Gradient Schemes for an Obstacle Problem

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Abstract The aim of this work is to adapt the gradient schemes, discretisations of weak variational formulations using independent approximations of functions and gradients, to obstacle problems modelled by linear and non-linear elliptic variational inequalities. It is highlighted in this paper that four properties which are coercivity, consistency, limit conformity and compactness are adequate to ensure the convergence of this scheme. Under some suitable assumptions, the error estimate for linear equations is also investigated.

Key words: Elliptic variational inequalities, Gradient scheme, An Obstacle Problem, Convergence and analysis.

1 Introduction

We are interested in obstacle problems formulated as linear and non-linear elliptic variational inequalities and their approximate solutions obtained by gradient schemes. In what follows, Ω is an open bounded subset of $\mathbb{R}^d$. The problem we consider is

$$(-\mathrm{div}(\Lambda(x,\bar{u})\nabla\bar{u}) - f(x))(g(x)-\bar{u}(x)) = 0, \quad x \in \Omega, \tag{1a}$$

$$\bar{u}(x) \le g(x), \quad x \in \Omega, \tag{1b}$$

$$\mathrm{div}(\Lambda(x,\bar{u})\nabla\bar{u}) + f(x) \ge 0, \quad x \in \Omega, \tag{1c}$$

$$\bar{u}(x) = 0, \quad x \in \partial\Omega, \tag{1d}$$

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under the following assumptions:

$$\begin{gathered}\Lambda \text{ is a Caratheodory function from } \Omega\times\mathbb{R} \text{ to } S_d(\mathbb{R})\\ \text{(the set of } d\times d \text{ symmetric matrices) such that,}\\ \text{for a.e. } x\in\Omega \text{ and all } s\in\mathbb{R},\ \Lambda(x,s) \text{ has eigenvalues in } (\overline{\lambda},\underline{\lambda})\subset(0,+\infty),\end{gathered} \tag{2a}$$

$$f\in L^2(\Omega),\ g\in H^1(\Omega) \text{ and } \gamma(g)\geq 0 \text{ on } \partial\Omega. \tag{2b}$$

Under these assumptions, the weak formulation of Problem (1) is written

$$\begin{aligned}&\text{Find } \bar{u}\in \mathrm{K}=\{v\in H_0^1(\Omega)\ :\ v\leq g \text{ in } \Omega\} \text{ such that, } \forall v\in \mathrm{K},\\ &\int_\Omega \Lambda(x,\bar{u})\nabla\bar{u}(x)\cdot\nabla(\bar{u}(x)-v(x))\mathrm{d}x \ \leq\ \int_\Omega f(x)(\bar{u}(x)-v(x))\mathrm{d}x.\end{aligned} \tag{3}$$

Note that K is a non-empty set since $v=\min(0,g)\in K$.

Variational inequalities with different boundary conditions have been employed to model several physical problems, such as lubrication phenomena and seepage of liquid in porous media (see [7] and references therein). Mathematical theories associated to existence, uniqueness and stability of the solution of obstacle problems have been extensively developed (see [4, 9], for example). From the numerical perspective, Herbin and Marchand [8] showed that if $\Lambda\equiv I_d$ the solution of the 2-points finite volume scheme converges in $L^2(\Omega)$ to the unique solution as the size mesh tends to zero. Under H^2 regularity conditions on the exact solution they provide $\mathscr{O}(h)$ error estimate. This 2-points finite volume method, however, requires grids to satisfy a strong orthogonality assumption. Under a number of assumptions, Falk [6] underlines that the convergence estimate of finite elements method is of order h. Both schemes are only applicable for $\Lambda\equiv I_d$ in Problem (1).

Our goal in this paper is to use gradient schemes to construct a general formulation of several discretisations of Problem (3). The gradient scheme has been developed analyse the convergence of numerical methods for diffusion equations (see [3, 5]). Furthermore, Droniou et al. [3] noticed that this framework contains various methods such as Galerkin and some MPFA schemes.

