# Convergence rate of the Allen-Cahn equation to generalized motion by mean curvature

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**Abstract.** We investigate the singular limit, as $\varepsilon \to 0$, of the Allen-Cahn equation $u_t^\varepsilon = \Delta u^\varepsilon + \varepsilon^{-2} f(u^\varepsilon)$, with $f$ a balanced bistable nonlinearity. We consider rather general initial data $u_0$ that is independent of $\varepsilon$. It is known that this equation converges to the generalized motion by mean curvature — in the sense of viscosity solutions— defined by Evans, Spruck and Chen, Giga, Goto. However the convergence rate has not been known. We prove that the transition layers of the solutions $u^\varepsilon$ are sandwiched between two sharp "interfaces" moving by mean curvature, provided that these "interfaces" sandwich at $t = 0$ an $\mathcal{O}(\varepsilon|\ln \varepsilon|)$ neighborhood of the initial layer. In some special cases, which allow both *extinction* and *pinches off* phenomenon, this enables to obtain an $\mathcal{O}(\varepsilon|\ln \varepsilon|)$ estimate of the location and the *thickness measured in space-time* of the transition layers. A result on the *regularity of the generalized motion by mean curvature* is also provided in the Appendix.

Key Words: Allen-Cahn equation, singular perturbation, generalized motion by mean curvature, viscosity solutions, location and thickness of the layers.
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## 1. Introduction

In this paper we study the behavior, as $\varepsilon \to 0$, of the solution $u^\varepsilon(x,t)$ of the Allen-Cahn type equation

$$(P^\varepsilon) \quad \begin{cases} u_t^\varepsilon = \Delta u^\varepsilon + \dfrac{1}{\varepsilon^2} f(u^\varepsilon) & \text{in } \mathbb{R}^N \times (0,\infty) \\ u^\varepsilon(x,0) = u_0(x) & \text{in } \mathbb{R}^N\,, \end{cases}$$

where $N \geq 2$. Here, the nonlinearity is given by $f(u) := -W'(u)$, where $W(u)$ is a double-well potential with equal well-depth, taking its global minimum value at $u = \alpha^*$, $u = \beta^*$. More precisely we assume that $f$ is $C^2$ and has exactly three zeros $\alpha^* < a < \beta^*$ such that

$$f'(\alpha^*) < 0, \quad f'(a) > 0, \quad f'(\beta^*) < 0 \quad \text{(bistable nonlinearity)}\,, \tag{1.1}$$

$$\int_{\alpha^*}^{\beta^*} f(u)\,du = 0 \quad \text{(balanced case)}\,. \tag{1.2}$$

The condition (1.1) implies that the potential $W(u)$ attains its local minima at $u = \alpha^*$, $u = \beta^*$, and (1.2) implies that $W(\alpha^*) = W(\beta^*)$. In other words, the two stable zeros of $f$, namely $\alpha^*$ and $\beta^*$, have "balanced" stability. A typical example is the cubic nonlinearity $f(u) = u(1-u^2)$.

As for the initial data $u_0$, we assume that it is bounded and of class $C^2$ on $\mathbb{R}^N$. Furthermore we define the "initial interface" $\Gamma_0$ by

$$\Gamma_0 := \{x \in \mathbb{R}^N : u_0(x) = a\}\,,$$

and suppose that

$$\begin{cases} \Gamma_0 \text{ is a smooth hypersurface without boundary of } \mathbb{R}^N\,, \\ \nabla u_0(x) \neq 0 \text{ for all } x \in \Gamma_0\,, \\ u_0 > a \text{ in } \Omega_0 \text{ and } u_0 < a \text{ in } (\Omega_0 \cup \Gamma_0)^c\,, \end{cases} \tag{1.3}$$

where $\Omega_0$ denotes the region enclosed by $\Gamma_0$. The non-zero gradient assumption in (1.3) is needed to obtain fine estimates for the development of steep transition layers at the very beginning period.

**Heuristics.** As $\varepsilon \to 0$, a formal asymptotic analysis shows the following: in the very early stage, the diffusion term $\Delta u^\varepsilon$ is negligible compared with the reaction term $\varepsilon^{-2} f(u^\varepsilon)$ so that, in the rescaled time scale $\tau = t/\varepsilon^2$, the equation is well approximated by the ordinary differential equation $u_\tau^\varepsilon = f(u^\varepsilon)$. Hence, in view of the profile of $f$, the value of $u^\varepsilon$ quickly becomes close to either $\beta^*$ or $\alpha^*$ in most part of $\mathbb{R}^N$, creating a steep interface (transition layer) between the regions $\{u^\varepsilon \approx \alpha^*\}$ and $\{u^\varepsilon \approx \beta^*\}$ (*Generation of interface*). Once such an interface develops, the diffusion term becomes large near the interface, and comes to balance with the reaction term. As a result, the interface ceases rapid development and starts to propagate in a much slower time scale (*Motion of interface*).

**Convergence to classical motion by mean curvature.** The singular limit of the Allen-Cahn equation was first studied in the pioneering work of Allen and Cahn

[2] and, slightly later, in Kawasaki and Ohta [18] from the point of view of physicists. They derived the interface equation by formal asymptotic analysis, thereby revealing that the interface moves by its mean curvature. More precisely, the limit solution $\tilde{u}(x,t)$ turns out to be a step function taking the value $\beta^*$ on one side of the interface, and $\alpha^*$ on the other side. This sharp interface, which we will denote by $\Gamma_t$, obeys the following law of motion:

$$(P^0_{\text{classical}}) \quad \begin{cases} V_n = -(N-1)\kappa & \text{on } \Gamma_t \\ \Gamma_t\big|_{t=0} = \Gamma_0\,, \end{cases}$$

where $V_n$ is the normal velocity of $\Gamma_t$ in the exterior direction, $\kappa$ the mean curvature at each point of $\Gamma_t$ (chosen to be positive when $\Gamma_t$ encloses a convex region). If $\Gamma_0$ is smooth enough, it is well known that $(P^0_{\text{classical}})$ possesses locally in time a unique smooth solution. For more details, see [11] and the references therein.

These early observations triggered a flow of mathematical studies aiming at rigorous justification of the above limiting procedure; see, for example, [19, 20], [9] and [10] for results on the convergence of the partial differential equation $(P^\varepsilon)$ to the free boundary Problem $(P^0_{\text{classical}})$. Later, in [1], the authors prove an improved estimate for this convergence for solutions with general initial data. By performing a rigorous analysis of both the generation and the motion of interface, they show that the solution develops a steep transition layer within the time scale of $\mathcal{O}(\varepsilon^2|\ln\varepsilon|)$, and that the layer obeys the law of motion that coincides with the formal asymptotic limit $(P^0_{\text{classical}})$ within an error margin of $\mathcal{O}(\varepsilon)$ (previously, the best thickness estimate in the literature was of $\mathcal{O}(\varepsilon|\ln\varepsilon|)$, [10]).

**Generalized motion by mean curvature.** Nevertheless, it is well-known that the classical motion by mean curvature may develop singularities in finite time, even if $\Gamma_0$ is smooth. In $\mathbb{R}^2$, an embedded curve evolving by its curvature can develop singularities only at the time of "shrinking to a point" [17]. In $\mathbb{R}^3$, singularities may even occur before "shrinking to a point": for instance, the boundary of a "dumbbell-shaped" region pinches off in finite time, if the neck is narrow enough. Therefore the classical framework is not sufficient for dealing with such phenomena. Thus, one has to introduce a generalized notion of the *motion by mean curvature* (MMC). This enables to define the MMC past the development of singularities and to study the singular limit of reaction-diffusion equations for all $t \geq 0$.

To define such a generalized MMC, the level set approach is quite convenient: one represents $\Gamma_t$ as the level set of an auxiliary function which solves (in the viscosity sense) a nonlinear partial differential equation. This direct partial differential equation approach has been developed by Evans and Spruck [15], and, independently, by Chen, Giga and Goto [12]. In this framework, the involved partial differential equation is the degenerate, and even singular in the points where $Dv = 0$, parabolic problem given by

$$(P^0) \quad \begin{cases} v_t - \operatorname{tr}\left[(I - \widehat{Dv}\otimes\widehat{Dv})D^2v\right] = 0 & \text{in } \mathbb{R}^N\times(0,\infty) \\ v = d_0 & \text{in } \mathbb{R}^N\times\{t=0\}\,, \end{cases}$$

with $\widehat{p} := \frac{p}{|p|}$, and $d_0$ the truncated signed distance function to $\Gamma_0$, which is positive in the set $\{u_0 > a\}$ and negative in the set $\{u_0 < a\}$. If $v$ is a viscosity solution of $(P^0)$, then each level set of $v$ evolves according to the mean curvature in a certain generalized sense, and also in the classical sense whenever $Dv$ does not vanish. Note that the equation can be written $v_t - \Delta v + D^2 v \widehat{Dv} \cdot \widehat{Dv} = 0$ or $v_t = |Dv| \operatorname{div}\left(\frac{Dv}{|Dv|}\right)$. We refer to Section 3 for a short overview of the techniques and results of [15] and [12].

**Convergence to generalized motion by mean curvature.** Let us make a brief overview of known results on the convergence of the Allen-Cahn equation to generalized MMC. Evans, Soner and Souganidis [14] prove that, as $\varepsilon \to 0$, the solution of $(P^\varepsilon)$ converges to $\beta^*$ locally uniformly in $\{v > 0\}$ and to $\alpha^*$ locally uniformly in $\{v < 0\}$, where $v$ is the solution of $(P^0)$. Since $\Gamma_t := \{x \in \mathbb{R}^N : v(x,t) = 0\}$ moves, in a weak sense, by mean curvature, this result is the natural generalization of the convergence to classical MMC mentioned above. Barles, Bronsard and Souganidis [5], Barles, Soner and Souganidis [7] generalized the result of [14] by allowing $(x,t)$-dependent nonlinearities and/or considering the unbalanced case instead of (1.2). Nevertheless, these early results consider only a very restricted class of initial data, namely those having a specific profile with well-developed transition layer. In other words the generation of interface from arbitrary initial data is not studied there.

Later, Soner [21, 22], Barles and Souganidis [8], Barles and Da Lio [6] study both the generation and the motion of interface; they prove the convergence of a large class of reaction-diffusion equations. By using the so-called "open set approach", the authors in [8] and [6] also provide a new definition for the global in time propagation of fronts; this definition turns out to be equivalent to the level set approach when there is no fattening of the interface.

From the above results, we know that the transition layers of $u^\varepsilon$ converge to a level set of $v$, the solution of $(P^0)$, as $\varepsilon \to 0$, for all $t \geq 0$. However, no fine estimate of the convergence rate nor the thickness of the transition layers of the solutions to $(P^\varepsilon)$, for all $t \geq 0$, exists. This is in contrast to the classical framework, for which $\mathcal{O}(\varepsilon)$ estimates are known, as long as the limit interface remains smooth.

**Overview of the main results.** In the present paper we obtain sharp estimates on the transition layers of solutions $u^\varepsilon$ to Problem $(P^\varepsilon)$, for all $t \geq 0$. Allowing arbitrariness of the initial data (i.e. not necessarily well-prepared initial data), we prove that — in a sense to be made precise later— the convergence rate is $\mathcal{O}(\varepsilon|\ln\varepsilon|)$. The body of this estimate is Section 4 where precise Allen-Cahn barriers are constructed by mixing and refining ideas from [14] and [1]. Then, under a geometric assumption on the "initial domain", we prove an $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ estimate of the location and the thickness of the transition layers. To our knowledge, these are the first sharp estimates — for the Allen-Cahn layers— which hold even after singularities have occurred in MMC. Note that, in order to deal with *extinction* and *pinches off* phenomenon, the thickness is measured in *space-time*. This is achieved in Section 5. In the next section, we discuss and precisely present these results.

## 2. Main results and comments

### 2.1. Results

We consider the solution $u^\varepsilon$ of the Allen-Cahn equation $(P^\varepsilon)$ with initial data $u_0$ independent of $\varepsilon$. As mentioned before, $u^\varepsilon$ quickly develops a steep transition layer. The time needed for such a generation (see Section 4) is

$$t^\varepsilon := f'(a)^{-1}\varepsilon^2|\ln\varepsilon| \quad \text{(generation time)}\,, \tag{2.1}$$

after which the transition layer starts to move approximately by the mean curvature. Theorem 2.1 is a first fine description of this motion of the Allen-Cahn transition layer: we show that it can be sandwiched between two sharp "interfaces" moving by mean curvature, provided that these "interfaces" sandwich at $t = 0$ an $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ neighborhood of the initial layer.

