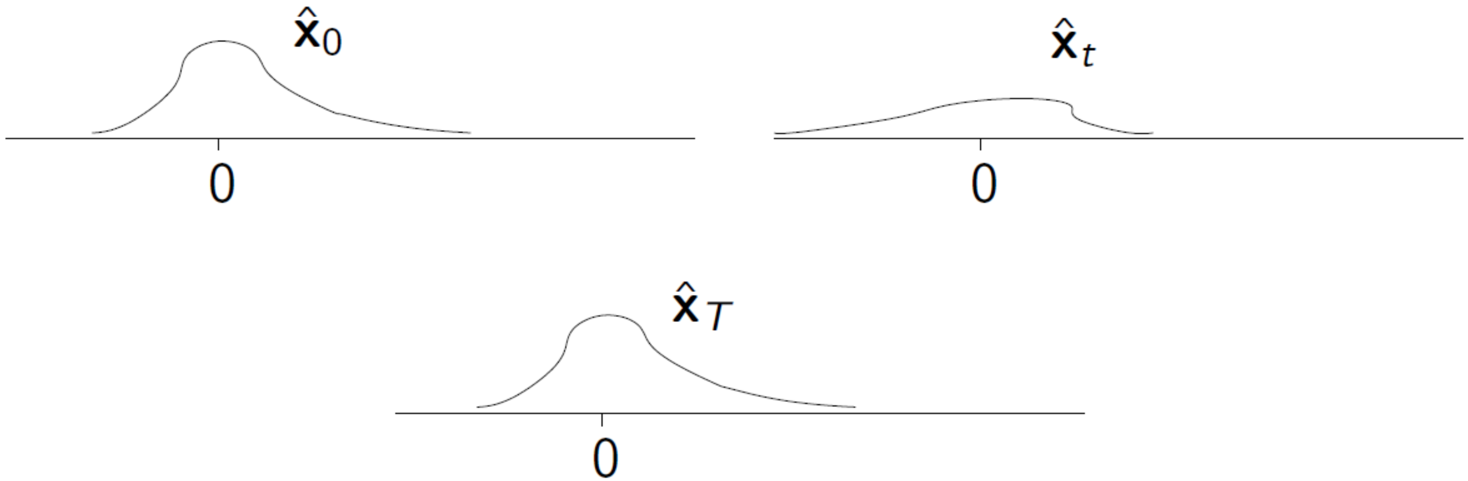


time-dependent shift l_t : $\hat{\mathbf{x}}_t = g(-l_t)\,\mathbf{x}_t$



‘Slicing’

time-dependent shift l_t : $\hat{\mathbf{x}}_t = g(-l_t)\,\mathbf{x}_t$



$$\hat{\mathbf{x}}_0 = \hat{\mathbf{x}}_T$$

relative periodic orbit (RPO) $\rightarrow$ periodic orbit (PO)

relative equilibrium (=TW) $\rightarrow$ equilibrium (=fixed pt.)

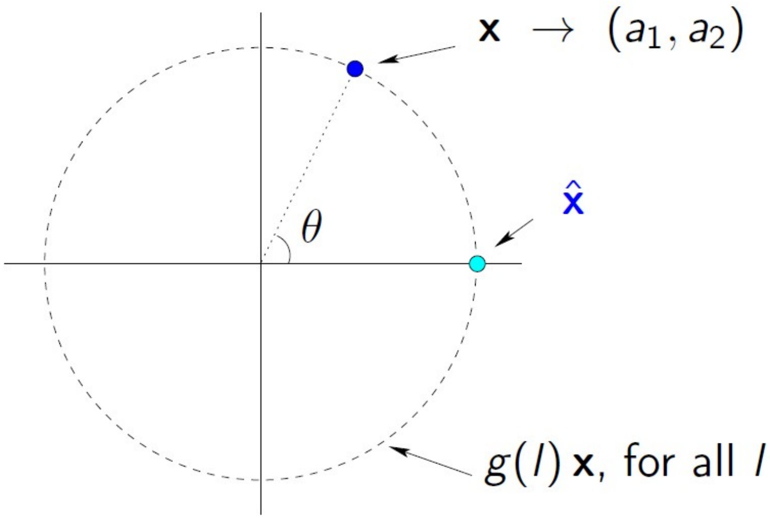
Fourier 'Slicing'