This paper is arranged as follows. In Section 2, we present the definitions of some concepts, which are necessary to construct gradient schemes and to prove their convergence. In Section 3, we give an error estimate and a convergence proof in the linear case. Since we deal here with nonconforming schemes, the technique used in [6] is not useful to obtain error estimates. Although we use a similar technique as in [5], dealing with variational inequalities in this nonconforming setting requires us to establish new preliminary estimates, which modify the final error estimate. Finally, Section 4 is devoted to prove a convergence result for non-linear equations. Numerical experiments will be the purpose of a future work.

2 Gradient discretisation and gradient schemes

Gradient schemes are based on gradient discretisations, which consist of discrete spaces and mappings, and provide a general formulation of different numerical methods. Except for the definition of consistency, the definitions presented here are the same as in [3].

Definition 1. A gradient discretisation $\mathscr{D}$ for homogeneous Dirichlet boundary conditions is defined by a triplet $\mathscr{D} = (X_{\mathscr{D},0}, \Pi_{\mathscr{D}}, \nabla_{\mathscr{D}})$, where

1. the set of discrete unknowns $X_{\mathscr{D},0}$ is a finite dimensional vector space of $\mathbb{R}$,
2. the linear mapping $\Pi_{\mathscr{D}} : X_{\mathscr{D},0} \longrightarrow L^2(\Omega)$ gives the reconstructed function,
3. the linear mapping $\nabla_{\mathscr{D}} : X_{\mathscr{D},0} \longrightarrow L^2(\Omega)^d$ gives a reconstructed discrete gradient, which must be defined such that $\|\cdot\|_{\mathscr{D}} := \|\nabla_{\mathscr{D}} \cdot\|_{L^2(\Omega)^d}$ is a norm on $X_{\mathscr{D},0}$.

Throughout this paper, $\mathscr{D}$ is a gradient discretisation in the sense of Definition 1. The gradient scheme associated to $\mathscr{D}$ for Problem (3) is given by

$$\begin{aligned}&\text{Find } u \in \mathrm{K}_{\mathscr{D}} = \{v \in X_{\mathscr{D},0} \;:\; \Pi_{\mathscr{D}} v \le g \text{ in } \Omega\} \text{ such that, } \forall v \in \mathrm{K}_{\mathscr{D}},\\ &\int_\Omega \Lambda(x, \Pi_{\mathscr{D}} u(x)) \nabla_{\mathscr{D}} u(x) \cdot \nabla_{\mathscr{D}}(u(x) - v(x))\mathrm{d}x \;\le\; \int_\Omega f(x) \Pi_{\mathscr{D}}(u(x) - v(x))\mathrm{d}x.\end{aligned} \tag{4}$$

Definition 2 (Coercivity, consistency, limit-conformity and compactness). Let $C_{\mathscr{D}}$ be the norm of linear mapping $\Pi_{\mathscr{D}}$, defined by

$$C_{\mathscr{D}} = \max_{v \in X_{\mathscr{D},0} \setminus \{0\}} \frac{\|\Pi_{\mathscr{D}} v\|_{L^2(\Omega)}}{\|\nabla_{\mathscr{D}} v\|_{L^2(\Omega)^d}}. \tag{5}$$

A sequence $(\mathscr{D}_m)_{m\in\mathbb{N}}$ is called *coercive* if there exits $C_P \in \mathbb{R}_+$ such that $C_{\mathscr{D}_m} \le C_P$ for all $m \in \mathbb{N}$.

