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SOLVING CONVECTION-DIFFUSION EQUATIONS WITH MIXED, NEUMANN AND FOURIER BOUNDARY CONDITIONS AND MEASURES AS DATA, BY A DUALITY METHOD

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Abstract. In this paper, we prove, following [1], existence and uniqueness of the solutions of convection-diffusion equations on an open subset of $\mathbb{R}^N$, with a measure as data and different boundary conditions: mixed, Neumann or Fourier. The first part is devoted to the proof of regularity results for solutions of convection-diffusion equations with these boundary conditions and data in $(W^{1,q}(\Omega))'$, when $q < N/(N-1)$. The second part transforms, thanks to a duality trick, these regularity results into existence and uniqueness results when the data are measures.

1. Introduction and notations. In all the sequel, Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$), with a Lipschitz continuous boundary. $\mathbf{n}$ is the unit normal to $\partial\Omega$ outward to Ω. We denote by $x \cdot y$ the usual Euclidean product of two vectors $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$; the associated Euclidean norm is written $|.|$. The Lebesgue measure of a measurable subset E in $\mathbb{R}^N$ is denoted by $|E|$; σ is the Lebesgue measure on $\partial\Omega$ (i.e., the $(N-1)$-dimensional Hausdorff measure).

For $q \in [1,+\infty]$, q' denotes the conjugate exponent of q (i.e., $1/q + 1/q' = 1$). $W^{1,q}(\Omega)$ is the usual Sobolev space, endowed with the norm $||u||_{W^{1,q}(\Omega)} = ||u||_{L^q(\Omega)} + || \, |\nabla u| \, ||_{L^q(\Omega)}$. $W^{1,q}_*(\Omega)$ denotes the space of functions of $W^{1,q}(\Omega)$ which have a null mean value on Ω. When Γ is a measurable subset of $\partial\Omega$, $W^{1,q}_\Gamma(\Omega)$ is the space of functions of $W^{1,q}(\Omega)$ which

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have a null trace on Γ. $W^{1-\frac{1}{q},q}(\partial\Omega)$ denotes the Banach space of the traces on $\partial\Omega$ of functions of $W^{1,q}(\Omega)$, endowed with the norm $||f||_{W^{1-1/q,q}(\partial\Omega)} = \inf\left\{||u||_{W^{1,q}(\Omega)} : u_{|\partial\Omega} = f\right\}$ (the trace of $u \in W^{1,q}(\Omega)$ on $\partial\Omega$ is also denoted by u). When Γ is a measurable subset of $\partial\Omega$, $W_\Gamma^{1-1/q,q}(\partial\Omega)$ denotes the space of functions of $W^{1-1/q,q}(\partial\Omega)$ which are null σ-a.e. on Γ (it is endowed with the same norm as $W^{1-1/q,q}(\partial\Omega)$).

In the following, we make the hypotheses

$$\begin{array}{c} A : \Omega \to M_N(\mathbb{R}) \text{ is a measurable function which satisfies:} \\ \exists \alpha > 0 \text{ such that } A(x)\xi \cdot \xi \geq \alpha|\xi|^2 \text{ for a.e. } x \in \Omega\,,\ \forall \xi \in \mathbb{R}^N\,, \\ \exists M > 0 \text{ such that } ||A(x)|| := \sup\left\{|A(x)\xi|,\ \xi \in \mathbb{R}^N,\ |\xi| = 1\right\} \leq M \\ \text{for a.e. } x \in \Omega \end{array} \tag{1.1}$$

(where $M_N(\mathbb{R})$ is the space of $N \times N$ real valued matrices), and

$$\mathbf{v} : \Omega \to \mathbb{R}^N \text{ is a Lipschitz continuous function.} \tag{1.2}$$

We take α_A a coercitivity constant for A, Λ_A an essential upper bound of $||A(.)||$ on Ω and $\Lambda_{\mathbf{v}}$ an upper bound of $|\mathbf{v}(.)|$ on Ω.

Hypothesis (1.2) on $\mathbf{v}$ may seem too strong since we have accepted a discontinuous diffusion matrix. Indeed, in the pure Dirichlet case, one can take far less regular convection terms (see Remarks 2.8 and 2.9). But, since we intend to handle different boundary conditions (mixed, Neumann and Fourier), we need such an hypothesis on $\mathbf{v}$ (it is necessary for the many integrations by parts we will have to do).

In the first part, we prove some regularity results on the solutions of

$$-\operatorname{div}(A\nabla u) + \operatorname{div}(u\mathbf{v}) + bu = L \tag{1.3}$$

when $L \in \bigcup_{p>N}(W^{1,p'}(\Omega))'$ and when we consider mixed (but "well distributed"), Neumann or Fourier conditions on the boundary of Ω; these conditions do not need to be homogeneous, but must be more regular than what is strictly necessary to apply the Lax-Milgram Theorem to the variational formulation of Problem (1.3). We obtain, in each of the three cases (mixed, Neumann or Fourier — in fact four cases, since the Fourier boundary conditions gather two different cases), a $\kappa \in]0, 1 - N/p]$ such that the solution is κ-Hölder continuous on Ω.

In [1], G. Stampacchia proved this result for the homogeneous Dirichlet boundary conditions; however, the way he handles the boundary problems (i.e., proving the Hölder continuity of the solution near the boundary of Ω) does not seem so clear, so we will give here an alternate proof, which is easily adaptable to many different boundary conditions. We will, in fact, use a result of [1] to show that the solutions to our problems are Hölder continuous on any compact subset of Ω, and we will see how, by some reflection tricks, the regularity near the boundary of Ω can be proved using the regularity result on the compact subsets of Ω. This Hölder continuity near $\partial\Omega$ could also be proved, in the Neumann and Fourier cases, by using the same tools that prove the regularity on the compact subsets of Ω, see [12] (this is due to the fact that, in the Neumann and Fourier cases, there is no condition on the values on $\partial\Omega$ of admissible test functions).

The fact that $\kappa \leq 1-N/p$ is foreseeable: thanks to [9] we know that, if Ω has a regular boundary ($\mathcal{C}^2$), if A is Lipschitz continuous and (for example) $\mathbf{v} = 0$, $b = 0$, the solution to the homogeneous Dirichlet problem with $L \in (W^{1,p'}(\Omega))'$ is in $W_0^{1,p}(\Omega)$ which is, when $p > N$, composed of $(1-N/p)$-Hölder continuous functions. Moreover, with the regularity we have chosen on A and Ω, if $\mathbf{v} = 0$ and $b = 0$, it is proved in [8] (in the homogeneous Dirichlet case), in [10] (in the homogeneous Neumann case) and in [11] (in the homogeneous mixed case) that this result is still true, provided that p is greater than but close enough to 2; thus, when $N = 2$, [8], [10] and [11] entails the results of the present paper in the homogeneous Dirichlet, Neumann and mixed cases.

In the second part of the paper, we show how the regularity results of the first part can be used, thanks to a duality method, to show existence and uniqueness of solutions to linear elliptic problems with less regular data, namely measures on $\overline{\Omega}$. Solving such problems, and even non-linear ones, has already been done in [2]; however, the tool used in [2] (approximating the problem by more regular problems) does not give the uniqueness of the solution. We will indeed see, thanks to a counter-example introduced in [6] and modified in [7], that the solution we find here by a duality method is strictly stronger than the solution in [2], except in the case $N = 2$; this last case is indeed particular since the results of [8], [10] and [11] show that the solution of [2] is, when $N = 2$, unique.

We will also show how the precise dependence of the constants (with respect to the data) we obtain in the first part of the paper can be used to obtain a stability result on the solutions to these dual elliptic problems and, thanks to the Leray-Schauder topological degree (see [4]), to solve some semi-linear elliptic problems with measures as data.

The results in the first part of this paper are new in the treatment of the boundary conditions. The L^∞ bound in the Neumann case demands more computations than in the Dirichlet case, but the main new idea with respect to [1] is to use a trick of transport and reflection, which allows us to handle many different boundary conditions and not only the Dirichlet one.

The main idea of the second part, that is to say using a duality method to transform regularity results into existence and uniqueness results for weaker data, is also contained in [1]; but getting the strong integral formulation of the equation from the duality formulation is not as straightforward in the Fourier or mixed case as in the Dirichlet case. The comparison between the solutions obtained by duality and those obtained by approximation (in [2]) had already been made in [7] in the Dirichlet case; we have adapted the methods used in this last paper to show that the same comparison can be made in the mixed or Fourier case. In this part, the main new results come from the application of the duality method to the *boundary conditions* (see section 3.2), not only to the right hand side of the equation.

The results of the last part of this paper, i.e., the stability result for the linear dual equation and the existence result for a non-linear dual problem, seem quite new. In particular, it does not seem possible to handle the kind of non-linearity of the right-hand side of (4.24) with an approximation method.

2. The regularity results.

2.1. Presentation of the problems.

We describe here the three problems for which a regularity result is proved.

2.1.1. Mixed boundary conditions.

The mixed problem is

$$\left\{\begin{array}{l} -\mathrm{div}(A\nabla u) + \mathrm{div}(u\mathbf{v}) + bu = L \quad \text{in} \quad \Omega, \\ u = g_d \quad \text{on} \quad \Gamma_d, \\ A\nabla u \cdot \mathbf{n} + \lambda u = g_n \quad \text{on} \quad \Gamma_n. \end{array}\right. \tag{2.1}$$

We make the following hypotheses on the data:

$$\Gamma_d \cup \Gamma_n = \partial\Omega\,,\ \sigma(\Gamma_d \cap \Gamma_n) = 0\,,\ \sigma(\Gamma_d) > 0, \tag{2.2}$$

$$\begin{array}{c} b \in L^r(\Omega) \text{ with } r = \frac{N}{2} \text{ if } N > 2 \text{ and } r > 1 \text{ if } N = 2, \\ \frac{1}{2}\mathrm{div}(\mathbf{v})(x) + b(x) \geq 0 \text{ for a.e. } x \in \Omega, \end{array} \tag{2.3}$$

$$\begin{array}{c}\lambda \in L^q(\Gamma_n) \text{ with } q = N-1 \text{ if } N>2 \text{ and } q>1 \text{ if } N=2,\\ \frac{1}{2}\mathbf{v}\cdot\mathbf{n} + \lambda \geq 0 \ \sigma\text{-a.e. on } \Gamma_n,\end{array} \tag{2.4}$$

$$L \in \left(H^1(\Omega)\right)', \ g_d \in H^{1/2}(\partial\Omega), \ g_n \in H^{-1/2}(\partial\Omega) := \left(H^{1/2}(\partial\Omega)\right)'. \tag{2.5}$$

Introducing $u_0 \in H^1(\Omega)$, a function with trace g_d on $\partial\Omega$, we can write the variational formulation of (2.1) as

$$\left\{\begin{array}{l} w = u - u_0 \in H^1_{\Gamma_d}(\Omega),\\ \displaystyle\int_\Omega A\nabla w\cdot\nabla\varphi + \int_{\Gamma_n}\lambda w\varphi\,d\sigma - \int_\Omega w\mathbf{v}\cdot\nabla\varphi + \int_{\Gamma_n} w\varphi\mathbf{v}\cdot\mathbf{n}\,d\sigma\\ \displaystyle+\int_\Omega bw\varphi = \langle L,\varphi\rangle_{(H^1(\Omega))',H^1(\Omega)} + \langle g_n,\varphi\rangle_{(H^{1/2}(\partial\Omega))',H^{1/2}(\partial\Omega)}\\ \displaystyle\quad-\int_\Omega A\nabla u_0\cdot\nabla\varphi - \int_{\Gamma_n}\lambda u_0\varphi\,d\sigma + \int_\Omega u_0\mathbf{v}\cdot\nabla\varphi\\ \displaystyle-\int_{\Gamma_n} u_0\varphi\mathbf{v}\cdot\mathbf{n}\,d\sigma - \int_\Omega bu_0\varphi\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega).\end{array}\right. \tag{2.6}$$

It is well known that, under Hypotheses (1.1), (1.2) and (2.2)—(2.5), Problem (2.6) has a unique solution.

Remark 2.1. Let us consider the simple case $A \equiv I_N$ (the identity matrix) on Ω, $\mathbf{v} = 0$, $b = 0$, $g_d = 0$, $\lambda = 0$ and $g_n = 0$. When u and L are regular (for example, $\mathcal{C}^\infty$), the solution of (2.6) is exactly the solution of (2.1). But, when the data are less regular, we must be careful with the meaning of (2.1); for example, if $F \in (\mathcal{C}^\infty(\overline{\Omega}))^N$ and $L \in (H^1(\Omega))'$ is defined by $\langle L,\varphi\rangle_{(H^1(\Omega))',H^1(\Omega)} = \int_\Omega F\cdot\nabla\varphi$ (notice that $L \notin \mathcal{C}^\infty$, unless $F\cdot\mathbf{n} = 0$ on $\partial\Omega$), the solution of the corresponding weak formulation

$$\left\{\begin{array}{l} u \in H^1_{\Gamma_d}(\Omega),\\ \displaystyle\int_\Omega \nabla u\cdot\nabla\varphi = \langle L,\varphi\rangle_{(H^1(\Omega))',H^1(\Omega)}\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega),\end{array}\right. \tag{2.7}$$

is in fact the solution $u \in H^1_{\Gamma_d}(\Omega)$ of

$$\left\{\begin{array}{lll} -\Delta u = -\mathrm{div}(F) & \text{in} & \mathcal{D}'(\Omega),\\ u = 0 & \text{on} & \Gamma_d,\\ \nabla u\cdot\mathbf{n} = F\cdot\mathbf{n} & \text{on} & \Gamma_n,\end{array}\right. \tag{2.8}$$

and not the solution of $-\Delta u = L$ in $(H^1(\Omega))'$ (this expression, anyway, has no defined sense since there is an infinity of extensions of Δu as an element of $(H^1(\Omega))'$).

The same kind of consideration can arise in the sequel, whenever there is a Neumann or Fourier condition on some part of $\partial\Omega$, but this will not cause us troubles, since we will only use the weak (i.e., variational) formulations of the problems.

By strenghtening Hypotheses (2.3)—(2.5) and adding an assumption on Γ_d and Γ_n, more regularity on the solution of (2.6) can be proved.

Let us introduce two kinds of mappings, that will describe the way Γ_d and Γ_n are distributed on $\partial\Omega$.

$$\begin{array}{c} O \text{ is an open subset of } \mathbb{R}^N, \\ h : O \to B := \{x \in \mathbb{R}^N \mid |x| < 1\} \text{ is a Lipschitz continuous} \\ \text{homeomorphism with a Lipschitz continuous inverse mapping,} \\ h(O \cap \Omega) = B_+ := \{x \in B \mid x_N > 0\}, \\ h(O \cap \partial\Omega) = B^{N-1} := \{x \in \partial B_+ \mid x_N = 0\} \end{array} \tag{2.9}$$

It is well known that, since Ω has a Lipschitz continuous boundary, there exists a finite number of $(O_i, h_i)_{i\in[1,m]}$, such that, for all $i \in [1, m]$, (O_i, h_i) satisfy (2.9) and $\partial\Omega \subset \cup_{i=1}^m O_i$.

But, to handle the mixed boundary conditions, we will need another kind of mapping, which tells that Γ_d and Γ_n are "well separated":

$$\begin{array}{c} O \text{ is an open subset of } \mathbb{R}^N, \\ h : O \to B \text{ is a Lipschitz continuous homeomorphism} \\ \text{with Lipschitz continuous inverse mapping,} \\ h(O \cap \Omega) = B_{++} := \{x \in B \mid x_N > 0\,,\ x_{N-1} > 0\}, \\ h(O \cap \Gamma_n) = \Gamma_1 := \{x \in \partial B_{++} \mid x_{N-1} = 0\}, \\ h(O \cap \Gamma_d) = \Gamma_2 := \{x \in \partial B_{++} \mid x_N = 0\}. \end{array} \tag{2.10}$$

The additional assumption we make on Γ_d and Γ_n is the following:

$$\begin{array}{c} \text{There exists a finite number of } (O_i, h_i)_{i\in[1,m]} \text{ such that } \partial\Omega \subset \cup_{i=1}^m O_i \\ \text{and, for all } i \in [1, m],\ (O_i, h_i) \text{ is of one of the following types:} \\ \left| \begin{array}{ll} (D) & O_i \cap \partial\Omega = O_i \cap \Gamma_d \text{ and } (O_i, h_i) \text{ satisfies (2.9)} \\ (F) & O_i \cap \partial\Omega = O_i \cap \Gamma_n \text{ and } (O_i, h_i) \text{ satisfies (2.9)} \\ (DF) & (O_i, h_i) \text{ satisfies (2.10).} \end{array} \right. \end{array} \tag{2.11}$$

Theorem 2.1. *Let $p > N$. Under Hypotheses* (1.1), (1.2), (2.2), (2.11) *and*

$$\begin{aligned} b \in L^{\frac{Np}{N+p}}(\Omega)\,,\ \tfrac{1}{2}\mathrm{div}(\mathbf{v})(x) + b(x) \geq 0 \text{ for a.e. } x \in \Omega, \\ \lambda \in L^{(N-1)\frac{p}{N}}(\Gamma_n)\,,\ \tfrac{1}{2}\mathbf{v}\cdot\mathbf{n} + \lambda \geq 0\ \sigma\text{-a.e. on } \Gamma_n, \end{aligned} \tag{2.12}$$

$$L \in (W^{1,p'}(\Omega))'\,,\ g_d \in W^{1-\frac{1}{p},p}(\partial\Omega)\,,\ g_n \in (W^{1-\frac{1}{p'},p'}(\partial\Omega))', \tag{2.13}$$

there exists $\kappa \in]0, 1 - N/p]$ only depending on $(\Omega, \alpha_A, \Lambda_A, p)$ such that the solution u of (2.6) is κ-Hölder continuous on Ω. Moreover, if Λ is such that

$$\begin{aligned} &||b||_{L^{\frac{Np}{N+p}}(\Omega)} + ||\lambda||_{L^{(N-1)\frac{p}{N}}(\Gamma_n)} \\ &+||L||_{(W^{1,p'}(\Omega))'} + ||g_d||_{W^{1-\frac{1}{p},p}(\partial\Omega)} + ||g_n||_{\left(W^{1-\frac{1}{p'},p'}(\partial\Omega)\right)'} \leq \Lambda, \end{aligned} \tag{2.14}$$

there exists $C > 0$ only depending on $(\Omega, \Gamma_d, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda)$ such that

$$||u||_{\mathcal{C}^{0,\kappa}(\Omega)} \leq C. \tag{2.15}$$

Remark 2.2. *Notice that (2.12) implies (2.3), (2.4) and that (2.13) implies (2.5); in fact, since $p' < N' \leq 2$, $H^1(\Omega)$ is continuously and densely imbedded in $W^{1,p'}(\Omega)$ and $H^{1/2}(\partial\Omega)$ is continuously and densely imbedded in $W^{1-\frac{1}{p'},p'}(\partial\Omega)$, so that $(W^{1,p'}(\Omega))'$ is continuously imbedded in $(H^1(\Omega))'$ and $(W^{1-\frac{1}{p'},p'}(\partial\Omega))'$ is continuously imbedded in $H^{-1/2}(\partial\Omega)$; moreover, since $p \geq 2$, $W^{1-\frac{1}{p},p}(\partial\Omega) \subset H^{1/2}(\partial\Omega)$.*

Remark 2.3. *An example of an element of $(W^{1-1/p',p'}(\partial\Omega))'$ is any element of $L^{(N-1)\frac{p}{N}}(\partial\Omega)$, since $W^{1-\frac{1}{p'},p'}(\partial\Omega)$ is continuously and densely imbedded in $L^{(N-1)p'/(N-p')}(\partial\Omega)$ (see [5]), and $((N-1)p'/(N-p'))' = (N-1)p/N$.*

Remark 2.4. *Recall that $\mathcal{C}^{0,\kappa}(\Omega)$ is the space of all κ-Hölder continuous functions endowed with the norm*

$$||f||_{\mathcal{C}^{0,\kappa}(\Omega)} = ||f||_{L^\infty(\Omega)} + \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\kappa}.$$

Remark 2.5. *This result has already been proved by G. Stampacchia in [1] in the case of the Dirichlet homogeneous equation (that is to say $\Gamma_d = \partial\Omega$ and $u_0 = 0$); in fact, we will use the results of [1] to prove, under the good hypotheses, the regularity of the solutions of (2.6), (2.19) and (2.28) within the domain Ω. Then, by a transport and reflection trick, we will prove that the regularity of these solutions near the boundary of Ω is also a consequence of the inner regularity of the solution of a Dirichlet problem.*

2.1.2. Neumann boundary conditions. The Neumann problem is

$$\begin{cases} -\mathrm{div}(A\nabla u) + \mathrm{div}(u\mathbf{v}) = L \quad \text{in} \quad \Omega, \\ A\nabla u \cdot \mathbf{n} = g \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.16}$$

The hypotheses on the data are:

$$\mathrm{div}(\mathbf{v})(x) = 0 \text{ for a.e. } x \in \Omega\,,\ \mathbf{v}\cdot\mathbf{n} = 0\ \sigma\text{-a.e. on } \partial\Omega, \tag{2.17}$$

$$\begin{array}{c} L \in (H^1(\Omega))'\,,\ g \in H^{-1/2}(\partial\Omega), \\ \langle L, 1\rangle_{(H^1(\Omega))', H^1(\Omega)} + \langle g, 1\rangle_{(H^{1/2}(\partial\Omega))', H^{1/2}(\partial\Omega)} = 0. \end{array} \tag{2.18}$$

Under Hypotheses (1.1), (1.2), (2.17) and (2.18), there exists a unique solution to (2.16) in the sense

$$\begin{cases} u \in H^1_*(\Omega), \\ \displaystyle\int_\Omega A\nabla u \cdot \nabla\varphi - \int_\Omega u\mathbf{v}\cdot\nabla\varphi = \langle L, \varphi\rangle_{(H^1(\Omega))', H^1(\Omega)} \\ \qquad + \langle g, \varphi\rangle_{(H^{1/2}(\partial\Omega))', H^{1/2}(\partial\Omega)}\,,\ \forall \varphi \in H^1(\Omega). \end{cases} \tag{2.19}$$

As for the mixed case, we will prove that, with stronger hypotheses on g and L, more regularity on the solution of (2.19) can be proved.

Theorem 2.2. *Let $p > N$. Under Hypotheses (1.1), (1.2), (2.17) and*

$$\begin{array}{c} L \in (W^{1,p'}(\Omega))'\,,\ g \in \left(W^{1-\frac{1}{p'},p'}(\partial\Omega)\right)', \\ \langle L, 1\rangle_{(W^{1,p'}(\Omega))', W^{1,p'}(\Omega)} + \langle g, 1\rangle_{(W^{1-1/p',p'}(\partial\Omega))', W^{1-1/p',p'}(\partial\Omega)} = 0, \end{array} \tag{2.20}$$

there exists $\kappa \in]0, 1 - N/p]$ only depending on $(\Omega, \alpha_A, \Lambda_A, p)$ such that the solution u to (2.19) is κ-Hölder continuous on Ω. Moreover, if Λ is such that

$$||L||_{(W^{1,p'}(\Omega))'} + ||g||_{(W^{1-1/p',p'}(\partial\Omega))'} \le \Lambda, \tag{2.21}$$

then there exists $C > 0$ only depending on $(\Omega, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda)$ such that u satisfies (2.15).

2.1.3. Fourier boundary conditions. The last problem we will study is the Fourier problem, obtained by taking $\Gamma_d = \emptyset$ in (2.1), that is to say:

$$\begin{cases} -\mathrm{div}(A\nabla u) + \mathrm{div}(u\mathbf{v}) + bu = L \quad \text{in} \quad \Omega, \\ A\nabla u \cdot \mathbf{n} + \lambda u = g \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.22}$$

with the hypotheses

$$\begin{gathered} b \in L^r(\Omega) \text{ with } r = \tfrac{N}{2} \text{ if } N > 2 \text{ and } r > 1 \text{ if } N = 2, \\ \tfrac{1}{2}\mathrm{div}(\mathbf{v})(x) + b(x) \geq 0 \text{ for a.e. } x \in \Omega, \end{gathered} \tag{2.23}$$

$$\begin{gathered} \lambda \in L^q(\partial\Omega) \text{ with } q = N-1 \text{ if } N > 2 \text{ and } q > 1 \text{ if } N = 2, \\ \tfrac{1}{2}\mathbf{v} \cdot \mathbf{n} + \lambda \geq 0 \ \sigma\text{-a.e. on } \partial\Omega \end{gathered} \tag{2.24}$$

and, to get the coercitivity of the bilinear form in the variational formulation, either one of the two following:

$$\exists b_0 > 0\,,\ \exists E \subset \Omega \text{ such that } |E| > 0 \text{ and } \tfrac{1}{2}\mathrm{div}(\mathbf{v}) + b \geq b_0 \text{ on } E, \tag{2.25}$$

$$\exists \lambda_0 > 0\,,\ \exists S \subset \partial\Omega \text{ such that } \sigma(S) > 0 \text{ and } \tfrac{1}{2}\mathbf{v} \cdot \mathbf{n} + \lambda \geq \lambda_0 \text{ on } S. \tag{2.26}$$

Finally, the hypotheses on L and g are

$$L \in (H^1(\Omega))'\,,\ g \in H^{-1/2}(\partial\Omega). \tag{2.27}$$

The variational formulation of (2.22), which has a unique solution under Hypotheses (1.1), (1.2), (2.23), (2.24), (2.27) and either (2.25) or (2.26), is

$$\begin{cases} u \in H^1(\Omega), \\ \displaystyle\int_\Omega A\nabla u \cdot \nabla\varphi + \int_{\partial\Omega} \lambda u\varphi \, d\sigma - \int_\Omega u\mathbf{v} \cdot \nabla\varphi + \int_{\partial\Omega} u\varphi\mathbf{v} \cdot \mathbf{n} \, d\sigma + \int_\Omega bu\varphi = \\ \qquad \langle L, \varphi\rangle_{(H^1(\Omega))', H^1(\Omega)} + \langle g, \varphi\rangle_{(H^{1/2}(\partial\Omega))', H^{1/2}(\partial\Omega)}\,,\ \forall \varphi \in H^1(\Omega). \end{cases} \tag{2.28}$$

The regularity result we will prove for this equation is the following.