Construct

$$\mathbf{x}' = \mathbf{x}_c \cos\alpha x + \mathbf{x}_s \sin\alpha x\,, \quad \text{any non-zero } \mathbf{x}_c,\ \mathbf{x}_s. \quad L = 2\pi/\alpha.$$

Calculate

$$a_1 = \langle \mathbf{x} | \mathbf{x}' \rangle, \qquad a_2 = \langle \mathbf{x} \,|\, g(L/4)\,\mathbf{x}' \rangle$$

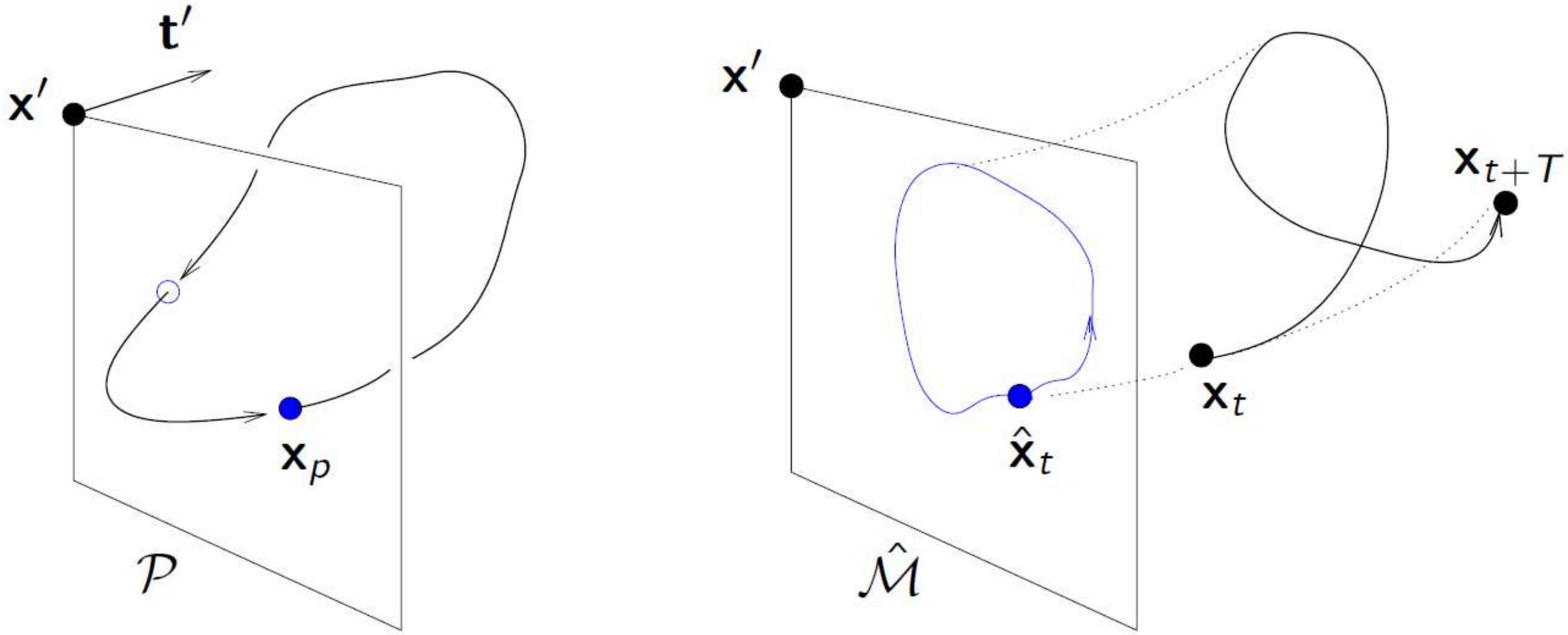


$$\hat{\mathbf{x}} = g(-(L/2\pi)\,\theta)\,\mathbf{x}$$

(Dynamics of TW
just goes around circle)

$$\mathbf{x}_t \;\rightarrow\; (a_1, a_2)_t \;\rightarrow\; \theta_t \;\rightarrow\; l_t = (L/2\pi)\,\theta_t \;\rightarrow\; \hat{\mathbf{x}} = g(-l_t)\,\mathbf{x}$$