We say that a sequence $(\mathscr{D}_m)_{m\in\mathbb{N}}$ is *consistent* if, for all $\varphi \in K$, $\lim_{m\to\infty} S_{\mathscr{D}_m}(\varphi) = 0$, where $S_{\mathscr{D}} : K \longrightarrow [0, +\infty)$ is defined by

$$\forall \varphi \in K,\; S_{\mathscr{D}}(\varphi) = \min_{v \in K_{\mathscr{D}}} (\|\Pi_{\mathscr{D}} v - \varphi\|_{L^2(\Omega)} + \|\nabla_{\mathscr{D}} v - \nabla\varphi\|_{L^2(\Omega)^d}). \tag{6}$$

A sequence $(\mathscr{D}_m)_{m\in\mathbb{N}}$ is called *limit-conforming* if $\lim_{m\to\infty} W_{\mathscr{D}_m}(\varphi) = 0$ for all $\varphi \in H_{\mathrm{div}}(\Omega)$, where $W_{\mathscr{D}} : H_{\mathrm{div}}(\Omega) \longrightarrow [0, +\infty)$ is defined by

$$\forall \varphi \in H_{\mathrm{div}}(\Omega),\; W(\varphi) = \sup_{v \in X_{\mathscr{D},0} \setminus \{0\}} \frac{\left| \int_\Omega (\nabla_{\mathscr{D}} v \cdot \varphi + \Pi_{\mathscr{D}} v \cdot \mathrm{div}(\varphi))\mathrm{d}x \right|}{\|\nabla_{\mathscr{D}} v\|_{L^2(\Omega)^d}}. \tag{7}$$

A sequence $(\mathscr{D}_m)_{m\in\mathbb{N}}$ is called *compact* if, for any sequence $(u_m)_{m\in\mathbb{N}}$ with $u_m \in \mathrm{K}_{\mathscr{D}_m}$ and such that $(\|u_m\|_{\mathscr{D}_m})_{m\in\mathbb{N}}$ is bounded, the sequence $(\|\Pi_{\mathscr{D}_m} u_m\|_{L^2(\Omega)})_{m\in\mathbb{N}}$ is relatively compact in $L^2(\Omega)$.

3 Convergence and Error Estimate in the Linear Case

We consider here $\Lambda(x,u)=\Lambda(x)$. Based on the previous properties, we give an error estimate that requires $\mathrm{div}(\Lambda\nabla\bar{u})\in L^2(\Omega)$. We note that Brezis and Stampacchia [1] establish an H^2 regularity result on $\bar{u}$ under proper assumption on the data. If we further assume that Λ is Lipschitz-continuous, then $\mathrm{div}(\Lambda\nabla\bar{u})\in L^2(\Omega)$.
In what follows, we define the interpolant $P_{\mathscr{D}}: K\longrightarrow K_{\mathscr{D}}$ as follows

$$P_{\mathscr{D}}\varphi=\arg\min_{v\in K_{\mathscr{D}}}(\|\Pi_{\mathscr{D}}v-\varphi\|_{L^2(\Omega)}+\|\nabla_{\mathscr{D}}v-\nabla\varphi\|_{L^2(\Omega)^d}). \tag{8}$$

Theorem 1. (Error estimate) *Under Assumptions* (2)*, let $\bar{u}\in K$ be the solution to Problem (1) and let $D=\{x\in\Omega:\ \bar{u}(x)=g(x)\text{ in }\Omega\}$. If we assume that $\mathscr{D}$ is a gradient discretisation and $K_{\mathscr{D}}$ is a non-empty set, then there exists a unique solution $u\in K_{\mathscr{D}}$ to the gradient scheme (4). Moreover, if* $\mathrm{div}(\Lambda\nabla\bar{u})\in L^2(\Omega)$ *then this solution satisfies the following inequalities:*

$$\|\nabla_{\mathscr{D}}u-\nabla\bar{u}\|_{L^2(\Omega)^d}\leq\sqrt{\frac{2}{\underline{\lambda}}E_{\mathscr{D}}(\bar{u})+\frac{1}{\underline{\lambda}^2}[W_{\mathscr{D}}(\Lambda\nabla\bar{u})+\overline{\lambda}S_{\mathscr{D}}(\bar{u})]^2}+S_{\mathscr{D}}(\bar{u}), \tag{9}$$