In the sequel, we take two families of (not necessarily smooth) hypersurfaces without boundaries $(\gamma^-_{\varepsilon,0})_{\varepsilon>0}$, $(\gamma^+_{\varepsilon,0})_{\varepsilon>0}$, which sandwich an $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ neighborhood of $\Gamma_0$, and such that

$$\gamma^-_{\varepsilon,0} << \Gamma_0 << \gamma^+_{\varepsilon,0}\,,$$

where $\Gamma_1 << \Gamma_2$ means that $\Gamma_1$ is enclosed by $\Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. More precisely, we consider two families of open sets $(\omega^-_{\varepsilon,0})_{\varepsilon>0}$, $(\omega^+_{\varepsilon,0})_{\varepsilon>0}$, such that

$$\{x \in \mathbb{R}^N : \operatorname{dist}(x, \omega^-_{\varepsilon,0}) \le C_0\varepsilon|\ln\varepsilon|\} \subset \Omega_0\,, \tag{2.2}$$

$$\{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega_0) \le C_0\varepsilon|\ln\varepsilon|\} \subset \omega^+_{\varepsilon,0}\,, \tag{2.3}$$

for some constant $C_0 > 0$ not depending on $\varepsilon$ and to be specified in (4.32). Then we define

$$\gamma^\pm_{\varepsilon,0} := \partial\omega^\pm_{\varepsilon,0} \quad \text{and} \quad \{\gamma^\pm_{\varepsilon,t}\}_{t\ge 0} := \text{ the generalized MMC starting from } \gamma^\pm_{\varepsilon,0}\,,$$

(see Section 3). In the same way as we define $\Omega_t$ as the "inside at time $t$" of $\Gamma_t$ in (3.4), we define $\omega^\pm_{\varepsilon,t}$ as the "inside" of $\gamma^\pm_{\varepsilon,t}$ by replacing $\Gamma_0$ in (3.1) by $\gamma^\pm_{\varepsilon,0}$.

**Theorem 2.1 ("Sandwiching" the Allen-Cahn layers).** *Let $f \in C^2(\mathbb{R})$ satisfy* (1.1) *and* (1.2), *and let $u_0 \in C^2_b(\mathbb{R}^N)$ be such that* (1.3) *holds. Let $(\omega^-_{\varepsilon,0})_{\varepsilon>0}$, respectively $(\omega^+_{\varepsilon,0})_{\varepsilon>0}$, be any family of open sets satisfying* (2.2), *respectively* (2.3). *Let $\{\omega^-_{\varepsilon,t}\}_{t\ge0}$ and $\{\omega^+_{\varepsilon,t}\}_{t\ge0}$ be defined as above. Fix $\zeta \in (0, \min(a-\alpha^*, \beta^*-a))$ arbitrarily. Then, for $\varepsilon > 0$ small enough,*

$$\begin{cases} \alpha^* - \zeta \le u^\varepsilon(x,t) \le \beta^* + \zeta & \text{for all } x \in \mathbb{R}^N \\ \beta^* - \zeta \le u^\varepsilon(x,t) \le \beta^* + \zeta & \text{for all } x \in \omega^-_{\varepsilon,t} \cup \gamma^-_{\varepsilon,t} \\ \alpha^* - \zeta \le u^\varepsilon(x,t) \le \alpha^* + \zeta & \text{for all } x \notin \omega^+_{\varepsilon,t}\,, \end{cases} \tag{2.4}$$

*for all $t \ge t^\varepsilon$, where $t^\varepsilon$ is as in* (2.1).

In the sequel, for $\alpha \in (\alpha^*, \beta^*)$, we define the sets

$$\Omega^\varepsilon_t(\alpha) := \{x \in \mathbb{R}^N : u^\varepsilon(x,t) > \alpha\} \quad \text{and} \quad \widetilde{\Omega}^\varepsilon_t(\alpha) := \{x \in \mathbb{R}^N : u^\varepsilon(x,t) \ge \alpha\}\,.$$

Roughly speaking, given $\alpha < \beta$ in $(\alpha^*, \beta^*)$, $\Gamma_t^\varepsilon(\alpha,\beta) := \widetilde{\Omega}_t^\varepsilon(\alpha) \setminus \Omega_t^\varepsilon(\beta)$ represents the transition layer of the Allen-Cahn solution $u^\varepsilon$, namely the "zone" $\alpha \le u^\varepsilon \le \beta$. As an immediate consequence of Theorem 2.1 we can localize these sets in terms of $\omega^-_{\varepsilon,t}$, $\gamma^-_{\varepsilon,t}$ and $\omega^+_{\varepsilon,t}$.

**Corollary 2.2 ("Sandwiching" the Allen-Cahn layers).** *Let the assumptions of Theorem* 2.1 *hold. Fix $\alpha < \beta$ in $(\alpha^*, \beta^*)$ arbitrarily. Then, for $\varepsilon > 0$ small enough,*

$$(\omega^-_{\varepsilon,t} \cup \gamma^-_{\varepsilon,t}) \subset \Omega_t^\varepsilon(\beta) \subset \widetilde{\Omega}_t^\varepsilon(\alpha) \subset \omega^+_{\varepsilon,t}\,, \tag{2.5}$$

*for all $t \ge t^\varepsilon$, where $t^\varepsilon$ is as in* (2.1).

The statement (2.5) gives lower and upper estimates for the Allen-Cahn layer $\Gamma_t^\varepsilon(\alpha,\beta)$, but it does not necessarily give fine estimate for the location nor the thickness of the layer. To explain this, let $\{\Gamma_t\}_{t\ge0}$ denote the generalized MMC starting from $\Gamma_0 = \partial\Omega_0$ (see Section 3) and define, as in (3.4), $\Omega_t$ as the "inside at time $t$" of $\Gamma_t$. Assume (which is natural) that (2.2) and (2.3) are sharp — in the sense that $\gamma^-_{\varepsilon,0}$ and $\gamma^+_{\varepsilon,0}$ actually lie in an $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ neighborhood of $\Gamma_0$. Then, for every $t \ge 0$ the property

$$\lim_{\varepsilon\to0} (\omega^-_{\varepsilon,t} \cup \gamma^-_{\varepsilon,t}) = \Omega_t \quad\text{and}\quad \lim_{\varepsilon\to0} \omega^+_{\varepsilon,t} = \Omega_t \cup \Gamma_t\,, \tag{2.6}$$

follows as an immediate consequence of the continuity of the viscosity solution of $(P^0)$ with respect to the initial data (see [15] or [3]). Thus (2.5) implies, in particular, that, for any $\alpha < \beta$ in $(\alpha^*, \beta^*)$,

$$\limsup_{\varepsilon\to0} \Gamma_t^\varepsilon(\alpha,\beta) \subset \Gamma_t\,. \tag{2.7}$$

However, no precise estimate of the convergence rate in (2.6) is known in general. Note that this is the case even if all the MMC starting from "neighbors"of $\Gamma_0$ are *regular* (in the sense of (A.1) in Appendix). Therefore Theorem 2.1 does not give good convergence rate for (2.7). Nevertheless, as we explain below, explicit fine estimates can be derived for *admissible* initial domains (see [7, Theorem 4.3] for the introduction of a similar notion).

**Definition 2.3 (Admissible domains).** Let $\Omega_0$ be a domain (=a bounded open set) in $\mathbb{R}^N$ whose boundary $\Gamma_0 := \partial\Omega_0$ is a smooth hypersurface without boundary. We say that $\Omega_0$ is admissible if there exists $a_1 \ge 0$, $a_2 \ge 0$ and a skew-symmetric matrix $Z$ such that, for all $x \in \Gamma_0$,

$$(-a_1 x + Zx - a_2(N-1)\kappa(x)n(x)) \cdot n(x) < 0\,, \tag{2.8}$$

where $n(x)$ the unit outer normal to $\Omega_0$ at $x$.

**Remark 2.4.** Assume $\Omega_0$ is admissible. For $t \ge 0$, define the evolution operator $\Phi_t : \Gamma_0 \mapsto \Gamma_t$, where $\{\Gamma_t\}_{t\ge0}$ denotes the generalized MMC starting from $\Gamma_0 = \partial\Omega_0$. Then, for $\nu \ge 0$, define

$$\psi_\nu(\Gamma_0) := e^{\nu Z}[e^{-a_1\nu}\Phi_{a_2\nu}(\Gamma_0)]\,,$$

obtained by letting $\Gamma_0$ evolve by its mean curvature for the time $a_2\nu$, then dilating by factor $e^{-a_1\nu}$, and rotating by the matrix $e^{\nu Z} \in SO_n(\mathbb{R})$. Since $e^{\nu Z}$, $e^{-a_1\nu}$,

$\Phi_{a_2\nu}$ are commutative, one can check that the collection $(\psi_\nu(\Gamma_0))_{0\leq\nu\leq\nu_0}$, with $\nu_0>0$ small enough, has the semigroup property $\psi_{\nu'}(\psi_\nu(\Gamma_0))=\psi_{\nu'+\nu}(\Gamma_0)$ when $\nu'+\nu\leq\nu_0$. Moreover the infinitesimal generator evaluated at $x\in\Gamma_0$ is $G(x):=-a_1x+Zx-a_2(N-1)\kappa(x)n(x)$. In view of (2.8) and the compactness of $\Gamma_0$, there is $\delta>0$ such that $G(x)\cdot n(x)\leq-\delta$ for all $x\in\Gamma_0$. If follows that, by choosing $\nu_0>0$ small if necessary,

$$0\leq\nu<\nu'\leq\nu_0 \implies \psi_{\nu'}(\Gamma_0)<<\psi_\nu(\Gamma_0)\,, \tag{2.9}$$

and that there is $\widehat{C}_0>0$ such that, for all $0\leq\nu\leq\nu_0$, $\mathrm{dist}(\psi_\nu(\Gamma_0),\Gamma_0)\geq\widehat{C}_0\nu$.

Before proceeding further, let us emphasize one important difference between the classical MMC and the generalized one. In the classical framework, one first defines $[0,T^{max})$ to be the maximal time-interval on which $\Gamma_t$ remains smooth, and then picks up an arbitrary closed sub-interval $0\leq t\leq T<T^{max}$, on which the derivatives of $\Gamma_t$ remain uniformly bounded. In this time range one can get a good convergence rate for (2.6) for each $0\leq t\leq T$, because of the smoothness of $\Gamma_t$. And we even have an optimal estimate as $d_{\mathcal{H}_{\mathbb{R}^N}}(\Gamma_t^\varepsilon(\alpha,\beta),\Gamma_t)=\mathcal{O}(\varepsilon)$, where $d_{\mathcal{H}_{\mathbb{R}^N}}$ denotes the Hausdorff distance (see [1], as mentioned before). However, in the generalized framework, such fine estimates collapse whenever $\Gamma_t$ develops a singularity. For example, consider two dumbbell-shaped hypersurfaces $\gamma_{\varepsilon,0}^-<<\Gamma_0$ whose Hausdorff distance $d_{\mathcal{H}_{\mathbb{R}^N}}(\gamma_{\varepsilon,0}^-,\Gamma_0)$ is very small, say of $\mathcal{O}(\varepsilon|\ln\varepsilon|)$. Within finite time the "neck" of the smaller dumbbell $\gamma_{\varepsilon,t}^-$ pinches off, splitting the hypersurface into two parts. Shortly after, before $\Gamma_t$ also pinches off, the Hausdorff distance $d_{\mathcal{H}_{\mathbb{R}^N}}(\gamma_{\varepsilon,t}^-,\Gamma_t)$ is rather large compared with the distance at $t=0$.

Therefore, in order to get fine quantitative estimates in the presence of singularities, the spatial distance at each fixed time slice is not the right measurement to use. It turns out that, by using the space-time distance, we can overcome this difficulty, at least for admissible initial domains. For this purpose, we define the "space-time insides"

$$\omega_\varepsilon^\pm:=\cup_{t\geq0}(\omega_{\varepsilon,t}^\pm\times\{t\})\,,\quad \Omega:=\cup_{t\geq0}(\Omega_t\times\{t\})\,,$$

and the "space-time interface"

$$\Gamma:=\{(x,t)\in\mathbb{R}^N\times[0,\infty):\,v(x,t)=0\}\,, \tag{2.10}$$

with $v$ the viscosity solution of $(P^0)$.