Theorem 2.3. *Let $p > N$. Under Hypotheses (1.1), (1.2),*

$$\begin{gathered} b \in L^{\frac{Np}{N+p}}(\Omega)\,,\ \tfrac{1}{2}\mathrm{div}(\mathbf{v})(x) + b(x) \geq 0 \textit{ for a.e. } x \in \Omega, \\ \lambda \in L^{(N-1)\frac{p}{N}}(\partial\Omega)\,,\ \tfrac{1}{2}\mathbf{v} \cdot \mathbf{n} + \lambda \geq 0 \ \sigma\textit{-a.e. on } \partial\Omega, \\ L \in (W^{1,p'}(\Omega))'\,,\ g \in \left(W^{1-\frac{1}{p'},p'}(\partial\Omega)\right)' \end{gathered} \tag{2.29}$$

and either (2.25) or (2.26), there exists $\kappa \in]0, 1 - N/p]$ *only depending on* $(\Omega, \alpha_A, \Lambda_A, p)$ *such that the solution* u *to (2.28) is* κ*-Hölder continuous on* Ω*. Moreover, if* Λ *is such that*

$$||b||_{L^{\frac{Np}{N+p}}(\Omega)} + ||\lambda||_{L^{(N-1)\frac{p}{N}}(\partial\Omega)} + ||L||_{(W^{1,p'}(\Omega))'} + ||g||_{(W^{1-1/p',p'}(\partial\Omega))'} \leq \Lambda, \tag{2.30}$$

then there exists $C > 0$ *only depending on*

$(\Omega, E, b_0, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda)$ *in the case where (2.25) is satisfied*
$(\Omega, S, \lambda_0, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda)$ *in the case where (2.26) is satisfied*

such that u *satisfies (2.15).*

2.2. Estimate in L^∞. Here, we take the first step forward the proof of Theorems 2.1, 2.2 and 2.3, by proving that, under Hypotheses of these theorems, their solutions are bounded on Ω.

Proposition 2.1. *Let* $p > N$*. Under Hypotheses (1.1), (1.2), (2.2), (2.12) and (2.13), the solution* u *of (2.6) is in* $L^\infty(\Omega)$ *and, if* Λ *satisfies the inequality (2.14), there exists* $C > 0$ *only depending on* $(\Omega, \Gamma_d, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda)$ *such that*

$$||u||_{L^\infty(\Omega)} \leq C. \tag{2.31}$$

Remark 2.6. *Notice that this result is true without Hypothesis (2.11).*

Remark 2.7. *We will see, in the course of the proof of this proposition, that when* $g_d = 0$ *and* $g_n = 0$*, we just need* Λ *to be an upper bound of* $||L||_{(W^{1,p'}(\Omega))'}$ *to have this estimate.*

Remark 2.8. *It is well known that, in the Dirichlet case (that is to say* $\Gamma_d = \partial\Omega$*), thanks to the Sobolev injection of* $H^1(\Omega)$ *(in* $L^{2N/(N-2)}(\Omega)$ *when* $N > 2$*, or* $L^r(\Omega)$ *for all* $r < \infty$ *when* $N = 2$*), one just needs, instead of (1.2),* $\mathbf{v} \in (L^N(\Omega))^N$ *if* $N > 2$ *or* $\mathbf{v} \in (L^q(\Omega))^N$ *with some arbitrary* $q > 2$ *if* $N = 2$ *to have existence and uniqueness of a solution to (2.6). We will see, in Remark 2.9, that the result of Proposition 2.1 si still true in the Dirichlet case when* $\mathbf{v} \in (L^p(\Omega))^N$*, in which case the condition "*$\frac{1}{2}\mathrm{div}(\mathbf{v}) + b \geq 0$*" is to be understood in the sense of the distributions on* Ω*.*

In fact, when $b \equiv 0$, this last condition "$\mathrm{div}(\mathbf{v}) \geq 0$" is not necessary to obtain the existence of a solution to the Dirichlet problem

$$\begin{cases} -\mathrm{div}(A\nabla u) + \mathrm{div}(\mathbf{v}u) = f \in L^\infty(\Omega), \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.32}$$

Though the corresponding bilinear form is not coercitive, an approximation method can be used to prove the existence of a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (2.32) (L. Boccardo, private communication).

Proposition 2.2. *Let $p > N$. Under Hypotheses (1.1), (1.2), (2.17) and (2.20), the solution u of (2.19) is in $L^\infty(\Omega)$ and, if Λ satisfies the inequality (2.21), there exists $C > 0$ only depending on $(\Omega, \alpha_A, p, \Lambda)$ such that u satisfies (2.31).*

Proposition 2.3. *Let $p > N$. Under Hypotheses (1.1), (1.2), (2.29) and either (2.25) or (2.26), the solution u of (2.28) is in $L^\infty(\Omega)$ and, if Λ satisfies the inequality (2.21), there exists $C > 0$ only depending on*

$(\Omega, E, b_0, \alpha_A, p, \Lambda_{\mathbf{v}}, \Lambda)$ *in the case where (2.25) is satisfied*

$(\Omega, S, \lambda_0, \alpha_A, p, \Lambda_{\mathbf{v}}, \Lambda)$ *in the case where (2.26) is satisfied*

such that u satisfies (2.31).

Let us begin with a technical lemma.

Lemma 2.1. *Let, for $k \geq 0$, $\varphi_k : \mathbb{R} \to \mathbb{R}$ be the function $\varphi_k(s) = \min(s + k, (s-k)^+)$ (where f^+ denotes the non-negative part of a function f, that is to say $f^+ = \max(f, 0)$). Under Hypotheses (1.2) and (2.2)—(2.4) or (2.23)—(2.24), if $\mathcal{U} \in H^1_{\Gamma_d}(\Omega)$ (with $\Gamma_d = \emptyset$ in the case of Hypotheses (2.23)—(2.24)), then*

$$\begin{aligned} &\int_{\Gamma_n} \lambda \mathcal{U} \varphi_k(\mathcal{U})\, d\sigma - \int_\Omega \mathcal{U}\mathbf{v} \cdot \nabla(\varphi_k(\mathcal{U})) \\ &+ \int_{\Gamma_n} \mathcal{U}\varphi_k(\mathcal{U})\mathbf{v}\cdot\mathbf{n}\, d\sigma + \int_\Omega b\mathcal{U}\varphi_k(\mathcal{U}) \\ &\geq \int_\Omega \left(\frac{1}{2}\mathrm{div}(\mathbf{v}) + b\right)(\varphi_k(\mathcal{U}))^2 + \int_{\Gamma_n} \left(\frac{1}{2}\mathbf{v}\cdot\mathbf{n} + \lambda\right)(\varphi_k(\mathcal{U}))^2\, d\sigma \\ &\quad -\frac{k\Lambda_{\mathbf{v}}}{2} || \, |\nabla(\varphi_k(\mathcal{U}))| \, ||_{L^1(\Omega)}. \end{aligned} \tag{2.33}$$

Proof of Lemma 2.1.

We know that $\varphi_k(\mathcal{U}) \in H^1_{\Gamma_d}(\Omega)$ and that $\nabla(\varphi_k(\mathcal{U})) = \chi_{\{|\mathcal{U}|\geq k\}}\nabla\mathcal{U}$ (where χ_A denotes the characteristic function of a measurable set A). Moreover, $|\varphi_k(\mathcal{U})| \in H^1_{\Gamma_d}(\Omega)$ and

$$\nabla(|\varphi_k(\mathcal{U})|) = (\chi_{\{\varphi_k(\mathcal{U})>0\}} - \chi_{\{\varphi_k(\mathcal{U})<0\}})\nabla(\varphi_k(\mathcal{U})) = (\chi_{\{\mathcal{U}>k\}} - \chi_{\{\mathcal{U}<k\}})\nabla\mathcal{U}, \tag{2.34}$$

so that

$$\begin{aligned}
\mathcal{U}\mathbf{v}\cdot\nabla(\varphi_k(\mathcal{U})) &= (\chi_{\{\mathcal{U}>k\}} + \chi_{\{\mathcal{U}<k\}})\mathcal{U}\mathbf{v}\cdot\nabla\mathcal{U}\\
&= ((\mathcal{U}-k)\chi_{\{\mathcal{U}>k\}} + (\mathcal{U}+k)\chi_{\{\mathcal{U}<k\}})\mathbf{v}\cdot\nabla\mathcal{U}\\
&\quad +(k\chi_{\{\mathcal{U}>k\}} - k\chi_{\{\mathcal{U}<k\}})\mathbf{v}\cdot\nabla\mathcal{U}\\
&= \varphi_k(\mathcal{U})\mathbf{v}\cdot\nabla\mathcal{U} + k\mathbf{v}\cdot\nabla(|\varphi_k(\mathcal{U})|)\\
&= \mathbf{v}\cdot\nabla\left(\frac{(\varphi_k(\mathcal{U}))^2}{2}\right) + k\mathbf{v}\cdot\nabla(|\varphi_k(\mathcal{U})|).
\end{aligned} \tag{2.35}$$

With an integration by parts, we get

$$\begin{aligned}
&-\int_\Omega \mathcal{U}\mathbf{v}\cdot\nabla(\varphi_k(\mathcal{U}))\\
&= -\int_{\Gamma_n}\frac{1}{2}\mathbf{v}\cdot\mathbf{n}(\varphi_k(\mathcal{U}))^2 + \int_\Omega \frac{1}{2}\mathrm{div}(\mathbf{v})(\varphi_k(\mathcal{U}))^2 - k\int_\Omega \mathbf{v}\cdot\nabla(|\varphi_k(\mathcal{U})|).
\end{aligned}$$

Since $\mathcal{U}\varphi_k(\mathcal{U}) - (\varphi_k(\mathcal{U}))^2 = k|\varphi_k(\mathcal{U})|$, we find

$$\begin{aligned}
&\int_{\Gamma_n}\lambda\mathcal{U}\varphi_k(\mathcal{U})\,d\sigma - \int_\Omega \mathcal{U}\mathbf{v}\cdot\nabla(\varphi_k(\mathcal{U}))\\
&+\int_{\Gamma_n}\mathcal{U}\varphi_k(\mathcal{U})\mathbf{v}\cdot\mathbf{n}\,d\sigma + \int_\Omega b\mathcal{U}\varphi_k(\mathcal{U})\\
&= \int_\Omega\left(\frac{1}{2}\mathrm{div}(\mathbf{v})+b\right)(\varphi_k(\mathcal{U}))^2 + \int_{\Gamma_n}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{n}+\lambda\right)(\varphi_k(\mathcal{U}))^2\,d\sigma +\\
&\quad k\int_\Omega b|\varphi_k(\mathcal{U})| + k\int_{\Gamma_n}\mathbf{v}\cdot\mathbf{n}|\varphi_k(\mathcal{U})| + k\int_{\Gamma_n}\lambda|\varphi_k(\mathcal{U})|\,d\sigma\\
&\quad -k\int_\Omega \mathbf{v}\cdot\nabla(|\varphi_k(\mathcal{U})|).
\end{aligned} \tag{2.36}$$

But $\frac{1}{2}\mathrm{div}(\mathbf{v}) + b \geq 0$ a.e. on Ω and $\frac{1}{2}\mathbf{v}\cdot\mathbf{n} + \lambda \geq 0$ σ-a.e. on Γ_n, so that

$$\int_\Omega b|\varphi_k(\mathcal{U})| + \int_{\Gamma_n}\mathbf{v}\cdot\mathbf{n}|\varphi_k(\mathcal{U})|\,d\sigma$$

$$
\begin{aligned}
&+\int_{\Gamma_n} \lambda|\varphi_k(\mathcal{U})|\,d\sigma - \int_\Omega \frac{1}{2}\mathbf{v}\cdot\nabla(|\varphi_k(\mathcal{U})|)\\
&= \int_\Omega \left(\frac{1}{2}\mathrm{div}(\mathbf{v})+b\right)|\varphi_k(\mathcal{U})| + \int_{\Gamma_n}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{n}+\lambda\right)|\varphi_k(\mathcal{U})|\,d\sigma\\
&\geq 0,
\end{aligned}
$$

and (2.36) give then

$$
\begin{aligned}
&\int_{\Gamma_n}\lambda\mathcal{U}\varphi_k(\mathcal{U})\,d\sigma - \int_\Omega \mathcal{U}\mathbf{v}\cdot\nabla(\varphi_k(\mathcal{U})) + \int_{\Gamma_n}\mathcal{U}\varphi_k(\mathcal{U})\mathbf{v}\cdot\mathbf{n}\,d\sigma + \int_\Omega b\mathcal{U}\varphi_k(\mathcal{U})\\
&\geq \int_\Omega\left(\frac{1}{2}\mathrm{div}(\mathbf{v})+b\right)(\varphi_k(\mathcal{U}))^2 + \int_{\Gamma_n}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{n}+\lambda\right)(\varphi_k(\mathcal{U}))^2\,d\sigma\\
&\quad -\frac{k}{2}\int_\Omega \mathbf{v}\cdot\nabla(|\varphi_k(\mathcal{U})|).
\end{aligned}
$$

With (2.34), we see that

$$
\int_\Omega \mathbf{v}\cdot\nabla(|\varphi_k(\mathcal{U})|) \leq \Lambda_{\mathbf{v}}||\,|\nabla(\varphi_k(\mathcal{U}))|\,||_{L^1(\Omega)}
$$

and (2.33) is thus proved. ■

Remark 2.9. *If* $\Gamma_d = \partial\Omega$, $\mathbf{v}\in(L^p(\Omega))^N$ *(instead of (1.2)) and* $\frac{1}{2}\mathrm{div}(\mathbf{v})+b\geq 0$ *in the sense of the distributions on* Ω, *then (2.35) gives*

$$
\begin{aligned}
&-\int_\Omega \mathcal{U}\mathbf{v}\cdot\nabla(\varphi_k(\mathcal{U})) + \int_\Omega b\mathcal{U}\varphi_k(\mathcal{U})\\
&= \int_\Omega\left(-\frac{1}{2}\mathbf{v}\cdot\nabla((\varphi_k(\mathcal{U}))^2) + b(\varphi_k(\mathcal{U}))^2\right)\\
&\quad +k\int_\Omega\left(-\frac{1}{2}\mathbf{v}\cdot\nabla(|\varphi_k(\mathcal{U})|)+b|\varphi_k(\mathcal{U})|\right) - \frac{k}{2}\int_\Omega \mathbf{v}\cdot\nabla(|\varphi_k(\mathcal{U})|)\\
&\geq -\frac{k}{2}||\,|\mathbf{v}|\,||_{L^p(\Omega)}||\,|\nabla(\varphi_k(\mathcal{U}))|\,||_{L^{p'}(\Omega)}.
\end{aligned}
$$

This inequality, used instead of (2.33) in the following proof, allows us to prove what we have claimed in Remark 2.8, that is to say that the result of Proposition 2.1 is still true in the Dirichlet case when $\mathbf{v}$ *is only in* $(L^p(\Omega))^N$.

Proof of Proposition 2.1

Since $g_d \in W^{1-\frac{1}{p},p}(\partial\Omega)$ (Hypothesis (2.13)), we can choose $u_0 \in W^{1,p}(\Omega)$ such that $u_{0\,|\partial\Omega} = g_d$ and, thanks to the definition of the norm on the space $W^{1-1/p,p}(\partial\Omega)$, such that $||u_0||_{W^{1,p}(\Omega)} \le 2||g_d||_{W^{1-1/p,p}(\partial\Omega)} \le 2\Lambda$.

By the Sobolev injection, since $p > N$, $u_0 \in L^\infty(\Omega)$ and there exists C_1 only depending on (Ω, p) such that $||u_0||_{L^\infty(\Omega)} \le C_1||u_0||_{W^{1,p}(\Omega)} \le 2C_1\Lambda$; thus, proving the result of Proposition 2.1 for $w = u - u_0$ will give us the same conclusion for the solution u of (2.6).

Let us introduce another useful notation: with this u_0, we denote by $\widetilde{L}_1 \in (W^{1,p'}(\Omega))'$ the linear form of the right hand side of (2.6), that is to say

$$\begin{aligned}
&\langle \widetilde{L}_1, \varphi\rangle_{(W^{1,p'}(\Omega))', W^{1,p'}(\Omega)}\\
&\quad = \langle L, \varphi\rangle_{(W^{1,p'}(\Omega))', W^{1,p'}(\Omega)} + \langle g_n, \varphi\rangle_{(W^{1-1/p',p'}(\partial\Omega))', W^{1-1/p',p'}(\partial\Omega)}\\
&\qquad - \int_\Omega A\nabla u_0 \cdot \nabla\varphi - \int_{\Gamma_n} \lambda u_0 \varphi \, d\sigma + \int_\Omega u_0 \mathbf{v}\cdot\nabla\varphi - \int_{\Gamma_n} u_0\varphi\mathbf{v}\cdot\mathbf{n}\, d\sigma\\
&\qquad - \int_\Omega b u_0 \varphi.
\end{aligned}$$

It is easy to show, thanks to Hypotheses (2.12), (2.13) and (2.14), that there exists C_2 only depending on $(\Omega, p, \Lambda_A, \Lambda_{\mathbf{v}}, \Lambda)$ such that

$$||\widetilde{L}_1||_{(W^{1,p'}(\Omega))'} \le C_2. \tag{2.37}$$

Notice that, when $g_n = 0$ and $g_d = 0$ (in which case we take $u_0 = 0$), $\widetilde{L}_1 = L$ and thus, with Λ being any upper bound of $||L||_{(W^{1,p'}(\Omega))'}$, $C_2 = \Lambda$ is correct.

Let $k > 0$.

Lemma 2.1 applied to $\mathcal{U} = w$ and Hypotheses (2.3)—(2.4) give us

$$\begin{aligned}
&\int_{\Gamma_n} \lambda w \varphi_k(w)\, d\sigma - \int_\Omega w\mathbf{v}\cdot\nabla(\varphi_k(w))\\
&+ \int_{\Gamma_n} w\varphi_k(w)\mathbf{v}\cdot\mathbf{n}\, d\sigma + \int_\Omega bw\varphi_k(w) \ge -\frac{k\Lambda_{\mathbf{v}}}{2}||\,|\nabla(\varphi_k(w))|\,||_{L^1(\Omega)}.
\end{aligned}$$

Thus, using $\varphi_k(w)$ as a test function in (2.6), we see that

$$\int_\Omega A\nabla(\varphi_k(w))\cdot\nabla(\varphi_k(w))$$

$$
\begin{aligned}
&= \int_\Omega A\nabla w \cdot \nabla(\varphi_k(w)) \\
&\leq \langle \widetilde{L}_1, \varphi_k(w)\rangle_{(W^{1,p'}(\Omega))', W^{1,p'}(\Omega)} + \frac{\Lambda_{\mathbf{v}}}{2} k || \, |\nabla(\varphi_k(w))| \, ||_{L^1(\Omega)} \\
&\leq \left(||\widetilde{L}_1||_{(W^{1,p'}(\Omega))'} + \frac{1}{2}\Lambda_{\mathbf{v}} |\Omega|^{1/p} k \right) ||\varphi_k(w)||_{W^{1,p'}(\Omega)},
\end{aligned}
$$

which gives, thanks to the coercitivity of A, to (2.37) and to the Poincaré inequality in $W^{1,p'}_{\Gamma_d}(\Omega)$,

$$
|| \, |\nabla(\varphi_k(w))| \, ||^2_{L^2(\Omega)} \leq C_3(1+k) || \, |\nabla(\varphi_k(w))| \, ||_{L^{p'}(\Omega)}, \tag{2.38}
$$

where C_3 only depends on $(\Omega, \Gamma_d, \alpha_A, \Lambda_A, C_2, \Lambda_{\mathbf{v}})$.

Defining $A_k = \{x \in \Omega \mid |w(x)| \geq k\}$, we see that $\nabla(\varphi_k(w)) = 0$ a.e. outside A_k, so that, thanks to Hölder inequality,

$$
|| \, |\nabla(\varphi_k(w))| \, ||_{L^{p'}(\Omega)} \leq |A_k|^{\frac{1}{2}-\frac{1}{p}} || \, |\nabla(\varphi_k(w))| \, ||_{L^2(\Omega)}.
$$

Using this in (2.38), we find

$$
|| \, |\nabla(\varphi_k(w))| \, ||_{L^{p'}(\Omega)} \leq C_3(1+k)|A_k|^{1-\frac{2}{p}}. \tag{2.39}
$$

The Sobolev injection $W^{1,p'}(\Omega) \hookrightarrow L^{\frac{Np'}{N-p'}}(\Omega)$ ($p' < N$) and the Poincaré inequality give us C_4 depending on (Ω, Γ_d, p) such that

$$
\begin{aligned}
\left(\int_\Omega |\varphi_k(w)|^{\frac{Np'}{N-p'}} \right)^{\frac{N-p'}{Np'}} &\leq C_4 || \, |\nabla \varphi_k(w)| \, ||_{L^{p'}(\Omega)} \\
&\leq C_4 C_3 (1+k)|A_k|^{1-\frac{2}{p}}.
\end{aligned}
$$

But $|\varphi_k(u)| \geq (h-k)$ on A_h, for all $h > k$, so that, with $C_5 = C_4 C_3$,

$$
(h-k)^{\frac{Np'}{N-p'}} |A_h| \leq (C_5(1+k))^{\frac{Np'}{N-p'}} |A_k|^{\frac{Np'}{N-p'}\left(1-\frac{2}{p}\right)} \qquad \text{for all } h > k \geq 0,
$$

that is to say, with $\beta = \frac{Np'}{N-p'}$ and $\gamma = \frac{Np'}{N-p'}(1-\frac{2}{p}) = \frac{Np-2N}{Np-N-p} > 1$,

$$
|A_h| \leq \frac{C_5^\beta (1+k)^\beta}{(h-k)^\beta} |A_k|^\gamma \qquad \text{for all } h > k \geq 0. \tag{2.40}
$$

Lemma 2.2 following this proof allows us to see that, if

$$H = \exp\left(\sum_{n\geq 0} \frac{C_5|\Omega|^{\frac{\gamma-1}{\beta}}}{\left(2^{\frac{\gamma-1}{\beta}}\right)^n}\right) < +\infty,$$

then $|A_H| = 0$, i.e., $|w| \leq H$ a.e. on Ω. ∎

Lemma 2.2. *Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function. If there exist $\beta > 0$, $\gamma > 1$ and $C > 0$ such that*

$$\forall h > k \geq 0\,,\ F(h) \leq \frac{C^\beta(1+k)^\beta}{(h-k)^\beta} F(k)^\gamma$$

and if

$$H = \exp\left(\sum_{n\geq 0} \frac{CF(0)^{\frac{\gamma-1}{\beta}}}{\left(2^{\frac{\gamma-1}{\beta}}\right)^n}\right) < +\infty,$$

then $F(H) = 0$.

This is a variant of the Lemma 4.1, i) in [1], which states the same result without the term $(1+k)$. In this paper, G. Stampacchia proves the result of Proposition 2.1 (in the homogeneous Dirichlet case, and with slightly different hypotheses on the convection term) with another method, involving a reasoning by induction. But we found it more readable to state this simple variant, which is the key to the proof of Propositions 2.1, 2.2 and 2.3.

Proof of Lemma 2.2.

If $F(0) = 0$, the lemma is trivial; we suppose thus that $F(0) > 0$.

Let $h_0 = 0$ and define, by induction,

$$h_{n+1} = \left(1 + \frac{CF(0)^{\frac{\gamma-1}{\beta}}}{\left(2^{\frac{\gamma-1}{\beta}}\right)^n}\right) h_n + \frac{CF(0)^{\frac{\gamma-1}{\beta}}}{\left(2^{\frac{\gamma-1}{\beta}}\right)^n} > h_n.$$

It is easy to see, by induction, that $F(h_n) \leq F(0)/2^n$ for all $n \geq 0$; indeed, it is true for $n = 0$ and, if $F(h_n) \leq F(0)/2^n$, one sees that

$$F(h_{n+1}) \quad \leq \quad \frac{C^\beta(1+h_n)^\beta}{(h_{n+1}-h_n)^\beta} F(h_n)^\gamma$$

$$\begin{aligned}
&\le \frac{C^\beta(1+h_n)^\beta}{C^\beta(1+h_n)^\beta F(0)^{\gamma-1}/2^{(\gamma-1)n}}\frac{F(0)^\gamma}{2^{\gamma n}}\\
&\le \frac{F(0)}{2^n}.
\end{aligned}$$

Moreover, if $l_n = h_n + 1$, one sees that, for all $n \ge 0$,

$$l_{n+1} = \left(1 + \frac{CF(0)^{\frac{\gamma-1}{\beta}}}{\left(2^{\frac{\gamma-1}{\beta}}\right)^n}\right) l_n,$$

so that, for all $n \ge 1$,

$$l_n = \prod_{k=0}^{n-1}\left(1 + \frac{CF(0)^{\frac{\gamma-1}{\beta}}}{\left(2^{\frac{\gamma-1}{\beta}}\right)^k}\right).$$

Taking the logarithm of this, we have

$$\begin{aligned}
\ln(l_n) &= \sum_{k=0}^{n-1}\ln\left(1 + \frac{CF(0)^{\frac{\gamma-1}{\beta}}}{\left(2^{\frac{\gamma-1}{\beta}}\right)^k}\right)\\
&\le \sum_{k=0}^{n-1}\frac{CF(0)^{\frac{\gamma-1}{\beta}}}{\left(2^{\frac{\gamma-1}{\beta}}\right)^k} \le \ln(H),
\end{aligned}$$

which means that, for all $n \ge 0$, $h_n \le l_n \le H$. Since F is non-negative and non-increasing, we get $0 \le F(H) \le F(h_n) \le F(0)/2^n$ for all $n \ge 0$, which implies, as $n \to \infty$, $F(H) = 0$. ■

Proof of Proposition 2.2

With the same φ_k as in Lemma (2.1), since

$$-\int_\Omega u\mathbf{v}\cdot\nabla(\varphi_k(u)) = -\int_\Omega \mathbf{v}\cdot\nabla(u\varphi_k(u)) + \int_\Omega \mathbf{v}\cdot\nabla\left(\frac{(\varphi_k(u))^2}{2}\right) = 0,$$

using $\varphi_k(u)$ as a test function in (2.19) allows us to see that

$$||\,|\nabla(\varphi_k(u))|\,||^2_{L^2(\Omega)} \le \frac{\Lambda}{\alpha_A}||\varphi_k(u)||_{W^{1,p'}(\Omega)}. \tag{2.41}$$

In order to continue this proof as the preceding one, we need a Poincaré inequality on $\varphi_k(u)$.