$$\|\Pi_{\mathscr{D}}u-\bar{u}\|_{L^2(\Omega)}\leq C_{\mathscr{D}}\sqrt{\frac{2}{\underline{\lambda}}E_{\mathscr{D}}(\bar{u})+\frac{1}{\underline{\lambda}^2}[W_{\mathscr{D}}(\Lambda\nabla\bar{u})+\overline{\lambda}S_{\mathscr{D}}(\bar{u})]^2}+S_{\mathscr{D}}(\bar{u}), \tag{10}$$

in which $E_{\mathscr{D}}(\bar{u})=\int_D(\mathrm{div}(\Lambda\nabla\bar{u})+f)(\bar{u}-\Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}))\mathrm{d}x.$

Remark 1. Note that $|E_{\mathscr{D}}(\bar{u})|\leq\|\mathrm{div}(\Lambda\nabla\bar{u})+f\|_{L^2(\Omega)}\|\bar{u}-\Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u})\|_{L^2(\Omega)}$.

Proof. The techniques used in [5] and [7] will be followed in this proof.
Since $K_{\mathscr{D}}$ is a closed convex set, we can apply Stampacchia's theorem which states that there exists a unique solution to Problem (4).
Under the assumption that $\mathrm{div}(\Lambda\nabla\bar{u})\in L^2(\Omega)$, we note that $\Lambda\nabla\bar{u}\in H_{\mathrm{div}}(\Omega)$. For any $v\in X_{\mathscr{D},0}$, replacing φ with $\Lambda\nabla\bar{u}$ in the definition of limit conformity (7) therefore implies

$$\int_\Omega\nabla_{\mathscr{D}}v\cdot\Lambda\nabla\bar{u}\mathrm{d}x+\int_\Omega\Pi_{\mathscr{D}}v\cdot\mathrm{div}(\Lambda\nabla\bar{u})\mathrm{d}x\leq\|\nabla_{\mathscr{D}}v\|_{L^2(\Omega)^d}W_{\mathscr{D}}(\Lambda\nabla\bar{u}). \tag{11}$$

It is obvious that

$$\begin{aligned}\int_\Omega\Pi_{\mathscr{D}}(u-P_{\mathscr{D}}\bar{u})\mathrm{div}(\Lambda\nabla\bar{u})\mathrm{d}x=&\int_\Omega(\Pi_{\mathscr{D}}u-g)(\mathrm{div}(\Lambda\nabla\bar{u})+f)\mathrm{d}x\\&+\int_\Omega(g-\Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}))(\mathrm{div}(\Lambda\nabla\bar{u})+f)\mathrm{d}x\\&-\int_\Omega(\Pi_{\mathscr{D}}u-\Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}))f\mathrm{d}x.\end{aligned}$$

Using (1) and $u\in K_{\mathscr{D}}$, we obtain $\int_\Omega(\Pi_{\mathscr{D}}u-g)(\mathrm{div}(\Lambda\nabla\bar{u})+f)\mathrm{d}x\leq 0$, so that

$$\begin{aligned}\int_\Omega \Pi_{\mathscr{D}}(u-P_{\mathscr{D}}\bar{u})\mathrm{div}(\Lambda\nabla\bar{u})\mathrm{d}x \leq& \int_\Omega (g-\Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}))(\mathrm{div}(\Lambda\nabla\bar{u})+f)\mathrm{d}x\\ &-\int_\Omega (\Pi_{\mathscr{D}}u-\Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}))f\mathrm{d}x\\ =& \int_\Omega (g-\bar{u})(\mathrm{div}(\Lambda\nabla\bar{u})+f)\mathrm{d}x\\ &+\int_\Omega (\bar{u}-\Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}))(\mathrm{div}(\Lambda\nabla\bar{u})+f)\mathrm{d}x\\ &-\int_\Omega (\Pi_{\mathscr{D}}u-\Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}))f\mathrm{d}x.\end{aligned}$$