When $\Omega_0$ is admissible, there is a constant $C>0$ such that, for the approximating domains $(\omega_{\varepsilon,0}^-)_{\varepsilon>0}$, $(\omega_{\varepsilon,0}^+)_{\varepsilon>0}$ satisfying (2.2), and (2.3) respectively, we have

$$d_{\mathcal{H}_{\mathbb{R}^{N+1}}}(\gamma_\varepsilon^\pm,\Gamma)\leq Cd_{\mathcal{H}_{\mathbb{R}^N}}(\gamma_{\varepsilon,0}^\pm,\Gamma_0)\,,$$

where $d_{\mathcal{H}_{\mathbb{R}^{N+1}}}$ denotes the Hausdorff distance in the space-time $\mathbb{R}^{N+1}$ (see Section 5 for details). Combining this and Theorem 2.1, we can obtain the following fine description of the Allen-Cahn layers for admissible initial domains.

**Theorem 2.5 (Fine estimates for admissible initial domains).** *Let $f\in C^2(\mathbb{R})$ satisfy* (1.1) *and* (1.2)*, and let $u_0\in C_b^2(\mathbb{R}^N)$ be such that* (1.3) *holds. Assume moreover* *that* $\Omega_0$ *is admissible. Fix* $\zeta \in (0, \min(a-\alpha^*, \beta^*-a))$ *arbitrarily. Then, there is* $C>0$ *such that, for* $\varepsilon>0$ *small enough, for all* $x\in\mathbb{R}^N$ *and all* $t \geq t^\varepsilon$,

$$u^\varepsilon(x,t) \in \begin{cases} [\alpha^*-\zeta, \beta^*+\zeta] & \text{if } (x,t)\in \mathbb{R}^N\times[t^\varepsilon,\infty) \\ [\beta^*-\zeta, \beta^*+\zeta] & \text{if } (x,t)\in \Omega\setminus \mathcal{N}_{C\varepsilon|\ln\varepsilon|}(\Gamma) \\ [\alpha^*-\zeta, \alpha^*+\zeta] & \text{if } (x,t)\in (\Omega\cup\Gamma)^c\setminus \mathcal{N}_{C\varepsilon|\ln\varepsilon|}(\Gamma) \end{cases} \tag{2.11}$$

*where* $\mathcal{N}_r(\mathcal{A}) := \{(x,t)\in\mathbb{R}^N\times[0,\infty) : \operatorname{dist}((x,t),\mathcal{A}) < r\}$ *denotes the* $r$*-neighborhood of the set* $\mathcal{A}$ *in* $\mathbb{R}^N\times[0,\infty)$*, and* $t^\varepsilon$ *is as in* (2.1).

Now, for $\alpha<\beta$ in $(\alpha^*,\beta^*)$, we define the "zone" $\alpha\le u^\varepsilon\le\beta$ by

$$\Gamma^\varepsilon(\alpha,\beta) := \{(x,t)\in\mathbb{R}^N\times[t^\varepsilon,\infty) : \alpha\le u^\varepsilon(x,t)\le\beta\},$$

which more or less represents the transition layer of the Allen-Cahn solution $u^\varepsilon$ in space-time. Then the following holds as a direct consequence of Theorem 2.5.

**Corollary 2.6 (Location and thickness of the layers).** *Let* $f\in C^2(\mathbb{R})$ *satisfy* (1.1) *and* (1.2)*, and let* $u_0\in C^2_b(\mathbb{R}^N)$ *be such that* (1.3) *holds. Assume moreover that* $\Omega_0$ *is admissible. Fix* $\alpha<\beta$ *in* $(\alpha^*,\beta^*)$ *arbitrarily. Then there is* $C>0$ *such that, for* $\varepsilon>0$ *small enough,*

$$\Gamma^\varepsilon(\alpha,\beta)\subset\mathcal{N}_{C\varepsilon|\ln\varepsilon|}(\Gamma). \tag{2.12}$$

Note that (2.12) does not only give fine estimates for the location of the Allen-Cahn layer $\Gamma^\varepsilon(\alpha,\beta)$, but it also gives fine estimates for its thickness. Indeed, under the assumption of $\Omega_0$ being admissible, $\Gamma$ is known to have no interior [7, Theorem 4.3]; in other words, the so-called *fattening phenomenon* does not occur for $\Gamma$. Therefore (2.12) provides an $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ estimate of the thickness of the Allen-Cahn layer.

Incidentally, if $\Omega_0$ is admissible, not only $\Gamma$ is known to have no interior, but it is also known to be *regular from inside*, that is, $\mathrm{Cl}_{\mathbb{R}^{N+1}}[\Omega]=\Omega\cup\Gamma$, where $\mathrm{Cl}_{\mathbb{R}^{N+1}}[A]$ denotes the closure of the set $A$ in $\mathbb{R}^{N+1}$ [16, Corollary 4.5.11]. In fact, one can also prove that $\Gamma$ is *regular* both from inside and from outside if the initial domain is admissible (see Appendix).

Note that Theorem 2.5 allows $\Omega_t$ to pinch off, as will be clear from Example 2.9 below. Note also that Theorem 2.5 holds even after the extinction time $t^*$ of the solution of the MMC starting from the initial interface $\Gamma_0$. Therefore, in the admissible case, we can provide an answer to the question: how quickly does $u^\varepsilon$ approach $\alpha^*$ after the extinction of the interface $\Gamma_t$?

**Corollary 2.7 (Behavior after the extinction time).** *Let* $f\in C^2(\mathbb{R})$ *satisfy* (1.1) *and* (1.2)*, and let* $u_0\in C^2_b(\mathbb{R}^N)$ *be such that* (1.3) *holds. Assume moreover that* $\Omega_0$ *is admissible. Fix* $\zeta\in(0,\min(a-\alpha^*,\beta^*-a))$ *arbitrarily. Then, there is* $C>0$ *such that, for* $\varepsilon>0$ *small enough,*

$$|u^\varepsilon(x,t)-\alpha^*|\le\zeta, \tag{2.13}$$

*for all* $x\in\mathbb{R}^N$*,* $t\ge t^*+C\varepsilon|\ln\varepsilon|$*, with* $t^*>0$ *the extinction time defined in* (3.7).

### 2.2. Examples of admissible and non-admissible domains

In what follows $\Omega_0$ is always assumed to be a domain (=a bounded open set) in $\mathbb{R}^N$ with smooth boundary $\Gamma_0$. Here are some examples of admissible domains but also an example of a domain which cannot satisfy (2.8).

**Example 2.8 (Strongly star-shaped domains).** A domain $\Omega_0$ is called strongly star-shaped with respect to the origin 0, if it is star-shaped with respect to 0, and if every ray emanating from 0 intersects $\Gamma_0$ transversely. The above condition is equivalent to $x \cdot n(x) > 0$, for all $x \in \Gamma_0$. Thus, any strongly star-shaped domain is admissible with $(a_1, a_2, Z) = (1, 0, 0)$.

**Example 2.9 (Dumbbell-shaped domains).** For $N \geq 3$, let $\Omega_0$ consist of a pair of disjoint bounded open sets $D_1$, $D_2$ and a narrow channel $D_3$ connecting $D_1$, $D_2$. For simplicity, we assume that $\Omega_0$ is rotationally symmetric around the $x_1$-axis and given in the form $\Omega_0 := \{0 \leq r < g(x_1)\}$, $r = ({x_2}^2 + \cdots + {x_N}^2)^{1/2}$, where $g$ is a function satisfying $g > 0$ on $(-L, L)$, $g(\pm L) = 0$ and $g'(\pm L) = \mp\infty$. Furthermore, for some $0 < L_1 < L_2 < L$ and $0 < a < 1$,

$$\begin{cases} g(x_1) = 1 & \text{if } \ |x_1| \leq L_1 \\ g(x_1) = \cosh(a(|x_1| - L_1)) & \text{if } \ L_1 \leq |x_1| < L_2 \\ g''(x_1) < 0 & \text{if } \ L_2 \leq |x_1| < L\,. \end{cases}$$

We then modify $g$ slightly around $|x_1| = L_1$ and $|x_1| = L_2$ so that $g$ is smooth for $|x_1| < L$ and that $\Gamma_0$ is a smooth hypersurface (see Figure 1 (left)). Then we can

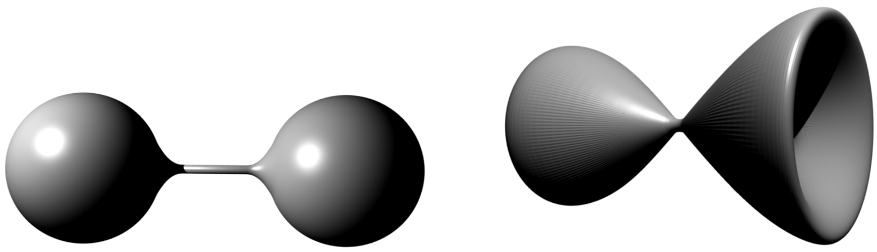

FIGURE 1. Dumbbell (left) and diabolo (right).

easily check that, for $a \in (0, 1)$ small enough and for some $\delta > 0$,

$$(N-1)\kappa(x) = \frac{1}{(1 + (g'(x))^2)^{1/2}} \left( \frac{N-2}{g(x)} - \frac{g''(x)}{1 + (g'(x))^2} \right) \geq \delta\,.$$

Hence $\Omega_0$ is admissible with $(a_1, a_2, Z) = (0, 1, 0)$. It is well known (see Angenent [4]) that the generalized MMC starting from such $\Gamma_0$ pinches off and splits into two parts if $L_1$ and $D_1$, $D_2$ are large enough — thus creating a singularity .

A different type of dumbbell-shaped domain, which we call a diabolo, can be constructed within the class of strongly star-shaped domains, that is $(a_1, a_2, Z) =$

$(1,0,0)$, see Figure 1 (right). The difference from the previous domain is that the center neck, namely $D_3$, cannot be too long; on the other hand, the outer end of $D_1$, $D_2$ need not to be of convex shape. This domain also leads to pinching off if the center neck is narrow enough.

**Example 2.10 (Galaxies).** Here $\Omega_0$ is a domain in $\mathbb{R}^3$ having the shape as in Figure 2. This is constructed by appropriately fattening the 2-dimensional skeleton in

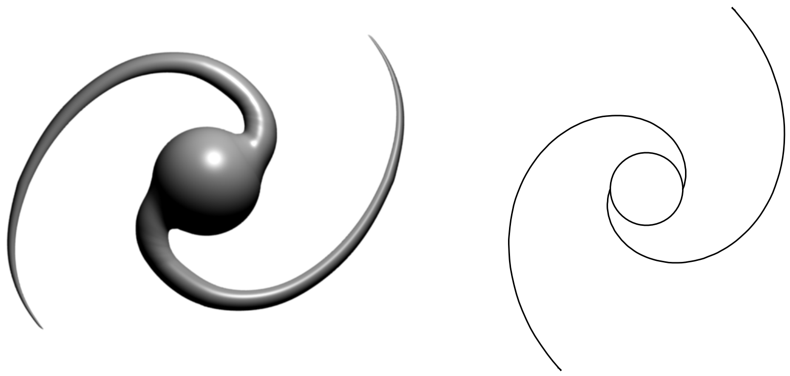

FIGURE 2. Galaxy and its skeleton.

Figure 2, which consists of a disk at the center and two arms both of which are a portion of the logarithmic spiral $r = e^{\beta(\theta-\theta_i)}$ $(i = 1, 2)$, where $\beta > 0$, $\theta_1$, $\theta_2$ are some constants and $r = \sqrt{x^2 + y^2}$. It is easily seen that $\Omega_0$ is admissible with

$$a_1 = 1\,, \quad a_2 = 0\,, \quad Z = \begin{bmatrix} 0 & -\beta^{-1} & 0 \\ \beta^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

**Example 2.11 (Gearwheels).** Here, $\Omega_0$ is a smooth 3-dimensional, but nearly flat, domain whose profile is as in Figure 3, with the origin 0 being the center of the inner circular hole, and with the $z$-axis perpendicular to this circle. The inner part of the boundary $\Gamma_0$ has positive mean curvature because of the large positive sectional curvature in the $z$ direction, compared with the small negative sectional curvature in the rotational direction around the $z$-axis. The outer part is strongly star-shaped with respect to 0. With an appropriate combination of the outer shape and the size of the sectional curvature around the inner part, we see that $\Omega_0$ is admissible with a suitable choice of $a_1 > 0$, $a_2 > 0$ and $Z = 0$.