The Sobolev injection $W^{1,p'}(\Omega) \hookrightarrow L^{\frac{Np'}{N-p'}}$ gives us C_1 depending on (Ω, p) such that

$$\left(\int_\Omega |\varphi_k(u)|^{\frac{Np'}{N-p'}}\right)^{\frac{N-p'}{Np'}} \leq C_1||\varphi_k(u)||_{L^{p'}(\Omega)} + C_1||\,|\nabla(\varphi_k(u))|\,||_{L^{p'}(\Omega)}. \quad (2.42)$$

Moreover, thanks to Hölder inequality, we have

$$||\varphi_k(u)||_{L^{p'}(\Omega)} \leq |A_k|^{\frac{1}{N}}||\varphi_k(u)||_{L^{\frac{Np'}{N-p'}}(\Omega)},$$

where $A_k = \{x \in \Omega \mid |u(x)| \geq k\}$, so that

$$||\varphi_k(u)||_{L^{p'}(\Omega)} \leq C_1|A_k|^{\frac{1}{N}}||\varphi_k(u)||_{L^{p'}(\Omega)} + C_1|A_k|^{\frac{1}{N}}||\,|\nabla(\varphi_k(u))|\,||_{L^{p'}(\Omega)}. \quad (2.43)$$

But the Tchebycheff inequality gives us

$$|A_k| \leq \frac{1}{k^2}||u||^2_{L^2(\Omega)} \leq \frac{C_2}{k^2},$$

with C_2 only depending on $(\Omega, \alpha_A, \Lambda)$ (because u is the solution of (2.19)).

Thus, if $k \geq k_0 = \sqrt{(2C_1)^N C_2}$, we have $C_1|A_k|^{\frac{1}{N}} \leq \frac{1}{2}$ which gives, in (2.43),

$$\begin{aligned} ||\varphi_k(u)||_{L^{p'}(\Omega)} &\leq 2C_1|A_k|^{\frac{1}{N}}||\,|\nabla(\varphi_k(u))|\,||_{L^{p'}(\Omega)} \\ &\leq 2C_1|\Omega|^{\frac{1}{N}}||\,|\nabla(\varphi_k(u))|\,||_{L^{p'}(\Omega)}, \end{aligned}$$

that is to say

$$||\varphi_k(u)||_{W^{1,p'}(\Omega)} \leq C_3||\,|\nabla(\varphi_k(u))|\,||_{L^{p'}(\Omega)}, \quad (2.44)$$

with $C_3 = 2C_1|\Omega|^{\frac{1}{N}} + 1$, i.e., the Poincaré inequality which we wanted to conclude the proof of this proposition.

Indeed, using (2.44) in (2.41), we get, forall $k \geq k_0$,

$$||\,|\nabla(\varphi_k(u))|\,||^2_{L^2(\Omega)} \leq \frac{\Lambda C_3}{\alpha_A}||\,|\nabla(\varphi_k(u))|\,||_{L^{p'}(\Omega)}$$

and, with the Hölder inequality,

$$|| \, |\nabla(\varphi_k(u))| \, ||_{L^{p'}(\Omega)} \leq C_4 |A_k|^{1-\frac{2}{p}},$$

where C_4 only depends on $(\Omega, \alpha_A, p, \Lambda)$.

Moreover, (2.44) used in (2.42) gives, for all $k \geq k_0$,

$$\left(\int_\Omega |\varphi_k(u)|^{\frac{Np'}{N-p'}} \right)^{\frac{N-p'}{Np'}} \leq C_1 C_3 || \, |\nabla(\varphi_k(u))| \, ||_{L^{p'}(\Omega)} \leq C_1 C_3 C_4 |A_k|^{1-\frac{2}{p}}.$$

Then, using $|\varphi_k(u)| \geq h - k$ on A_h, when $h > k \geq k_0$, we find

$$|A_h| \leq \frac{C_5^\beta}{(h-k)^\beta} |A_k|^\gamma$$

(with the same β and γ as in the proof of Proposition 2.1), with C_5 only depending on $(\Omega, \alpha_A, p, \Lambda)$.

We can thus use Lemma 4.1 of [1], or Lemma 2.2, to conclude (here, the term $(1+k)$ we found in the course of the demonstration of Proposition 2.1 does not appear, so that the classical Stampacchia Lemma applies).

For example, by noticing that

$$|A_h| \leq \frac{C_5^\beta}{(h-k)^\beta} |A_k|^\gamma \leq \frac{C_5^\beta (1+k)^\beta}{(h-k)^\beta} |A_k|^\gamma \qquad \text{for all } h > k \geq k_0$$

and defining

$$H = k_0 + \exp\left(\sum_{n \geq 0} \frac{C_5 |\Omega|^{\frac{\gamma-1}{\beta}}}{\left(2^{\frac{\gamma-1}{\beta}}\right)^n} \right) < +\infty,$$

which only depends on $(\Omega, \alpha_A, p, \Lambda)$, Lemma 2.2 applied to $F(k) = |A_{k_0+k}|$ gives us $|A_H| = 0$. ■

Proof of Proposition 2.3

Thanks to Lemma 2.1 applied to $\mathcal{U} = u$, to the coercitivity of A and to Hypothesis (2.25) or (2.26), we find, using $\varphi_k(u)$ as a test function in (2.28), C_1 only depending on $(\Omega, \alpha_A, \Lambda)$ and

(E, b_0)	if (2.25) is satisfied
(S, λ_0)	if (2.26) is satisfied

such that

$$\begin{aligned}||\varphi_k(u)||^2_{H^1(\Omega)} &\leq C_1||\varphi_k(u)||_{W^{1,p'}(\Omega)} + C_1\frac{\Lambda_{\mathbf{v}}k}{2}||\,|\nabla(\varphi_k(u))|\,||_{L^1(\Omega)}\\ &\leq C_1||\varphi_k(u)||_{W^{1,p'}(\Omega)} + \frac{\Lambda_{\mathbf{v}}|\Omega|^{1/p}}{2}k||\,|\nabla(\varphi_k(u))|\,||_{L^{p'}(\Omega)}\\ &\leq C_2(1+k)||\varphi_k(u)||_{W^{1,p'}(\Omega)},\end{aligned} \tag{2.45}$$

where C_2 only depends on $(C_1,\Omega,p,\Lambda_{\mathbf{v}})$.

Thanks to Hölder inequality (and since $\varphi_k(u)=|\nabla(\varphi_k(u))|=0$ outside A_k), we have

$$||\varphi_k(u)||_{W^{1,p'}(\Omega)} \leq |A_k|^{\frac{1}{2}-\frac{1}{p}}||\varphi_k(u)||_{H^1(\Omega)},$$

which gives

$$||\varphi_k(u)||_{W^{1,p'}(\Omega)} \leq C_2(1+k)|A_k|^{1-\frac{2}{p}}.$$

With the Sobolev injection $W^{1,p'}(\Omega)\hookrightarrow L^{\frac{Np'}{N-p'}}(\Omega)$, we find thus C_3 only depending on (Ω,p) such that, for all $k\geq 0$,

$$\left(\int_\Omega |\varphi_k(u)|^{\frac{Np'}{N-p'}}\right)^{\frac{N-p'}{Np'}} \leq C_3||\varphi_k(u)||_{W^{1,p'}(\Omega)} \leq C_2C_3(1+k)|A_k|^{1-\frac{2}{p}}$$

and the conclusion is similar to that of the proof of Proposition 2.1. ■

2.3. Continuity within the domain Ω. In [1], the author proves the following result. The precise dependence of the "constants" C and κ can be found in [12].

Theorem 2.4. *Let Ω_0 be an open subset of $\mathbb{R}^N$ and $A_0:\Omega_0\to M_N(\mathbb{R})$ be a measurable bounded uniformly elliptic matrix valued function; we denote by α_{A_0} a coercitivity constant for A_0 and Λ_{A_0} an essential upper bound of $||A_0(.)||$ on Ω_0. If $p>N$ and $L_0\in W^{-1,p}(\Omega_0):=(W_0^{1,p'}(\Omega_0))'$, then there exists $\kappa\in]0,1-N/p]$ only depending on $(N,\alpha_{A_0},\Lambda_{A_0},p)$ such that any solution u of*

$$\left\{\begin{array}{l} u\in H^1(\Omega),\\ \displaystyle\int_{\Omega_0} A_0\nabla u\cdot\nabla\varphi = \langle L_0,\varphi\rangle_{(H_0^1(\Omega_0))',H_0^1(\Omega_0)}\,,\ \forall\varphi\in H_0^1(\Omega_0)\end{array}\right. \tag{2.46}$$

is κ-Hölder continuous on any compact subset of Ω_0. Moreover, for all compact subset K of Ω_0, if Λ_{L_0} is an upper bound of $||L_0||_{W^{-1,p}(\Omega_0)}$ and Λ_u is an upper bound for $||u||_{L^2(\Omega_0)}$, there exists C only depending on

$$(\Omega_0, \alpha_{A_0}, \Lambda_{A_0}, p, \Lambda_{L_0}, \Lambda_u, K)$$

such that

$$||u||_{\mathcal{C}^{0,\kappa}(K)} \leq C. \tag{2.47}$$

Remark 2.10. *We have chosen, on $W_0^{1,p'}(\Omega_0)$, the norm $||\varphi||_{W_0^{1,p'}(\Omega_0)} = ||\,|\nabla\varphi|\,||_{L^{p'}(\Omega_0)}$. The norm on $W^{-1,p}(\Omega_0)$ is the associated dual norm.*

All the following regularity results come from this one, as we shall see.

In fact, since we already know that the solutions of (2.6), (2.19) and (2.28) are bounded, we can eliminate the terms involving $\mathbf{v}$, b and λ from the equations satisfied by these solutions.

Lemma 2.3. *Let $p > N$. Suppose Hypotheses (1.1), (1.2), (2.2), (2.12), (2.13) and (2.14). Let $u_0 \in W^{1,p}(\Omega)$ such that $u_{0\,|\partial\Omega} = g_d$ and $||u_0||_{W^{1,p}(\Omega)} \leq 2||g_d||_{W^{1-1/p,p}(\Omega)}$. Then there exists $\widetilde{L} \in (W^{1,p'}(\Omega))'$ such that the solution u of (2.6) is solution of*

$$\left\{ \begin{array}{l} w = u - u_0 \in H^1_{\Gamma_d}(\Omega), \\ \displaystyle\int_\Omega A\nabla w \cdot \nabla\varphi = \langle \widetilde{L}, \varphi\rangle_{(H^1(\Omega))', H^1(\Omega)}\,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega) \end{array} \right. \tag{2.48}$$

and such that $||\widetilde{L}||_{(W^{1,p'}(\Omega))'} \leq C$, where C only depends on

$$(\Omega, \Gamma_d, \alpha_A, p, \Lambda_A, \Lambda_{\mathbf{v}}, \Lambda).$$

Proof of Lemma 2.3

We have already seen that $w = u - u_0$ satisfies

$$\left\{ \begin{array}{l} w = u - u_0 \in H^1_{\Gamma_d}(\Omega), \\ \displaystyle\int_\Omega A\nabla w \cdot \nabla\varphi = \langle \widetilde{L}_1, \varphi\rangle_{(H^1(\Omega))', H^1(\Omega)} - \int_{\Gamma_n} \lambda w\varphi\, d\sigma \\ \displaystyle\qquad + \int_\Omega w\mathbf{v}\cdot\nabla\varphi - \int_{\Gamma_n} w\varphi\mathbf{v}\cdot\mathbf{n}\, d\sigma - \int_\Omega bw\varphi\,,\ \forall\varphi \in H^1_{\Gamma_d}(\Omega) \end{array} \right. \tag{2.49}$$

with $\widetilde{L}_1 \in (W^{1,p'}(\Omega))'$ satisfying (2.37).

But we know that $w \in H^1(\Omega) \cap L^\infty(\Omega)$, and that the norm of w in this space is bounded by C_1 only depending on $(\Omega, \Gamma_d, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda)$; one can show that this implies $w \in L^\infty(\partial\Omega)$, with $||w||_{L^\infty(\partial\Omega)} \leq C_1$.

Thus, $\lambda w \in L^{(N-1)\frac{p}{N}}(\Gamma_n)$, $w\mathbf{v} \in (L^p(\Omega))^N$, $w\mathbf{v}\cdot\mathbf{n} \in L^p(\partial\Omega)$, $bw \in L^{\frac{Np}{N+p}}(\Omega)$, and all these functions have their norms in these respective spaces bounded by C_2 only depending on $(\Omega, \Gamma_d, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda)$; for all $\varphi \in W^{1,p'}(\Omega)$, we have then, using Hölder inequalities,

$$\begin{aligned}
&-\int_{\Gamma_n} \lambda w\varphi \, d\sigma + \int_\Omega w\mathbf{v}\cdot\nabla\varphi - \int_{\Gamma_n} w\varphi\mathbf{v}\cdot\mathbf{n}\, d\sigma - \int_\Omega bw\varphi \\
&\leq \ ||w\lambda||_{L^{(N-1)\frac{p}{N}}(\Gamma_n)} ||\varphi||_{L^{\frac{(N-1)p'}{N-p'}}(\partial\Omega)} + ||w\mathbf{v}||_{(L^p(\Omega))^N} ||\varphi||_{W^{1,p'}(\Omega)} \\
&\qquad + ||w\mathbf{v}\cdot\mathbf{n}||_{L^p(\partial\Omega)} ||\varphi||_{L^{p'}(\partial\Omega)} + ||bw||_{L^{\frac{Np}{N+p}}(\Omega)} ||\varphi||_{L^{\frac{Np'}{N-p'}}(\Omega)} \\
&\leq \ (C_2C_3 + C_2 + C_2C_4 + C_2C_5)||\varphi||_{W^{1,p'}(\Omega)},
\end{aligned}$$

with C_3, C_4 and C_5 only depending on (Ω, p) (C_4 is the norm of the trace operator $W^{1,p'}(\Omega) \to L^{p'}(\partial\Omega)$ and C_3, C_5 are the norms of the Sobolev injections $W^{1-1/p',p'}(\partial\Omega) \hookrightarrow L^{\frac{(N-1)p'}{N-p'}}(\partial\Omega)$ and $W^{1,p'}(\Omega) \hookrightarrow L^{\frac{Np'}{N-p'}}(\Omega)$, see [5]). The result of the lemma is proved. ■

Lemma 2.4. *Let $p > N$. Under Hypotheses (1.1), (1.2), (2.17), (2.20) and (2.21), there exists $\widetilde{L} \in (W^{1,p'}(\Omega))'$ such that the solution u of (2.19) is solution of*

$$\left\{\begin{array}{l} u \in H^1_*(\Omega), \\ \displaystyle\int_\Omega A\nabla u\cdot\nabla\varphi = \langle \widetilde{L}, \varphi\rangle_{(H^1(\Omega))',H^1(\Omega)}, \ \forall \varphi \in H^1(\Omega) \end{array}\right. \tag{2.50}$$

and such that $||\widetilde{L}||_{(W^{1,p'}(\Omega))'} \leq C$, where C only depends on $(\Omega, \alpha_A, p, \Lambda_{\mathbf{v}}, \Lambda)$.

Lemma 2.5. *Let $p > N$. Under Hypotheses (1.1), (1.2), (2.29), (2.30) and either (2.25) or (2.26), there exists $\widetilde{L} \in (W^{1,p'}(\Omega))'$ such that the solution u of (2.28) is solution of*

$$\left\{\begin{array}{l} u \in H^1(\Omega), \\ \displaystyle\int_\Omega A\nabla u\cdot\nabla\varphi = \langle \widetilde{L}, \varphi\rangle_{(H^1(\Omega))',H^1(\Omega)}, \ \forall \varphi \in H^1(\Omega) \end{array}\right. \tag{2.51}$$

and such that $||\widetilde{L}||_{(W^{1,p'}(\Omega))'} \leq C$*, where* C *only depends on*

$(\Omega, E, b_0, \alpha_A, p, \Lambda_{\mathbf{v}}, \Lambda)$	*in the case where (2.25) is satisfied,*
$(\Omega, S, \lambda_0, \alpha_A, p, \Lambda_{\mathbf{v}}, \Lambda)$	*in the case where (2.26) is satisfied.*

The demonstrations of these lemmas are of the same kind as the proof of Lemma 2.3, and we omit them.

Corollary 2.1. *Under the hypotheses of Lemma 2.3, 2.4 or 2.5, there exists* $\kappa \in]0, 1 - N/p]$ *only depending on* $(N, \alpha_A, \Lambda_A, p)$ *such that the solution* u *of the corresponding problem (i.e., the mixed, Neumann or Fourier problem) is* κ*-Hölder continuous on any compact subset of* Ω*. Moreover, if* K *is a compact subset of* Ω*, there exists* C *only depending on*

$(\Omega, \Gamma_d, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda, K)$	*in the case of the mixed problem,*
$(\Omega, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda, K)$	*in the case of the Neumann problem,*
$(\Omega, E, b_0, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda, K)$	*in the case of the Fourier problem with Hypothesis (2.25),*
$(\Omega, S, \lambda_0, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda, K)$	*in the case of the Fourier problem with Hypothesis (2.26),*

such that u *satisfies (2.47).*

Proof of Corollary 2.1

The fact that the solutions of these problems are in $\mathcal{C}^{0,\kappa}(K)$ for a $\kappa \in]0, 1 - N/p]$ is clear from Lemmas 2.3, 2.4, 2.5 and Theorem 2.4.

The upper bound on $||u||_{\mathcal{C}^{0,\kappa}(\Omega)}$ is a consequence of the following remark: if $l \in (W^{1,p'}(\Omega))'$ then

$$\begin{aligned}
&||l||_{W^{-1,p}(\Omega)} \\
&:= \sup\left\{\langle l, \varphi\rangle_{(W^{1,p'}(\Omega))', W^{1,p'}(\Omega)}\,,\ \varphi \in W_0^{1,p'}(\Omega)\,,\ ||\,|\nabla\varphi|\,||_{L^p(\Omega)} \leq 1\right\} \\
&\leq (1 + \operatorname{diam}(\Omega)^{1/p'})||l||_{(W^{1,p'}(\Omega))'},
\end{aligned}$$

which is clear once we recall the Poincaré inequality:

$$\forall \varphi \in W_0^{1,p'}(\Omega)\,,\ ||\varphi||_{W^{1,p'}(\Omega)} \leq (1 + \operatorname{diam}(\Omega)^{1/p'})||\,|\nabla\varphi|\,||_{L^{p'}(\Omega)}.$$

■

2.4. Continuity near the boundary of the domain. We will now prove that the solutions of the mixed, Neumann and Fourier problems are, in fact, Hölder continuous on Ω (which implies that they are Hölder continuous on $\overline{\Omega}$). Since we already know that they are Hölder continuous on any compact subset of Ω, it is sufficient to prove the Hölder continuity near the boundary of Ω and this is where the boundary conditions (especially Hypothesis (2.11), which we have not used up to now) come into play.

Proposition 2.4. *Let $p > N$. Under Hypotheses (1.1), (1.2), (2.2), (2.11), (2.12), (2.13) and (2.14), for all $i \in [1, m]$, there exists κ only depending on $(N, \alpha_A, \Lambda_A, p, h_i)$ such that the solution u of (2.6) is κ-Hölder continuous on any compact subset of $O_i \cap \overline{\Omega}$. Moreover, for all such compact subset K, there exists C only depending on $(\Omega, \Gamma_d, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda, h_i, K)$ such that*

$$||u||_{\mathcal{C}^{0,\kappa}(K)} \le C. \tag{2.52}$$

Proof of Proposition 2.4

Let $u_0 \in W^{1,p}(\Omega) \subset \mathcal{C}^{0,1-N/p}(\Omega)$ as in Lemma 2.3; it is sufficient to prove the result for $w = u - u_0$, which satisfies (2.48).

Thanks to Hypothesis (2.11), we see that

$$\left\{ \begin{array}{rcl} H^1(O_i) & \longrightarrow & H^1(B) \\ \varphi & \longrightarrow & \varphi \circ h_i^{-1} \end{array} \right.$$

and

$$\left\{ \begin{array}{rcl} H^1_{\Gamma_d}(\Omega \cap O_i) & \longrightarrow & H^1_{h_i(\Gamma_d)}(B_+) \\ \varphi & \longrightarrow & \varphi \circ h_i^{-1} \end{array} \right.$$

are isomorphisms. It is also well known that we can compute the derivatives of $\varphi \circ h_i^{-1}$ (or $\psi \circ h_i$, when $\psi \in H^1(B)$ or $\psi \in H^1(B_+)$) by using the classical chain rule.

The case when (O_i, h_i) is of type (F) is the easier; we thus begin by this one. Then, we will handle the more difficult case when (O_i, h_i) is of type (D), and we will quickly see how we can deduce the case (DF) from the two preceding ones.

Step 1: The mapping (O_i, h_i) is of type (F).

Taking $\phi \in \mathcal{C}_c^\infty(B)$, we notice that $\varphi = \phi \circ h_i : O_i \cap \Omega \to \mathbb{R}$ is in $H^1(O_i \cap \Omega)$ and has a compact support in O_i; thus, the extension of φ, still denoted by

φ, to Ω by zero outside $O_i \cap \Omega$ is in $H^1_{\Gamma_d}(\Omega)$ (since $\Gamma_d \cap O_i = \emptyset$). We have, thanks a change of variable (Jh_i denotes the Jacobian determinant of h_i), and by denoting $w_{\rm tr} = w \circ h_i^{-1}$,

$$
\begin{aligned}
\langle \widetilde{L}, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} &= \int_\Omega A \nabla w \cdot \nabla \varphi \\
&= \int_{O_i \cap \Omega} A \nabla (w_{\rm tr} \circ h_i) \cdot \nabla (\phi \circ h_i) \\
&= \int_{O_i \cap \Omega} h_i' A (h_i')^T (\nabla w_{\rm tr}) \circ h_i \cdot (\nabla \phi) \circ h_i \\
&= \int_{B_+} \left(\frac{h_i' A (h_i')^T}{|J h_i|} \circ h_i^{-1} \right) \nabla w_{\rm tr} \cdot \nabla \phi,
\end{aligned}
$$

with $\widetilde{L} \in W^{1,p'}(\Omega)$ given by Lemma 2.3; we denote by C_1 the upper bound on $||\widetilde{L}||_{(W^{1,p'}(\Omega))'}$ given by this lemma (i.e., C_1 only depends on Ω, Γ_d, α_A, p, Λ_A, $\Lambda_{\mathbf{v}}$ and Λ).

We notice that $A_{\rm tr} = \big((h_i' A (h_i')^T)/|Jh_i|\big) \circ h_i^{-1}$ has the same properties as A, with a coercitivity constant $\alpha_{A_{\rm tr}}$ only depending on (α_A, h_i) and an upper bound $\Lambda_{A_{\rm tr}}$ only depending on (Λ_A, h_i). If $\Gamma_0 = \{x \in \partial B_+ \mid x_N > 0\}$, we also notice that

$$
\left\{ \begin{array}{ccc} \mathcal{C}_c^\infty(B) & \longrightarrow & \mathbb{R}, \\ \phi & \longrightarrow & \langle \widetilde{L}, \phi \circ h_i \rangle_{(H^1(\Omega))', H^1(\Omega)} \end{array} \right.
$$

(where $\phi \circ h_i$ has been naturally extented to Ω by 0 outside O_i) defines a continuous linear form for the norm of $W^{1,p'}(B_+)$ and can thus be extented to $W^{1,p'}_{\Gamma_0}(B_+)$ in a continuous linear form, denoted by $\widetilde{L}_{\rm tr}$, whose norm is bounded by C_2 only depending on $(\Omega, h_i, \Gamma_d, \alpha_A, p, \Lambda_A, \Lambda_{\mathbf{v}}, \Lambda)$.

We thus have proven that $w_{\rm tr}$ satisfies

$$
\left\{ \begin{array}{l} w_{\rm tr} \in H^1(B_+), \\ \displaystyle\int_{B_+} A_{\rm tr} \nabla w_{\rm tr} \cdot \nabla \phi = \langle \widetilde{L}_{\rm tr}, \phi \rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)} \,,\ \forall \phi \in H^1_{\Gamma_0}(B_+) \end{array} \right. \tag{2.53}
$$

and, since $w_{\rm tr} = w \circ h_i^{-1}$ (with h_i^{-1} Lipschitz continuous), it is sufficient, to conclude this step, to prove that $w_{\rm tr}$ is κ-Hölder continuous on any compact subset of $B \cap \overline{B_+}$.

To see this point, we will show that $w_{\rm tr}$ is the restriction to B_+ of a function $W \in H^1(B)$ which satisfies an equation of the kind (2.46).

Let us define the reflection $\tau : \mathbb{R}^N \to \mathbb{R}^N$, $\tau(x', x_N) = (x', -x_N)$ (where $x' = (x_1, \ldots, x_{N-1})$); we will make no difference between τ and the matrix which represent τ in the canonical base of $\mathbb{R}^N$. We also define

$$\begin{array}{l}\mathcal{A} : B \to M_N(\mathbb{R}) \text{ by} \\ \mathcal{A}(x) = A_{\rm tr}(x) \text{ if } x \in B_+\,,\ \mathcal{A}(x) = \tau A_{\rm tr}(\tau(x))\tau \text{ if } x \in \tau(B_+)\end{array} \tag{2.54}$$

and

$$\begin{array}{l}\mathcal{L} \in W^{-1,p}(B) \text{ by, } \forall \phi \in W_0^{1,p'}(B)\,, \\ \langle \mathcal{L}, \phi\rangle_{(W_0^{1,p'}(B))', W_0^{1,p'}(B)} = \langle \widetilde{L}_{\rm tr}, \phi_{|B_+}\rangle_{(W_{\Gamma_0}^{1,p'}(B_+))', W_{\Gamma_0}^{1,p'}(B_+)} \\ \qquad + \langle \widetilde{L}_{\rm tr}, \phi\circ\tau_{|B_+}\rangle_{(W_{\Gamma_0}^{1,p'}(B_+))', W_{\Gamma_0}^{1,p'}(B_+)}.\end{array} \tag{2.55}$$

It is clear (no matter the definition of $\mathcal{A}$ on B^{N-1}) that $\mathcal{A}$ satisfies the same hypotheses as $A_{\rm tr}$, with the same constants (since τ is a symmetric isometry), and that $\mathcal{L}$ such defined is in $W^{-1,p}(B) = (W_0^{1,p'}(B))'$, with a norm in this space bounded by C_3 only depending of Ω, Γ_d, α_A, p, Λ_A, $\Lambda_{\mathbf{v}}$, Λ, and h_i.

We then define $W \in H^1(B)$, and show that it satisfies Problem (2.46) with $\Omega_0 = B$, $A_0 = \mathcal{A}$ and $L_0 = \mathcal{L}$.

$$\left\{\begin{array}{l} W(x) = w_{\rm tr}(x) \text{ if } x \in B_+, \\ W(x) = w_{\rm tr}\circ\tau(x) \text{ if } x \in \tau(B_+) := B_-\end{array}\right.$$

(it is a classical result that W, such defined, is in $H^1(B)$).

