Benchmark on anisotropic problems

Use of the mixed finite volume method

C. Chainais-Hillairet[1], J. Droniou[2] and R. Eymard[3]

[1] *Université Blaise Pascal, Clermont-Ferrand, France*
Claire.Chainais@math.univ-bpclermont.fr
[2] *Université de Montpellier, France*
droniou@math.univ-montp2.fr
[3] *Université Paris-Est, France*
Robert.Eymard@univ-mlv.fr

ABSTRACT. We present the numerical results obtained for the FVCA5 benchmark of numerical schemes for heterogeneous and anisotropic diffusion problems, using the mixed finite volume method.

KEYWORDS: Anisotropy benchmark, mixed finite volume method

1. Presentation of the scheme

We consider in this paper the use of the mixed finite volume scheme for handling the FVCA5 benchmark test cases. This scheme has been first introduced in [DE06]. It has been shown efficient and accurate in two applications: miscible fluid flows in porous media [CD07], Navier-Stokes equations [DE08]. This scheme has also been shown to apply in the case of nonlinear Leray-Lions operators [DRO06]. Its main properties are the following:

1) It provides an approximation of the solution of a linear or nonlinear diffusion problem on any general non matching grid.

2) It provides an approximation of the unknown, of its gradient and of the fluxes at the interfaces of the mesh.

3) A proof of convergence and error estimate holds for this scheme.

We briefly recall the principles of this scheme in the case of the homogeneous Dirichlet problem $-\mathrm{div}\mathbb{K}\nabla u = f$, where f is a function and $\mathbb{K}$ is the diffusion field, satisfying hypotheses of measurability, positivity and boundedness. We look for

• u_K and $\mathbf{v}_K$, the respective approximate value of u and ∇u at the point $x_K \in K$, for any control volume K of the mesh $\mathcal{M}$ (x_K can be any point of K but, in the numerical experiments, we take the barycenter of the vertices of K),

• $F_{K,\sigma}$, the approximate value of the flux $\int_\sigma \mathbb{K}\nabla u \cdot \mathbf{n}_{K,\sigma}\,\mathrm{d}s$ at any face $\sigma \in \mathcal{E}_K$ of a control volume K ($\mathbf{n}_{K,\sigma}$ denotes the unit normal to σ outwards to K),

• u_σ, the approximate value of u at the center of gravity x_σ of any interface $\sigma \in \mathcal{E}$ ($\mathcal{E}$ is the union of the set of the interior edges $\mathcal{E}_{\rm int}$ and of the set of the boundary edges $\mathcal{E}_{\rm ext}$).

These values are then found as the solution, for given values $\nu_0 > 0$ and $\alpha > 0$ and denoting by $h > 0$ the size of the mesh, of the following system of equations:

$$\mathbf{v}_K \cdot (x_\sigma - x_K) + \nu_0 h^\alpha \frac{\mathrm{diam}(K)}{\mathrm{meas}(\sigma)} F_{K,\sigma} = u_\sigma - u_K, \quad \forall K \in \mathcal{M},\ \forall \sigma \in \mathcal{E}_K, \quad [1a]$$

$$F_{K,\sigma} + F_{L,\sigma} = 0, \quad \forall \sigma \in \mathcal{E}_{\rm int} \text{ between } K \text{ and } L, \quad [1b]$$

$$u_\sigma = 0, \quad \forall \sigma \in \mathcal{E}_{\rm ext}, \quad [1c]$$

$$\left(\int_K \mathbb{K}(x)\mathrm{d}x\right) \mathbf{v}_K = \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(x_\sigma - x_K), \quad \forall K \in \mathcal{M}, \quad [1d]$$

$$-\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} = \int_K f(x)\mathrm{d}x, \quad \forall K \in \mathcal{M}. \quad [1e]$$

The values chosen for $\nu_0 h^\alpha$ are discussed in Section 3.2 (recall that the convergence is proved for $\alpha \in]0,2[$). For Tests 3 and 4, we have to define the numerical approximation of the energies

$$E_1 = \int_\Omega \mathbb{K}\nabla u \cdot \nabla u \, dx\,, \qquad E_2 = \int_{\partial\Omega} \mathbb{K}\nabla u \cdot \mathbf{n}\, u \, dx.$$

Therefore, we set :

$$\mathrm{ener1} = \sum_{K \in \mathcal{M}} \left(\int_K \mathbb{K}(x)\mathrm{d}x\right) \mathbf{v}_K \cdot \mathbf{v}_K\,, \qquad \mathrm{ener2} = \sum_{\sigma \in \mathcal{E}_{\rm ext}} F_{K,\sigma} u_\sigma.$$

Due to [1a], [1b], [1d] and [1e], we have :

$$\mathrm{ener1} = \mathrm{ener2} + \sum_{K \in \mathcal{M}} \left(\int_K f(x)\mathrm{d}x\right) u_K - \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \nu_0 h^\alpha \frac{\mathrm{diam}(K)}{\mathrm{meas}(\sigma)} (F_{K,\sigma})^2.$$

Therefore, when $f = 0$ (as in Tests 3 and 4), we have $\mathrm{ener1} - \mathrm{ener2} \leq 0$.