**Example 2.12 (Non-admissible domain).** An example of non-admissible domain in dimension $N = 2$ is given in Figure 4. This domain is symmetric with respect to 0 and has two linear parts $A$ and $B$ which are aligned with the axes. Assuming that

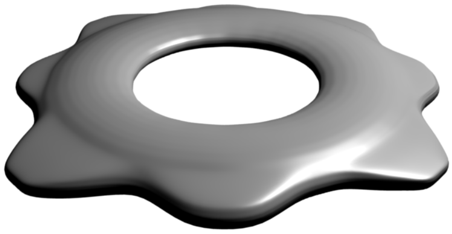

FIGURE 3. Gearwheel.

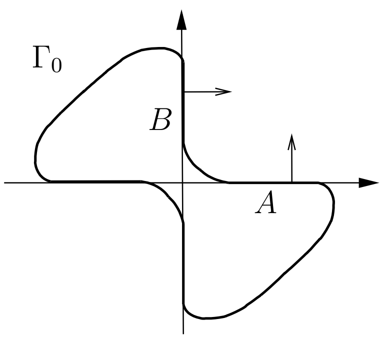


FIGURE 4. Domain not satisfying Definition 2.3.

(2.8) is satisfied for some $a_1 \geq 0$, $a_2 \geq 0$ and $Z = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}$, we would obtain, for $x \in A$, $-\beta < 0$ and, for $x \in B$, $\beta < 0$, which is impossible. Note that, for all $x_0$, this domain is also "not-admissible with respect to $x_0$", in the sense that it cannot satisfy (2.8) even if $-a_1 x + Zx$ is replaced by $-a_1(x - x_0) + Z(x - x_0)$. Indeed, in this case by symmetry with respect to 0 it would also satisfy the same inequality with $-x_0$ instead of $x_0$ and, adding both inequalities with $x_0$ and $-x_0$, we would see that it satisfies the original (2.8).

**Organization of the paper.** In Section 3, we recall the basic ideas of the level set approach together with some useful known properties. Section 4 is devoted to the construction of refined barriers (sub- and super-solutions) for the Allen-Cahn equation. By quoting a generation of interface result from [1] and using these barriers, we prove Theorem 2.1. In Section 5, we prove Theorem 2.5. In Appendix, we present some results on the *regularity of the generalized MMC* which, to the best of our knowledge, are not explicitly stated in the literature. They are related to our singular limit problem but are also interesting by themselves.

## 3. Generalized motion by mean curvature

For the convenience of the reader, we briefly recall here the level set approach which enables to define uniquely a generalized MMC. We also recall some useful properties of the associated signed distance function. For more details and proofs, we refer to Evans and Spruck [15], Chen, Giga and Goto [12], Evans, Soner and Souganidis [14] (from whom we borrow the notations) and the references therein.

Given a compact set $\Gamma_0 \subset \mathbb{R}^N$, we choose a continuous function $g : \mathbb{R}^N \to \mathbb{R}$, constant outside some ball and such that

$$\Gamma_0 = \{x \in \mathbb{R}^N : \, g(x) = 0\} \,. \tag{3.1}$$

Then, we consider the *mean curvature evolution partial differential equation* $v_t - \mathrm{tr}[(I - \widehat{Dv} \otimes \widehat{Dv}) D^2 v] = 0$ on $\mathbb{R}^N \times (0, \infty)$, which is nonlinear, degenerate and even undefined in the points where $Dv$ vanishes (recall $\widehat{p} := \frac{p}{|p|}$). Nevertheless, Problem

$$(P^0) \quad \begin{cases} v_t - \mathrm{tr}\left[(I - \widehat{Dv} \otimes \widehat{Dv}) D^2 v\right] = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ v = g & \text{in } \mathbb{R}^N \times \{t = 0\} \,, \end{cases}$$

admits a unique viscosity solution $v \in C(\mathbb{R}^N \times [0, \infty))$, constant outside some large enough ball, and each level set of $v$ evolves according to the mean curvature in a generalized sense. As far as viscosity solutions are concerned, we refer the reader to the *User's guide* of Crandall, Ishii and Lions [13] and the references therein.

Now, for each $t \geq 0$, we define the "interface at time $t$" by

$$\Gamma_t := \{x \in \mathbb{R}^N : \, v(x, t) = 0\} \,, \tag{3.2}$$

which is a compact set in $\mathbb{R}^N$. Then the collection $\{\Gamma_t\}_{t \geq 0}$ does not depend on the choice of the function $g$. The family $\{\Gamma_t\}_{t \geq 0}$ is called the *generalized motion by mean curvature* starting from $\Gamma_0$. The "space-time interface" $\Gamma$, which is defined in (2.10), can be expressed as $\Gamma = \cup_{t \geq 0} (\Gamma_t \times \{t\})$.

Assume moreover that $\Gamma_0$ is the boundary of a domain $\Omega_0 \subset \mathbb{R}^N$, and choose a continuous function $g$ such that

$$g(x) > 0 \;\text{ if } x \in \Omega_0 \,, \;\; g(x) < 0 \;\text{ if } x \in (\Omega_0 \cup \Gamma_0)^c \,. \tag{3.3}$$

If $v$ denotes the solution of $(P^0)$, we then define, for each $t \geq 0$, the "inside at time $t$" by

$$\Omega_t := \{x \in \mathbb{R}^N : \, v(x, t) > 0\} \,, \tag{3.4}$$

which is an open set in $\mathbb{R}^N$. We also define the "space-time inside" by $\Omega := \{(x, t) \in \mathbb{R}^N \times [0, \infty) : \, v(x, t) > 0\} = \cup_{t \geq 0} (\Omega_t \times \{t\})$.

For the generalized MMC, the following comparison principle is known to holds (see [15] or [3, Lemma 3.2]):

$$\gamma_0 << \widetilde{\gamma}_0 \implies (\omega_t \cup \gamma_t) \subset \widetilde{\omega}_t \,, \forall t \geq 0 \,. \tag{3.5}$$

Here $\{\gamma_t\}_{t\geq 0}$, respectively $\{\widetilde{\gamma}_t\}_{t\geq 0}$, denotes the generalized MMC starting from $\gamma_0$, respectively $\widetilde{\gamma}_0$, and $\omega_t$, respectively $\widetilde{\omega}_t$, denotes the "inside at time $t$" of $\gamma_t$, respectively $\widetilde{\gamma}_t$. As a consequence,

$$\gamma_0 << \widetilde{\gamma}_0 \implies (\omega \cup \gamma) \subset \widetilde{\omega}\,, \tag{3.6}$$

where $\omega$, respectively $\widetilde{\omega}$, denotes the "space-time inside" associated with $\gamma_t$, respectively $\widetilde{\gamma}_t$, and $\gamma$ the "space-time interface".

Next, let $t^*$ denote the extinction time, namely

$$t^* := \inf\{t > 0 : \Gamma_t = \emptyset\}\,. \tag{3.7}$$

Finally, we let $d(x,t)$ be the signed distance function to $\Gamma_t$, defined by

$$d(x,t) = \begin{cases} \operatorname{dist}(x,\Gamma_t) & \text{if } x \in \Omega_t \\ 0 & \text{if } x \in \Gamma_t \\ -\operatorname{dist}(x,\Gamma_t) & \text{if } x \in (\Omega_t \cup \Gamma_t)^c\,, \end{cases} \tag{3.8}$$

for all $x \in \mathbb{R}^N$, $0 \leq t \leq t^*$. Note that $d$ is well-defined at time $t = t^*$ (because of the continuity of $v$), and that $d$ may be not continuous in time (for instance, if $\Gamma_t$ is made of two pieces, one "disappearing" before the other).

As proved in [14], the signed distance function $d$ is a viscosity super-solution, sub-solution, of the heat equation in the set $\{d > 0\}$, $\{d < 0\}$ respectively.

**Lemma 3.1.** *We have, in the viscosity sense,*

$$d_t - \Delta d \geq 0 \quad \text{in } \Omega \cap (\mathbb{R}^N \times (0,t^*])\,, \tag{3.9}$$

$$d_t - \Delta d \leq 0 \quad \text{in } (\Omega \cup \Gamma)^c \cap (\mathbb{R}^N \times (0,t^*])\,. \tag{3.10}$$

## 4. Refined Allen-Cahn barriers

The goal of this section is to show that, for *any* generalized MMC $\{\gamma_t\}_{t\geq 0}$, there is a super-solution of $(P^\varepsilon)$ whose transition layer lies *inside* of $\gamma_t$ within distance of $\mathcal{O}(\varepsilon|\ln\varepsilon|)$, and a sub-solution of $(P^\varepsilon)$ whose transition layer lies *outside* of $\gamma_t$ within distance of $\mathcal{O}(\varepsilon|\ln\varepsilon|)$. The results are stated in the following propositions which are fundamental for our analysis. Combining these propositions with a generation of interface result proved in [1], we will prove Theorem 2.1 in subsection 4.3. Note that the constant $\lambda > 0$ which appears below is completely determined by the underlying travelling wave solution (see Lemma 4.4).

**Proposition 4.1 (Super-solutions).** *Let $(\gamma_0,\omega_0)$ be an arbitrary pair with $\gamma_0$ the boundary of the domain $\omega_0 \subset \mathbb{R}^N$. Denote by $\{\gamma_t\}_{t\geq 0}$, $\{\omega_t\}_{t\geq 0}$ the associated "interface at time $t$", "inside at time $t$" respectively. Denote by $d(x,t)$ the signed distance function to $\gamma_t$ (see Section 3). Fix $\zeta > 0$ arbitrarily small and $T > 0$ arbitrarily. Then, for all $\varepsilon > 0$ small enough, there is a function $w_\varepsilon^+(x,t)$ such that*

(i) *$w_\varepsilon^+$ is a viscosity super-solution of the Allen-Cahn equation on $\mathbb{R}^N \times (0,T]$*

(ii) $w_\varepsilon^+$ *has, for all* $t \geq 0$*, the following upper bounds:*

$$\begin{cases} w_\varepsilon^+(x,t) \leq \beta^* + \zeta & \text{for all } x \in \mathbb{R}^N \\ w_\varepsilon^+(x,t) \leq \alpha^* + \zeta & \text{for all } x \notin \omega_t \end{cases} \tag{4.1}$$

(iii) $w_\varepsilon^+(\cdot,0)$ *has the following lower bounds:*

$$\begin{cases} \alpha^* + \frac{\zeta}{3} \leq w_\varepsilon^+(x,0) & \text{for all } x \in \mathbb{R}^N \\ \beta^* + \frac{\zeta}{3} \leq w_\varepsilon^+(x,0) & \text{for all } x \text{ such that } d(x,0) \geq \frac{8}{\lambda}\varepsilon|\ln\varepsilon|\,. \end{cases} \tag{4.2}$$

**Proposition 4.2 (Sub-solutions).** *Let the notations of Proposition* 4.1 *hold. Fix* $\zeta > 0$ *arbitrarily small and* $T > 0$ *arbitrarily. Then, for all* $\varepsilon > 0$ *small enough, there is a function* $w_\varepsilon^-(x,t)$ *such that*

(i) $w_\varepsilon^-$ *is a viscosity sub-solution of the Allen-Cahn equation on* $\mathbb{R}^N \times (0,T]$
(ii) $w_\varepsilon^-$ *has, for all* $t \geq 0$*, the following lower bounds:*

$$\begin{cases} \alpha^* - \zeta \leq w_\varepsilon^-(x,t) & \text{for all } x \in \mathbb{R}^N \\ \beta^* - \zeta \leq w_\varepsilon^-(x,t) & \text{for all } x \in \omega_t \cup \gamma_t \end{cases} \tag{4.3}$$

(iii) $w_\varepsilon^-(\cdot,0)$ *has the following upper bounds:*

$$\begin{cases} w_\varepsilon^-(x,0) \leq \beta^* - \frac{\zeta}{3} & \text{for all } x \in \mathbb{R}^N \\ w_\varepsilon^-(x,0) \leq \alpha^* - \frac{\zeta}{3} & \text{for all } x \text{ such that } d(x,0) \leq -\frac{8}{\lambda}\varepsilon|\ln\varepsilon|\,. \end{cases} \tag{4.4}$$

One of the role of such a pair of sub- and super-solution shall be to control the solution $u^\varepsilon$ to $(P^\varepsilon)$ during the latter time range — after the generation of interface— when the motion of interface occurs. In the sequel we prove Proposition 4.1, the proof of Proposition 4.2 being similar. We begin with some preparations.