Let $\phi \in H_0^1(B)$: since $(\phi_{|B_+}, \phi\circ\tau_{|B_+}) \in H^1_{\Gamma_0}(B_+)$, we have

$$\int_{B_+} \mathcal{A}\nabla W\cdot\nabla\phi = \int_{B_+} A_{\rm tr}\nabla w_{\rm tr}\cdot\nabla\phi = \langle \widetilde{L}_{\rm tr}, \phi\rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)} \tag{2.56}$$

and

$$\begin{array}{rcl}\displaystyle\int_{B_-} \mathcal{A}\nabla W\cdot\nabla\phi &=& \displaystyle\int_{B_+} \tau A_{\rm tr}\tau(\tau\nabla w_{\rm tr})\cdot(\tau\nabla(\phi\circ\tau)) \\ &=& \displaystyle\int_{B_+} A_{\rm tr}\nabla w_{\rm tr}\cdot\nabla(\phi\circ\tau) \\ &=& \langle \widetilde{L}_{\rm tr}, \phi\circ\tau_{|B_+}\rangle_{(W_{\Gamma_0}^{1,p'}(B_+))', W_{\Gamma_0}^{1,p'}(B_+)}\end{array} \tag{2.57}$$

(using the fact that τ is symmetric and involutive).

The sum of (2.56) and (2.57) shows us that W satisfies (2.46) with $\Omega_0 = B$, $A_0 = \mathcal{A}$ and $L_0 = \mathcal{L}$; we deduce from Theorem 2.4 that there exists $\kappa \in]0, 1-N/p]$ only depending on $(N, \alpha_{A_{\rm tr}}, \Lambda_{A_{\rm tr}}, p)$, i.e., on $(h_i, \alpha_A, \Lambda_A, p)$ [notice that a dependence on h_i, as well as a dependence on Ω, takes into account a dependence on N], such that W is κ-Hölder continuous on any compact subset K of B, with a norm in $\mathcal{C}^{0,\kappa}(K)$ bounded by C_4 only depending on $(h_i, \alpha_A, \Lambda_A, p, C_3, K)$, i.e., on $(\Omega, \Gamma_d, h_i, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda, K)$, and the result of the proposition is proved in this case.

Step 2: The mapping (O_i, h_i) is of type (D).

With the same $A_{\rm tr}$ and the same $\widetilde{L}_{\rm tr} \in (W^{1,p'}_{\Gamma_0}(B_+))'$ as in step 1 (recall that $\Gamma_0 = \partial B_+ \backslash B^{N-1}$), using $\phi \in \mathcal{C}^\infty_c(B_+)$ and $\varphi = \phi \circ h_i \in H^1_{\Gamma_d}(\Omega)$ (extented by 0 outside O_i), we notice that $w_{\rm tr}$ (where $w_{tr} = w \circ h_i^{-1}$, as in step 1) satisfies

$$\left\{ \begin{array}{l} w_{\rm tr} \in H^1_{B^{N-1}}(B_+), \\ \displaystyle\int_{B_+} A_{\rm tr} \nabla w_{\rm tr} \cdot \nabla \phi = \langle \widetilde{L}_{\rm tr}, \phi \rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)}, \ \forall \phi \in H^1_0(B_+). \end{array} \right. \tag{2.58}$$

We will also prove that $w_{\rm tr}$ is the restriction of a $W \in H^1(B)$ satisfying a problem of the type (2.46)... but this is here somehow more difficult than in the case (F), because of the necessity to choose test functions in (2.58) which satisfy $\phi_{|B^{N-1}} = 0$.

We define W, whose restriction to B_+ is $w_{\rm tr}$, by

$$\left\{ \begin{array}{l} W(x) = w_{\rm tr}(x) \text{ if } x \in B_+, \\ W(x) = -w_{\rm tr} \circ \tau(x) \text{ if } x \in B_-, \end{array} \right.$$

$\mathcal{A}$ by (2.54) and $\mathcal{L} \in W^{-1,p}(B)$ by: $\forall \phi \in W^{1,p'}_0(B)$,

$$\begin{array}{rcl} \langle \mathcal{L}, \phi \rangle_{(W^{1,p'}_0(B))', W^{1,p'}_0(B)} & = & \langle \widetilde{L}_{\rm tr}, \phi_{|B_+} \rangle_{(W^{1,p'}_{\Gamma_0}(B_+))', W^{1,p'}_{\Gamma_0}(B_+)} \\ & & - \langle \widetilde{L}_{\rm tr}, \phi \circ \tau_{|B_+} \rangle_{(W^{1,p'}_{\Gamma_0}(B_+))', W^{1,p'}_{\Gamma_0}(B_+)} \end{array}$$

(this time, we make an odd reflection of $w_{\rm tr}$ and $\widetilde{L}_{\rm tr}$), and we prove that W satisfies (2.46) for $\Omega_0 = B$, $A_0 = \mathcal{A}$ and $L_0 = \mathcal{L}$, which will allow us to conclude as in step 1.

For $\theta \in \mathcal{C}_c^\infty(B)$, define $w_{\mathrm{tr},\theta} = \theta w_{\mathrm{tr}}$: $w_{\mathrm{tr},\theta}$ satisfies

$$\left\{\begin{array}{l} w_{\mathrm{tr},\theta} \in H_0^1(B_+), \\ \displaystyle\int_{B_+} A_{\mathrm{tr}} \nabla(w_{\mathrm{tr},\theta}) \cdot \nabla\phi = \langle \widetilde{L}_{\mathrm{tr}}, \theta\phi\rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)} \\ \qquad\qquad \displaystyle -\int_{B_+} \phi A_{\mathrm{tr}} \nabla w_{\mathrm{tr}} \cdot \nabla\theta + \int_{B_+} w_{\mathrm{tr}} A_{\mathrm{tr}} \nabla\theta \cdot \nabla\phi\,,\ \forall \phi \in H_0^1(B_+). \end{array}\right. \tag{2.59}$$

Define $\widetilde{L}_{\mathrm{tr},\theta} \in (H^1_{\Gamma_0}(B_+))'$ by

$$\begin{aligned} &\langle \widetilde{L}_{\mathrm{tr},\theta}, \phi\rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)} \\ &\quad = \langle \widetilde{L}_{\mathrm{tr}}, \theta\phi\rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)} - \int_{B_+} \phi A_{\mathrm{tr}} \nabla w_{\mathrm{tr}} \cdot \nabla\theta + \int_{B_+} w_{\mathrm{tr}} A_{\mathrm{tr}} \nabla\theta \cdot \nabla\phi \end{aligned}$$

(notice that $\widetilde{L}_{\mathrm{tr},\theta}$ is *not*, in general, in $(W^{1,p'}_{\Gamma_0}(\Omega))'$, because of the term $\int_{B_+} \phi A_{\mathrm{tr}} \nabla w_{\mathrm{tr}} \cdot \nabla\theta$; this is why we have not multiplied w by θ before transporting the problem into B_+) and $\mathcal{L}_\theta \in H^{-1}(B)$ by

$$\begin{aligned} &\langle \mathcal{L}_\theta, \phi\rangle_{(H_0^1(B))', H_0^1(B)} \\ &\quad = \langle \widetilde{L}_{\mathrm{tr},\theta}, \phi_{|B_+}\rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)} - \langle \widetilde{L}_{\mathrm{tr},\theta}, \phi\circ\tau_{|B_+}\rangle_{(H^1_{\Gamma_0})', H^1_{\Gamma_0}} \\ &\quad = \langle \mathcal{L}, \theta\phi\rangle_{(H_0^1(B))', H_0^1(B)} - \int_{B_+} \phi A_{\mathrm{tr}} \nabla w_{\mathrm{tr}} \cdot \nabla\theta + \int_{B_+} w_{\mathrm{tr}} A_{\mathrm{tr}} \nabla\theta \cdot \nabla\phi \\ &\qquad + \int_{B_+} \phi\circ\tau A_{\mathrm{tr}} \nabla w_{\mathrm{tr}} \cdot \nabla\theta - \int_{B_+} w_{\mathrm{tr}} A_{\mathrm{tr}} \nabla\theta \cdot \nabla(\phi\circ\tau). \end{aligned}$$

Let V_θ be the unique solution to

$$\left\{\begin{array}{l} V_\theta \in H_0^1(B), \\ \displaystyle\int_B \mathcal{A}\nabla V_\theta \cdot \nabla\phi = \langle \mathcal{L}_\theta, \phi\rangle_{(H_0^1(B))', H_0^1(B)}\,,\ \forall \phi \in H_0^1(B). \end{array}\right. \tag{2.60}$$

We want to show, using the uniqueness of the solution of (2.59), that

$$\begin{array}{l} V_\theta = w_{\mathrm{tr},\theta} \text{ on } B_+, \\ V_\theta = -w_{\mathrm{tr},\theta}\circ\tau \text{ on } B_-. \end{array} \tag{2.61}$$

Let us first verify that V_θ satisfies the equation of (2.59): for all $\phi \in H_0^1(B_+)$, by denoting also $\phi \in H_0^1(B)$ the extension of ϕ by 0 to B_-, we have

$$\begin{aligned}
&\int_{B_+} A_{\mathrm{tr}} \nabla V_\theta \cdot \nabla \phi \\
&\quad = \int_B \mathcal{A} \nabla V_\theta \cdot \nabla \phi \\
&\quad = \langle \mathcal{L}_\theta, \phi \rangle_{(H_0^1(B))', H_0^1(B)} \\
&\quad = \langle \widetilde{L}_{\mathrm{tr},\theta}, \phi_{|B_+} \rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)} - \langle \widetilde{L}_{\mathrm{tr},\theta}, \phi \circ \tau_{|B_+} \rangle_{(H^1_{\Gamma_0})', H^1_{\Gamma_0}} \\
&\quad = \langle \widetilde{L}_{\mathrm{tr},\theta}, \phi_{|B_+} \rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)},
\end{aligned}$$

since $\phi \circ \tau_{|B_+} = 0$.

To have $V_\theta = w_{\mathrm{tr},\theta}$ on B_+, it remains to show that $V_{\theta\,|B^{N-1}} = 0$. In fact, we will show that $V_\theta = -V_\theta \circ \tau$ on B, which will immediately give us $V_{\theta\,|B^{N-1}} = 0$, so that $V_\theta = w_{\mathrm{tr},\theta}$ on B_+ (uniqueness of the solution to (2.59)) and, using $V_\theta = -V_\theta \circ \tau$ once again, $V_\theta = -w_{\mathrm{tr},\theta} \circ \tau$ on B_-.

Let $\mathcal{V} = -V_\theta \circ \tau \in H_0^1(B)$: proving that $\mathcal{V}$ satisfy (2.60) is enough, thanks to the uniqueness of the solution to this problem, to have $\mathcal{V} = V_\theta$. Let $\phi \in H_0^1(B)$: by noticing that $\mathcal{A}(x) = \tau \mathcal{A}(\tau(x))\tau$ for a.e. $x \in B$, we can compute

$$\begin{aligned}
&\int_B \mathcal{A} \nabla \mathcal{V} \cdot \nabla \phi \\
&\quad = -\int_B \tau \mathcal{A}(\tau(x)) \tau (\tau(\nabla V_\theta) \circ \tau(x)) \cdot (\tau(\nabla(\phi \circ \tau)) \circ \tau(x))\, dx \\
&\quad = -\int_B \mathcal{A}(\tau(x)) \nabla V_\theta(\tau(x)) \cdot (\nabla(\phi \circ \tau))(\tau(x))\, dx \\
&\quad = -\int_B \mathcal{A} \nabla V_\theta \cdot \nabla(\phi \circ \tau) \\
&\quad = -\langle \mathcal{L}_\theta, \phi \circ \tau \rangle_{(H_0^1(B))', H_0^1(B)} \\
&\quad = -\langle \widetilde{L}_{\mathrm{tr},\theta}, \phi \circ \tau_{|B_+} \rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)} + \langle \widetilde{L}_{\mathrm{tr},\theta}, \phi_{|B_+} \rangle_{(H^1_{\Gamma_0}(B_+))', H^1_{\Gamma_0}(B_+)} \\
&\quad = \langle \mathcal{L}_\theta, \phi \rangle_{(H_0^1(B))', H_0^1(B)},
\end{aligned}$$

which is exactly what we wanted.

We can now conclude this step, by proving that $W \in H^1(B)$ satisfies (2.46) with $\Omega_0 = B$, $A_0 = \mathcal{A}$, and $L_0 = \mathcal{L}$.

Let $\phi \in \mathcal{C}_c^\infty(B)$ and choose $\theta \in \mathcal{C}_c^\infty(B)$ such that $\theta \equiv 1$ on a neighbourhood of the support of ϕ and θ is invariant by τ (that is to say, $\theta \circ \tau = \theta$); we have then $V_\theta = \theta W$ (cf. the definition of W and property (2.61) of V_θ). By (2.60), we have

$$\int_B \mathcal{A}\nabla(\theta W) \cdot \nabla\phi = \langle \mathcal{L}_\theta, \phi\rangle_{(H_0^1(B))', H_0^1(B)} \tag{2.62}$$

Since $\theta \equiv 1$ on a neighbourhood of $\mathrm{supp}(\phi) \cup \mathrm{supp}(\phi \circ \tau)$,

$$\int_B \mathcal{A}\nabla(\theta W) \cdot \nabla\phi = \int_B \mathcal{A}\nabla W \cdot \nabla\phi \tag{2.63}$$

and, with the definition of $\mathcal{L}_\theta$,

$$\langle \mathcal{L}_\theta, \phi\rangle_{(H_0^1(B))', H_0^1(B)} = \langle \mathcal{L}, \phi\rangle_{(H_0^1(B))', H_0^1(B)}. \tag{2.64}$$

(2.62), (2.63) and (2.64) give us

$$\int_B \mathcal{A}\nabla W \cdot \nabla\phi = \langle \mathcal{L}, \phi\rangle_{(H_0^1(B))', H_0^1(B)}$$

i.e., exactly what we wanted to prove, which concludes this step.

Step 3: The mapping (O_i, h_i) is of the type (DF).

This case, as we said before, can be handled thanks to the tools we have introduced in the two preceding steps.

Let $A_{\mathrm{tr}} = \big((h_i' A (h_i')^T)/|Jh_i|\big) \circ h_i^{-1} : B_{++} \to M_N(\mathbb{R})$.

The linear form

$$\left\{\begin{array}{rcl} \mathcal{C}_c^\infty(B) & \longrightarrow & \mathbb{R} \\ \phi & \longrightarrow & \langle \widetilde{L}, \phi \circ h_i\rangle_{(W^{1,p'}(\Omega))', W^{1,p'}(\Omega)} \end{array}\right.$$

(with $\phi \circ h_i$ extented by 0 outside $O_i \cap \Omega$) is continuous for the norm of $W^{1,p'}(B_{++})$ and can thus be extented in a continuous linear form $\widetilde{L}_{\mathrm{tr}}$ on $W^{1,p'}_{\Gamma_{0,+}}(B_{++})$, where $\Gamma_{0,+} = \Gamma_0 \cap \partial B_{++} = \{x \in \partial B_{++} \mid x_N > 0\,,\ x_{N-1} > 0\}$, whose norm is bounded by a constant only depending on Ω, h_i, Γ_d, α_A, Λ_A, p, $\Lambda_{\mathbf{v}}$, and Λ.

If $\phi \in \mathcal{C}_c^\infty(B_+)$, $\phi \circ h_i \in H^1_{\Gamma_d \cap O_i}(O_i \cap \Omega)$ and has a compact support in O_i, so that its extension to Ω by zero outside O_i is in $H^1_{\Gamma_d}(\Omega)$, and the same kind of calculus as in steps 1 or 2 allows us to see that $w_{\rm tr}$ is a solution to

$$\left\{\begin{array}{l} w_{\rm tr} \in H^1_{\Gamma_2}(B_{++}), \\ \displaystyle\int_{B_{++}} A_{\rm tr}\nabla w_{\rm tr}\cdot\nabla\phi = \langle \widetilde{L}_{\rm tr},\phi\rangle_{(H^1_{\Gamma_3}(B_{++}))',H^1_{\Gamma_3}(B_{++})}\,,\ \forall \phi\in H^1_{\Gamma_3}(B_{++}), \end{array}\right. \tag{2.65}$$

where $\Gamma_3 = \Gamma_2 \cup \Gamma_{0,+}$ (cf (2.10) for the definition of Γ_2).

If $\tau_1 : \mathbb{R}^N \to \mathbb{R}^N$ is the reflection with respect to Γ_1, i.e.,

$$\tau_1(x'', x_{N-1}, x_N) = (x'', -x_{N-1}, x_N) \quad (\text{with } x'' = (x_1, \dots, x_{N-2})),$$

we define then $W \in H^1_{B^{N-1}}(B_+)$ by

$$\left\{\begin{array}{l} W(x) = w_{\rm tr}(x) \text{ on } B_{++}, \\ W(x) = w_{\rm tr}\circ\tau_1(x) \text{ on } \tau_1(B_{++}) \end{array}\right.$$

and we see, as in step 1, that W is a solution to

$$\left\{\begin{array}{l} W \in H^1_{B^{N-1}}(B_+), \\ \displaystyle\int \mathcal{A}\nabla W\cdot\nabla\phi = \langle \mathcal{L},\phi\rangle_{(H^1_0(B_+))',H^1_0(B_+)}\,,\ \forall\phi\in H^1_0(B_+), \end{array}\right. \tag{2.66}$$

with $\mathcal{A} = A$ on B_{++}, $\mathcal{A} = \tau_1(A_{\rm tr}\circ\tau_1)\tau_1$ on $\tau_1(B_{++})$ and $\mathcal{L} \in (W^{1,p'}_{\Gamma_0}(B_+))'$ defined by

$$\begin{aligned} \langle \mathcal{L},\phi\rangle_{(W^{1,p'}_{\Gamma_0}(B_+))',W^{1,p'}_{\Gamma_0}(B_+)} &= \langle \widetilde{L}_{\rm tr},\phi_{|B_{++}}\rangle_{(W^{1,p'}_{\Gamma_{0,+}}(B_{++}))',W^{1,p'}_{\Gamma_{0,+}}(B_{++})} \\ &\quad + \langle \widetilde{L}_{\rm tr},\phi\circ\tau_{1\,|B_{++}}\rangle_{(W^{1,p'}_{\Gamma_{0,+}}(B_{++}))',W^{1,p'}_{\Gamma_{0,+}}(B_{++})}. \end{aligned}$$

Since W satisfies (2.66), a problem of the same form as (2.58), the step 2 allows us to see that W is κ-Hölder continuous on any compact subset of $B\cap\overline{B_+}$, and since $W_{|B_{++}} = w$, this concludes this step and the demonstration of this proposition. ■

Proposition 2.5. *Let $p > N$. Under Hypotheses (1.1), (1.2), (2.17), (2.20) and (2.21), if $(O_i, h_i)_{i\in[1,m]}$ are mappings of the boundary of Ω (i.e., such that $(O_i)_{i\in[1,m]}$ is a covering of $\partial\Omega$ by open subsets of $\mathbb{R}^N$ and, for all $i \in [1,m]$, (O_i, h_i) satisfies (2.9)), then for all $i \in [1,m]$, there exists $\kappa > 0$ only depending on $(N, \alpha_A, \Lambda_A, p, h_i)$ such that the solution u of (2.19) is κ-Hölder continuous on any compact subset of $O_i \cap \overline{\Omega}$. Moreover, for such a compact subset K, there exists C only depending on $(\Omega, h_i, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda, K)$ such that u satisfies (2.52).*

Proposition 2.6. *Let $p > N$. Under Hypotheses (1.1), (1.2), (2.29), (2.30) and either (2.25) or (2.26), if $(O_i, h_i)_{i\in[1,m]}$ are mappings of the boundary of Ω, then for all $i \in [1, m]$, there exists κ only depending on $(N, \alpha_A, \Lambda_A, p, h_i)$ such that the solution u of (2.28) is κ-Hölder continuous on any compact subset of $O_i \cap \overline{\Omega}$. Moreover, for such a compact subset K, there exists C only depending on $(\Omega, h_i, \alpha_A, \Lambda_A, p, \Lambda_{\mathbf{v}}, \Lambda, K)$ and*

$$\begin{array}{ll} (E, b_0) & \text{if (2.25) is verified,} \\ (S, \lambda_0) & \text{if (2.26) is verified} \end{array}$$

such that u satisfies (2.52).

The proofs of these propositions are of the same kind as that of Proposition 2.4, except that they are far easier, since the only case that appears is the case studied in step 1 (this is due to the fact that there is no condition on the values on $\partial\Omega$ of the test functions). We omit them.

Theorems 2.1, 2.2 and 2.3 are now easy consequences of Corollary 2.1 and Propositions 2.4, 2.5 and 2.6; for example, the κ of Theorem 2.1 can be chosen as the infimum between the κ of Corollary 2.1 (for the mixed case) and the κ's corresponding to each mapping (O_i, h_i) in Proposition 2.4.

We omit the details of these proofs, which consist in putting together the preceding results, but the reader can find them in [12].

3. The dual problem. We study the dual problem only in the mixed and Fourier cases, and we refer the reader to [12] for the study in the case of Neumann boundary conditions.

In the following, we suppose either one of the two series of hypotheses:

$$\text{In the mixed case: Hypotheses (1.1), (1.2), (2.2), (2.3), (2.4) and (2.11),} \tag{3.1}$$

$$\begin{array}{c} \text{In the Fourier case: } \Gamma_d = \emptyset,\ \Gamma_n = \partial\Omega \text{ and} \\ \text{Hypotheses (1.1), (1.2), (2.23), (2.24) and either (2.25) or (2.26).} \end{array} \tag{3.2}$$

We also take $p \in]N, +\infty[$ and suppose (2.12).

3.1. Solving the problem with a measure as data. For each $l \in (H^1_{\Gamma_d}(\Omega))'$, there exists an extension $L \in (H^1(\Omega))'$. We can thus solve, in the mixed case, Problem (2.6) for this L, with $g_d = 0$ and $g_n = 0$, or, in the Fourier case, Problem (2.28) for this L, with $g = 0$; it is easy to see that the solution u of this problem only depends on the values of L on $H^1_{\Gamma_d}(\Omega)$, i.e., on l. This allows us to define the continuous (because the norm of L in $(H^1(\Omega))'$ can be chosen equal to the norm of l in $(H^1_{\Gamma_d}(\Omega))'$) linear operator

$$\mathcal{T}_1 \left\{ \begin{array}{rcl} \left(H^1_{\Gamma_d}(\Omega)\right)' & \longrightarrow & H^1_{\Gamma_d}(\Omega), \\ l & \longrightarrow & u \text{ such defined.} \end{array} \right. \tag{3.3}$$

When $l \in (W^{1,p'}_{\Gamma_d}(\Omega))'$, since there exists an extension L of l to $W^{1,p'}(\Omega)$ whose norm in $(W^{1,p'}(\Omega))'$ is the norm of l in $(W^{1,p'}_{\Gamma_d}(\Omega))'$, Theorem 2.1 (in the mixed case) or 2.3 (in the Fourier case) allows us to see that $\mathcal{T}_1(l) \in \mathcal{C}(\overline{\Omega})$; we define thus

$$\mathcal{T}_{1,p} \left\{ \begin{array}{rcl} \left(W^{1,p'}_{\Gamma_d}(\Omega)\right)' & \longrightarrow & \mathcal{C}(\overline{\Omega}), \\ l & \longrightarrow & \mathcal{T}_1(l), \end{array} \right. \tag{3.4}$$

and estimate (2.15) gives us, since $\mathcal{T}_1$ is linear, the continuity of $\mathcal{T}_{1,p}$.

We denote by $\mathcal{M}(\overline{\Omega})$ the space of bounded measures on $\overline{\Omega}$, which is, thanks to the Riesz representation theorem, identified to the dual of $\mathcal{C}(\overline{\Omega})$. The adjoint operator of $\mathcal{T}_{1,p}$, $\mathcal{T}^*_{1,p} : \mathcal{M}(\overline{\Omega}) \to ((W^{1,p'}_{\Gamma_d}(\Omega))')' = W^{1,p'}_{\Gamma_d}(\Omega)$ (since $p' \in]1, N/(N-1)[$, $W^{1,p'}_{\Gamma_d}(\Omega)$ is a reflexive space), is such that, for all $\mu \in \mathcal{M}(\overline{\Omega})$, $f = \mathcal{T}^*_{1,p}(\mu)$ is the unique solution to

$$\left\{ \begin{array}{l} f \in W^{1,p'}_{\Gamma_d}(\Omega), \\ \forall l \in (W^{1,p'}_{\Gamma_d}(\Omega))', \ \langle l, f \rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))', W^{1,p'}_{\Gamma_d}(\Omega)} = \langle \mu, \mathcal{T}_1(l) \rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})}. \end{array} \right. \tag{3.5}$$

Remark 3.1. *If we take stronger hypotheses on b and λ, namely $b \in L^\infty(\Omega)$ and $\lambda \in L^\infty(\Gamma_n)$, we can solve (3.5) for each $p \in]N, +\infty[$; since the spaces $W^{1,p'}_{\Gamma_d}(\Omega)$ are imbedded one into another, and since the solution to each (3.5) is unique, we have in fact a unique solution to*

$$\left\{ \begin{array}{l} f \in \bigcap_{q < \frac{N}{N-1}} W^{1,q}_{\Gamma_d}(\Omega), \\ \forall q < \frac{N}{N-1}, \ \forall l \in (W^{1,q}_{\Gamma_d}(\Omega))', \\ \qquad \langle l, f \rangle_{(W^{1,q}_{\Gamma_d}(\Omega))', W^{1,q}_{\Gamma_d}(\Omega)} = \langle \mu, \mathcal{T}_1(l) \rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})}. \end{array} \right. \tag{3.6}$$

We will say that f is the solution to the equation

$$\left\{\begin{array}{l} -\mathrm{div}(A^T\nabla f) - \mathrm{div}(f\mathbf{v}) + (\mathrm{div}(\mathbf{v}) + b)f = \mu \quad \text{in} \quad \Omega, \\ f = 0 \quad \text{on} \quad \Gamma_d, \\ A^T\nabla f\cdot\mathbf{n} + (\lambda + \mathbf{v}\cdot\mathbf{n})f = 0 \quad \text{on} \quad \Gamma_n. \end{array}\right. \tag{3.7}$$

The following subsections will give us evidences that the solution to (3.5) can be considered, in a certain way, as a solution to (3.7).