It follows, since $\bar{u}$ is the solution to Problem (1),

$$\begin{aligned}\int_\Omega \Pi_{\mathscr{D}}(u-P_{\mathscr{D}}\bar{u})\mathrm{div}(\Lambda\nabla\bar{u})\mathrm{d}x \leq& \int_\Omega (\bar{u}-\Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}))(\mathrm{div}(\Lambda\nabla\bar{u})+f)\mathrm{d}x\\ &-\int_\Omega (\Pi_{\mathscr{D}}u-\Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}))f\mathrm{d}x.\end{aligned}$$

Because $\mathrm{div}(\Lambda\nabla\bar{u})+f=0$ in $\Omega\setminus D$, the above inequality becomes

$$\begin{aligned}\int_\Omega \Pi_{\mathscr{D}}(u-P_{\mathscr{D}}\bar{u})\mathrm{div}(\Lambda\nabla\bar{u})\mathrm{d}x \leq& \int_D (\bar{u}-\Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}))(\mathrm{div}(\Lambda\nabla\bar{u})+f)\mathrm{d}x\\ &-\int_\Omega (\Pi_{\mathscr{D}}u-\Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}))f\mathrm{d}x.\end{aligned}$$

Using the definition of $E_{\mathscr{D}}(\bar{u})$, one has

$$\int_\Omega \Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}-u)\mathrm{div}(\Lambda\nabla\bar{u})\mathrm{d}x \geq -E_{\mathscr{D}}(\bar{u})-\int_\Omega \Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}-u)f\mathrm{d}x.$$

From this inequality and setting $v=P_{\mathscr{D}}\bar{u}-u\in X_{\mathscr{D},0}$ in (11), we obtain

$$\begin{aligned}\int_\Omega \nabla_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}-u)\cdot\Lambda\nabla\bar{u}\mathrm{d}x-\int_\Omega f\Pi_{\mathscr{D}}\ (P_{\mathscr{D}}\bar{u}-u)\mathrm{d}x \leq& E_{\mathscr{D}}(\bar{u})\\ &+\|\nabla_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}-u)\|_{L^2(\Omega)^d}W_{\mathscr{D}}(\Lambda\nabla\bar{u}).\end{aligned}$$

Since u is the solution to Problem (4), we get

$$\int_\Omega \Lambda\nabla_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}-u)[\nabla\bar{u}-\nabla_{\mathscr{D}}u]\mathrm{d}x \leq \|\nabla_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}-u)\|_{L^2(\Omega)^d}W_{\mathscr{D}}(\Lambda\nabla\bar{u})+E_{\mathscr{D}}(\bar{u})$$

and, thanks to the definition of $P_{\mathscr{D}}$, we obtain

$$\begin{aligned}&\underline{\lambda}\|\nabla_{\mathscr{D}}(P_{\mathscr{D}}\bar{u})-\nabla_{\mathscr{D}}u\|^2_{L^2(\Omega)^d}\\ &\qquad\leq \|\nabla_{\mathscr{D}}(P_{\mathscr{D}}\bar{u})-\nabla_{\mathscr{D}}u\|_{L^2(\Omega)^d}[W_{\mathscr{D}}(\Lambda\nabla\bar{u})+\overline{\lambda}S_{\mathscr{D}}(\bar{u})]+E_{\mathscr{D}}(\bar{u}).\end{aligned}$$

Applying Young's inequality leads to

$$\|\nabla_{\mathscr{D}}(P_{\mathscr{D}}\bar{u})-\nabla_{\mathscr{D}}u)\| \leq \sqrt{\frac{2}{\underline{\lambda}}E_{\mathscr{D}}(\bar{u})+\frac{1}{\underline{\lambda}^2}[W_{\mathscr{D}}(\Lambda\nabla\bar{u})+\overline{\lambda}S_{\mathscr{D}}(\bar{u})]^2}$$

and, from $\|\nabla_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}) - \nabla\bar{u}\| \le S_{\mathscr{D}}(\bar{u})$, Estimate (9) is achieved. Using (5), we obtain