### 4.1. Some preliminaries

**A modified signed distance function.** Let $d$ be the signed distance function to an *arbitrary* generalized MMC. In order to construct super-solutions of $(P^\varepsilon)$ involving the signed distance function, it is necessary to cut-off $d$ in the set $\{d < 0\}$, where it is a sub-solution of the heat equation (Lemma 3.1). To that purpose, we slightly improve the cut-off argument used in [14].

In the following, $\theta(\varepsilon)$ is a positive function defined for $\varepsilon \in (0,\varepsilon_0)$, $\varepsilon_0 > 0$ small enough; its possible explicit forms will be indicated later. Consider a smooth auxiliary function $\eta = \eta^\varepsilon : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} \eta(z) = -\theta(\varepsilon) & \text{for all } -\infty < z \leq \frac{1}{4}\theta(\varepsilon)\,, \\ \eta(z) = z - \theta(\varepsilon) & \text{for all } z \geq \frac{1}{2}\theta(\varepsilon)\,, \\ 0 \leq \eta' \leq C \quad \text{and} \quad |\eta''| \leq C/\theta(\varepsilon)\,, \end{cases} \tag{4.5}$$

where $C$ is a constant independent of $\varepsilon$. Rather than $d$ we shall use $\eta(d)$, for the construction of our super-solutions.

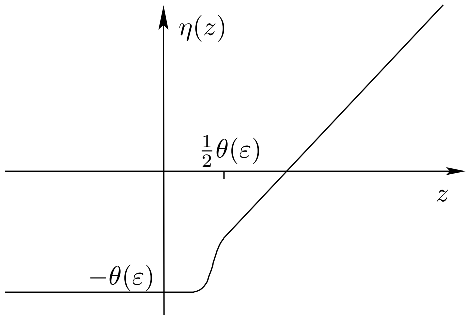


FIGURE 5. Graph of $\eta$.

From [14, Lemma 3.1], there is a constant $C > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$\eta(d)_t - \Delta\eta(d) \geq -\frac{C}{\theta(\varepsilon)} \quad \text{in } \mathbb{R}^N \times (0, t^*], \tag{4.6}$$

$$\eta(d)_t - \Delta\eta(d) \geq 0 \quad \text{in } \left\{d > \frac{1}{2}\theta(\varepsilon)\right\} \subset \mathbb{R}^N \times (0, t^*], \tag{4.7}$$

in the viscosity sense (which, in particular, contains the fact that $\eta(d)$ is lower semi continuous). For our results to hold after the extinction time, we need to extend $\eta(d(x,t))$ to all times. By abusing the notations slightly, we define

$$\eta(d(x,t)) = \begin{cases} \eta(d(x,t)) & \text{if } t \leq t^* \\ -\theta(\varepsilon) & \text{if } t > t^* . \end{cases} \tag{4.8}$$

Let us notice that, from the definition of $t^*$, $d(x, t^*) \leq 0$ for all $x \in \mathbb{R}^N$. But, as proved in [14], $d$ is continuous from below (with respect to time) on $\mathbb{R}^N \times (0, t^*]$; therefore there is a neighborhood of $(x, t^*)$ in $\mathbb{R}^N \times (0, t^*]$ on which $\eta(d) \equiv -\theta(\varepsilon)$. It is thus clear that (4.6)—(4.7) still hold for $t > t^*$, for the extension (4.8).

**Lemma 4.3.** *There is a constant $C > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\eta(d)_t - \Delta\eta(d) \geq -\frac{C}{\theta(\varepsilon)} \quad \textit{in } \mathbb{R}^N \times (0, \infty), \tag{4.9}$$

$$\eta(d)_t - \Delta\eta(d) \geq 0 \quad \textit{in } \left\{d > \frac{1}{2}\theta(\varepsilon)\right\} \subset \mathbb{R}^N \times (0, \infty). \tag{4.10}$$

**A standing wave.** We shall also need $U_0(z)$ the unique solution of the stationary problem

$$\begin{cases} {U_0}'' + f(U_0) = 0 \\ U_0(-\infty) = \alpha^* \quad U_0(0) = a \quad U_0(+\infty) = \beta^* . \end{cases} \tag{4.11}$$

This solution represents the first approximation of the profile of a transition layer around the interface observed in the stretched coordinates; it naturally arises when performing a formal asymptotic expansion of the solution (see [1] and the references therein). Note that the "balanced stability assumption", namely the integral condition (1.2), guarantees the existence of a solution of (4.11). In the simple case where $f(u) = u(1-u^2)$, we know that $U_0(z) = \tanh(z/\sqrt{2})$. In the general case, the following standard estimates hold.

**Lemma 4.4.** *There are positive constants $C$ and $\lambda$ such that*

$$\begin{aligned} 0 < \beta^* - U_0(z) \leq Ce^{-\lambda|z|} \quad & \textit{for } z \geq 0\,, \\ 0 < U_0(z) - \alpha^* \leq Ce^{-\lambda|z|} \quad & \textit{for } z \leq 0\,. \end{aligned}$$

*In addition, $U_0$ is a strictly increasing function and, for $j = 1, 2$,*

$$|D^j U_0(z)| \leq Ce^{-\lambda|z|} \quad \textit{for } z \in \mathbb{R}\,. \tag{4.12}$$

**4.2. Construction of super-solutions**

We look for super-solutions $w_\varepsilon^+$ for Problem $(P^\varepsilon)$ in the form

$$w_\varepsilon^+(x,t) = U_0\left(\frac{\eta(d(x,t)) + \varepsilon p(t)}{\varepsilon}\right) + q(t)\,, \tag{4.13}$$

for all $(x,t) \in \mathbb{R}^N \times [0,\infty)$, where

$$p(t) = -e^{-\beta t/\varepsilon^2} + e^{Lt} + K\,, \quad q(t) = \sigma\left(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}\right)\,, \tag{4.14}$$

and where $d$ is the signed distance function to an *arbitrary* generalized MMC $\{\gamma_t\}_{t\geq 0}$. Note that by $\eta(d(x,t))$ we understand the extension (4.8).

Let us first specify the choice of $\beta$ and $\sigma$ and give a useful inequality. Note that these choices are reminiscent of the ones in [1] where the convergence to a classical solution of the MMC is studied. By assumption (1.1), there are positive constants $b$, $m$ such that

$$f'(U_0(z)) \leq -m \quad \text{if} \quad U_0(z) \in [\alpha^*, \alpha^* + b] \cup [\beta^* - b, \beta^*]\,. \tag{4.15}$$

On the other hand, since the region $\{z \in \mathbb{R} : U_0(z) \in [\alpha^* + b, \beta^* - b]\,\}$ is compact and since ${U_0}' > 0$ on $\mathbb{R}$, there is a constant $\delta_1 > 0$ such that

$${U_0}'(z) \geq \delta_1 \quad \text{if} \quad U_0(z) \in [\alpha^* + b, \beta^* - b]\,. \tag{4.16}$$

We set

$$\beta := \frac{m}{4} \quad \text{and} \quad \sigma := \frac{\zeta}{2\beta}\,.$$

By reducing $\zeta > 0$ if necessary, we can assume $\sigma \leq \min(\sigma_0, \sigma_1, \sigma_2)$, where

$$\sigma_0 := \frac{\delta_1}{m + F_1} \qquad \sigma_1 := \frac{1}{\beta + 1} \qquad \sigma_2 := \frac{4\beta}{F_2(\beta+1)}$$

$$F_1 := \|f'\|_{L^\infty(\alpha^*,\beta^*)} \qquad F_2 := \|f''\|_{L^\infty(\alpha^*-1,\beta^*+1)}\,.$$

Combining (4.15) and (4.16), and considering that $\sigma \leq \sigma_0$, we obtain

$${U_0}'(z) - \sigma f'(U_0(z)) \geq \sigma m \quad \text{for} \quad -\infty < z < \infty\,. \tag{4.17}$$

**Proof of Proposition** 4.1 $(i)$**.** For ease of notation we here denote $w_\varepsilon^+$ by $w$. Let $K>1$ be arbitrary. What we shall prove is that, for all $\varepsilon\in(0,\varepsilon_0)$, the inequality

$$\mathcal{L}w := w_t - \Delta w - \frac{1}{\varepsilon^2}f(w) \geq 0 \quad \text{in } \mathbb{R}^N\times(0,T] \tag{4.18}$$

holds in the viscosity sense, provided that the constants $\varepsilon_0>0$ and $L>0$ are appropriately chosen. Note that the remaining freedom for the choice of $K>1$ is crucial for the proof of Proposition 4.1 $(iii)$.

We recall that $\alpha^*<U_0<\beta^*$ and go on under the following assumption

$$\varepsilon_0^2 L e^{LT} \leq 1\,. \tag{4.19}$$

Then, given any $\varepsilon\in(0,\varepsilon_0)$, since $\sigma\leq\sigma_1$, we have $0\leq q(t)\leq 1$, so that

$$\alpha^*\leq w(x,t)\leq \beta^*+1\,. \tag{4.20}$$

In order to prove (4.18), choose $\phi\in C^\infty(\mathbb{R}^N\times(0,\infty))$ such that

$$w-\phi \quad \text{has a minimum at } (x_0,t_0)\in\mathbb{R}^N\times(0,T]\,. \tag{4.21}$$

Subtracting if necessary a constant from $\phi$ we can assume that

$$w-\phi=0 \quad \text{at point } (x_0,t_0)\,. \tag{4.22}$$

What we have to prove is

$$\mathcal{L}\phi = \phi_t-\Delta\phi-\frac{1}{\varepsilon^2}f(\phi)\geq 0 \quad \text{at point } (x_0,t_0)\,, \tag{4.23}$$

for all $\varepsilon\in(0,\varepsilon_0)$, with $\varepsilon_0$ small enough, $L$ large enough, both independent on $\phi$. In view of (4.22), we have $\phi(x_0,t_0)-q(t_0)=U_0\left(\frac{\eta(d(x_0,t_0))+\varepsilon p(t_0)}{\varepsilon}\right)\in(\alpha^*,\beta^*)$, and one can define a smooth function $\psi$ in a neighborhood of $(x_0,t_0)$ by

$$\psi(x,t):=\varepsilon {U_0}^{-1}(\phi(x,t)-q(t))\,, \tag{4.24}$$

so that (4.21), (4.22) transfer to

$$\eta(d)-(\psi-\varepsilon p) \quad \text{has a zero minimum at } (x_0,t_0)\,. \tag{4.25}$$

It follows from Lemma 4.3 applied to test functions $\psi-\varepsilon p$ that

$$\psi_t-\Delta\psi\geq\varepsilon p_t-\frac{C}{\theta(\varepsilon)} \quad \text{at point } (x_0,t_0)\,, \tag{4.26}$$

$$\psi_t-\Delta\psi\geq\varepsilon p_t \quad \text{at point } (x_0,t_0) \text{ if } d(x_0,t_0)>\frac{1}{2}\theta(\varepsilon) \text{ holds}\,. \tag{4.27}$$

Using $\phi=U_0(\frac{\psi}{\varepsilon})+q$, we have the expansion $f(\phi)=f(U_0(\frac{\psi}{\varepsilon}))+qf'(U_0(\frac{\psi}{\varepsilon}))+\frac12 q^2 f''(\theta)$ for some $U_0<\theta<U_0+q$. In view of the ordinary differential equation (4.11), some straightforward computations yield, at point $(x_0,t_0)$,

$$\mathcal{L}\phi=E_1+E_2+E_3\,,$$

with

$$E_1 = -\frac{1}{\varepsilon^2} q \left( f'(U_0) + \frac{1}{2} q f''(\theta) \right) + {U_0}' p_t + q_t \,,$$

$$E_2 = \frac{{U_0}''}{\varepsilon^2}(1 - |\nabla \psi|^2) \quad \text{and} \quad E_3 = \frac{{U_0}'}{\varepsilon}(\psi_t - \Delta\psi - \varepsilon p_t)\,.$$

**The term $E_1$.** Plugging the expressions (4.14) for $p$ and $q$ in $E_1$, we obtain

$$E_1 = \frac{\beta}{\varepsilon^2}\, e^{-\beta t/\varepsilon^2}(I - \sigma\beta) + Le^{Lt}(I + \varepsilon^2 \sigma L)\,,$$

with

$$I := {U_0}' - \sigma f'(U_0) - \frac{\sigma^2}{2} f''(\theta)(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}) \geq \sigma m - \frac{\sigma^2}{2} F_2(\beta + \varepsilon^2 L e^{LT})\,,$$

where we have used (4.17) and (4.20). Combining this, (4.19) and the inequality $\sigma \leq \sigma_2$, we obtain $I \geq 2\sigma\beta$. Consequently, we have

$$E_1 \geq \frac{\sigma\beta^2}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + 2\sigma\beta L e^{Lt}\,.$$

**The term $E_2$.** First, assume $d(x_0,t_0) > \frac{1}{2}\theta(\varepsilon)$. From the definition of $\eta$, we have $\eta(d) = d - \theta(\varepsilon)$ in a neighborhood of $(x_0, t_0)$. Arguing as in the proof of [14, Theorem 2.2], we see that $|\nabla\psi(x_0,t_0)| = 1$ so that $E_2 = 0$.