Remark 3.2. *By denoting*

$$\widehat{\ } : (A, \mathbf{v}, b, \lambda) \to (\widehat{A}, \widehat{\mathbf{v}}, \widehat{b}, \widehat{\lambda}) = (A^T, -\mathbf{v}, \mathrm{div}(\mathbf{v}) + b, \lambda + \mathbf{v}\cdot\mathbf{n})$$

($\widehat{\ }$ is an involution), we notice that the data $(A, \mathbf{v}, b, \lambda)$ satisfy (3.1) in the mixed case (respectively (3.2) in the Fourier case) if and only if the data $(\widehat{A}, \widehat{\mathbf{v}}, \widehat{b}, \widehat{\lambda})$ satisfy (3.1) in the mixed case (respectively (3.2) in the Fourier case). Thus, $\widehat{\ }$ defines a bijection between "well-posed" problems of the form

$$\left\{\begin{array}{l} -\mathrm{div}(\widehat{A}\nabla f) + \mathrm{div}(f\widehat{\mathbf{v}}) + \widehat{b}f = \mu \quad \textit{in} \quad \Omega, \\ f = 0 \quad \textit{on} \quad \Gamma_d, \\ \widehat{A}\nabla f\cdot\mathbf{n} + \widehat{\lambda}f = 0 \quad \textit{on} \quad \Gamma_n, \end{array}\right. \tag{3.8}$$

and "well-posed" problems of the form (3.7).

3.1.1. Link with the variational formulation. When $\mu \in (H^1(\Omega))'$, there exists a unique solution to (3.7) in the variational sense

$$\left\{\begin{array}{l} \widehat{f} \in H^1_{\Gamma_d}(\Omega), \\ \displaystyle\int_\Omega A^T\nabla\widehat{f}\cdot\nabla\psi + \int_{\Gamma_n}(\lambda + \mathbf{v}\cdot\mathbf{n})\widehat{f}\psi\,d\sigma - \int_\Omega \psi\mathbf{v}\cdot\nabla\widehat{f} + \int_\Omega b\widehat{f}\psi \\ \qquad = \langle\mu, \psi\rangle_{(H^1(\Omega))', H^1(\Omega)}\,,\ \forall\psi \in H^1_{\Gamma_d}(\Omega). \end{array}\right. \tag{3.9}$$

We will see that the solution f to (3.5) is, when $\mu \in (H^1(\Omega))' \cap \mathcal{M}(\overline{\Omega})$, the solution $\widehat{f}$ to (3.9) (which is the first reason toward the fact that f is the solution of (3.7)).

Indeed, since the solution f to (3.5) is unique, it is sufficient to prove that $\widehat{f}$ satisfies (3.5); but $\widehat{f} \in H^1_{\Gamma_d}(\Omega) \subset W^{1,p'}_{\Gamma_d}(\Omega)$ and, for all $l \in (W^{1,p'}_{\Gamma_d}(\Omega))'$, by definition of $\mathcal{T}_1(l) \in H^1_{\Gamma_d}(\Omega) \cap \mathcal{C}(\overline{\Omega})$,

$$\langle\mu, \mathcal{T}_1(l)\rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})} \quad = \quad \langle\mu, \mathcal{T}_1(l)\rangle_{(H^1(\Omega))', H^1(\Omega)}$$

$$
\begin{aligned}
&= \int_\Omega A^T\nabla \widehat{f}\cdot\nabla(\mathcal{T}_1(l)) + \int_{\Gamma_n}(\lambda+\mathbf{v}\cdot\mathbf{n})\widehat{f}\mathcal{T}_1(l)\,d\sigma \\
&\quad -\int_\Omega \mathcal{T}_1(l)\mathbf{v}\cdot\nabla\widehat{f} + \int_\Omega b\widehat{f}\mathcal{T}_1(l) \\
&= \langle l,\widehat{f}\rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))',W^{1,p'}_{\Gamma_d}(\Omega)}.
\end{aligned}
$$

Thus, $\widehat{f}$ satisfies (3.5) and $\widehat{f}=f$ in this case.

3.1.2. Strong integral formulation of (3.5). It is maybe easier to see why f can be considered as the solution of (3.7) once we have put (3.5) in an equivalent formulation which involves (as the variational formulations of classical elliptic problems) integrals.

However, to do that, we need some preliminaries.

Since $p\in]N,+\infty[$, one can see that the application

$$
\left\{\begin{array}{rcl} \mathcal{C}^\infty_c(\Omega) & \longrightarrow & \left(W^{1,p'}_{\Gamma_d}(\Omega)\right)' \\ \varphi & \longrightarrow & \left(\psi\longrightarrow\int_\Omega\varphi\psi\right)\end{array}\right. \tag{3.10}
$$

is an dense imbedding (the density comes from the classical characterisation of elements of $(W^{1,p'}_{\Gamma_d}(\Omega))'$ as sums of elements of $L^p(\Omega)$ and of "divergences" — in a certain sense — of elements of $(L^p(\Omega))^N$).

We define, for $q\in]1,+\infty[$, the continuous linear function

$$
\Theta_q\left\{\begin{array}{rcl} W^{1,q}_{\Gamma_d}(\Omega) & \longrightarrow & \left(W^{1,q'}_{\Gamma_d}(\Omega)\right)' \\ \varphi & \longrightarrow & \Theta_q(\varphi)\end{array}\right.
$$

such that, for all $\psi\in W^{1,q'}_{\Gamma_d}(\Omega)$,

$$
\begin{aligned}
\langle\Theta_q(\varphi),\psi\rangle_{(W^{1,q'}_{\Gamma_d}(\Omega))',W^{1,q'}_{\Gamma_d}(\Omega)} &= \int_\Omega A\nabla\varphi\cdot\nabla\psi + \int_{\Gamma_n}\lambda\varphi\psi\,d\sigma \\
&\quad -\int_\Omega\varphi\mathbf{v}\cdot\nabla\psi + \int_{\Gamma_n}\varphi\psi\mathbf{v}\cdot\mathbf{n}\,d\sigma + \int_\Omega b\varphi\psi
\end{aligned}
$$

and we notice two properties of this function Θ_q, which will be useful to put (3.5) under an integral formulation:

i) for all $\varphi\in W^{1,q}_{\Gamma_d}(\Omega)$, $\Theta_q(\varphi)_{|\mathcal{D}(\Omega)} = -\mathrm{div}(A\nabla\varphi)+\mathrm{div}(\varphi\mathbf{v})+b\varphi$ in the sense of $\mathcal{D}'(\Omega)$,

ii) $\Theta_2 \circ \mathcal{T}_1 = Id_{(H^1_{\Gamma_d}(\Omega))'}$ and, for all $q \ge 2$, $\mathcal{T}_1 \circ \Theta_q = Id_{|W^{1,q}_{\Gamma_d}(\Omega)}$.

Remark 3.3. *In the Dirichlet case, that is to say $\Gamma_d = \partial\Omega$ and $\Gamma_n = \emptyset$, $(W^{1,q'}_0(\Omega))' \subset \mathcal{D}'(\Omega)$ (because $\mathcal{D}(\Omega)$ is densely imbedded in $W^{1,q'}_0(\Omega)$), that is to say that any element of $(W^{1,q'}_0(\Omega))'$ is fully known by its values on $\mathcal{D}(\Omega)$. We have thus, for all $\varphi \in W^{1,q}_0(\Omega)$, $\Theta_q(\varphi) = -\mathrm{div}(A\nabla\varphi) + \mathrm{div}(\varphi\mathbf{v}) + b\varphi$ in $(W^{1,q'}_0(\Omega))'$.*

This is not, in general, the case (but this is true if $\Theta_q(\varphi) \in \mathcal{C}^\infty_c(\Omega)$, see below).

Proof of property i): By definition, we have, for all $\psi \in \mathcal{D}(\Omega) \subset W^{1,q'}_{\Gamma_d}(\Omega)$,

$$\begin{aligned}\langle \Theta_q(\varphi)_{|\mathcal{D}(\Omega)}, \psi\rangle_{(\mathcal{D}(\Omega))',\mathcal{D}(\Omega)} &= \langle \Theta_q(\varphi), \psi\rangle_{(W^{1,q'}_{\Gamma_d}(\Omega))',W^{1,q'}_{\Gamma_d}(\Omega)} \\ &= \int_\Omega A\nabla\varphi \cdot \nabla\psi - \int_\Omega \varphi\mathbf{v}\cdot\nabla\psi + \int_\Omega b\varphi\psi\end{aligned}$$

(since $\psi = 0$ on Γ_n when $\psi \in \mathcal{D}(\Omega)$), which exactly means $\Theta_q(\varphi)_{|\mathcal{D}(\Omega)} = -\mathrm{div}(A\nabla\varphi) + \mathrm{div}(\varphi\mathbf{v}) + b\varphi$ in $\mathcal{D}'(\Omega)$. ■

Proof of property ii): For all $l \in (H^1_{\Gamma_d}(\Omega))'$, we have, by definition of $\mathcal{T}_1$, for all $\psi \in H^1_{\Gamma_d}(\Omega)$,

$$\begin{aligned}&\langle l, \psi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)} \\ &= \int_\Omega A\nabla(\mathcal{T}_1(l))\cdot\nabla\psi + \int_{\Gamma_n} \lambda\mathcal{T}_1(l)\psi\,d\sigma - \int_\Omega \mathcal{T}_1(l)\mathbf{v}\cdot\nabla\psi \\ &\quad + \int_{\Gamma_n} \mathcal{T}_1(l)\psi\mathbf{v}\cdot\mathbf{n}\,d\sigma + \int_\Omega b\mathcal{T}_1(l)\psi \\ &= \langle \Theta_2(\mathcal{T}_1(l)), \psi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\end{aligned}$$

which gives $\Theta_2(\mathcal{T}_1(l)) = l$ in $(H^1_{\Gamma_d}(\Omega))'$, i.e., $\Theta_2 \circ \mathcal{T}_1 = Id_{(H^1_{\Gamma_d}(\Omega))'}$.

If $\varphi \in W^{1,q}_{\Gamma_d}(\Omega)$ with $q \ge 2$, we have, by definition of Θ_q,

$$\left\{\begin{array}{l}\varphi \in H^1_{\Gamma_d}(\Omega), \\ \displaystyle\int_\Omega A\nabla\varphi\cdot\nabla\psi + \int_{\Gamma_n}\lambda\varphi\psi\,d\sigma - \int_\Omega \varphi\mathbf{v}\cdot\nabla\psi + \int_{\Gamma_n}\varphi\psi\mathbf{v}\cdot\mathbf{n}\,d\sigma + \int_\Omega b\varphi\psi \\ \qquad = \langle\Theta_q(\varphi),\psi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\,,\ \forall\psi \in H^1_{\Gamma_d}(\Omega) \subset W^{1,q'}_{\Gamma_d}(\Omega).\end{array}\right.$$

But the unique solution to this problem is $\mathcal{T}_1(\Theta_q(\varphi))$, so that $\mathcal{T}_1(\Theta_q(\varphi)) = \varphi$ in $H^1_{\Gamma_d}(\Omega)$, which gives $\mathcal{T}_1 \circ \Theta_q = Id_{|W^{1,q}_{\Gamma_d}(\Omega)}$. ∎

We can now give the strong integral formulation of (3.5).

Since $\mathcal{C}^\infty_c(\Omega)$ is densely imbedded by (3.10) in $(W^{1,p'}_{\Gamma_d}(\Omega))'$, (3.5) is equivalent to

$$\begin{cases} f \in W^{1,p'}_{\Gamma_d}(\Omega), \\ \forall l \in \mathcal{C}^\infty_c(\Omega)\,,\ \langle l, f\rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))', W^{1,p'}_{\Gamma_d}(\Omega)} = \displaystyle\int_\Omega lf = \langle \mu, \mathcal{T}_1(l)\rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})} \end{cases} \tag{3.11}$$

By the preliminaries up above, there is a bijection

$$\mathcal{C}^\infty_c(\Omega) \longrightarrow \left\{\varphi \in H^1_{\Gamma_d}(\Omega)\,,\ \Theta_2(\varphi) \in \mathcal{C}^\infty_c(\Omega)\right\}$$

which, to any $l \in \mathcal{C}^\infty_c(\Omega)$, associates $\varphi = \mathcal{T}_1(l)$ and, to any $\varphi \in H^1_{\Gamma_d}(\Omega)$ such that $\Theta_2(\varphi) \in \mathcal{C}^\infty_c(\Omega)$ associates $l = \Theta_2(\varphi)$. We also notice that, if $\Theta_2(\varphi) \in \mathcal{C}^\infty_c(\Omega) \subset (W^{1,p'}_{\Gamma_d}(\Omega))'$, since $\mathcal{T}_1(\Theta_2(\varphi)) = \mathcal{T}_{1,p}(\Theta_2(\varphi)) = \varphi$ in $H^1_{\Gamma_d}(\Omega)$, one has $\varphi \in \mathcal{C}(\overline{\Omega})$.

Thus, (3.11) is equivalent to

$$\begin{cases} f \in W^{1,p'}_{\Gamma_d}(\Omega), \\ \forall \varphi \in H^1_{\Gamma_d} \text{ such that } \Theta_2(\varphi) \in \mathcal{C}^\infty_c(\Omega), \\ \qquad \langle \Theta_2(\varphi), f\rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))', W^{1,p'}_{\Gamma_d}(\Omega)} = \displaystyle\int_\Omega \Theta_2(\varphi) f = \langle \mu, \varphi\rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})} \end{cases} \tag{3.12}$$

But, when $\Theta_2(\varphi) \in \mathcal{C}^\infty_c(\Omega)$, we see that $\Theta_2(\varphi)$, as an element of the dual space of $W^{1,p'}_{\Gamma_d}(\Omega)$ through (3.10), is fully known by its values on $\mathcal{D}(\Omega)$, i.e., by $\Theta_2(\varphi)_{|\mathcal{D}(\Omega)}$ (this is the fondamental lemma of the distributions, which states that we can consider $L^1_{loc}(\Omega)$ as a subspace of $\mathcal{D}'(\Omega)$). Thus, when $\Theta_2(\varphi) \in \mathcal{C}^\infty_c(\Omega)$, one has, thanks to property i), $\Theta_2(\varphi) = -\mathrm{div}(A\nabla\varphi) + \mathrm{div}(\varphi\mathbf{v}) + b\varphi$ in $\mathcal{D}'(\Omega)$, i.e., on Ω since these are functions.

Thus, we have proven the following theorem:

Theorem 3.1. *Let $p > N$. Under Hypotheses (3.1) and (2.12) in the mixed case or (3.2) and (2.12) in the Fourier case, there exists a unique solution* *f to (3.7) in the sense*

$$\begin{cases} f \in W^{1,p'}_{\Gamma_d}(\Omega), \\ \forall \varphi \in H^1_{\Gamma_d}(\Omega) \text{ such that } \Theta_2(\varphi) \in \mathcal{C}^\infty_c(\Omega)\,, \\ \qquad \displaystyle\int_\Omega f(-\mathrm{div}(A\nabla\varphi) + \mathrm{div}(\varphi\mathbf{v}) + b\varphi) = \int_{\overline{\Omega}} \varphi\, d\mu. \end{cases} \tag{3.13}$$

Remark 3.4. *In the Dirichlet case, thanks to Remark 3.3, the condition "$\Theta_2(\varphi) \in \mathcal{C}^\infty_c(\Omega)$" is equivalent to "$-\mathrm{div}(A\nabla\varphi) + \mathrm{div}(\varphi\mathbf{v}) + b \in \mathcal{C}^\infty_c(\Omega)$", where the derivatives are taken in the sense of the distributions on Ω.*

Remark 3.5. *If $b \in L^\infty(\Omega)$ and $\lambda \in L^\infty(\Gamma_n)$, the same reasoning shows that (3.6) is equivalent to*

$$\begin{cases} f \in \displaystyle\bigcap_{q<\frac{N}{N-1}} W^{1,q}_{\Gamma_d}(\Omega), \\ \forall \varphi \in H^1_{\Gamma_d}(\Omega) \text{ such that } \Theta_2(\varphi) \in \mathcal{C}^\infty_c(\Omega)\,, \\ \qquad \displaystyle\int_\Omega f(-\mathrm{div}(A\nabla\varphi) + \mathrm{div}(\varphi\mathbf{v}) + b\varphi) = \int_{\overline{\Omega}} \varphi\, d\mu. \end{cases} \tag{3.14}$$

3.1.3. Weak integral formulation of (3.5). In fact, what we will see here is not really a formulation of (3.5), since it is not equivalent to this problem, but it is the third and last reason which allows us to say that f solves (3.7).

Let $\varphi \in W^{1,p}_{\Gamma_d}(\Omega) \subset \mathcal{C}(\overline{\Omega})$; since $L = \Theta_p(\varphi) \in (W^{1,p'}_{\Gamma_d}(\Omega))'$ and $\mathcal{T}_1 \circ \Theta_p(\varphi) = \varphi$ ($p \geq 2$), we see that f satisfies

$$\langle \Theta_p(\varphi), f \rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))', W^{1,p'}_{\Gamma_d}(\Omega)} = \langle \mu, \varphi \rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})}.$$

Since $f \in W^{1,p'}_{\Gamma_d}(\Omega)$, we have $f\varphi \in W^{1,1}_{\Gamma_d}(\Omega)$, and some integrations by parts allows us to see that

$$\begin{aligned} \langle \Theta_p(\varphi), f \rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))', W^{1,p'}_{\Gamma_d}(\Omega)} &= \int_\Omega A\nabla\varphi \cdot \nabla f + \int_{\Gamma_n} \lambda\varphi f\, d\sigma \\ &\quad - \int_\Omega \varphi\mathbf{v} \cdot \nabla f + \int_{\Gamma_n} \varphi f \mathbf{v} \cdot \mathbf{n}\, d\sigma + \int_\Omega b f \varphi \\ &= \int_\Omega A^T \nabla f \cdot \nabla\varphi + \int_{\Gamma_n} f\lambda\varphi\, d\sigma \\ &\quad + \int_\Omega f\mathbf{v} \cdot \nabla\varphi + \int_\Omega (\mathrm{div}(\mathbf{v}) + b) f\varphi. \end{aligned}$$

Thus, the solution f to (3.5) is also a solution to

$$\left\{\begin{array}{l} f \in W^{1,p'}_{\Gamma_d}(\Omega), \\ \displaystyle\int_\Omega A^T \nabla f \cdot \nabla\varphi + \int_{\Gamma_n} \lambda f \varphi \, d\sigma + \int_\Omega f \mathbf{v} \cdot \nabla\varphi + \int_\Omega (\mathrm{div}(\mathbf{v}) + b) f \varphi \\ \qquad \displaystyle = \int_{\overline{\Omega}} \varphi \, d\mu \,, \; \forall \varphi \in W^{1,p}_{\Gamma_d}(\Omega). \end{array}\right. \tag{3.15}$$

Remark 3.6. *This is a "natural" integral formulation of (3.7), i.e., one obtained by multiplying the equation of (3.7) by φ null on Γ_d and integrating formally by parts. However, (3.15) and (3.7) are not equivalent since, in general, the solution to (3.15) is not unique (see Remark 3.7 and the following section).*

Remark 3.7. *If we suppose $b \in L^\infty(\Omega)$ and $\lambda \in L^\infty(\partial\Omega)$, we can see the same way that the solution f to (3.6) is also a solution to*

$$\left\{\begin{array}{l} f \in \displaystyle\bigcap_{q<\frac{N}{N-1}} W^{1,q}_{\Gamma_d}(\Omega), \\ \displaystyle\int_\Omega A^T \nabla f \cdot \nabla\varphi + \int_{\Gamma_n} f \lambda \varphi \, d\sigma + \int_\Omega f \mathbf{v} \cdot \nabla\varphi + \int_\Omega (\mathrm{div}(\mathbf{v}) + b) f \varphi \\ \qquad \displaystyle = \int_{\overline{\Omega}} \varphi \, d\mu \,, \; \forall \varphi \in \bigcup_{p>N} W^{1,p}_{\Gamma_d}(\Omega) \end{array}\right. \tag{3.16}$$

which is, in fact, the solution L. Boccardo and T. Gallouët found in [2], using approximation methods. But, even for this equation (which is stronger than (3.15), because of the space to which the solution belongs), there is not uniqueness, in general, of the solution as soon as $N > 2$ (see the following section).

3.1.4. Non-uniqueness of the solution of (3.16). In the case $N = 2$, a result by A. Monier ([11], from the work of N.G. Meyers in [8]) shows that the solution of (3.15) is unique, which implies that (3.5) and (3.15) are equivalent (since *the* solution to (3.5) is *the* solution to (3.15)).

In the case $N > 2$ (and $\mathbf{v} = 0$, $b = 0$, $\Gamma_d = \partial\Omega$, i.e., the Dirichlet homogeneous case), a result by J. Serrin [6] modified by A. Prignet [7] shows that the solution to (3.15) is not unique, and so that (3.5) and (3.15) are not equivalent. Indeed, if we would have wanted to show that a solution of (3.15) is also a solution of (3.5), we would have taken, for any $l \in (W^{1,p'}_{\Gamma_d}(\Omega))'$, the test function $\varphi = \mathcal{T}_1(l)\dots$ and we would have not been able to go further, since (except in the case where the theorem by Agmon, Douglis and Niremberg applies — i.e., with A and Ω more regular, cf [9] — or in the case where the theorem by Meyers applies — i.e., with $N = 2$, see [11]) $\mathcal{T}_1(l)$ is not an element of $W^{1,p}_{\Gamma_d}(\Omega)$.

We show here how the counter-example by J. Serrin can be adapted to prove, in the mixed and Fourier cases with $\mathbf{v} = 0$, $b \equiv 1$ and $\lambda = 0$, that the solution of (3.16) in the case $N \geq 3$ is not unique (since a solution to (3.16) is also a solution of (3.15) for any $p \in]N, +\infty[$, this proves that the solution to any Problem (3.15) is not unique). Since (3.16) is a linear problem, it is sufficient to find a domain Ω of $\mathbb{R}^N$ with a Lispchitz continuous boundary, a A satisfying (1.1) and a $\dot{f} \neq 0$ solution of (3.16) when $\mu = 0$.

We take $\Omega = B := \{x \in \mathbb{R}^N \mid |x| < 1\}$, with $N \geq 3$. We notice (that will be useful in the following) that, when a function $F : B \to \mathbb{R}$ only depends on the first two coordinates (that is to say $F(x) = \widetilde{F}(x_1, x_2)$), F is integrable on B if and only if $\widetilde{F}$ is integrable on $D := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$.

Let $\varepsilon \in]0, 1[$ and define $A : B \to M_N(\mathbb{R})$ by

$$A(x) = \begin{pmatrix} 1 + \left(\frac{1}{\varepsilon^2} - 1\right)\frac{x_1^2}{r^2} & \left(\frac{1}{\varepsilon^2} - 1\right)\frac{x_1x_2}{r^2} & 0 & \cdots & 0 \\ \left(\frac{1}{\varepsilon^2} - 1\right)\frac{x_1x_2}{r^2} & 1 + \left(\frac{1}{\varepsilon^2} - 1\right)\frac{x_2^2}{r^2} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $x = (x_1, \dots, x_N)^T$ and $r = \sqrt{x_1^2 + x_2^2}$; A satisfies (1.1) (with $\alpha_A = 1$). Notice also that $A^T = A$.

Define $\overline{f} : B \to \mathbb{R}$ by

$$\overline{f}(x) = \frac{x_1}{r^{1+\varepsilon}}.$$

$\overline{f}$ is $\mathcal{C}^\infty$ on $B\backslash\{r = 0\}$ and $\overline{f} \in L^q(B)$ for all $q < 2/(1+\varepsilon)$ (in fact for all $q < 2/\varepsilon$). Moreover, on $B\backslash\{r = 0\}$,

$$\begin{aligned} \frac{\partial \overline{f}}{\partial x_1} &= \frac{1}{r^{1+\varepsilon}} - \frac{(1+\varepsilon)x_1^2}{r^{3+\varepsilon}}, \\ \frac{\partial \overline{f}}{\partial x_2} &= \frac{-(1+\varepsilon)x_1x_2}{r^{3+\varepsilon}}, \\ \frac{\partial \overline{f}}{\partial x_j} &= 0 \qquad \text{if } j \geq 3, \end{aligned} \tag{3.17}$$

and, using the fact that these functions are regular on $B\backslash\{r=0\}$ and bounded by $M/r^{1+\varepsilon}$ on the neighbourhood of $r=0$, we see that (3.17) also gives the derivatives of $\overline{f}$ in the sense of the distributions on B (see [7]). We see thus that $\overline{f}\in W^{1,q}(B)$ for all $q<2/(1+\varepsilon)$, and that $\overline{f}$ is not in $H^1(B)$ (to see this last point, compute $\int_D |\widetilde{D_2\overline{f}}|^2$ by using the polar coordinates).

Let us study $A\nabla\overline{f}\cdot\mathbf{n}:\partial B\to\mathbb{R}$: it is a regular function except at the points $(0,0,x_3,\dots,x_N)\in\partial B$, i.e., σ-a.e. on ∂B: $A\nabla\overline{f}\cdot\mathbf{n}$ is thus measurable on $\partial\Omega$; the only componants of $A\nabla\overline{f}$ which are not null being the first two components, we have $A\nabla\overline{f}\cdot\mathbf{n}=(A\nabla\overline{f})_1x_1+(A\nabla\overline{f})_2x_2$, so that $A\nabla\overline{f}\cdot\mathbf{n}$ is bounded by M/r^{ε} on ∂B and is thus a function of $L^2(\partial B)$ (the boundary of B is of dimension $N-1\geq 2$).

Moreover, a rather long computation (see [7] or [12]) allows us to see that $\operatorname{div}(A\nabla\overline{f})=0$ on $B\backslash\{r=0\}$ (this function is regular on this set) and, using once again the estimates on $A\nabla\overline{f}\cdot\mathbf{n}$ near $r=0$, one can see that, for all $\varphi\in\mathcal{C}^\infty(\overline{B})$,

$$\int_B A\nabla\overline{f}\cdot\nabla\varphi=\int_{\partial B}A\nabla\overline{f}\cdot\mathbf{n}\varphi.$$

By density, this expression is also true when $\varphi\in\bigcup_{p>(2/(1+\varepsilon))'}W^{1,p}(B)$.

Since $A\nabla\overline{f}\cdot\mathbf{n}\in L^2(\partial B)$, we deduce from the preceding inequality that

$$\varphi\in\mathcal{C}^\infty(\overline{B})\longrightarrow\int_B A\nabla\overline{f}\cdot\nabla\varphi$$

can be extented to $H^1(B)$ in a continuous linear form $L\in(H^1(B))'$.