1.1. *Practical implementation*

The system [1] can be reduced by hybridization to a system of size $\mathrm{Card}(\mathcal{E}_{\mathrm{int}})$ on $(u_\sigma)_{\sigma\in\mathcal{E}_{\mathrm{int}}}$. This is done by eliminating $\mathbf{v}_K$ thanks to [1d] and writing [1a]–[1e] as

$$\begin{pmatrix} B_K & (1)_{\sigma\in\mathcal{E}_K} \\ (1)^T_{\sigma\in\mathcal{E}_K} & 0 \end{pmatrix} \begin{pmatrix} (F_{K,\sigma})_{\sigma\in\mathcal{E}_K} \\ u_K \end{pmatrix} = \begin{pmatrix} (u_\sigma)_{\sigma\in\mathcal{E}_K} \\ -\int_K f(x)\mathrm{d}x \end{pmatrix} \qquad [2]$$

where $(1)_{\sigma\in\mathcal{E}_K}$ is the vector of size $\mathrm{Card}(\mathcal{E}_K)$ with all components equal to 1 and, $\delta_{\sigma,\sigma'}$ being Kronecker's symbol, B_K is the $\mathrm{Card}(\mathcal{E}_K)\times\mathrm{Card}(\mathcal{E}_K)$ matrix defined by

$$(B_K)_{\sigma,\sigma'} = \left(\int_K \mathbb{K}(x)\mathrm{d}x\right)^{-1}(x_{\sigma'}-x_K)\cdot(x_\sigma-x_K)+\delta_{\sigma,\sigma'}\nu_0 h^\alpha\frac{\mathrm{diam}(K)}{\mathrm{meas}(\sigma)}.$$

The matrix C_K in the left-hand side of [2] is invertible and allows to write $(F_{K,\sigma})_{\sigma\in\mathcal{E}_K}$ in terms of $(u_\sigma)_{\sigma\in\mathcal{E}_K}$; plugging the resulting expressions in [1b], the system [1] is reduced to the following $\mathrm{Card}(\mathcal{E}_{\mathrm{int}})$ equations:

$$\sum_{\sigma'\in\mathcal{E}_K}(C_K^{-1})_{\sigma,\sigma'}u_{\sigma'}+\sum_{\sigma'\in\mathcal{E}_L}(C_K^{-1})_{\sigma,\sigma'}u_{\sigma'} = (C_K^{-1})_{\sigma,K}\int_K f(x)\mathrm{d}x+(C_L^{-1})_{\sigma,L}\int_L f(x)\mathrm{d}x\,, \quad \forall\sigma\in\mathcal{E}_{\mathrm{int}}\text{ between }K\text{ and }L$$

(the subscripts σ,σ' in C_K^{-1} and C_L^{-1} denote the first $\mathrm{Card}(\mathcal{E}_K)$ rows and columns of the matrices, and K and L their last row and last column).

2. Numerical results

• **Test 1.1 Mild anisotropy,** $u(x,y)=16x(1-x)y(1-y)$**,** $\min=0,\ \max=1$**, regular triangular mesh,** `mesh1`

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	76	348	6.22E-15	1.57E-02	1.72E-01		
2	320	1536	-7.11E-15	3.74E-03	8.55E-02	2.00	0.97
3	1312	6432	2.04E-14	9.14E-04	4.26E-02	2.00	0.99
4	5312	26304	2.03E-13	2.27E-04	2.13E-02	1.99	0.99
5	21376	106368	2.29E-13	5.64E-05	1.06E-02	2.00	1.00
6	85760	427776	-2.64E-12	1.39E-05	5.32E-03	2.02	0.99
7	343552	1715712	1.62E-11	3.30E-06	2.66E-03	2.07	1.00

ocvl2$=2.08$, **ocvgradl2**$=1.00$.