Now assume $d(x_0,t_0) \leq \frac{1}{2}\theta(\varepsilon)$, which implies $\eta(d(x_0,t_0)) \leq -\frac{1}{2}\theta(\varepsilon)$. In view of statement (4.25) and the definition of $\eta$, we have $|\nabla\psi| \leq C$ at point $(x_0,t_0)$. We deduce from Lemma 4.4 that

$$|E_2| \leq \frac{C}{\varepsilon^2} e^{-\lambda|\eta(d)+\varepsilon p|/\varepsilon} \leq \frac{C}{\varepsilon^2} e^{-\lambda(\frac{1}{2}\theta(\varepsilon) - \varepsilon p)/\varepsilon}\,.$$

We remark that $0 < K - 1 \leq p \leq e^{LT} + K$. Consequently, if we assume

$$e^{LT} + K \leq \frac{\theta(\varepsilon)}{4\varepsilon}\,, \tag{4.28}$$

then

$$|E_2| \leq \frac{C}{\varepsilon^2} e^{-\lambda \frac{\theta(\varepsilon)}{4\varepsilon}}\,.$$

**The term $E_3$.** If $d(x_0,t_0) > \frac{1}{2}\theta(\varepsilon)$ it directly follows from (4.27) that $E_3 \geq 0$.

Now assume $d(x_0,t_0) \leq \frac{1}{2}\theta(\varepsilon)$, which implies $\eta(d)(x_0,t_0) \leq -\frac{1}{2}\theta(\varepsilon)$. It follows from (4.26) that

$$E_3 \geq -\frac{C}{\theta(\varepsilon)} \frac{{U_0}'}{\varepsilon}\,.$$

Using again Lemma 4.4 and arguing as above for the term $E_2$, we see that

$$E_3 \geq -\frac{C}{\theta(\varepsilon)} \frac{1}{\varepsilon} e^{-\lambda\frac{\theta(\varepsilon)}{4\varepsilon}}.$$

**Assumptions on $\theta(\varepsilon)$.** We now specify a possible choice for $\theta(\varepsilon)$. Assume that, as $\varepsilon \to 0$,

$$\theta(\varepsilon)|\ln\varepsilon| \leq C \quad \text{and} \quad \frac{1}{\varepsilon^2} e^{-\lambda\frac{\theta(\varepsilon)}{4\varepsilon}} \leq C\,, \tag{4.29}$$

for some constant $C > 0$; we remark that the latter assumption implies that $\frac{\theta(\varepsilon)}{\varepsilon} \to \infty$ and that $\frac{1}{\theta(\varepsilon)} \frac{1}{\varepsilon} e^{-\lambda \frac{\theta(\varepsilon)}{4\varepsilon}}$ is also bounded (so that $E_3$ is bounded from below). In the following we select

$$\theta(\varepsilon) = \frac{8}{\lambda} \varepsilon |\ln \varepsilon| \,, \tag{4.30}$$

so that (4.29) holds. As easily understood, the above possible choice is related to our improved estimate of the convergence rate of the Allen-Cahn equation to generalized MMC (see Section 2).

**Completion of the proof.** Collecting all these estimates gives

$$\mathcal{L}\phi \geq \frac{\sigma\beta^2}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + 2\sigma\beta L e^{Lt} - C \geq 2\sigma\beta L - C \,.$$

Now we set $L := \frac{1}{T} \ln \frac{\theta(\varepsilon_0)}{8\varepsilon_0}$. If $\varepsilon_0$ is chosen small enough, the assumptions on $\theta(\varepsilon)$ combined with the above choice for $L$ validate assumptions (4.19) and (4.28) and insure $\mathcal{L}\phi \geq 0$. The proof of Proposition 4.1 $(i)$ is now complete. □

**Proof of Proposition** 4.1 $(ii)$ **and** $(iii)$**.** For ease of notation we denote $w_\varepsilon^+$ by $w$.

In view of $\sigma\beta = \zeta/2$ and (4.19), we have, for $\varepsilon > 0$ small enough, $q(t) \leq \zeta$ for all $t \geq 0$. Hence $w(x,t) \leq \beta^* + \zeta$ holds true for all $x \in \mathbb{R}^N$. Next, choose $x \in \mathbb{R}^N$ such that $x \notin \omega_t$, that is $d(x,t) \leq 0$. In view of the graph of $\eta$ we then have $\eta(d(x,t)) = -\theta(\varepsilon) = -\frac{8}{\lambda}\varepsilon|\ln\varepsilon|$. Therefore, for $t \geq 0$, we have

$$\begin{aligned} w(x,t) &= U_0\left(-\tfrac{8}{\lambda}|\ln\varepsilon| + p(t)\right) + q(t) \\ &\leq U_0\left(-\tfrac{8}{\lambda}|\ln\varepsilon| + e^{LT} + K\right) + \sigma(\beta + \varepsilon^2 L e^{LT}) \,. \end{aligned}$$

Then it follows from $\sigma\beta = \zeta/2$ and from $U_0(-\infty) = \alpha^*$ that, for $\varepsilon > 0$ small enough (not depending on $x \notin \omega_t$), the inequality $w(x,t) \leq \alpha^* + \zeta$ holds true. The proof of Proposition 4.1 $(ii)$ is now complete.

We now prove $(iii)$. Since

$$w_\varepsilon^+(x,0) = U_0\left(\frac{\eta(d(x,0))}{\varepsilon} + K\right) + \frac{\zeta}{2} + \sigma\varepsilon^2 L \,, \tag{4.31}$$

it is immediate that $w_\varepsilon^+(x,0) \geq \alpha^* + \zeta/3$ for all $x \in \mathbb{R}^N$. Last, choose $K > 1$ large enough so that $U_0(K) \geq \beta^* - \frac{\zeta}{6}$. If $x$ is such that $d(x,0) \geq \frac{8}{\lambda}\varepsilon|\ln\varepsilon| = \theta(\varepsilon)$, the graph of $\eta$ shows that $\eta(d(x,0)) \geq 0$ so that $w_\varepsilon^+(x,0) \geq U_0(K) + \frac{\zeta}{2} \geq \beta^* + \frac{\zeta}{3}$. Proposition 4.1 $(iii)$ is proved. □

**Remark 4.5.** Our super-solutions $w_\varepsilon^+$ actually prove more than (4.1). $w_\varepsilon^+(x,t) \leq \alpha^* + \zeta$ is valid not only for $d(x,t) \leq 0$, but also for $d(x,t) \leq \frac{8}{\lambda}\varepsilon|\ln\varepsilon| - C\varepsilon$, with $C > 0$ large enough (because, in this case, $\eta(d(x,t)) \leq -C\varepsilon$). See Figure 7. □

### 4.3. Proof of Theorem 2.1

Let $(\omega_{\varepsilon,0}^+)_{\varepsilon>0}$ be any family of open sets satisfying (2.3), with

$$C_0 > \frac{8}{\lambda} \,, \tag{4.32}$$

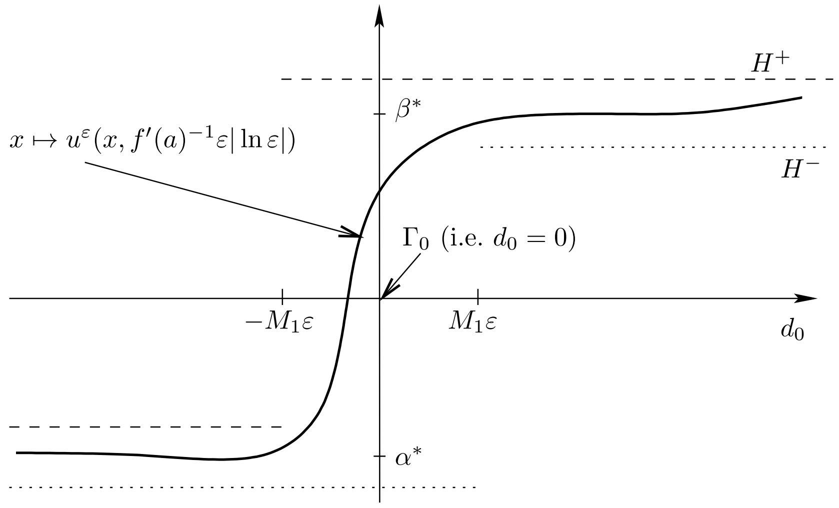


FIGURE 6. Prepared initial condition.

where $\lambda > 0$ is the constant that appears in Lemma 4.4. Fix $\zeta \in (0, \min(a - \alpha^*, \beta^* - a))$ arbitrarily. The strategy is the following. By quoting a generation of interface result from [1] and then using the super-solutions $w_\varepsilon^+$ associated with the pair $(\gamma_{\varepsilon,0}^+, \omega_{\varepsilon,0}^+) := (\partial\omega_{\varepsilon,0}^+, \omega_{\varepsilon,0}^+)$ (see Proposition 4.1), we will show that

$$\begin{cases} u^\varepsilon(x,t) \le \beta^* + \zeta & \text{for all } x \in \mathbb{R}^N \\ u^\varepsilon(x,t) \le \alpha^* + \zeta & \text{for all } x \notin \omega_{\varepsilon,t}^+ , \end{cases} \tag{4.33}$$

for all $t \ge t^\varepsilon$, with $t^\varepsilon$ the generation time that appears in (2.1). Since sub-solutions $w_\varepsilon^-$ can be used in an analogous way, this will be enough to prove the theorem.

The rapid formation of internal layers that takes place in a neighborhood of $\Gamma_0 = \{x \in \mathbb{R}^N : u_0(x) = a\}$ is studied in [1]: from an arbitrary initial data $u_0 \in C_b^2(\mathbb{R}^N)$ satisfying (1.3), an interface is fully developed at time $t^\varepsilon := f'(a)^{-1}\varepsilon^2|\ln\varepsilon|$. In particular, there is $M_1 > 0$ such that, for $\varepsilon > 0$ small enough,

$$H^-(x) \le u^\varepsilon(x, t^\varepsilon) \le H^+(x) , \tag{4.34}$$

for all $x \in \mathbb{R}^N$, where the functions $H^+(x), H^-(x)$ are given by

$$H^+(x) = \begin{cases} \beta^* + \frac{\zeta}{3} & \text{if } d_0(x) > -M_1\varepsilon \\ \alpha^* + \frac{\zeta}{3} & \text{if } d_0(x) \le -M_1\varepsilon \end{cases}$$
$$H^-(x) = \begin{cases} \beta^* - \frac{\zeta}{3} & \text{if } d_0(x) \ge M_1\varepsilon \\ \alpha^* - \frac{\zeta}{3} & \text{if } d_0(x) < M_1\varepsilon , \end{cases}$$

with $d_0(x) := d(x, 0)$ the signed distance function to $\Gamma_0$ (see Figure 6).

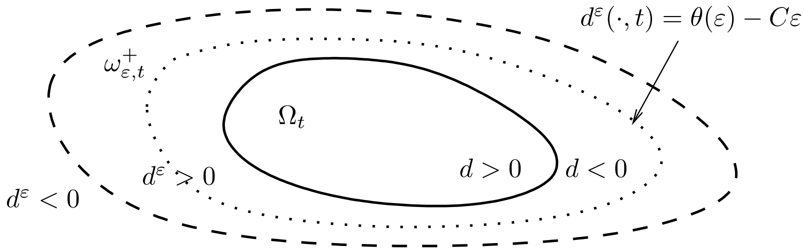


FIGURE 7. $\Omega_t$, $\omega^+_{\varepsilon,t}$ and related signed distances.