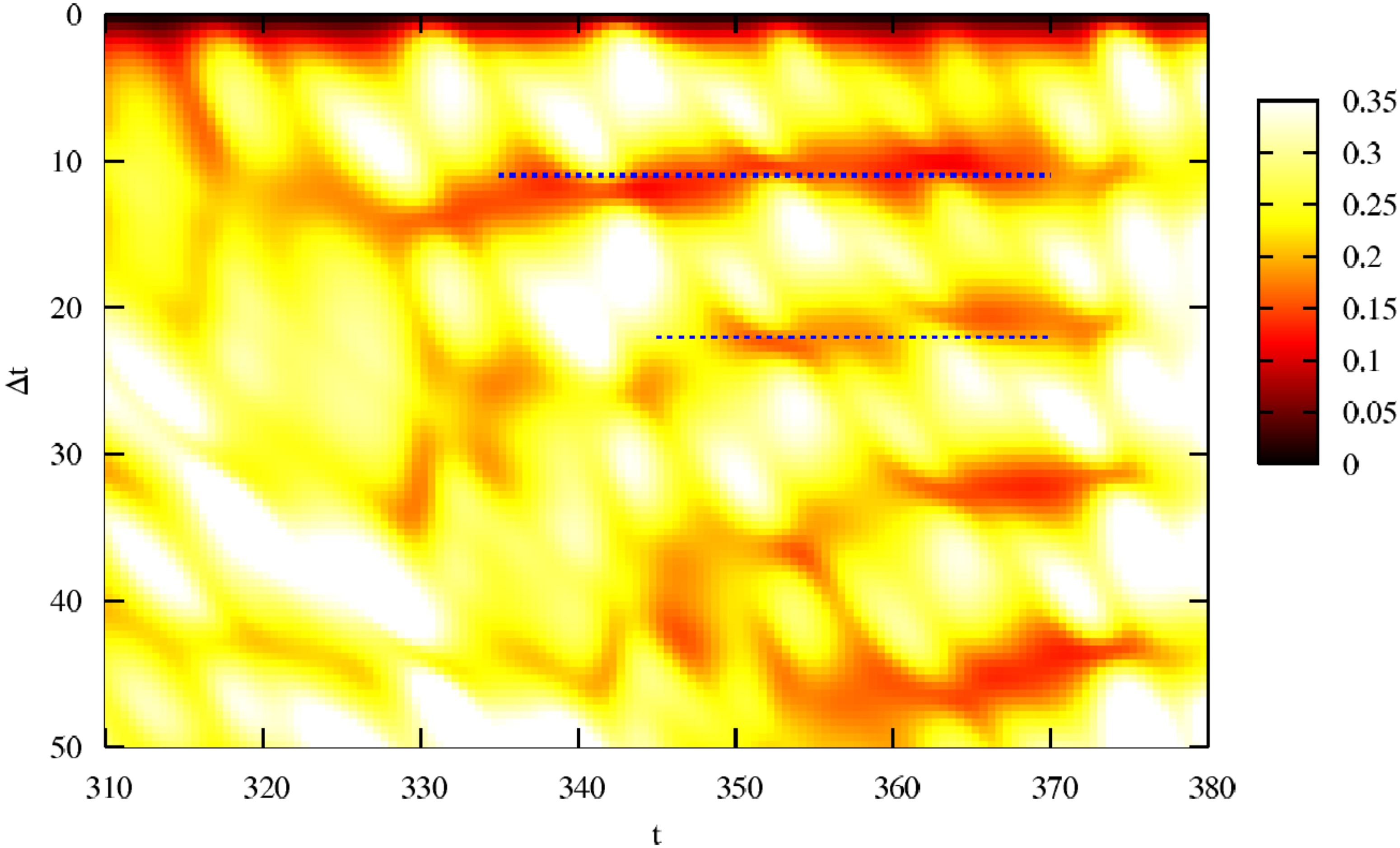
Slice vs Poincaré section



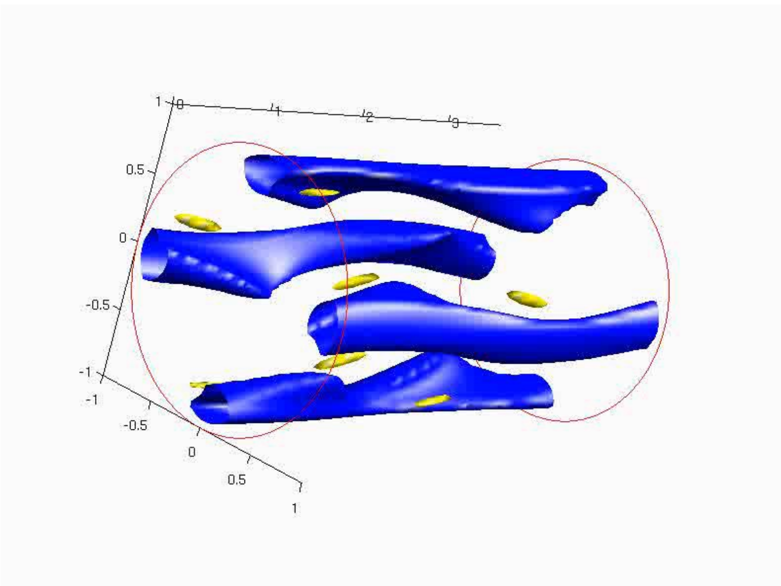
- Poincaré section $\mathcal{P}$ pierced by trajectories.