$$\|\Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u} - u)\| \le C_{\mathscr{D}}\sqrt{\frac{2}{\underline{\lambda}}E_{\mathscr{D}}(\bar{u}) + \frac{1}{\underline{\lambda}^2}[W_{\mathscr{D}}(\Lambda\nabla\bar{u}) + \overline{\lambda}S_{\mathscr{D}}(\bar{u})]^2}$$

which shows that (10) holds, owing to $\|\Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u}) - \bar{u}\|_{L^2(\Omega)} \le S_{\mathscr{D}}(\bar{u})$. □

Remark 2. It can be seen in [5] that for most gradient schemes based on meshes, $W_{\mathscr{D}}$ and $S_{\mathscr{D}}$ are $\mathscr{O}(h)$ (where h is the mesh size) if $\bar{u} \in H^2(\Omega) \cap H^1_0(\Omega)$ and Λ is Lipschitz-continuous. In these cases, Theorem 1 gives an $\mathscr{O}(\sqrt{h})$ error estimate. Given that $\mathrm{div}(\Lambda\nabla\bar{u}) + f = 0$ outside D and $u = g$ on D, there is potential, if g is constant or smooth, for the interplant $P_{\mathscr{D}}$ to give a better approximation of $\bar{u}$ on D. The term $\bar{u} - \Pi_{\mathscr{D}}(P_{\mathscr{D}}\bar{u})$ therefore may be much lower on D than $S_{\mathscr{D}}(\bar{u})$. This means that $E_{\mathscr{D}}$ is expected to be lower than $\mathscr{O}(h)$ and therefore that the error estimate could be indeed better than $\mathscr{O}(\sqrt{h})$ in practice.

From the above theorem and Remark, we can obtain the following convergence of the scheme.

Corollary 1. (Convergence) *Let $(\mathscr{D}_m)_{m\in\mathbb{N}}$ be a sequence of gradient discretisation which is coercive, consistent and limit-conforming. Let $\bar{u}$ be the exact solution to Problem (3). Assume that $K_{\mathscr{D}_m}$ is non-empty set for any $m \in \mathbb{N}$. If $u_m \in K_{\mathscr{D}_m}$ is the solution to gradient scheme (4), then $\Pi_{\mathscr{D}_m}u_m$ converges strongly to $\bar{u}$ in $L^2(\Omega)$ and $\nabla_{\mathscr{D}_m}u_m$ strongly converges in $L^2(\Omega)^d$ to $\nabla\bar{u}$.*

Remark 3. It is noted that the convergence proof and error estimate for linear equations are obtained without using compactness property.

4 Convergence in Non-Linear Case

In this section, we study the convergence of non-linear case written as Problem (1). Such this non-linear equation can be seen in the seepage problems (see [10]).

Theorem 2. (Convergence) *Under Hypotheses* (2), *let $(\mathscr{D}_m)_{m\in\mathbb{N}}$ be a sequence of gradient discretisations, which is coercive, consistent, limit-conforming and compact, and such that $K_{\mathscr{D}_m}$ is a non-empty set for any m. Then, for any $m \in \mathbb{N}$, the gradient scheme (4) has at least one solution $u_m \in K_{\mathscr{D}_m}$ and, up to a subsequence, $\Pi_{\mathscr{D}_m}u_m$ converges strongly in $L^2(\Omega)$ to a weak solution $\bar{u}$ of Problem (3), and $\nabla_{\mathscr{D}_m}u_m$ strongly converges in $L^2(\Omega)^d$ to $\nabla\bar{u}$.*

Proof. We follow here the same approach used in [3].
Define the mapping $T : v \longrightarrow u$ where for any $v \in X_{\mathscr{D},0}$, $u \in K_{\mathscr{D}}$ is defined as the solution to

$$\text{for all } w \in \mathrm{K}_{\mathscr{D}},\ \int_\Omega \Lambda(x,\Pi_{\mathscr{D}}v)\nabla_{\mathscr{D}}u \cdot \nabla_{\mathscr{D}}(u-w)\mathrm{d}x \ \le\ \int_\Omega f\Pi_{\mathscr{D}}(u-w)\mathrm{d}x.$$