For $T>0$, we denote by $w^+_\varepsilon$ the super-solution associated with the pair $(\gamma^+_{\varepsilon,0},\omega^+_{\varepsilon,0})$ in the sense of Proposition 4.1. We denote by $d^\varepsilon$ the signed distance function associated with $\{\gamma^+_{\varepsilon,t}\}_{t\geq 0}$, the generalized MMC starting from $\gamma^+_{\varepsilon,0}:=\partial\omega^+_{\varepsilon,0}$ (see Figure 7).

We claim that, for $\varepsilon>0$ small enough,

$$H^+(x)\leq w^+_\varepsilon(x,0)\,, \tag{4.35}$$

for all $x\in\mathbb{R}^N$. In the range where $d_0(x)\leq -M_1\varepsilon$ this follows from (4.2). Now assume $d_0(x)>-M_1\varepsilon$. Since the constant $C_0$ which appears in (2.3) is such that $C_0>\frac{8}{\lambda}$, we see that $d^\varepsilon(x,0)\geq\theta(\varepsilon)=\frac{8}{\lambda}\varepsilon|\ln\varepsilon|$, for $\varepsilon>0$ small enough. Therefore, (4.2) implies (4.35).

From (4.35) and the comparison principle, we have

$$u^\varepsilon(x,t+t^\varepsilon)\leq w^+_\varepsilon(x,t)\quad \text{ for } 0\leq t\leq T-t^\varepsilon\,. \tag{4.36}$$

From this and (4.1) (with $\omega^+_{\varepsilon,t}$ playing the role of $\omega_t$) we immediately infer that, for all $T>0$, (4.33) is true on the time interval $[t^\varepsilon,T]$. If we choose $T>0$ large enough so that $\omega^+_{\varepsilon,t}=\emptyset$ for all $t\geq T$ (that is the generalized MMC starting from $\gamma^+_{\varepsilon,0}$ have become extinct), we see that $u^\varepsilon(x,T)\leq\alpha^*+\zeta$ for all $x\in\mathbb{R}^N$; the comparison principle then shows that this inequality persists for all $t\geq T$ and thus that (4.33) remains true on the time interval $[t^\varepsilon,\infty)$. □

## 5. Proof of Theorem 2.5

In this section, we prove Theorem 2.5. Assume $\Omega_0$ is admissible in the sense of Definition 2.3. Fix $\zeta>0$ arbitrarily small. After making two key observations in subsection 5.1, we split the proof into the lower bounds and the upper bounds appearing in Theorem 2.5.

### 5.1. Two key observations

A first observation is that the mean curvature evolution partial differential equation is invariant under time-shifts, dilations, rotations. More precisely if $v(x,t)$

solves

$$v_t = |Dv| \operatorname{div}\left(\frac{Dv}{|Dv|}\right), \tag{5.1}$$

so do $v(x, t+s)$ with $s \geq 0$, $v(\lambda x, \lambda^2 t)$ with $\lambda > 0$ and $v(Rx, t)$ with $R \in SO_n(\mathbb{R})$.

Next, for $\nu \geq 0$, define the invertible map $\Pi_\nu : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}$ by

$$\Pi_\nu((x,t)) := \left(e^{\nu Z} e^{-a_1 \nu} x, e^{-2a_1\nu}(t - a_2\nu)\right). \tag{5.2}$$

For a given compact set $K \subset \mathbb{R}^N \times \mathbb{R}$ it is obvious that there is $C_K > 0$ such that, for all $\nu \geq 0$,

$$\sup_K \|(\Pi_\nu - Id)\| \leq C_K \nu \quad \text{ and } \quad \sup_K \|({\Pi_\nu}^{-1} - Id)\| \leq C_K \nu. \tag{5.3}$$

**5.2. The lower bounds**

For $\varepsilon \geq 0$ small enough, we construct "inner approximations of $\Gamma_0$" by

$$\gamma^-_{\varepsilon,0} := \psi_{\varepsilon|\ln\varepsilon|}(\Gamma_0) = e^{\varepsilon|\ln\varepsilon| Z}\left[e^{-a_1\varepsilon|\ln\varepsilon|}\Phi_{a_2\varepsilon|\ln\varepsilon|}(\Gamma_0)\right], \tag{5.4}$$

see Remark 2.4. Here we remark that

$$\gamma^-_{\varepsilon,0} = P\left(\Pi_{\varepsilon|\ln\varepsilon|}(\Gamma_0)\right),$$

where $P : (x,t) \mapsto x$ is the projection from $\mathbb{R}^{N+1}$ onto $\mathbb{R}^N$. We then define $\omega^-_{\varepsilon,0}$ as the domain enclosed by $\gamma^-_{\varepsilon,0}$. From Remark 2.4, we deduce that

$$0 \leq \varepsilon < \varepsilon' \implies \gamma^-_{\varepsilon',0} << \gamma^-_{\varepsilon,0}, \tag{5.5}$$

$$\operatorname{dist}(\gamma^-_{\varepsilon,0}, \Gamma_0) \geq \widehat{C}_0 \varepsilon|\ln\varepsilon|, \tag{5.6}$$

for some constant $\widehat{C}_0 > 0$. By replacing if necessary $\psi_{\varepsilon|\ln\varepsilon|}(\Gamma_0)$ in (5.4) by $\psi_{C\varepsilon|\ln\varepsilon|}$ with $C >> 1$, we can assume $\widehat{C}_0 > \frac{8}{\lambda}$, so that (2.2) is satisfied with $C_0 = \widehat{C}_0$. Therefore the lower bounds in Theorem 2.1 hold.

Next, it follows from (5.5) and the comparison principle (3.6) that $(\omega^-_\varepsilon \cup \gamma^-_\varepsilon) \subset \Omega$. Since (see proof below)

$$d_{\mathcal{H}_{\mathbb{R}^{N+1}}}(\gamma^-_\varepsilon, \Gamma) \leq C\varepsilon|\ln\varepsilon|, \tag{5.7}$$

for some $C > 0$, it follows that

$$\Omega \setminus \mathcal{N}_{C\varepsilon|\ln\varepsilon|}(\Gamma) \subset (\omega^-_\varepsilon \cup \gamma^-_\varepsilon), \tag{5.8}$$

so that the lower bounds in Theorem 2.5 follow from the ones in Theorem 2.1. More precisely, if $(x,t) \in \Omega \setminus \mathcal{N}_{C\varepsilon|\ln\varepsilon|}(\Gamma)$, we deduce from (5.8) and the lower bounds in Theorem 2.1 that $u^\varepsilon(x,t) \geq \beta^* - \zeta$.

It remains to prove (5.7). We use the observations made in subsection 5.1. The function

$$v^-_\varepsilon(x,t) := v\left({\Pi_{\varepsilon|\ln\varepsilon|}}^{-1}(x,t)\right) \tag{5.9}$$

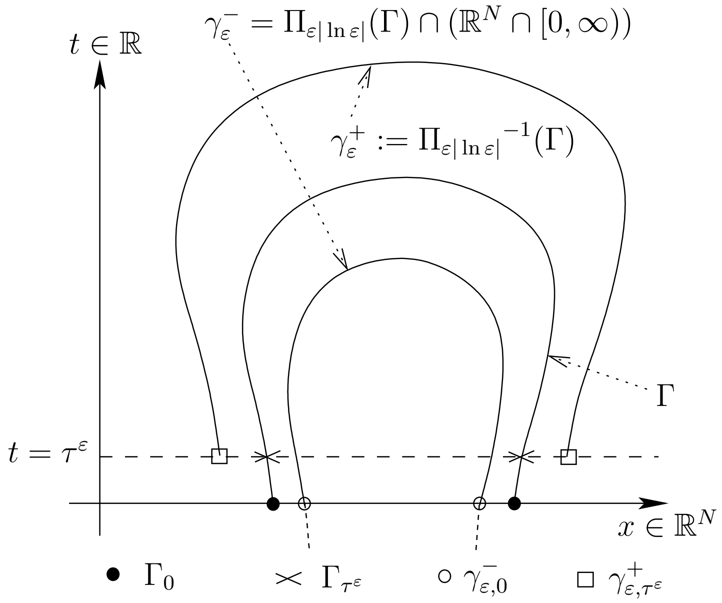


FIGURE 8. Sets used in the proof of Theorem 2.5.

solves (5.1), and $v_\varepsilon^-(x,0) = 0$ if and only if $x \in \gamma_{\varepsilon,0}^-$. Hence $v_\varepsilon^-$ "describes" the generalized MMC $\{\gamma_{\varepsilon,t}^-\}_{t\geq 0}$ starting from $\gamma_{\varepsilon,0}^-$. Therefore the "space-time interface" $\gamma_\varepsilon^-$ is given by

$$\gamma_\varepsilon^- = \{(x,t) \in \mathbb{R}^N \times [0,\infty) : \, v_\varepsilon^-(x,t) = 0\}\,.$$

In view of (5.9) this yields (see Figure 8)

$$\gamma_\varepsilon^- = \Pi_{\varepsilon|\ln\varepsilon|}(\Gamma) \cap (\mathbb{R}^N \times [0,\infty))\,. \tag{5.10}$$

Therefore (5.7) follows from (5.10) and (5.3).

**5.3. The upper bounds**

Since the evolution operator $\Phi_t : \Gamma_0 \mapsto \Gamma_t$ is not invertible, the argument for the upper bounds is more involved.

First, choose $\varepsilon > 0$ small enough so that the MMC starting from $\Gamma_0$ remains smooth on the time interval $[0, \tau^\varepsilon]$, where $\tau^\varepsilon := a_2\varepsilon|\ln\varepsilon|$. From the classical framework analysis [1, Theorem 1.3], there is $M > 0$ such that, for all $t^\varepsilon \leq t \leq \tau^\varepsilon$,

$$u^\varepsilon(x,t) \in \begin{cases} [\alpha^* - \frac{\zeta}{3}, \alpha^* + \frac{\zeta}{3}] & \text{if} \quad d(x,t) \leq -M\varepsilon \\ [\alpha^* - \frac{\zeta}{3}, \beta^* + \frac{\zeta}{3}] & \text{if} \quad -M\varepsilon < d(x,t) < M\varepsilon \\ [\beta^* - \frac{\zeta}{3}, \beta^* + \frac{\zeta}{3}] & \text{if} \quad d(x,t) \geq M\varepsilon\,, \end{cases} \tag{5.11}$$

with $d(x,t)$ the signed distance function to $\Gamma_t$ defined in (3.8). We recall that $t^\varepsilon := f'(a)^{-1}\varepsilon^2|\ln\varepsilon|$ denotes the generation time.

Next, using the map ${\Pi_{\varepsilon|\ln\varepsilon|}}^{-1}$, we define the space-time sets (see Figure 8) $\gamma_\varepsilon^+ := {\Pi_{\varepsilon|\ln\varepsilon|}}^{-1}(\Gamma)$ and $\omega_\varepsilon^+ := {\Pi_{\varepsilon|\ln\varepsilon|}}^{-1}(\Omega)$. From (5.3) we deduce that there is $C>0$ such that

$$d_{\mathcal{H}_{\mathbb{R}^{N+1}}}(\gamma_\varepsilon^+,\Gamma) \le C\varepsilon|\ln\varepsilon|\,. \tag{5.12}$$

Since the function

$$v_\varepsilon^+(x,t) := v\left(\Pi_{\varepsilon|\ln\varepsilon|}(x,t)\right) \tag{5.13}$$

solves (5.1) for $t \ge \tau^\varepsilon$, and $v_\varepsilon^+(x,\tau^\varepsilon)=0$ if and only if $x \in e^{-\varepsilon|\ln\varepsilon|Z}e^{a_1\varepsilon|\ln\varepsilon|}\Gamma_0$, the set $\gamma_\varepsilon^+$ is actually the "space-time interface" associated with the generalized MMC $\{\gamma_{\varepsilon,t}^+\}_{t\ge\tau^\varepsilon}$ starting from

$$\gamma_{\varepsilon,\tau^\varepsilon}^+ := e^{-\varepsilon|\ln\varepsilon|Z}e^{a_1\varepsilon|\ln\varepsilon|}\Gamma_0\,. \tag{5.14}$$

Hence Proposition 4.1 provides an Allen-Cahn super-solution $w_\varepsilon^+$ on $(\tau^\varepsilon, T]$ such that, for all $t \ge \tau^\varepsilon$,

$$w_\varepsilon^+(x,t) \le \alpha^* + \zeta \quad \text{for all } x \notin \omega_{\varepsilon,t}^+\,, \tag{5.15}$$

with $\omega_{\varepsilon,t}^+$ the "inside at time t" associated with $\gamma_{\varepsilon,t}^+$, and

$$\begin{cases} \alpha^* + \frac{\zeta}{3} \le w_\varepsilon^+(x,\tau^\varepsilon) & \text{for all } x \in \mathbb{R}^N \\ \beta^* + \frac{\zeta}{3} \le w_\varepsilon^+(x,\tau^\varepsilon) & \text{for all } x \text{ such that } d^\varepsilon(x,\tau^\varepsilon) \ge \frac{8}{\lambda}\varepsilon|\ln\varepsilon|\,, \end{cases} \tag{5.16}$$

with $d^\varepsilon(x,t)$ the signed distance function to $\gamma_{\varepsilon,t}^+$.