- Sliced trajectory lies within $\hat{\mathcal{M}}$.

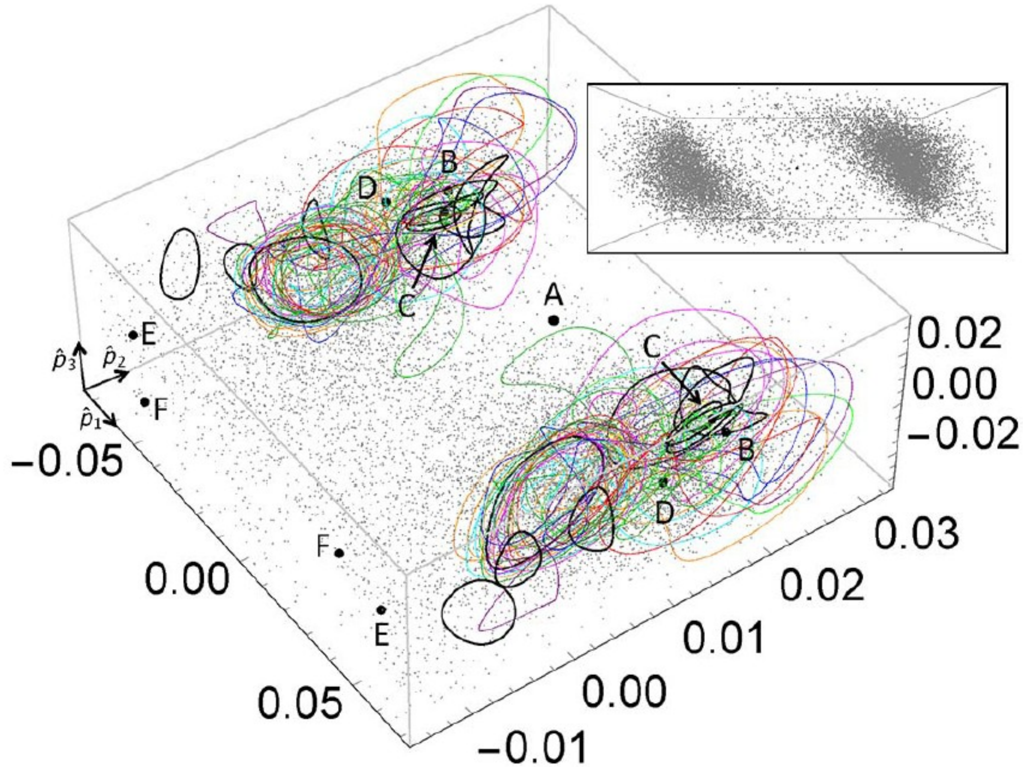
Recurrence plot (sliced dynamics)