That is u is the solution to the variational inequality with the non-linearity in Λ frozen to v. There is only one such u, so the mapping T is well defined, and it is clearly continuous from $X_{\mathscr{D},0}$ into $X_{\mathscr{D},0}$. Since it sends all of $X_{\mathscr{D},0}$ inside a fixed ball of this space (see estimate to follow), Brouwer's theorem ensures the existence of a fixed point $u = T(u)$, which is a solution to the non-linear variational inequality.
Let $\varphi \in K$. Thanks to consistency, we can choose $v_m \in K_{\mathscr{D}_m}$ defined as $v_m = P_{\mathscr{D}_m}\varphi$ (see (8)). Setting $u := u_m$ and $v := v_m \in K_{\mathscr{D}_m}$ in (4) and applying the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}\underline{\lambda}\|\nabla_{\mathscr{D}_m} u_m\|^2_{L^2(\Omega)^d} &\leq \|f\|(\|\Pi_{\mathscr{D}_m} u_m\|_{L^2(\Omega)} + \|\Pi_{\mathscr{D}_m} v_m\|_{L^2(\Omega)}) \\ &+\overline{\lambda}\|\nabla_{\mathscr{D}_m} v_m\|_{L^2(\Omega)^d}\|\nabla_{\mathscr{D}_m} u_m\|_{L^2(\Omega)^d}.\end{aligned} \tag{12}$$

Since $\|v_m\|_{\mathscr{D}_m}$ is bounded, (12) can be written as

$$\|\nabla_{\mathscr{D}_m} u_m\|_{L^2(\Omega)^d} \leq C$$

in which $C > 0$ is constant. Using Lemma 1.13 in [2] (see also the proof of Theorem 3.5 in [3]), there exists a subsequence, still denoted by $(\mathscr{D}_m)_{m\in\mathbb{N}}$, and $\bar{u} \in H^1_0(\Omega)$, such that $\Pi_{\mathscr{D}_m} u_m$ converges weakly to $\bar{u}$ in $L^2(\Omega)$ and $\nabla_{\mathscr{D}_m} u_m$ converges weakly to $\nabla\bar{u}$ in $L^2(\Omega)^d$. Since $u_m \in K_{\mathscr{D}_m}$, passing to the limit in $\Pi_{\mathscr{D}_m} u_m$ shows that $\bar{u}$ is in K. Using the compactness hypothesis, we see that the convergence of $\Pi_{\mathscr{D}_m} u_m$ to $\bar{u}$ is actually strong $L^2(\Omega)$. Up to another subsequence, we can therefore assume that this convergence is also true almost everywhere. To complete the proof, it remains to prove the strong convergence of $\nabla_{\mathscr{D}_m} u_m$ and that $\bar{u}$ is the solution to (3).
It is classical that if $U_m \to U$ in $L^2(\Omega)^d$, then $\|U\|_{L^2(\Omega)^d} \leq \liminf_{m\longrightarrow\infty} \|U_m\|_{L^2(\Omega)^d}$. Using the positiveness of Λ, the strong convergence of $\Pi_{\mathscr{D}_m} u_m$ to $\bar{u}$ and the weak convergence of $\nabla_{\mathscr{D}_m} u_m$ to $\nabla\bar{u}$, we can adapt the proof of this classical result to see that

$$\int_\Omega \Lambda(x,\bar{u})\nabla\bar{u}\cdot\nabla\bar{u}\mathrm{d}x \leq \liminf_{m\longrightarrow\infty} \int_\Omega \Lambda(x,\Pi_{\mathscr{D}_m} u_m)\nabla_{\mathscr{D}_m} u_m \cdot \nabla_{\mathscr{D}_m} u_m \mathrm{d}x. \tag{13}$$