One can prove (see [7]) that, when $\varepsilon<1/(N-1)$, $\overline{f}\in H^{1/2}(\partial B)$ (but without $\overline{f}\in H^1(B)$).

Since $\overline{f}\in L^2(B)$ ($\overline{f}\in L^q(\Omega)$ for all $q<2/\varepsilon$), we can solve the following mixed (if $\sigma(\Gamma_d)>0$) or Fourier (if $\Gamma_d=\emptyset$) variational problem

$$\begin{cases}-\operatorname{div}(A\nabla\check{f})+\check{f}=L+\overline{f}\text{ in }B,\\ \check{f}=\overline{f}\text{ on }\Gamma_d,\\ A\nabla\check{f}\cdot\mathbf{n}=0\text{ on }\Gamma_n\end{cases}\tag{3.18}$$

that is to say, using $\overline{f}_0\in H^1(B)$, a function with trace $\overline{f}$,

$$\begin{cases}\check{w}=\check{f}-\overline{f}_0\in H^1_{\Gamma_d}(B)\\ \displaystyle\int_B A\nabla\check{w}\cdot\nabla\varphi+\int_B\check{w}\varphi=\langle L,\varphi\rangle_{(H^1(B))',H^1(B)}\\ \displaystyle\qquad+\int_B\overline{f}\varphi-\int_B A\nabla\overline{f}_0\cdot\nabla\varphi-\int_B\overline{f}_0\varphi\,,\ \forall\varphi\in H^1_{\Gamma_d}(B).\end{cases}\tag{3.19}$$

Choosing $\varepsilon < 1/(N-1)$ small enough so that $2/(1+\varepsilon) \geq N/(N-1)$, we define $\dot{f} = \overline{f} - \check{f} \in \bigcap_{q<2/(1+\varepsilon)} W^{1,q}_{\Gamma_d}(B) \subset \bigcap_{q<N/(N-1)} W^{1,p}_{\Gamma_d}(B)$; $\dot{f}$ is not null (since $\check{f} \in H^1(B)$, $\overline{f} \notin H^1(B)$) and $\dot{f}$ satisfies, for all $\varphi \in \bigcup_{p>N} W^{1,p}_{\Gamma_d}(B) \subset H^1_{\Gamma_d}(B) \cap \bigcup_{p>(2/(1+\varepsilon))'} W^{1,p}(B)$,

$$\int_B A\nabla\dot{f} \cdot \nabla\varphi + \int_B \dot{f}\varphi = 0.$$

We have thus find a non null solution of (3.16) when $\mu = 0$, which is what we wanted.

3.2. Getting the traces of the solution of (3.5).

3.2.1. The trace of f. Under Hypothesis (3.1) or (3.2), by the same tricks as in the preceding section, we can define

$$\mathcal{T}_2 \left\{ \begin{array}{rcl} \left(H^{1/2}_{\Gamma_d}(\partial\Omega)\right)' & \longrightarrow & H^1_{\Gamma_d}(\Omega) \\ g_n & \longrightarrow & u \text{ solution of (2.6) with this } g_n,\ L = 0,\ g_d = 0. \end{array} \right.$$

If we suppose (2.12) with $p \in]N, +\infty[$, then, for all $g_n \in (W^{1-1/p',p'}_{\Gamma_d}(\Omega))'$, we see as before that $\mathcal{T}_2(g_n) \in \mathcal{C}(\overline{\Omega})$ and that

$$\mathcal{T}_{2,p} \left\{ \begin{array}{rcl} \left(W^{1-\frac{1}{p'},p'}_{\Gamma_d}(\partial\Omega)\right)' & \longrightarrow & \mathcal{C}(\overline{\Omega}), \\ g_n & \longrightarrow & \mathcal{T}_2(g_n) \end{array} \right.$$

is linear continuous.

Thus, we can also study the adjoint operator of $\mathcal{T}_{2,p}$, i.e., $\mathcal{T}^*_{2,p} : \mathcal{M}(\overline{\Omega}) \to W^{1-1/p',p'}_{\Gamma_d}(\partial\Omega)$ ($W^{1-1/p',p'}_{\Gamma_d}(\partial\Omega)$ is reflexive, since $p' \in]1, +\infty[$), such that, for all $\mu \in \mathcal{M}(\overline{\Omega})$, $f_\partial = \mathcal{T}^*_{2,p}(\mu)$ is the unique solution to

$$\left\{ \begin{array}{l} f_\partial \in W^{1-\frac{1}{p'},p'}_{\Gamma_d}(\partial\Omega), \\ \forall g_n \in \left(W^{1-\frac{1}{p'},p'}_{\Gamma_d}(\partial\Omega)\right)', \\ \qquad \langle g_n, f_\partial \rangle_{(W^{1-1/p',p'}_{\Gamma_d}(\partial\Omega))', W^{1-1/p',p'}_{\Gamma_d}(\partial\Omega)} = \langle \mu, \mathcal{T}_2(g_n) \rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})}. \end{array} \right. \tag{3.20}$$

In fact, we will show that f_∂ is the trace, on $\partial\Omega$, of the solution f to (3.5).

For $g_n \in (W^{1-1/p',p'}_{\Gamma_d}(\Omega))'$, define $\widetilde{g_n} \in (W^{1,p'}_{\Gamma_d}(\Omega))'$ by

$$\langle \widetilde{g_n}, \varphi\rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))', W^{1,p'}_{\Gamma_d}(\Omega)} = \langle g_n, \varphi\rangle_{(W^{1-1/p',p'}_{\Gamma_d}(\partial\Omega))', W^{1-1/p',p'}_{\Gamma_d}(\partial\Omega)},$$

and notice that $\mathcal{T}_2(g_n) = \mathcal{T}_1(\widetilde{g_n})$ (it is immediate on (2.6)). We have thus, for all $g_n \in (W^{1-1/p',p'}_{\Gamma_d}(\Omega))'$,

$$\begin{aligned}
\langle g_n, f_\partial\rangle_{(W^{1-1/p',p'}_{\Gamma_d}(\partial\Omega))', W^{1-1/p',p'}_{\Gamma_d}(\partial\Omega)} &= \langle \mu, \mathcal{T}_2(g_n)\rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})} \\
&= \langle \mu, \mathcal{T}_1(\widetilde{g_n})\rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})} \\
&= \langle \widetilde{g_n}, f\rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))', W^{1,p'}_{\Gamma_d}(\Omega)} \\
&= \langle g_n, f\rangle_{(W^{1-1/p',p'}_{\Gamma_d}(\partial\Omega))', W^{1-1/p',p'}_{\Gamma_d}(\partial\Omega)},
\end{aligned}$$

that is to say $f = f_\partial$ on $\partial\Omega$.

3.2.2. The trace of $A^T\nabla f \cdot \mathbf{n}$. Under Hypothesis (3.1) or (3.2), we define

$$\mathcal{T}_3 \left\{ \begin{array}{rcl} H^{1/2}(\partial\Omega) & \longrightarrow & H^1(\Omega), \\ g_d & \longrightarrow & u \text{ solution of (2.6) with this } g_d,\ L = 0 \text{ and } g_n = 0, \end{array} \right.$$

and we notice that

$$\mathcal{T}_{3,p} \left\{ \begin{array}{rcl} W^{1-\frac{1}{p}}(\partial\Omega) & \longrightarrow & \mathcal{C}(\overline{\Omega}), \\ g_d & \longrightarrow & \mathcal{T}_3(g_d) \end{array} \right.$$

is, thanks to Theorem 2.1, well defined, linear and continuous. Moreover, it is easy to see that the kernel of $\mathcal{T}_{3,p}$ is $W^{1-1/p,p}_{\Gamma_d}(\partial\Omega)$ (the solution u of (2.6) only depends of the values of g_d on Γ_d); thus, if

$$E = W^{1-1/p,p}(\partial\Omega)) \Big/ \left(W^{1-1/p,p}_{\Gamma_d}(\partial\Omega)\right),$$

we can define

$$\overline{\mathcal{T}_{3,p}} \left\{ \begin{array}{rcl} E & \longrightarrow & \mathcal{C}(\overline{\Omega}), \\ \pi(g_d) & \longrightarrow & \mathcal{T}_3(g_d) \end{array} \right.$$

(where $\pi(g_d)$ denotes the class of $g_d \in W^{1-1/p,p}(\partial\Omega)$ in E).

The adjoint operator of $\overline{\mathcal{T}_{3,p}}$ is $\overline{\mathcal{T}_{3,p}}^* : \mathcal{M}(\overline{\Omega}) \to E'$, such that, for all $\mu \in \mathcal{M}(\overline{\Omega})$, $f_{\nabla,\partial} = \overline{\mathcal{T}_{3,p}}^*(\mu)$ is the unique solution to

$$\left\{\begin{array}{l} f_{\nabla,\partial} \in E', \\ \forall \pi(g_d) \in E\,,\ \langle f_{\nabla,\partial}, \pi(g_d)\rangle_{(E)',E} = \langle \mu, \mathcal{T}_3(g_d)\rangle_{(\mathcal{C}(\overline{\Omega}))',\mathcal{C}(\overline{\Omega})}. \end{array}\right. \tag{3.21}$$

But it is a classical result that E' is isomorphic to $(W^{1-1/p,p}_{\Gamma_d}(\partial\Omega))^\circ$, the space of linear forms on $W^{1-1/p,p}(\partial\Omega)$ which are null on $W^{1-1/p,p}_{\Gamma_d}(\partial\Omega)$, by the following isomorphism:

$$\left\{\begin{array}{rcl} \left(W^{1-1/p,p}_{\Gamma_d}(\partial\Omega)\right)^\circ & \longrightarrow & \left((W^{1-1/p,p}(\partial\Omega))/(W^{1-1/p,p}_{\Gamma_d}(\partial\Omega))\right)', \\ l & \longrightarrow & l\circ\pi. \end{array}\right.$$

Thus, $f_{\nabla,\partial}$ is the unique solution to

$$\left\{\begin{array}{l} f_{\nabla,\partial} \in (W^{1-1/p,p}_{\Gamma_d}(\partial\Omega))^\circ \subset \left(W^{1-1/p,p}(\partial\Omega)\right)', \\ \forall g_d \in W^{1-1/p,p}(\partial\Omega)\,, \\ \qquad \langle f_{\nabla,\partial}, g_d\rangle_{(W^{1-1/p,p}(\partial\Omega))',W^{1-1/p,p}(\partial\Omega)} = \langle \mu, \mathcal{T}_3(g_d)\rangle_{(\mathcal{C}(\overline{\Omega}))',\mathcal{C}(\overline{\Omega})}. \end{array}\right. \tag{3.22}$$

We will see that $f_{\nabla,\partial}$ is, in fact, the trace on $\partial\Omega$ of $-A^T\nabla f\cdot\mathbf{n}-(\lambda+\mathbf{v}\cdot\mathbf{n})f$, with a coherent definition of this expression.

Let us first define $\left\{A^T\nabla f\cdot\mathbf{n}+\mathbf{v}\cdot\mathbf{n}f\right\} \in (W^{1-1/p,p}(\partial\Omega))'$. Since f satisfies (3.15), we have

$$-\operatorname{div}(A^T\nabla f)-\operatorname{div}(f\mathbf{v})+(\operatorname{div}(\mathbf{v})+b)f = \mu \quad \text{in the sense of } \mathcal{D}'(\Omega). \tag{3.23}$$

Define $\left\{A^T\nabla f\cdot\mathbf{n}+\mathbf{v}\cdot\mathbf{n}f\right\}$ as an element of $(W^{1-1/p,p}(\partial\Omega))'$ by: $\forall g_d \in W^{1-1/p,p}(\partial\Omega)$,

$$\begin{array}{l} \langle\left\{A^T\nabla f\cdot\mathbf{n}+\mathbf{v}\cdot\mathbf{n}f\right\}, g_d\rangle_{(W^{1-1/p,p}(\partial\Omega))',W^{1-1/p,p}(\partial\Omega)} \\ \quad = \ \langle(\operatorname{div}(\mathbf{v})+b)f-\mu, u_0\rangle_{(\mathcal{C}(\overline{\Omega}))',\mathcal{C}(\overline{\Omega})} + \int_\Omega A^T\nabla f\cdot\nabla u_0 + \int_\Omega f\mathbf{v}\cdot\nabla u_0 \\ \quad = \ \int_\Omega (\operatorname{div}(\mathbf{v})+b)fu_0 - \int_{\overline{\Omega}} u_0\,d\mu + \int_\Omega A^T\nabla f\cdot\nabla u_0 \\ \qquad + \int_\Omega f\mathbf{v}\cdot\nabla u_0, \end{array} \tag{3.24}$$

where u_0 is any function of $W^{1,p}(\Omega)$ (recall that $p > N$, so that $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$) with trace on $\partial\Omega$ equal to g_d (this definition only depends on g_d

because, when $u_0 \in \mathcal{D}(\Omega)$, thanks to (3.23), the right hand side of this expression is null and, by density of $\mathcal{D}(\Omega)$ in $W_0^{1,p}(\Omega)$, is still null when $u_0 \in W_0^{1,p}(\Omega)$); with the norm we have put on $W^{1-1/p,p}(\partial\Omega)$, it is clear that (3.24) defines a continuous linear form on $W^{1-1/p,p}(\partial\Omega)$. Notice that we can not define separately $A^T\nabla f \cdot \mathbf{n}$ or $\mathbf{v}\cdot\mathbf{n}f$, since $\operatorname{div}(A^T\nabla f)$ or $\operatorname{div}(f\mathbf{v})$ are not, in general, measures on $\overline{\Omega}$; we must thus always use the whole expression $\{A^T\nabla f \cdot \mathbf{n} + \mathbf{v}\cdot\mathbf{n}f\}$ and the + of this expression is not a sum in $(W^{1-1/p,p}(\partial\Omega))'$ (that is why we put this expression into brackets).

Remark 3.8. *Of course, we have denoted the linear form of (3.24) by this way because, when the data $(A,\mathbf{v},b,\lambda,\mu)$ are regular (say of class $\mathcal{C}^\infty(\overline{\Omega})$) and f is a classical $\mathcal{C}^\infty(\overline{\Omega})$ solution of (3.7), we have*

$$\begin{aligned}&\langle\{A^T\nabla f\cdot\mathbf{n}+\mathbf{v}\cdot\mathbf{n}\},g_d\rangle_{(W^{1-1/p,p}(\partial\Omega))',W^{1-1/p,p}(\partial\Omega)}\\ &\quad= \int_\Omega(\operatorname{div}(A^T\nabla f)+\operatorname{div}(f\mathbf{v}))u_0+\int_\Omega A^T\nabla f\cdot\nabla u_0+\int_\Omega f\mathbf{v}\cdot\nabla u_0,\end{aligned}$$

and some integrations by parts allow us to see that this linear form is $A^T\nabla f\cdot\mathbf{n}+\mathbf{v}\cdot\mathbf{n}f$, when this expression is understood in the classical sense.

Since $\lambda \in L^{(N-1)\frac{p}{N}}(\partial\Omega)$ and $f \in W^{1-1/p',p'}(\partial\Omega)$, we have, by a Sobolev injection (see [5]), $f \in L^{(N-1)p/(Np-N-p)}(\partial\Omega)$, which gives $\lambda f \in L^1(\partial\Omega) \hookrightarrow (W^{1-1/p,p}(\partial\Omega))'$ (because $W^{1-1/p,p}(\partial\Omega)$ is densely imbedded in $\mathcal{C}(\partial\Omega)$).

We have thus defined $\{A^T\nabla f\cdot\mathbf{n}+\mathbf{v}\cdot\mathbf{n}f\}+\lambda f \in (W^{1-1/p,p}(\partial\Omega))'$.

Let us now show that $f_{\nabla,\partial} = -\{A^T\nabla f\cdot\mathbf{n}+\mathbf{v}\cdot\mathbf{n}f\}-\lambda f$. For all $g_d \in W^{1-1/p,p}(\partial\Omega)$, with $u_0 \in W^{1,p}(\Omega)$, a function with trace g_d, we have

$$\begin{aligned}&\langle\{A^T\nabla f\cdot\mathbf{n}+\mathbf{v}\cdot\mathbf{n}f\}+\lambda f,g_d\rangle_{(W^{1-1/p,p}(\partial\Omega))',W^{1-1/p,p}(\partial\Omega)}\\ &\quad= -\langle\mu,u_0\rangle_{(\mathcal{C}(\overline{\Omega}))',\mathcal{C}(\overline{\Omega})}+\int_\Omega(\operatorname{div}(\mathbf{v})+b)fu_0+\int_\Omega A^T\nabla f\cdot\nabla u_0\\ &\qquad+\int_\Omega f\mathbf{v}\cdot\nabla u_0+\int_{\Gamma_n}\lambda f u_0\,d\sigma.\end{aligned}$$

Since $f \in W^{1,p'}_{\Gamma_d}(\Omega)$ and $u_0 \in W^{1,p}(\Omega)$, an integration by parts gives

$$\begin{aligned}&\int_\Omega(\operatorname{div}(\mathbf{v})+b)fu_0+\int_\Omega A^T\nabla f\cdot\nabla u_0+\int_\Omega f\mathbf{v}\cdot\nabla u_0+\int_{\Gamma_n}\lambda f u_0\,d\sigma\\ &\quad= \int_{\Gamma_n}u_0 f\mathbf{v}\cdot\mathbf{n}\,d\sigma-\int_\Omega u_0\mathbf{v}\cdot\nabla f+\int_\Omega b u_0 f+\int_\Omega A\nabla u_0\cdot\nabla f\end{aligned}$$

$$+\int_{\Gamma_n} \lambda u_0 f\, d\sigma.$$

This last term is a linear continuous form l_{u_0} in $f \in W^{1,p'}_{\Gamma_d}(\Omega)$, so that, by (3.5),

$$\begin{aligned}\langle l_{u_0}, f\rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))', W^{1,p'}_{\Gamma_d}(\Omega)} &= \int_\Omega A\nabla u_0 \cdot \nabla f + \int_{\Gamma_n} \lambda u_0 f\, d\sigma - \int_\Omega u_0 \mathbf{v}\cdot\nabla f \\ &\quad + \int_{\Gamma_n} f u_0 \mathbf{v}\cdot\mathbf{n}\, d\sigma + \int_\Omega b u_0 f \\ &= \langle \mu, \mathcal{T}_1(l_{u_0})\rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})},\end{aligned}$$

and we finally have

$$\begin{aligned}&\langle \{A^T\nabla f\cdot\mathbf{n} + \mathbf{v}\cdot\mathbf{n} f\} + \lambda f, g_d\rangle_{(W^{1-1/p,p}(\partial\Omega))', W^{1-1/p,p}(\partial\Omega)} \\ &\quad = \langle \mu, \mathcal{T}_1(l_{u_0}) - u_0\rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})}.\end{aligned}$$

But, by definition of $\mathcal{T}_1$, $\mathcal{T}_3$ and l_{u_0}, $\mathcal{T}_3(g_d) - u_0 \in H^1_{\Gamma_d}(\Omega)$ and $\mathcal{T}_1(-l_{u_0}) \in H^1_{\Gamma_d}(\Omega)$ are both solutions of (2.6) when $L = 0$, $g_n = 0$; thus, $\mathcal{T}_3(g_d) - u_0 = -\mathcal{T}_1(l_{u_0})$ and we have, for all $g_d \in W^{1-1/p,p}(\partial\Omega)$,

$$\begin{aligned}&\langle \{A^T\nabla f\cdot\mathbf{n} + \mathbf{v}\cdot\mathbf{n} f\} + \lambda f, g_d\rangle_{(W^{1-1/p,p}(\partial\Omega))', W^{1-1/p,p}(\partial\Omega)} \\ &\quad = -\langle \mu, \mathcal{T}_3(g_d)\rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})} \\ &\quad = -\langle f_{\nabla,\partial}, g_d\rangle_{(W^{1-1/p,p}(\partial\Omega))', W^{1-1/p,p}(\partial\Omega)},\end{aligned}$$

i.e., exactly $f_{\nabla,\partial} = -\{A^T\nabla f\cdot\mathbf{n} + \mathbf{v}\cdot\mathbf{n} f\} - \lambda f$ in $(W^{1-1/p,p}(\partial\Omega))'$; notice that the condition $f_{\nabla,\partial} \in (W^{1-1/p,p}_{\Gamma_d}(\partial\Omega))^\circ$ is the equivalent of $A^T\nabla f\cdot\mathbf{n} + (\lambda + \mathbf{v}\cdot\mathbf{n}) f = 0$ on Γ_n (cf (3.7)).

4. Applications. As before, we study the mixed and Fourier problems, thus supposing Hypotheses (3.1) or Hypotheses (3.2). We also take $p \in]N, +\infty[$ and we suppose Hypothesis (2.12).

In the preceding section, we have only used Theorems 2.1 and 2.3 to say that the solutions of (2.1) and (2.22) are (when the data are more regular than usual) continuous on $\overline{\Omega}$; but these theorems state much more than this: indeed, the solutions of (2.1) and (2.22) are *Hölder continuous*, and we have an estimate on the Hölder spaces to which these solutions belong, as well as a bound on their norms in these spaces. We will show here how these estimates can be used to obtain a stability result on the solution of (3.5) and to solve non-linear elliptic problems with measures as data.

4.1. A stability result. We prove here a stability result on the solution of (3.5).

We make the following hypotheses:

$$\begin{gathered}\forall m \geq 1\,,\ A_m : \Omega \to M_N(\mathbb{R}) \text{ is a measurable function,}\\ \exists \alpha_A > 0 \text{ such that } A_m(x)\xi\cdot\xi \geq \alpha_A|\xi|^2 \text{ for all } m\geq 1,\\ \text{for a.e. } x\in\Omega \text{ and for all } \xi\in\mathbb{R}^N,\\ \exists \Lambda_A \geq 0 \text{ such that } ||A_m(x)|| \leq \Lambda_A \text{ for all } m\geq 1 \text{ and for a.e. } x\in\Omega,\\ A_m \longrightarrow A \text{ a.e. on } \Omega \text{ as } m\to\infty,\end{gathered} \tag{4.1}$$

$$\begin{gathered}\forall m\geq 1\,,\ \mathbf{v}_m : \Omega \to \mathbb{R}^N \text{ is a Lipschitz continuous function,}\\ (\mathbf{v}_m)_{m\geq 1} \text{ is bounded in } \mathcal{C}^{0,1}(\Omega;\mathbb{R}^N),\\ \mathbf{v}_m \longrightarrow \mathbf{v} \text{ uniformly on } \Omega,\end{gathered} \tag{4.2}$$

$$\begin{gathered}\forall m\geq 1\,,\ b_m \in L^{\frac{Np}{N+p}}(\Omega),\\ b_m \longrightarrow b \text{ weakly in } L^{\frac{Np}{N+p}}(\Omega),\end{gathered} \tag{4.3}$$

$$\begin{gathered}\forall m\geq 1\,,\ \lambda_m \in L^{(N-1)\frac{p}{N}}(\Gamma_n),\\ \lambda_m \longrightarrow \lambda \text{ weakly in } L^{(N-1)\frac{p}{N}}(\Gamma_n),\end{gathered} \tag{4.4}$$

$$\begin{gathered}\forall m\geq 1\,,\ \mu_m \in \mathcal{M}(\overline{\Omega}),\\ \mu_m \longrightarrow \mu \text{ in } \mathcal{M}(\overline{\Omega}) \text{ weak-}*.\end{gathered} \tag{4.5}$$

We take $\Lambda \geq 0$ such that, for all $m\geq 1$,

$$||\mathbf{v}_m||_{\mathcal{C}^{0,1}(\Omega;\mathbb{R}^N)} + ||b_m||_{L^{\frac{Np}{N+p}}(\Omega)} + ||\lambda_m||_{L^{(N-1)\frac{p}{N}}(\Gamma_n)} + ||\mu_m||_{\mathcal{M}(\overline{\Omega})} \leq \Lambda.$$

We also suppose that our problems are "well-posed", that is to say

$$\forall m\geq 1\,,\ \tfrac{1}{2}\mathrm{div}(\mathbf{v}_m) + b_m \geq 0 \text{ a.e. on } \Omega,\ \tfrac{1}{2}\mathbf{v}_m\cdot\mathbf{n} + \lambda_m \geq 0\ \sigma\text{-a.e. on } \Gamma_n, \tag{4.6}$$

and

i) in the mixed case:

$$\text{Hypotheses (2.2), (2.11),} \tag{4.7}$$

ii) in the Fourier case, one of the following:

$$\begin{array}{c}\exists b_0 > 0\,,\ \exists E \subset \Omega \text{ such that } |E| > 0 \text{ and,}\\ \text{for all } m \geq 1,\ \frac{1}{2}\mathrm{div}(\mathbf{v}_m) + b_m \geq b_0 \text{ on } E,\\ \text{or}\\ \exists \lambda_0 > 0\,,\ \exists S \subset \partial\Omega \text{ such that } \sigma(S) > 0 \text{ and,}\\ \text{for all } m \geq 1,\ \frac{1}{2}\mathbf{v}_m \cdot \mathbf{n} + \lambda_m \geq b_0 \text{ on } S.\end{array} \tag{4.8}$$

Theorem 4.1. *Under Hypotheses (4.1)—(4.6) and (4.7) in the mixed case or (4.8) in the Fourier case, by denoting f_m the solution to (3.5) for the data $(A_m, \mathbf{v}_m, b_m, \lambda_m, \mu_m)$ and f the solution to (3.5) for the data $(A, \mathbf{v}, b, \lambda, \mu)$, we have*

$$f_m \overset{m\to\infty}{\longrightarrow} f \text{ strongly in } W^{1,q}_{\Gamma_d}(\Omega) \text{ for all } q < p', \text{ and weakly in } W^{1,p'}_{\Gamma_d}(\Omega). \tag{4.9}$$

Remark 4.1. *We also have a stability result for the solution of (3.6): under the hypotheses of Theorem 4.1, if $b_m \to b$ in $L^\infty(\Omega)$ for the weak-$*$ topology and if $\lambda_m \to \lambda$ in $L^\infty(\Gamma_n)$ for the weak-$*$ topology, then the solution f_m to (3.6) for $(A_m, \mathbf{v}_m, b_m, \lambda_m, \mu_m)$ converges to the solution f to (3.6) for $(A, \mathbf{v}, b, \lambda, \mu)$ strongly in $W^{1,q}_{\Gamma_d}(\Omega)$ for all $q < N/(N-1)$ (this is an easy consequence of Theorem 4.1).*

We need, to make the proof of Theorem 4.1 more readable, some technical lemmas.