$$\frac{||\hat{\mathbf{x}}_t - \hat{\mathbf{x}}_{t-\Delta t}||_c}{||\hat{\mathbf{x}}_{t-\Delta t}||_c}.$$



Sliced pipe





Model of pipe flow $\mathbf{x}_t \in \mathbb{R}^n$, $n = 154755$.

Sliced dynamics – eliminate axial shifts.

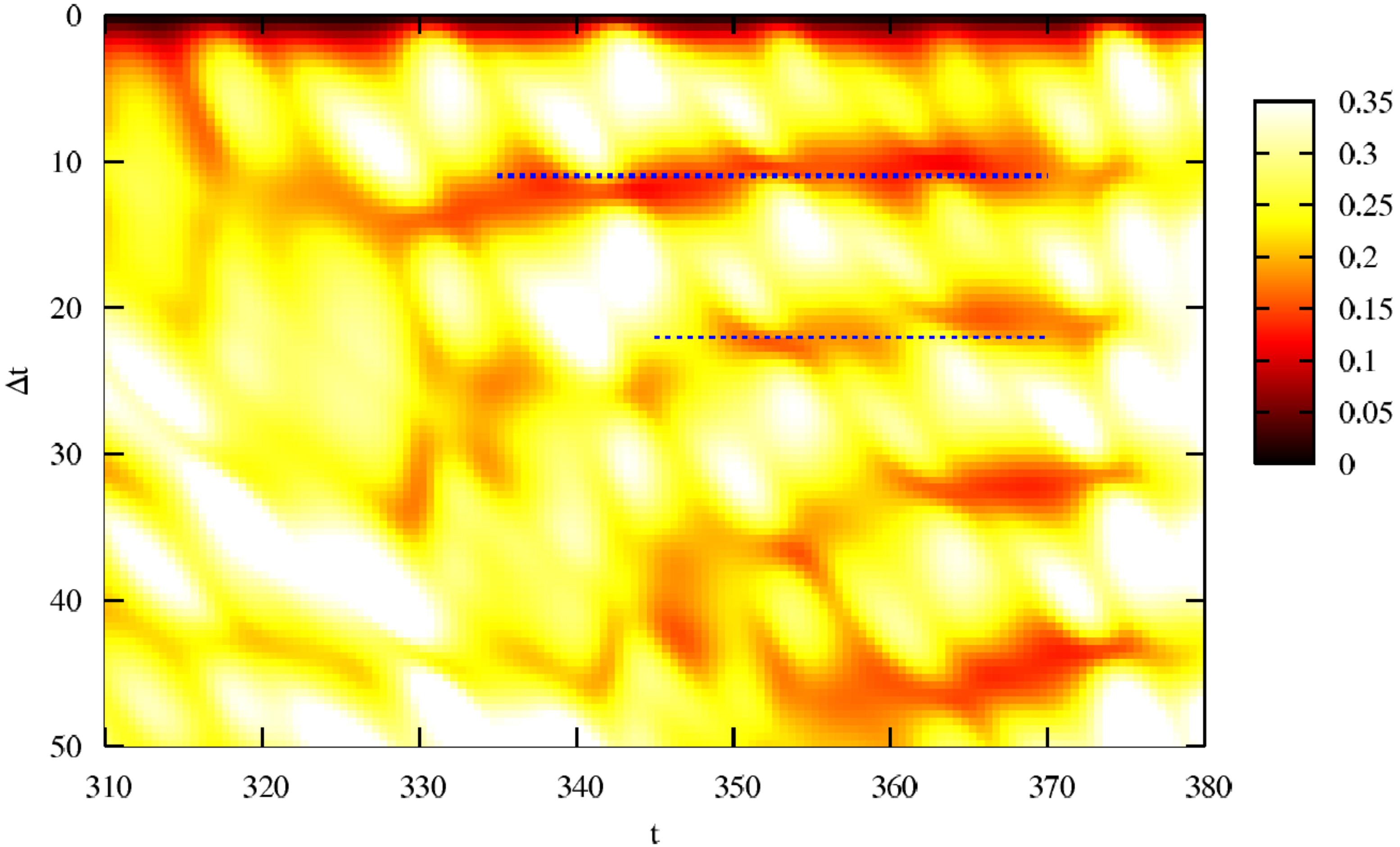
(Willis, Short and Cvitanović 2016)