Thanks to the consistency of the gradient discretisations, $\Pi_{\mathscr{D}_m}(P_{\mathscr{D}_m}\varphi) \longrightarrow \varphi$ strongly in $L^2(\Omega)$ and $\nabla_{\mathscr{D}_m}(P_{\mathscr{D}_m}\varphi) \longrightarrow \nabla\varphi$ strongly in $L^2(\Omega)^d$. This later convergence and the a.e. convergence of $\Lambda(.,\Pi_{\mathscr{D}_m} u_m)$ show that $\Lambda(\cdot,\Pi_{\mathscr{D}_m} u_m)\nabla_{\mathscr{D}_m}(P_{\mathscr{D}_m}\varphi)$ converges to $\Lambda(\cdot,\bar{u})\nabla\varphi$ in $L^2(\Omega)$. Using (13) and the fact that u_m is a solution to (4), we get

$$\begin{aligned}\int_\Omega \Lambda(x,\bar{u})\nabla\bar{u}\cdot\nabla\bar{u}\mathrm{d}x &\leq \liminf_{m\longrightarrow\infty}\Big[\int_\Omega f\Pi_{\mathscr{D}_m}(u_m - P_{\mathscr{D}_m}\varphi)\mathrm{d}x \\ &\quad+\int_\Omega \Lambda(x,\Pi_{\mathscr{D}_m} u_m)\nabla_{\mathscr{D}_m} u_m\cdot\nabla_{\mathscr{D}_m}(P_{\mathscr{D}_m}\varphi)\mathrm{d}x\Big] \\ &= \int_\Omega f(\bar{u}(x) - \varphi(x)) + \int_\Omega \Lambda(x)\nabla\bar{u}(x)\cdot\nabla\varphi(x)\mathrm{d}x.\end{aligned}$$

This shows that $\bar{u}$ is a weak solution to (3). Now, we prove the strong convergence of the discrete gradients. For a given $v_m \in K_{\mathscr{D}_m}$, we have

$$
\begin{aligned}
0 &\leq \limsup_{m\longrightarrow\infty} \underline{\lambda} \|\nabla_{\mathcal{D}_m} u_m(x) - \nabla\bar{u}(x)\|^2_{L^2(\Omega)^d} \\
&\leq \limsup_{m\longrightarrow\infty} \int_\Omega \Lambda(x,\Pi_{\mathcal{D}_m} u_m)(\nabla_{\mathcal{D}_m} u_m(x) - \nabla\bar{u}(x))(\nabla_{\mathcal{D}_m} u_m(x) - \nabla\bar{u}(x))\mathrm{d}x \\
&\leq \limsup_{m\longrightarrow\infty} \Big[\int_\Omega f\Pi_{\mathcal{D}_m}(u_m(x) - v_m(x))\mathrm{d}x + \int_\Omega \Lambda(x,\Pi_{\mathcal{D}_m} u_m)\nabla\bar{u}(x)\cdot\nabla\bar{u}(x)\mathrm{d}x \\
&\quad -2\int_\Omega \Lambda(x,\Pi_{\mathcal{D}_m} u_m)\nabla_{\mathcal{D}_m} u_m(x)\cdot\nabla\bar{u}(x)\mathrm{d}x + \int_\Omega \Lambda(x)\nabla_{\mathcal{D}_m} u_m(x)\cdot\nabla_{\mathcal{D}_m} v_m(x)\mathrm{d}x\Big]
\end{aligned}
$$

since u_m is a solution to (4). Choosing $v_m = P_{\mathcal{D}_m}\bar{u}$ in this inequality and passing to the limit leads to $\limsup_{m\longrightarrow\infty} \|_{L^2(\Omega)^d} \nabla_{\mathcal{D}_m} u_m - \nabla\bar{u}\| \leq 0$ and concludes the proof. □

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