Lemma 4.1. *Under the notations and hypotheses of Theorem 4.1, $f_m \to f$ weakly in $W^{1,p'}_{\Gamma_d}(\Omega)$ and strongly in $L^{p'}(\Omega)$.*

Let us define, for all $k \in \mathbb{R}^+$, the function $T_k : \mathbb{R} \to \mathbb{R}$ by $T_k(s) = \min(k, \max(-k, s))$. We notice that T_k is a continuous piecewise $\mathcal{C}^1$ function, with a derivative $T_k'(s) = \chi_{]-k,k[}(s)$ in $L^\infty(\mathbb{R})$.

Lemma 4.2. *Let $\underline{A}$ satisfy (1.1), $\underline{\mathbf{v}} : \Omega \to \mathbb{R}$ a Lipschitz continuous function, $\underline{b} \in L^{\frac{Np}{N+p}}(\Omega)$, $\underline{\lambda} \in L^{(N-1)\frac{p}{N}}(\Gamma_n)$ and $\underline{\mu} \in \mathcal{M}(\overline{\Omega})$. We suppose that these data satisfy Hypotheses (2.3) and (2.4); we also suppose that they satisfy (2.2) and (2.11) in the mixed case or either (2.25) or (2.26) in the Fourier case. If $\underline{f}$ is the solution to (3.5) for $(\underline{A}, \underline{\mathbf{v}}, \underline{b}, \underline{\lambda}, \underline{\mu})$, then, for all $k \in \mathbb{R}^+$, $T_k(\underline{f}) \in H^1_{\Gamma_d}(\Omega)$.*

Lemma 4.3. *Under Hypothesis (4.1), if* $(a_m)_{m\geq 1} \in H^1(\Omega)$ *and* $a_m \to a$ *weakly in* $H^1(\Omega)$*, then*

$$\int_\Omega A^T \nabla a \cdot \nabla a \leq \liminf_{m\to\infty} \int_\Omega A_m^T \nabla a_m \cdot \nabla a_m.$$

Lemma 4.4. *Under the notations of Lemma 4.2, if* $\underline{\alpha}$ *is a coercitivity constant for* $\underline{A}$*,* $\underline{\Lambda}$ *is an essential bound for* $\{||\underline{A}(x)||\,,\ x \in \Omega\}$*, and if* Λ_0 *is such that*

$$||\underline{\mathbf{v}}||_{\mathcal{C}^{0,1}(\Omega)} + ||\underline{b}||_{L^{\frac{Np}{N+p}}(\Omega)} + ||\underline{\lambda}||_{L^{(N-1)\frac{p}{N}}(\Gamma_n)} + ||\underline{\mu}||_{\mathcal{M}(\overline{\Omega})} \leq \Lambda_0,$$

then there exists $C > 0$ *only depending on* $(\Omega, \Gamma_d, \Lambda_0, \underline{\alpha}, \underline{\Lambda}, p)$ *in the mixed case,* $(\Omega, \Lambda_0, \underline{\alpha}, \underline{\Lambda}, p)$ *and* (b_0, E) *or* (λ_0, S) *in the Fourier case such that, for all* $\delta \in]0, 1[$*,* $k \in \mathbb{R}^+$ *and* $\psi \in H^1_{\Gamma_d}(\Omega)$*,*

$$\int_\Omega \underline{A}^T \nabla(T_{k+1}(\underline{f})) \cdot \nabla(T_\delta(T_{k+1}(\underline{f}) - T_k(\psi)) \leq C\delta. \tag{4.10}$$

Proof of Lemma 4.1.

Denote by $\mathcal{T}_1^{(m)}$ (respectively by $\mathcal{T}_1$) the application defined by (3.3) for $(A_m, \mathbf{v}_m, b_m, \lambda_m)$ (respectively for $(A, \mathbf{v}, b, \lambda)$).

Let us first notice that, for all $l \in (W^{1,p'}_{\Gamma_d}(\Omega))'$, $\mathcal{T}_1^{(m)}(l) \to \mathcal{T}_1(l)$ in $\mathcal{C}(\overline{\Omega})$: to see this, we notice that, thanks to Hypotheses (4.1)—(4.3) (which imply that $(\mathbf{v}_m)_{m\geq 1}$, $(b_m)_{m\geq 1}$ and $(\lambda_m)_{m\geq 1}$ are bounded in their respective spaces) and to Theorem 2.1 or 2.3, there exists $\kappa > 0$ such that $(\mathcal{T}_1^{(m)}(l))_{m\geq 1}$ is bounded in $\mathcal{C}^{0,\kappa}(\Omega)$, and thus relatively compact in $\mathcal{C}(\overline{\Omega})$; we thus just have to prove that, if a subsequence of $(\mathcal{T}_1^{(m)}(l))_{m\geq 1}$ converges in $\mathcal{C}(\overline{\Omega})$, the limit must be $\mathcal{T}_1(l)$; but it is a classical result that $\mathcal{T}_1^{(m)}(l) \to \mathcal{T}_1(l)$ in $H^1_{\Gamma_d}(\Omega)$, and the convergence in $\mathcal{C}(\overline{\Omega})$ is thus proved.

We can now see that $f_m \to f$ weakly in $W^{1,p'}_{\Gamma_d}(\Omega)$: for all $l \in (W^{1,p'}_{\Gamma_d}(\Omega))'$, we have

$$\begin{aligned}&\langle l, f_m - f\rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))', W^{1,p'}_{\Gamma_d}(\Omega)} \\ &\quad = \quad \langle \mu_m, \mathcal{T}_1^{(m)}(l)\rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})} - \langle \mu, \mathcal{T}_1(l)\rangle_{(\mathcal{C}(\overline{\Omega}))', \mathcal{C}(\overline{\Omega})} \to 0 \quad \text{ as } m \to \infty,\end{aligned}$$

since $\mu_m \to \mu$ in $\mathcal{M}(\overline{\Omega})$ weak-$*$ and $\mathcal{T}_1^{(m)}(l) \to \mathcal{T}_1(l)$ strongly in $\mathcal{C}(\overline{\Omega})$.

The end of the proof is a classical argument: by the Rellich theorem, and since the only limit of the subsequences of $(f_m)_{m\geq 1}$ in $L^{p'}(\Omega)$ is f, we deduce that $f_m \to f$ in $L^{p'}(\Omega)$. ■

Proof of Lemma 4.2. Let $\mu_j \in L^2(\Omega)$ such that $\mu_j \to \underline{\mu}$ in $\mathcal{M}(\overline{\Omega})$ weak-$*$. Let $\underline{f}^{(j)}$ be the solution of (3.5) for $(\underline{A}, \underline{\mathbf{v}}, \underline{b}, \underline{\lambda}, \mu_j)$: we know that $\underline{f}^{(j)}$ is, in fact, the solution of the variational problem (3.9) for theses data.

Thus, $T_k(\underline{f}^{(j)}) \in H^1_{\Gamma_d}(\Omega)$ and, using this function in the problem satisfied by $\underline{f}^{(j)}$, we find

$$\int_\Omega \underline{A}^T\nabla(T_k(\underline{f}^{(j)}))\cdot\nabla(T_k(\underline{f}^{(j)})) + \int_{\Gamma_n}(\underline{\mathbf{v}}\cdot\mathbf{n}+\underline{\lambda})\underline{f}^{(j)}T_k(\underline{f}^{(j)})\,d\sigma$$
$$-\int_\Omega T_k(\underline{f}^{(j)})\mathbf{v}_0\cdot\nabla\underline{f}^{(j)} + \int_\Omega \underline{b}\underline{f}^{(j)}T_k(\underline{f}^{(j)}) = \int_\Omega \mu_j T_k(\underline{f}^{(j)}). \tag{4.11}$$

But we have

$$\begin{aligned}
&-\int_\Omega T_k(\underline{f}^{(j)})\underline{\mathbf{v}}\cdot\nabla\underline{f}^{(j)}\\
&= -\int_\Omega \underline{\mathbf{v}}\cdot\nabla(\underline{f}^{(j)}T_k(\underline{f}^{(j)})) + \int_\Omega \underline{f}^{(j)}\underline{\mathbf{v}}\cdot\nabla(T_k(\underline{f}^{(j)})\\
&= -\int_{\Gamma_n}\underline{f}^{(j)}T_k(\underline{f}^{(j)})\underline{\mathbf{v}}\cdot\mathbf{n}\,d\sigma + \int_\Omega \mathrm{div}(\underline{\mathbf{v}})\underline{f}^{(j)}T_k(\underline{f}^{(j)})\\
&\quad + \int_\Omega \underline{\mathbf{v}}\cdot\nabla\left(\frac{(T_k(\underline{f}^{(j)}))^2}{2}\right)\\
&= -\int_{\Gamma_n}\underline{f}^{(j)}T_k(\underline{f}^{(j)})\underline{\mathbf{v}}\cdot\mathbf{n}\,d\sigma + \int_\Omega \mathrm{div}(\underline{\mathbf{v}})\underline{f}^{(j)}T_k(\underline{f}^{(j)})\\
&\quad + \int_{\Gamma_n}\frac{1}{2}(T_k(\underline{f}^{(j)}))^2\underline{\mathbf{v}}\cdot\mathbf{n}\,d\sigma - \int_\Omega \frac{1}{2}(T_k(\underline{f}^{(j)}))^2\mathrm{div}(\underline{\mathbf{v}}),
\end{aligned}$$

so that

$$\begin{aligned}
&\int_{\Gamma_n}(\underline{\mathbf{v}}\cdot\mathbf{n}+\underline{\lambda})\underline{f}^{(j)}T_k(\underline{f}^{(j)})\,d\sigma - \int_\Omega T_k(\underline{f}^{(j)})\underline{\mathbf{v}}\cdot\nabla\underline{f}^{(j)} + \int_\Omega \underline{b}\underline{f}^{(j)}T_k(\underline{f}^{(j)})\\
&= \int_{\Gamma_n}\left(\frac{1}{2}\underline{\mathbf{v}}\cdot\mathbf{n}+\underline{\lambda}\right)(T_k(\underline{f}^{(j)}))^2\,d\sigma + \int_\Omega\left(\frac{1}{2}\mathrm{div}(\underline{\mathbf{v}})+\underline{b}\right)(T_k(\underline{f}^{(j)}))^2
\end{aligned}$$

$$+\int_{\Gamma_n} \underline{\lambda}(\underline{f}^{(j)}T_k(\underline{f}^{(j)}) - (T_k(\underline{f}^{(j)}))^2)\,d\sigma$$
$$+\int_\Omega (\mathrm{div}(\underline{\mathbf{v}}) + \underline{b})(\underline{f}^{(j)}T_k(\underline{f}^{(j)}) - (T_k(\underline{f}^{(j)}))^2). \tag{4.12}$$

We also have

$$\begin{aligned}
&\int_\Omega (\underline{f}^{(j)}T_k(\underline{f}^{(j)}) - (T_k(\underline{f}^{(j)}))^2)\mathrm{div}(\underline{\mathbf{v}})\\
&-\int_{\Gamma_n} (\underline{f}^{(j)}T_k(\underline{f}^{(j)}) - (T_k(\underline{f}^{(j)}))^2)\underline{\mathbf{v}}\cdot\mathbf{n}\,d\sigma\\
&\quad = \quad -\int_\Omega \underline{\mathbf{v}}\cdot\nabla(\underline{f}^{(j)}T_k(\underline{f}^{(j)}) - (T_k(\underline{f}^{(j)}))^2)\\
&\quad \geq \quad -||\,|\underline{\mathbf{v}}|\,||_{L^\infty(\Omega)}||\,|\nabla(\underline{f}^{(j)}T_k(\underline{f}^{(j)}) - (T_k(\underline{f}^{(j)}))^2)|\,||_{L^1(\Omega)}.
\end{aligned} \tag{4.13}$$

By denoting $\widetilde{T}_k : \mathbb{R} \to \mathbb{R}$ the function $\widetilde{T}_k(s) = k(|s| - k)^+$, whose derivative is $\widetilde{T}_k'(s) = k\chi_{\mathbb{R}\setminus[-k,k]}(s)$, we have

$$\underline{f}^{(j)}T_k(\underline{f}^{(j)}) - (T_k(\underline{f}^{(j)}))^2 = k\widetilde{T}_k(\underline{f}^{(j)}) \geq 0, \tag{4.14}$$

so that, using $\frac{1}{2}\mathrm{div}(\underline{\mathbf{v}}) + \underline{b} \geq 0$ a.e. on Ω and $\frac{1}{2}\underline{\mathbf{v}}\cdot\mathbf{n} + \underline{\lambda} \geq 0$ σ-a.e. on Γ_n, we obtain

$$\begin{aligned}
&\int_{\Gamma_n} \underline{\lambda}(\underline{f}^{(j)}T_k(\underline{f}^{(j)}) - (T_k(\underline{f}^{(j)}))^2)\,d\sigma\\
&+\int_\Omega (\mathrm{div}(\underline{\mathbf{v}}) + \underline{b})(\underline{f}^{(j)}T_k(\underline{f}^{(j)}) - (T_k(\underline{f}^{(j)}))^2)\\
&\quad = \quad \int_{\Gamma_n}\left(\frac{1}{2}\underline{\mathbf{v}}\cdot\mathbf{n} + \underline{\lambda}\right)(\underline{f}^{(j)}T_k(\underline{f}^{(j)}) - (T_k(\underline{f}^{(j)}))^2)\\
&\qquad\quad +\int_\Omega\left(\frac{1}{2}\mathrm{div}(\underline{\mathbf{v}}) + \underline{b}\right)(\underline{f}^{(j)}T_k(\underline{f}^{(j)}) - (T_k(\underline{f}^{(j)}))^2)\\
&\qquad\quad +\frac{1}{2}\int_\Omega \mathrm{div}(\underline{\mathbf{v}})(\underline{f}^{(j)}T_k(\underline{f}^{(j)}) - (T_k(\underline{f}^{(j)}))^2)\\
&\qquad\quad -\frac{1}{2}\int_{\Gamma_n}(\underline{f}^{(j)}T_k(\underline{f}^{(j)}) - (T_k(\underline{f}^{(j)}))^2)\underline{\mathbf{v}}\cdot\mathbf{n}\,d\sigma\\
&\quad \geq \quad -\frac{||\,|\underline{\mathbf{v}}|\,||_{L^\infty(\Omega)}}{2}||\,|\nabla(\widetilde{T}_k(\underline{f}^{(j)}))|\,||_{L^1(\Omega)}\\
&\quad \geq \quad -\frac{k||\,|\underline{\mathbf{v}}|\,||_{L^\infty(\Omega)}}{2}||\,|\nabla\underline{f}^{(j)}|\,||_{L^1(\Omega)}.
\end{aligned} \tag{4.15}$$

(4.11), (4.12) and (4.15) give thus

$$\int_\Omega \underline{A}^T\nabla(T_k(\underline{f}^{(j)}))\cdot\nabla(T_k(\underline{f}^{(j)})) + \int_{\Gamma_n}\left(\frac{1}{2}\underline{\mathbf{v}}\cdot\mathbf{n}+\underline{\lambda}\right)(T_k(\underline{f}^{(j)}))^2\,d\sigma$$
$$+\int_\Omega\left(\frac{1}{2}\mathrm{div}(\underline{\mathbf{v}})+\underline{b}\right)(T_k(\underline{f}^{(j)}))^2$$
$$\leq \frac{k||\,|\underline{\mathbf{v}}|\,||_{L^\infty(\Omega)}}{2}||\,|\nabla\underline{f}^{(j)}|\,||_{L^1(\Omega)} + k||\mu_j||_{L^1(\Omega)}.$$

Thanks to the hypotheses on the data (i.e., the coercitivity of the bilinear form on the left hand side of this expression), we obtain C such that, for all $j\geq 1$,

$$||T_k(\underline{f}^{(j)})||_{H^1_{\Gamma_d}(\Omega)} \leq C(||\underline{f}^{(j)}||_{W^{1,1}_{\Gamma_d}(\Omega)} + ||\mu_j||_{L^1(\Omega)}). \tag{4.16}$$

Lemma 4.1 (in fact a simplier version of this lemma, since $(\underline{A},\underline{\mathbf{v}},\underline{b},\underline{\lambda})$ are fixed here) allows us to see that $\underline{f}^{(j)}\to\underline{f}$ weakly in $W^{1,1}_{\Gamma_d}(\Omega)$ and strongly in $L^1(\Omega)$. We see thus that $(\underline{f}^{(j)})_{j\geq1}$ is bounded in $W^{1,1}_{\Gamma_d}(\Omega)$ and, since $(\mu_j)_{j\geq1}$ is bounded in $\mathcal{M}(\overline{\Omega})$ (it converges weakly-$*$ in this space), we find, thanks to (4.16), that $(T_k(\underline{f}^{(j)}))_{j\geq1}$ is bounded in $H^1_{\Gamma_d}(\Omega)$ (recall that, when $\mu_j\in L^1(\Omega)$, $||\mu_j||_{L^1(\Omega)} = ||\mu_j||_{\mathcal{M}(\overline{\Omega})}$).

Up to a subsequence, we can suppose that $(T_k(\underline{f}^{(j)}))_{j\geq1}$ weakly converges in $H^1_{\Gamma_d}(\Omega)$ and a.e. on Ω. Since, up to a subsequence, $\underline{f}^{(j)}\to\underline{f}$ a.e. on Ω (because of the convergence in $L^1(\Omega)$), so that $T_k(\underline{f}^{(j)})\to T_k(\underline{f})$ a.e. on Ω, we have proven that, for all $k\in\mathbb{R}^+$, $T_k(\underline{f})$ is in $H^1_{\Gamma_d}(\Omega)$, as the weak limit in this space of $(T_k(\underline{f}^{(j)}))_{j\geq1}$. ■

Proof of Lemma 4.3.

Let B_m be the symetric bilinear form defined on $H^1(\Omega)$ by $(A_m^T+A_m)/2$, that is to say

$$\forall(w,\widetilde{w})\in H^1(\Omega)\,,\ B_m(a,\widetilde{a}) = \int_\Omega \frac{A_m^T+A_m}{2}\nabla w\cdot\nabla\widetilde{w}$$

Hypothesis (4.1) allows us to see that, for all $w\in H^1(\Omega)$,

$$B_m(w,w) = \int_\Omega A_m^T\nabla w\cdot\nabla w\geq 0.$$

Thus, B_m being a non-negative symetric bilinear form, we can apply the Cauchy-Schwartz inegality to find, for all $m \geq 1$,

$$B_m(a, a_m)^2 \leq B_m(a_m, a_m) B_m(a, a). \tag{4.17}$$

Since $A_m \to A$ a.e. on Ω and $(A_m)_{m\geq 1}$ is bounded in $L^\infty(\Omega; M_N(\mathbb{R}))$, we have $\frac{1}{2}(A_m^T + A_m)\nabla a \to \frac{1}{2}(A^T + A)\nabla a$ in $(L^2(\Omega))^N$; we obtain thus

$$\int_\Omega \frac{1}{2}(A_m^T + A_m)\nabla a \cdot \nabla a \to \int_\Omega \frac{1}{2}(A^T + A)\nabla a \cdot \nabla a = \int_\Omega A^T \nabla a \cdot \nabla a$$

and, using the fact that $a_m \to a$ weakly in $H^1(\Omega)$,

$$\int_\Omega \frac{1}{2}(A_m^T + A_m)\nabla a \cdot \nabla a_m \to \int_\Omega \frac{1}{2}(A^T + A)\nabla a \cdot \nabla a = \int_\Omega A^T \nabla a \cdot \nabla a.$$

Taking the lim inf as $m \to \infty$ in (4.17), we get

$$\left(\int_\Omega A^T \nabla a \cdot \nabla a \right)^2 \leq \left(\liminf_{m\to\infty} \int_\Omega A_m^T \nabla a_m \cdot \nabla a_m \right) \int_\Omega A^T \nabla a \cdot \nabla a,$$

which concludes the proof of this lemma. ■

Proof of Lemma 4.4.

Let $\mu_j \in L^2(\Omega)$ which converges to $\underline{\mu}$ in $\mathcal{M}(\overline{\Omega})$ weak-$*$ and such that $||\mu_j||_{L^1(\Omega)} \leq ||\underline{\mu}||_{\mathcal{M}(\overline{\Omega})} \leq \Lambda_0$. Let $\underline{f}^{(j)}$ as in the proof of Lemma 4.2; up to a subsequence, we can suppose that $\underline{f}^{(j)} \to \underline{f}$ a.e. on Ω and, since $(T_\delta(T_{k+1}(\underline{f}^{(j)} - T_k(\psi)))_{j\geq 1}$ is bounded in $H^1_{\Gamma_d}(\Omega)$ $((T_{k+1}(\underline{f}^{(j)}))_{j\geq 1}$ is bounded in this space, thanks to the proof of Lemma 4.2), we can suppose that $T_\delta(T_{k+1}(\underline{f}^{(j)} - T_k(\psi)) \to T_\delta(T_{k+1}(\underline{f}) - T_k(\psi))$ weakly in $H^1_{\Gamma_d}(\Omega)$.

Using $T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi)) \in H^1_{\Gamma_d}(\Omega)$ as a test function in the variational problem satisfied by $\underline{f}^{(j)}$, we find

$$\begin{aligned} &\int_\Omega \underline{A}^T \nabla \underline{f}^{(j)} \cdot \nabla(T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi)) \\ &\quad = \int_\Omega (\mu_j - \underline{b}\underline{f}^{(j)} + \underline{\mathbf{v}} \cdot \nabla \underline{f}^{(j)}) T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi)) \\ &\qquad - \int_{\Gamma_n} (\underline{\lambda} + \underline{\mathbf{v}} \cdot \mathbf{n}) \underline{f}^{(j)} T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi))\, d\sigma. \end{aligned} \tag{4.18}$$

If we denote by $\underline{\mathcal{T}_{1,p}}$ the application defined by (3.4) for $(\underline{A}, \underline{\mathbf{v}}, \underline{b}, \underline{\lambda})$, then Theorem 2.1 or 2.3 gives us C_1 only depending on $(\Omega, \alpha_A, \Lambda_A, p, \Lambda_0)$ and

Γ_d in the mixed case,
(b_0, E) or (λ_0, S) in the Fourier case,

such that

$$||\underline{\mathcal{T}_{1,p}}^*||_{\mathcal{L}(\mathcal{M}(\overline{\Omega}), W^{1,p'}_{\Gamma_d}(\Omega))} = ||\underline{\mathcal{T}_{1,p}}||_{\mathcal{L}((W^{1,p'}_{\Gamma_d}(\Omega))', \mathcal{C}(\overline{\Omega}))} \le C_1.$$

Since $\underline{f}^{(j)} = \underline{\mathcal{T}_{1,p}}^*(\mu_j)$, we notice that $(\underline{f}^{(j)})_{j\ge 1}$ is bounded in $W^{1,p'}_{\Gamma_d}(\Omega)$ by $C_1\Lambda_0$, and thus in $W^{1,1}_{\Gamma_d}(\Omega)$ by $C_1\Lambda_0|\Omega|^{1/p}$.

By denoting C_2 the norm of the Sobolev injection $W^{1,p'}_{\Gamma_d}(\Omega) \hookrightarrow L^{\frac{Np'}{N-p'}}(\Omega)$ and C_3 the norm of the Sobolev injection $W^{1,p'}_{\Gamma_d}(\Omega) \hookrightarrow L^{\frac{(N-1)p'}{N-p'}}(\Gamma_n)$ (C_2 and C_3 only depend on (Ω, p)), we thus obtain that $(\underline{f}^{(j)})_{j\ge 1}$ is bounded in $L^{\frac{Np'}{N-p'}}(\Omega)$ by $C_1C_2\Lambda_0$ and in $L^{\frac{(N-1)p'}{N-p'}}(\Gamma_n)$ by $C_1C_3\Lambda_0$ (and thus in $L^1(\Gamma_n)$ by $C_1C_3\Lambda_0\sigma(\Gamma_n)^{\frac{(N-1)p'}{N(p'-1)}}$).

Using the hypotheses on $\underline{b}$ and $\underline{\lambda}$, we see that $(\underline{b}\,\underline{f}^{(j)})_{j\ge 1}$ is bounded in $L^1(\Omega)$ by $C_1C_2\Lambda_0^2$ and that $(\underline{\lambda}\,\underline{f}^{(j)})_{j\ge 1}$ is bounded in $L^1(\Gamma_n)$ by $C_1C_3\Lambda_0^2$.

Since $||T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi))||_{L^\infty(\Omega)} \le \delta$, we deduce that

$$\begin{aligned}
&\int_\Omega (\mu_j - \underline{b}\underline{f}^{(j)} + \underline{\mathbf{v}} \cdot \nabla \underline{f}^{(j)}) T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi)) \\
&- \int_{\Gamma_n} (\underline{\lambda} + \underline{\mathbf{v}} \cdot \mathbf{n}) \underline{f}^{(j)} T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi))\, d\sigma \\
&\quad \le \Big(||\mu_j||_{L^1(\Omega)} + C_1C_2\Lambda_0^2 + ||\,|\underline{\mathbf{v}}|\,||_{L^\infty(\Omega)} C_1\Lambda_0 |\Omega|^{1/p} \\
&\qquad + C_1C_3\Lambda_0^2 + ||\underline{\mathbf{v}} \cdot \mathbf{n}||_{L^\infty(\Gamma_n)} C_1C_3\Lambda_0\sigma(\Gamma_n)^{\frac{(N-1)p'}{N(p'-1)}} \Big) \delta \\
&\quad \le C_4\delta,
\end{aligned} \tag{4.19}$$

where C_4 only depends on $(\Omega, \alpha_A, \Lambda_A, p, \Lambda_0)$ and

Γ_d in the mixed case,
(b_0, E) or (λ_0, S) in the Fourier case.