Recurrence plot

$$\frac{||\hat{\mathbf{x}}_t - \hat{\mathbf{x}}_{t-\Delta t}||_c}{||\hat{\mathbf{x}}_{t-\Delta t}||_c}.$$

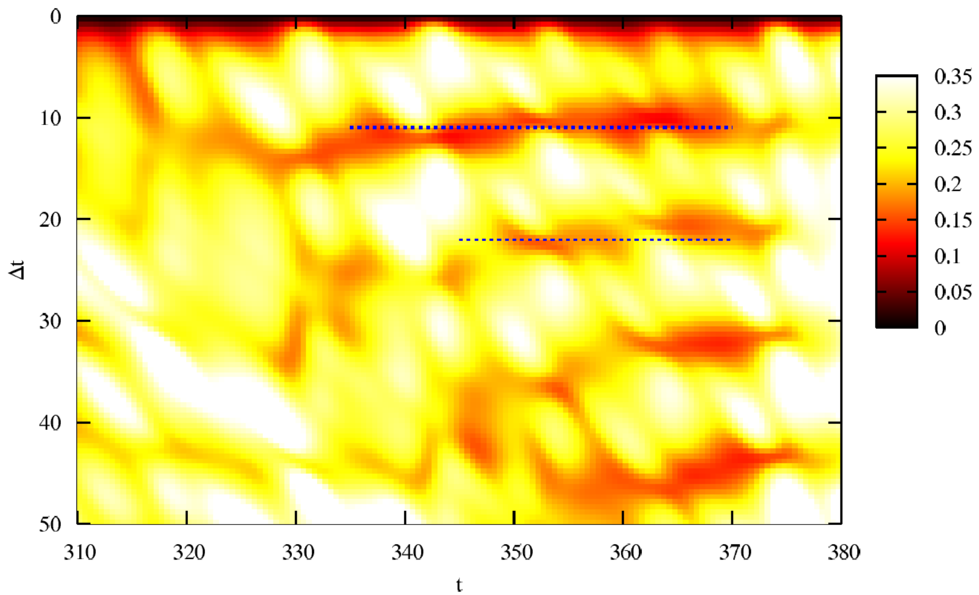
'Compensatory' norm.

'Crap' norm ?



Recurrence plot

$$\frac{||\hat{\mathbf{x}}_t - \hat{\mathbf{x}}_{t-\Delta t}||_c}{||\hat{\mathbf{x}}_{t-\Delta t}||_c}.$$



DMD/Koopman analysis ? Page & Kerswell (arXiv:1906.01310)

Machine learning ? Page & Kerswell

CSC methods ? Marensi & Willis

Summary

- Jacobian-free Newton-Krylov (- Hookstep): workhorse for dynamical systems approach. arxiv:1908.06730
- Relative equilibria (TWs) and relative periodic orbits (RPOs) embedded in laminar-turbulent boundary.
- Getting intital guesses for JFNK main issue.
- Bisection / ‘Surfing’ edge.
- ‘Slicing’ (symmetry reduction)
- RPOs embedded in turbulence → proxy for turbulence
- Norm problem.