From (5.4) and (5.14) we have $e^{\varepsilon|\ln\varepsilon|Z}e^{-a_1\varepsilon|\ln\varepsilon|}\Gamma_{\tau^\varepsilon} = \gamma_{\varepsilon,0}^-$ and $\gamma_{\varepsilon,\tau^\varepsilon}^+ = [e^{\varepsilon|\ln\varepsilon|Z}e^{-a_1\varepsilon|\ln\varepsilon|}]^{-1}\Gamma_0$, and thus

$$\operatorname{dist}(\Gamma_{\tau^\varepsilon}\,,\,\gamma_{\varepsilon,\tau^\varepsilon}^+) \ge c_\varepsilon \operatorname{dist}(\gamma_{\varepsilon,0}^-\,,\,\Gamma_0)\,,$$

where $c_\varepsilon \to 1$, as $\varepsilon \to 0$. In view of (5.6) it follows that, for $\varepsilon>0$ small enough, $\operatorname{dist}(\Gamma_{\tau^\varepsilon}\,,\,\gamma_{\varepsilon,\tau^\varepsilon}^+) \ge \widetilde{C}_0\varepsilon|\ln\varepsilon|$, with $\widetilde{C}_0 > \frac{8}{\lambda}$. It follows that, for $\varepsilon>0$ small enough,

$$d(x,\tau^\varepsilon) \ge -M\varepsilon \implies d^\varepsilon(x,\tau^\varepsilon) \ge \frac{8}{\lambda}\varepsilon|\ln\varepsilon|\,.$$

Combining this with (5.11) and (5.16), we infer that $u^\varepsilon(x,\tau^\varepsilon) \le w_\varepsilon^+(x,\tau^\varepsilon)$ for all $x \in \mathbb{R}^N$. The comparison principle now implies

$$u^\varepsilon(x,\tau^\varepsilon + t) \le w_\varepsilon^+(x,\tau^\varepsilon + t)\,, \tag{5.17}$$

for all $x \in \mathbb{R}^N$, all $t \in [0, T-\tau^\varepsilon]$.

Finally, for $C>0$ as in (5.12), we take

$$(x,t) \in (\Omega\cup\Gamma)^c \setminus \mathcal{N}_{C\varepsilon|\ln\varepsilon|}(\Gamma)\,, \tag{5.18}$$

with $t \ge t^\varepsilon$, and prove that $u^\varepsilon(x,t) \le \alpha^* + \zeta$. If $t \ge \tau^\varepsilon$, we deduce from (5.18) and (5.12) that $x \notin \omega_{\varepsilon,t}^+$ so that conclusion follows from (5.15). If $t^\varepsilon \le t \le \tau^\varepsilon$, (5.18) shows that $d(x,t) \le -C\varepsilon|\ln\varepsilon| \le -M\varepsilon$ and the conclusion follows from (5.11).

## Appendix A. On the generic regularity of generalized MMC

Let $\Omega_0$ be a domain (=a bounded open set) in $\mathbb{R}^N$, whose boundary $\Gamma_0 := \partial\Omega_0$ is a smooth hypersurface without boundary. Let $\{\Gamma_t\}_{t\geq 0}$ be the generalized MMC starting from $\Gamma_0$, and let $\Omega_t$ denote the "inside at time $t$" as defined in (3.4). We denote by $\Gamma$ the "space-time interface" and by $\Omega$ the "space-time inside". Let us recall some classical definitions in the "generalized MMC literature" (see [7] for instance). We say that the motion is *regular from inside* if $\mathrm{Cl}_{\mathbb{R}^{N+1}}\,[\Omega] = \Omega \cup \Gamma$, and *regular* if

$$\mathrm{Cl}_{\mathbb{R}^{N+1}}\,[\Omega] = \Omega \cup \Gamma \quad \text{and} \quad \mathrm{Cl}_{\mathbb{R}^{N+1}}\,[(\Omega \cup \Gamma)^c] = (\Omega \cup \Gamma)^c \cup \Gamma\,. \tag{A.1}$$

It is clear that regularity (or regularity from inside) implies non fattening.

In this Appendix we state a result on the regularity, Proposition A.2, which does not seem to exist in the literature. We state without proof the following lemma which is well-known in general topology.

**Lemma A.1.** *Assume $S \subset (0,\infty)$ is an uncountable set. Then*

$$\exists a > 0\,, \forall b > a\,, S \cap [a,b) \text{ is uncountable}\,. \tag{A.2}$$

**Proposition A.2 (Generic regularity).** *Let $(\omega_{\nu,0})_{\nu>0}$ be a family of domains such that $\gamma_{\nu,0} := \partial\omega_{\nu,0}$ is a hypersurface without boundary. Assume that*

$$0 < \nu < \nu' \implies \mathrm{Cl}_{\mathbb{R}^N}\,[\omega_{\nu',0}] \subset \omega_{\nu,0}\,. \tag{A.3}$$

*For $\nu > 0$, let $\{\gamma_{\nu,t}\}_{t\geq 0}$ be the generalized MMC starting from $\gamma_{\nu,0}$, $\gamma_\nu$ and $\omega_\nu$ the associated "space-time interface" and "space-time inside". Then the sets*

$$\mathcal{J}^- := \{\nu > 0 :\, \mathrm{Cl}_{\mathbb{R}^{N+1}}\,[\omega_\nu] \neq \omega_\nu \cup \gamma_\nu\}$$
$$\mathcal{J}^+ := \{\nu > 0 :\, \mathrm{Cl}_{\mathbb{R}^{N+1}}\,[(\omega_\nu \cup \gamma_\nu)^c] \neq (\omega_\nu \cup \gamma_\nu)^c \cup \gamma_\nu\}$$

*are at most countable.*

**Proof.** We only prove the assertion for $\mathcal{J}^-$. First note that it follows from assumption (A.3) and the comparison principle (3.6) that

$$0 < \nu < \nu' \implies (\omega_{\nu'} \cup \gamma_{\nu'}) \subset \omega_\nu\,. \tag{A.4}$$

For $\nu > 0$, define $\delta_\nu := \sup_{(y,\tau)\in\gamma_\nu} \mathrm{dist}\,((y,\tau), \mathrm{Cl}_{\mathbb{R}^{N+1}}\,[\omega_\nu])$. Let us observe that $\mathrm{Cl}_{\mathbb{R}^{N+1}}\,[\omega_\nu] \subset (\omega_\nu \cup \gamma_\nu)$ and thus $\mathcal{J}^- = \{\nu > 0 :\, \delta_\nu > 0\}$. Assume by contradiction that $\mathcal{J}^-$ is uncountable. Then there is an integer $n_0$ such that the set $\mathcal{J}_0^- := \{\nu > 0 :\, \delta_\nu \geq \frac{1}{n_0}\}$ is uncountable. From Lemma A.1, there is $\nu^* > 0$ such that, for all $\nu > \nu^*$, the set $\mathcal{J}_0^- \cap [\nu^*, \nu)$ is uncountable. Therefore we can construct a decreasing sequence $(\nu_n)$ of elements of $\mathcal{J}_0^-$ which tends to $\nu^*$. From the definition of $\mathcal{J}_0^-$, we deduce the existence of $(x_n, t_n) \in \gamma_{\nu_n}$ such that

$$\mathrm{dist}\,((x_n,t_n), \mathrm{Cl}_{\mathbb{R}^{N+1}}\,[\omega_{\nu_n}]) \geq \frac{1}{n_0}\,. \tag{A.5}$$

On the one hand, if $j < k$, the decreasing of the sequence $(\nu_n)$ and (A.4) imply that $(x_j, t_j) \in \omega_{\nu_k}$. Therefore (A.5) yields $d\,((x_k,t_k),(x_j,t_j)) \geq \frac{1}{n_0}$. But, on the other hand, (A.4) implies that $(x_n, t_n) \in \omega_{\nu^*}$ so that we can extract a convergent subsequence of $(x_n, t_n)$. This is a contradiction. □

It is known that, for an admissible initial domain $\Omega_0$, the evolution $\{\Gamma_t\}_{t\geq 0}$ starting from $\Gamma_0 = \partial\Omega_0$ is regular from inside [16, Corollary 4.5.11]. Proposition A.2 provides a simple proof of the regularity both from inside and from outside.

**Corollary A.3.** *Let $\Omega_0$ be an admissible domain in the sense of Definition* 2.3. *Then the generalized MMC starting from $\Gamma_0 := \partial\Omega_0$ is regular.*

**Proof.** For $0 \leq \nu \leq \nu_0$, define, as in Remark 2.4,

$$\gamma_{\nu,0} := \psi_\nu(\Gamma_0) = e^{\nu Z}\left[e^{-a_1\nu}\Phi_{a_2\nu}(\Gamma_0)\right],$$

and denote by $\omega_{\nu,0}$ the domain enclosed by $\gamma_{\nu,0}$. It is clear from (2.9) that the family of domains $(\omega_{\nu,0})_{0<\nu<\nu_0}$ satisfies assumption (A.3) of Proposition A.2. Therefore, for almost all $\nu \in (0,\nu_0)$, the generalized MMC $\{\gamma_{\nu,t}\}_{t\geq 0}$ starting from $\gamma_{\nu,0}$ is regular, that is,

$$\mathrm{Cl}_{\mathbb{R}^{N+1}}\left[\omega_\nu\right] = \omega_\nu \cup \gamma_\nu \quad \text{and} \quad \mathrm{Cl}_{\mathbb{R}^{N+1}}\left[(\omega_\nu \cup \gamma_\nu)^c\right] = (\omega_\nu \cup \gamma_\nu)^c \cup \gamma_\nu\,. \tag{A.6}$$

Since $\omega_\nu = \Pi_\nu(\Omega) \cap (\mathbb{R}^N \times [0,\infty)) = \Pi_\nu\left(\Omega \cap (\mathbb{R}^N \times [a_2\nu,\infty))\right)$, with $\Pi_\nu$ as in (5.2), we have $\Omega \cap (\mathbb{R}^N \times [a_2\nu,\infty)) = {\Pi_\nu}^{-1}(\omega_\nu)$. Since $\Pi_\nu$ is a homeomorphism on $\mathbb{R}^{N+1}$, we see from (A.6) that, for almost all $\nu \in (0,\nu_0)$,

$$\mathrm{Cl}_{\mathbb{R}^{N+1}}\left[\Omega \cap (\mathbb{R}^N \times [a_2\nu,\infty))\right] = (\Omega \cup \Gamma) \cap (\mathbb{R}^N \times [a_2\nu,\infty))\,.$$

Letting $\nu \to 0$, we obtain

$$(\Omega \cup \Gamma) \cap (\mathbb{R}^N \times (0,\infty)) \subset \mathrm{Cl}_{\mathbb{R}^{N+1}}\left[\Omega \cap (\mathbb{R}^N \times (0,\infty))\right].$$

Combining this and $\mathrm{Cl}_{\mathbb{R}^N}\left[\Omega_0\right] = \Omega_0 \cup \Gamma_0$, we obtain $\mathrm{Cl}_{\mathbb{R}^{N+1}}\left[\Omega\right] = \Omega \cup \Gamma$. Similarly, we obtain $\mathrm{Cl}_{\mathbb{R}^{N+1}}\left[(\Omega \cup \Gamma)^c\right] = (\Omega \cup \Gamma)^c \cup \Gamma$. □

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