We have, for all $\varphi \in H^1_{\Gamma_d}(\Omega)$,

$$\nabla\varphi \cdot \nabla(T_\delta(\varphi - T_k(\psi)) = \nabla(T_{k+1}(\varphi)) \cdot \nabla(T_\delta(\varphi - T_k(\psi)) \qquad (4.20)$$

(recall that $\delta \in]0,1[$) and

$$\nabla(T_\delta(\varphi - T_k(\psi))) \cdot \nabla(T_\delta(\varphi - T_k(\psi)) = \nabla(\varphi - T_k(\psi)) \cdot \nabla(T_\delta(\varphi - T_k(\psi)). \quad (4.21)$$

Thus, applying (4.20) with $\varphi = \underline{f}^{(j)}$ and (4.21) with $\varphi = T_{k+1}(\underline{f}^{(j)})$, we find

$$\begin{aligned}
&\int_\Omega \underline{A}^T \nabla \underline{f}^{(j)} \cdot \nabla(T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi)) = \\
&\int_\Omega \underline{A}^T \nabla(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi)) \cdot \nabla(T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi)) \\
&\qquad + \int_\Omega \underline{A}^T \nabla(T_k(\psi)) \cdot \nabla(T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi)) = \\
&\int_\Omega \underline{A}^T \nabla(T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi))) \cdot \nabla(T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi)) \\
&\qquad + \int_\Omega \underline{A}^T \nabla(T_k(\psi)) \cdot \nabla(T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi)). \qquad (4.22)
\end{aligned}$$

(4.18), (4.19) and (4.22) give thus

$$\begin{aligned}
&\int_\Omega \underline{A}^T \nabla(T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi))) \cdot \nabla(T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi)) \\
&\quad \leq \; C_4\delta - \int_\Omega \underline{A}^T \nabla(T_k(\psi)) \cdot \nabla(T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi)).
\end{aligned}$$

By taking the lim inf of this as $j \to \infty$, using the fact that $T_\delta(T_{k+1}(\underline{f}^{(j)}) - T_k(\psi)) \to T_\delta(T_{k+1}(\underline{f}) - T_k(\psi))$ in $H^1_{\Gamma_d}(\Omega)$ weak-$*$ and Lemma 4.3, we find

$$\begin{aligned}
&\int_\Omega \underline{A}^T \nabla(T_\delta(T_{k+1}(\underline{f}) - T_k(\psi))) \cdot \nabla(T_\delta(T_{k+1}(\underline{f}) - T_k(\psi))) \\
&\quad \leq \; C_4\delta - \int_\Omega \underline{A}^T \nabla(T_k(\psi)) \cdot \nabla(T_\delta(T_{k+1}(\underline{f}) - T_k(\psi)))
\end{aligned}$$

which gives, since $\underline{A}^T \nabla(T_\delta(T_{k+1}(\underline{f}) - T_k(\psi))) \cdot \nabla(T_\delta(T_{k+1}(\underline{f}) - T_k(\psi))) = \underline{A}^T \nabla(T_{k+1}(\underline{f}) - T_k(\psi)) \cdot \nabla(T_\delta(T_{k+1}(\underline{f}) - T_k(\psi)))$, the result of Lemma 4.4. ■

Proof of Theorem 4.1.

We already know that $f_m \to f$ weakly in $W^{1,p'}_{\Gamma_d}(\Omega)$ on strongly in $L^{p'}(\Omega)$ (Lemma 4.1). We thus deduce that $(f_m)_{m\geq 1}$ is bounded in $W^{1,p'}_{\Gamma_d}(\Omega)$: if we prove that $\nabla f_m \to \nabla f$ in measure on Ω, then we have $\nabla f_m \to \nabla f$ a.e. up to a subsequence and, thanks a classical lemma, $\nabla f_m \to \nabla f$ in $L^q(\Omega)$, for all $q < p'$, up to a subsequence; since the only possible limit for subsequences of $(\nabla f_m)_{m\geq 1}$ in $L^q(\Omega)$ is ∇f, the whole sequence $(\nabla f_m)_{m\geq 1}$ converges in these spaces.

We thus have to show that $\nabla f_m \to \nabla f$ in measure, i.e., by denoting $\{F > r\}$ the subset of Ω where a function $F : \Omega \to \mathbb{R}$ is greater than $r \in \mathbb{R}$, we have to prove that, for all $\eta > 0$, $|\{|\nabla f_m - \nabla f| > \eta\}| \to 0$ as $m \to \infty$.

We write, following [3],

$$\{|\nabla f_m - \nabla f| > \eta\} \subset \{|f| > k\} \cup \{|f_m - f| > \delta\} \cup E_{k,m,\delta}, \tag{4.23}$$

with $\delta \in]0,1[$ and $E_{k,m,\delta} = \{|\nabla f_m - \nabla f| > \eta\} \cap \{|f| \leq k\} \cap \{|f_m - f| \leq \delta\}$.

Let $\varepsilon > 0$ and choose $k \in \mathbb{R}^+$ such that $|\{|f| > k\}| \leq \varepsilon$.

By Lemma 4.4, there exists C only depending on $(\Omega, \alpha_A, \Lambda_A, \Lambda, p)$ and

Γ_d in the mixed case,
(b_0, E) or (λ_0, S) in the Fourier case

such that, for all $m \geq 1$,

$$\int_\Omega A_m^T \nabla(T_{k+1}(f_m)) \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f)) \leq C\delta.$$

We also choose $\delta \in]0,1[$ such that $C\delta < \varepsilon$.

We prove now that, with these choices of k and δ, we can find $m_1 \geq 1$ such that, for all $m \geq m_1$, $|\{|\nabla f_m - \nabla f| > \eta\}| \leq M\varepsilon$, where M does not depend on m or ε. Let $m_0 \geq 1$ such that, for all $m \geq m_0$, $|\{|f_m - f| > \delta\}| \leq \varepsilon$ (recall that $f_m \to f$ in $L^q(\Omega)$, thus also in measure).

We have

$$\begin{aligned}
\alpha_A \eta^2 |E_{k,m,\delta}| &\leq \int_\Omega A_m^T \nabla(T_\delta(T_{k+1}(f_m) - T_k(f)) \cdot (T_\delta(T_{k+1}(f_m) - T_k(f)) \\
&= \int_\Omega A_m^T \nabla(T_{k+1}(f_m)) \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f)) \\
&\quad - \int_\Omega A_m^T \nabla(T_k(f)) \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f)) \\
&\leq C\delta - \int_\Omega A_m^T \nabla(T_k(f)) \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f)).
\end{aligned}$$

But $T_\delta(T_{k+1}(f_m) - T_k(f)) \to T_\delta(T_{k+1}(f) - T_k(f))$ weakly in $H^1_{\Gamma_d}(\Omega)$, so that

$$\int_\Omega A_m^T \nabla(T_k(f)) \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f)) \\ \overset{m\to\infty}{\longrightarrow} \int_\Omega A^T \nabla(T_k(f)) \cdot \nabla(T_\delta(T_{k+1}(f) - T_k(f)) = 0$$

(since $\nabla(T_k(f))\cdot\nabla(T_\delta(T_{k+1}(f) - T_k(f))) = \nabla f \cdot \nabla(T_{k+1}(f) - T_k(f))\chi_{\{|f|<k\}} = 0$ a.e.).

We can thus find $m_1 \geq m_0$ such that, for all $m \geq m_1$,

$$\left| \int_\Omega A_m^T \nabla(T_k(f)) \cdot \nabla(T_\delta(T_{k+1}(f_m) - T_k(f)) \right| \leq \varepsilon,$$

which gives, thanks to (4.23) and the choices of k and δ,

$$|\{|\nabla f_m - \nabla f| > \eta\}| \leq \left(2 + \frac{2}{\alpha_A \eta^2}\right)\varepsilon \quad \text{for all } m \geq m_1,$$

and the theorem is proved. ■

4.2. Solving non-linear problems. We use here the stability result that has just been proved and the Leray-Schauder topological degree to obtain the existence of a solution to a semi-linear problem with a measure as data.

The problem we want to solve is

$$\left\{ \begin{array}{l} -\mathrm{div}(A(f)^T \nabla f) - \mathrm{div}(f\mathbf{v}) + (\mathrm{div}(\mathbf{v}) + b(f))f = \mu[f] \quad \text{in} \quad \Omega, \\ f = 0 \quad \text{on} \quad \Gamma_d, \\ A(f)^T \nabla f + (\lambda(f) + \mathbf{v}\cdot\mathbf{n})f = 0 \quad \text{on} \quad \Gamma_n, \end{array} \right. \tag{4.24}$$

and we make the following hypotheses.

$$\begin{array}{c} A : \Omega \times \mathbb{R} \to M_N(\mathbb{R}) \text{ is a Caratheodory function,} \\ \exists \alpha_A > 0 \text{ such that } A(x,s)\xi\cdot\xi \geq \alpha_A |\xi|^2 \text{ for a.e. } x \in \Omega, \\ \text{for all } s \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^N, \\ \exists \Lambda_A > 0 \text{ such that } ||A(x,s)|| \leq \Lambda_A \text{ for a.e. } x \in \Omega, \text{ for all } s \in \mathbb{R}, \end{array} \tag{4.25}$$

$$\mathbf{v} : \Omega \to \mathbb{R} \text{ is a Lipschitz continuous function.} \tag{4.26}$$

There exists $q\in[1,p'[$, $r_1\in[\frac{Np}{N+p},\frac{Np}{N+p-Np}]$ and $r_2\in[\frac{(N-1)p}{N},\frac{(N-1)p}{N-(N-1)p}]$ such that, by denoting

$$q^*=\frac{Nq}{N-q}\,,\ \overline{q}=\frac{(N-1)q}{N-q}\,,\ \delta=q^*\left(\frac{N+p}{Np}-\frac{1}{r_1}\right)\in[0,q^*]$$
$$\text{and}\quad \zeta=\overline{q}\left(\frac{N}{(N-1)p}-\frac{1}{r_2}\right)\in[0,\overline{q}],$$

we have

$$\begin{array}{c} b:\Omega\times\mathbb{R}\to\mathbb{R}\text{ is a Caratheodory function,}\\ \exists C_0\in L^{r_1}(\Omega)\,,\ \exists C_1\in L^{\frac{Np}{N+p}}(\Omega)\text{ satisfying}\\ |b(x,s)|\leq C_0(x)|s|^\delta+C_1(x)\text{ for a.e. }x\in\Omega\text{, for all }s\in\mathbb{R},\\ \frac{1}{2}\mathrm{div}(\mathbf{v})(x)+b(x,s)\geq 0\text{ for a.e. }x\in\Omega\text{, for all }s\in\mathbb{R}, \end{array}\tag{4.27}$$

$$\begin{array}{c} \lambda:\partial\Omega\times\mathbb{R}\to\mathbb{R}\text{ is a Caratheodory function,}\\ \exists C_2\in L^{r_2}(\Omega)\,,\ \exists C_3\in L^{(N-1)\frac{p}{N}}(\partial\Omega)\text{ satisfying}\\ |\lambda(x,s)|\leq C_2(x)|s|^\zeta+C_3(x)\text{ for }\sigma\text{-a.e. }x\in\partial\Omega\text{, for all }s\in\mathbb{R},\\ \frac{1}{2}\mathbf{v}\cdot\mathbf{n}(x)+\lambda(x,s)\geq 0\text{ for }\sigma\text{-a.e. }x\in\partial\Omega\text{, for all }s\in\mathbb{R}. \end{array}\tag{4.28}$$

In the mixed case, we add Hypotheses (2.2) and (2.11).

In the Fourier case, we add either Hypothesis (2.25) or (2.26) uniform with respect to $s\in\mathbb{R}$, that is to say

$$\begin{array}{c} \exists b_0>0\,,\ \exists E\subset\Omega\text{ such that }|E|>0\\ \text{and }\frac{1}{2}\mathrm{div}(\mathbf{v})(x)+b(x,s)\geq b_0\text{ for all }x\in E\text{, for all }s\in\mathbb{R}\\ \text{or}\\ \exists\lambda_0>0\,,\ \exists S\subset\Omega\text{ such that }\sigma(S)>0\\ \text{and }\frac{1}{2}\mathbf{v}\cdot\mathbf{n}(x)+\lambda(x,s)\geq\lambda_0\text{ for all }x\in S\text{, for all }s\in\mathbb{R} \end{array}\tag{4.29}$$

The hypothesis on the right-hand side is:

$$\begin{array}{c} \mu:W^{1,q}_{\Gamma_d}(\Omega)\to\mathcal{M}(\overline{\Omega})\text{ is a sequentially continuous function}\\ \text{(when }\mathcal{M}(\overline{\Omega})\text{ is endowed with its weak-}*\text{ topology) which satisfies:}\\ \text{there exists }C_4>0,\ C_5>0\text{ and }\nu\in[0,1[\text{ such that, for all }f\in W^{1,q}_{\Gamma_d}(\Omega),\\ ||\mu[f]||_{\mathcal{M}(\overline{\Omega})}\leq C_4||f||^\nu_{W^{1,q}_{\Gamma_d}(\Omega)}+C_5. \end{array}\tag{4.30}$$

Remark 4.2. *Two examples of such functions:*

1) *If $S \subset \Omega$ is a measurable subset of an hyperplan or S is a measurable subset of $\partial\Omega$, $\nu \in [0,1[$ and $c : S \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that there exists $d_1 \in L^{(\overline{q}/\nu)'}(S, \mathcal{L}_{N-1})$ and $d_2 \in L^1(S, \mathcal{L}_{N-1})$ ($\mathcal{L}_{N-1}$ denotes the Lebesgue measure on S, i.e., the $(N–1)$-dimensional Hausdorff measure) satisfying*

$$|c(x,s)| \le d_1(x)|s|^\nu + d_2(x) \text{ for } \sigma\text{-a.e. } x \in S, \text{ for all } s \in \mathbb{R},$$

then $\mu[f] = c(., f(.))\mathcal{L}_{N-1}$ satisfies Hypothesis (4.30).

2) *If $G : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function such that there exists $\nu \in [0,1[$, $E_1 \in L^{(q^*/\nu)'}(\Omega)$, $E_2 \in L^{(q/\nu)'}(\Omega)$ and $E_3 \in L^1(\Omega)$ satisfying*

$$|G(x,s,\xi)| \le E_1(x)|s|^\nu + E_2(x)|\xi|^\nu + E_3(x)$$
$$\text{for a.e. } x \in \Omega, \text{ for all } s \in \mathbb{R}, \text{ for all } \xi \in \mathbb{R}^N,$$

then $\mu[f] = G(., f(.), \nabla f(.))\mathcal{L}_N$ (with $\mathcal{L}_N$ the Lebesgue measure on Ω) satisfies Hypothesis (4.30).

Theorem 4.2. *Under Hypotheses (4.25)—(4.28), (4.30) and*

(2.2) and (2.11) in the mixed case,
(4.29) in the Fourier case,

there exists at least one solution to (4.24) in the sense

$$\left\{ \begin{array}{l} f \in W^{1,p'}_{\Gamma_d}(\Omega), \\ \displaystyle\int_\Omega A(f)^T \nabla f \cdot \nabla\varphi + \int_{\Gamma_n} \lambda(f) f \varphi \, d\sigma + \int_\Omega f \mathbf{v} \cdot \nabla\varphi \\ \qquad \displaystyle + \int_\Omega (\operatorname{div}(\mathbf{v}) + b(f)) f\varphi = \int_{\overline{\Omega}} \varphi \, d(\mu[f]) \,,\ \forall \varphi \in W^{1,p}_{\Gamma_d}(\Omega). \end{array} \right. \tag{4.31}$$

Remark 4.3. *We have chosen an integral formulation of the kind (3.15) because the uniqueness of the solution we had in (3.5) or (3.13) is lost here. But we will see in the course of the proof that, in fact, we find a solution f to (4.24) in the sense of formulations of the kind (3.5) or (3.13)... which are here far more difficult to write than in the linear case (since $\mathcal{T}_1$ or Θ_2 must now depend on f through the non-linearity of (4.24) in A).*

Proof of Theorem 4.2.

Step 1: a sub-linear function ...

If $f \in W^{1,q}_{\Gamma_d}(\Omega) \subset L^{q^*}(\Omega)$, $(A(f), \mathbf{v}, b(.,f(.)), \lambda(.,f(.)))$ satisfy (3.1) in the mixed case and (3.2) in the Fourier case; denote by $\mathcal{T}_1^f$ the application defined by (3.3) for $(A(f), \mathbf{v}, b(f), \lambda(f))$.

Thanks to Theorem 2.1 or 2.3, to Proposition 2.1 (with Remark 2.7) or 2.3 and to Hypotheses (4.29), there exists $M_1 > 0$ not depending on $f \in W^{1,q}_{\Gamma_d}(\Omega)$ neither on $l \in (W^{1,p'}_{\Gamma_d}(\Omega))'$ such that

$$||\mathcal{T}_1^f(l)||_{\mathcal{C}(\overline{\Omega})} \leq M_1 ||l||_{(W^{1,p'}_{\Gamma_d}(\Omega))'}$$

(we have used the linearity of $\mathcal{T}_1^f$ and applied Propositions 2.1 or 2.3 with $\Lambda = 1$; recall that $\mathcal{C}(\overline{\Omega})$ is endowed with the same norm as $L^\infty(\Omega)$), that is to say

$$||\mathcal{T}_1^f||_{\mathcal{L}((W^{1,p'}_{\Gamma_d}(\Omega))', \mathcal{C}(\overline{\Omega}))} \leq M_1.$$

But it is well known that

$$||(\mathcal{T}_1^f)^*||_{\mathcal{L}(\mathcal{M}(\overline{\Omega}), W^{1,p'}_{\Gamma_d}(\Omega))} = ||\mathcal{T}_1^f||_{\mathcal{L}((W^{1,p'}_{\Gamma_d}(\Omega))', \mathcal{C}(\overline{\Omega}))},$$

so that

$$||(\mathcal{T}_1^f)^*||_{\mathcal{L}(\mathcal{M}(\overline{\Omega}), W^{1,p'}_{\Gamma_d}(\Omega))} \leq M_1. \tag{4.32}$$

Define

$$\Phi \left\{ \begin{array}{ccc} W^{1,q}_{\Gamma_d}(\Omega) & \longrightarrow & W^{1,p'}_{\Gamma_d}(\Omega) \hookrightarrow W^{1,q}_{\Gamma_d}(\Omega), \\ f & \longrightarrow & (\mathcal{T}_1^f)^*(\mu[f]). \end{array} \right.$$

Thanks to Hypothesis (4.30) and by denoting M_2 the norm of the injection $W^{1,p'}_{\Gamma_d}(\Omega) \hookrightarrow W^{1,q}_{\Gamma_d}(\Omega)$, Φ satisfies, for all $f \in W^{1,q}_{\Gamma_d}(\Omega)$,

$$||\Phi(f)||_{W^{1,q}_{\Gamma_d}(\Omega)} \leq M_2 ||\Phi(f)||_{W^{1,p'}_{\Gamma_d}(\Omega)} \leq M_1 M_2 C_4 ||f||^\nu_{W^{1,q}_{\Gamma_d}(\Omega)} + M_1 M_2 C_5. \tag{4.33}$$

Step 2: ... which is also continuous ...

We show here that $\Phi : W^{1,q}_{\Gamma_d}(\Omega) \to W^{1,q}_{\Gamma_d}(\Omega)$ is continuous.

Suppose that $(f_m)_{m\geq 1}$ converges to f in $W^{1,q}_{\Gamma_d}(\Omega)$; by a classical trick, it is sufficient to show that there exists a subsequence of $(f_m)_{m\geq 1}$, still denoted by $(f_m)_{m\geq 1}$, such that $(\Phi(f_m))_{m\geq 1}$ converges to $\Phi(f)$ in $W^{1,q}_{\Gamma_d}(\Omega)$.

Thus, up to a subsequence, we can suppose that $f_m \to f$ a.e. on Ω by being dominated by $F \in W^{1,q}_{\Gamma_d}(\Omega)$ and that $\nabla f_m \to \nabla f$ a.e. on Ω by being dominated by $\widetilde{F} \in L^q(\Omega)$.

$$(A_m, \mathbf{v}_m, b_m, \lambda_m, \mu_m) = (A(f_m), \mathbf{v}, b(., f_m(.)), \lambda(., f_m(.)), \mu[f_m])$$

satisfy then Hypotheses (4.1)—(4.6) and (4.7) or (4.8) with

$$(A(f), \mathbf{v}, b(., f(.)), \lambda(., f(.)), \mu[f])$$

as a limit (recall that μ is sequentially continuous for the weak-$*$ topology or $\mathcal{M}(\overline{\Omega})$), and Theorem 4.1 gives thus the convergence of $\Phi(f_m)$ toward $\Phi(f)$ in $W^{1,q}_{\Gamma_d}(\Omega)$ (since $q < p'$).

Step 3: …and such that $\Phi(\{f \in W^{1,q}_{\Gamma_d}(\Omega) \mid ||f||_{W^{1,q}_{\Gamma_d}(\Omega)} \le R\})$ is relatively compact in $W^{1,q}_{\Gamma_d}(\Omega)$, for all $R > 0$.

Indeed, if $(f_m)_{m\ge1}$ is bounded in $W^{1,q}_{\Gamma_d}(\Omega)$, we can suppose, up to a subsequence, that $f_m \to f$ a.e. on Ω: we have then $A(f_m) \to A(f)$ a.e. on Ω.

Since

$$\begin{aligned}
&(b(., f_m(.)))_{m\ge1} \text{ is bounded in } L^{\frac{Np}{N+p}}(\Omega),\\
&(\lambda(., f_m(.)))_{m\ge1} \text{ is bounded in } L^{(N-1)\frac{p}{N}}(\partial\Omega),\\
&(\mu[f_m])_{m\ge1} \text{ is bounded in } \mathcal{M}(\overline{\Omega}) \text{ (Hypothesis (4.30))},
\end{aligned}$$

there exist $b_\infty \in L^{\frac{Np}{N+p}}(\Omega)$, $\lambda_\infty \in L^{(N-1)\frac{p}{N}}(\partial\Omega)$ and $\mu_\infty \in \mathcal{M}(\overline{\Omega})$ such that, up to subsequences,

$$\begin{aligned}
&b(., f_m(.)) \to b_\infty \text{ weakly in } L^{\frac{Np}{N+p}}(\Omega),\\
&\lambda(., f_m(.)) \to \lambda_\infty \text{ weakly in } L^{(N-1)\frac{p}{N}}(\partial\Omega),\\
&\mu[f_m] \to \mu_\infty \text{ in } \mathcal{M}(\overline{\Omega}) \text{ for the weak-}* \text{ topology.}
\end{aligned}$$

Thus,

$$(A_m, \mathbf{v}_m, b_m, \lambda_m, \mu_m) = (A(f_m), \mathbf{v}, b(., f_m(.)), \lambda(., f_m(.)), \mu[f_m])$$

satisfy Hypotheses (4.1)—(4.6) and (4.7) or (4.8) with $(A(f), \mathbf{v}, b_\infty, \lambda_\infty, \mu_\infty)$ as a limit, and we have, thanks to Theorem 4.1, the convergence in $W^{1,q}_{\Gamma_d}(\Omega)$

of $(\Phi(f_m))_{m\geq 1}$ toward the solution $\widetilde{f}$ of (3.5) for $(A(f), \mathbf{v}, b_\infty, \lambda_\infty, \mu_\infty)$, since $q < p'$.

Step 4: Conclusion.

Lemma 4.5 just after this proof shows us that Φ has a fixed point in $W^{1,q}_{\Gamma_d}(\Omega)$, i.e., a $f \in W^{1,q}_{\Gamma_d}(\Omega)$ such that $f = \Phi(f)$; since Φ takes its values into $W^{1,p'}_{\Gamma_d}(\Omega)$, we see that f is in fact in $W^{1,p'}_{\Gamma_d}(\Omega)$.

Thus, f is the unique solution to (3.5) for the data $A(f)$, $\mathbf{v}$, $b(., f(.))$, $\lambda(., f(.))$ and $\mu[f]$.

But we have already proven that the solution to (3.5) is a solution to (3.15), and this concludes the demonstration of this theorem. ■

Lemma 4.5. *Let E be a Banach space. Let $F : E \to E$ be a compact operator, that is to say, F is continuous and $F(\{x \in E \mid ||x|| \leq R\})$ is relatively compact in E, for all $R \geq 0$. If F is sub-linear, that is to say there exists $K_1 > 0$, $K_2 > 0$ and $\omega \in [0, 1[$ such that, for all $x \in E$,*

$$||F(x)|| \leq K_1 ||x||^\omega + K_2$$

then F has a fixed point in E, i.e., a $x \in E$ such that $F(x) = x$.

The demonstration of this lemma is a straightforward application of the Leray-Schauder topological degree (see [4]); recall that the topological degree is an application $d : \mathcal{A} \to \mathbb{Z}$, defined on

$$\mathcal{A} = \{(Id - J, U, y)\,,\ U \text{ bounded open set of } E\,,\ J : \overline{U} \to E \text{ compact operator}\,,\ y \notin (Id - J)(\partial U)\}$$

and such that

i) $d(Id, U, y) = 1$ if $y \in U$,

ii) If $h : [0, 1] \times \overline{U} \to E$ is a compact operator and if $y \notin (Id - h(t, .))(\partial U)$ for all $t \in [0, 1]$, then $d(Id - h(0, .), U, y) = d(Id - h(1, .), U, y)$,

iii) If $d(Id - J, U, y) \neq 0$, then there exists $x \in U$ such that $x - J(x) = y$.

There are others properties to this degree, but we will use only these three.

Proof of Lemma 4.5.

Let $R > 0$ (that we will precise later) and denote by B_R the open ball of radius R and center 0 in E.

We want to prove that we can choose R large enough such that $d(Id - F, B_R, 0) = 1$; thanks to Property iii) of the topological degree, this will give us $x \in B_R$ such that $x - F(x) = 0$, i.e., a fixed point for F.

To prove this, we introduce the natural homotopy h between F and the null function, and we prove that, if R is large enough, $0 \notin (Id - h(t, .))(\partial B_R)$ for all $t \in [0, 1]$; applying then Properties ii) and i) of the topological degree, we deduce that $d(Id - F, B_R, 0) = d(Id - h(1, .), B_R, 0) = d(Id - h(0, .), B_R, 0) = d(Id, B_r, 0) = 1$.

Let $h : [0, 1] \times \overline{B_R} \to E$ be $h(t, x) = tF(x)$. h is continuous on $[0, 1] \times \overline{B_R}$ and, if $(t_n)_{n\geq 1} \in [0, 1]$, $(x_n)_{n\geq 1} \in \overline{B_R}$, then by compacity of $[0, 1]$ and F, there exists subsequences, still denoted by $(t_n)_{n\geq 1}$ and $(x_n)_{n\geq 1}$ such that $t_n \to t \in [0, 1]$ and $F(x_n) \to x_\infty \in E$; thus $h(t_n, x_n) \to tx_\infty$, and we have proved that h is a compact operator.

Suppose that there exists $t \in [0, 1]$ such that $0 \in (Id - h(t, .))(\partial B_R)$, i.e., such that there exists $x \in E$, with $||x|| = R$ and $x - tF(x) = 0$. Thanks to the sublinear property of F, we have then

$$||x|| = R \leq t||F(x)|| \leq K_1 t||x||^\omega + K_2 t \leq K_1 R^\omega + K_2$$

Since $\omega \in [0, 1[$, there exists $R_0 > 0$ such that $R_0 > K_1 R_0^\omega + K_2$. For this R_0, we thus have $0 \notin (Id - h(t, .))(\partial B_{R_0})$ for all $t \in [0, 1]$. This is exactly what we needed to conclude this proof. ■

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