i	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1	1.06E-02	1.06E-02	1.06E-02	1.06E-02	2.87E-01	7.46E-02	9.11E-01
2	2.65E-03	2.65E-03	2.65E-03	2.65E-03	1.69E-01	2.02E-02	9.78E-01
3	6.63E-04	6.63E-04	6.63E-04	6.63E-04	9.17E-02	5.25E-03	9.95E-01
4	1.66E-04	1.66E-04	1.66E-04	1.66E-04	4.77E-02	1.33E-03	9.99E-01
5	4.14E-05	4.14E-05	4.14E-05	4.14E-05	2.43E-02	3.36E-04	1.00E+00
6	1.04E-05	1.04E-05	1.04E-05	1.04E-05	1.23E-02	8.43E-05	1.00E+00
7	2.59E-06	2.59E-06	2.59E-06	2.59E-06	6.16E-03	2.11E-05	1.00E+00

• **Test 1.1 Mild anisotropy,** $u(x, y) = 16x(1-x)y(1-y)$**,** $\min = 0,\ \max = 1$**, coarse (C) and fine (F) distorted quadrangular meshes,** `mesh4_1` **and** `mesh4_2`

grid	nunkw	nnmat	sumflux	erl2	ergrad
C	544	3612	4.25E-10	4.38E-02	4.32E-02
F	2112	14396	1.56E-08	1.22E-02	1.20E-02

grid	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
C	1.81E-04	1.05E-03	4.18E-03	1.85E-03	4.10E-01	1.08E-02	9.42E-01
F	2.39e-04	6.06e-04	7.44e-04	2.71e-04	1.06e-01	3.34e-03	9.82e-01

• **Test 1.2 Mild anisotropy,** $u(x, y) = \sin((1-x)(1-y)) + (1-x)^3(1-y)^2$**,** $\min = 0,\ \max = 1 + \sin 1$**, regular triangular mesh,** `mesh1`

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	76	348	7.11E-15	3.38E-03	1.26E-01		
2	320	1536	-2.33E-15	7.95E-04	6.17E-02	2.01	0.99
3	1312	6432	-1.78E-15	1.94E-04	3.07E-02	2.00	0.99
4	5312	26304	-6.03E-14	4.82E-05	1.53E-02	1.99	1.00
5	21376	106368	2.10E-13	1.20E-05	7.66E-03	2.00	0.99
6	85760	427776	8.63E-14	3.00E-06	3.83E-03	2.00	1.00
7	343552	1715712	-1.46E-12	7.51E-07	1.91E-03	2.00	1.00

ocvl2$= 2.00$, **ocvgradl2**$= 1.01$.

i	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1	3.81E-03	4.28E-04	7.03E-04	8.42E-03	1.23E-01	5.38E-03	1.37E+00
2	1.01E-03	2.73E-04	3.72E-04	2.64E-03	8.26E-02	1.35E-03	1.60E+00
3	2.65E-04	8.62E-05	1.61E-04	8.26E-04	4.75E-02	3.38E-04	1.72E+00
4	6.99E-05	2.35E-05	5.97E-05	2.53E-04	2.54E-02	8.46E-05	1.78E+00
5	1.84E-05	6.09E-06	2.01E-05	7.53E-05	1.31E-02	2.12E-05	1.81E+00
6	4.82E-06	1.55E-06	6.37E-06	2.19E-05	6.67E-03	5.29E-06	1.83E+00
7	1.26E-06	3.90E-07	1.93E-06	6.27E-06	3.38E-03	1.32E-06	1.83E+00

• **Test 1.2 Mild anisotropy,** $u(x, y) = \sin((1-x)(1-y)) + (1-x)^3(1-y)^2$**,** $\min = 0,\ \max = 1 + \sin 1$**, locally refined nonconforming rectangular mesh,** `mesh3`

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	72	472	4.09E-13	4.93E-03	8.98E-02		
2	304	2064	-1.44E-11	1.60E-03	4.64E-02	1.56	0.92
3	1248	8608	-7.46E-11	4.55E-04	2.35E-02	1.78	0.96
4	5056	35136	-1.47E-11	1.23E-04	1.18E-02	1.87	0.98
5	20352	141952	1.09E-08	3.25E-05	5.90E-03	1.91	1.00

ocvl2$= 1.92$, **ocvgradl2**$= 1.00$.

i	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1	3.14E-02	5.97E-02	8.30E-03	1.21E-01	3.08E+00	1.59E-02	1.66E+00
2	1.04E-02	1.57E-02	5.67E-06	3.49E-02	3.01E+00	3.90E-03	1.75E+00
3	3.74E-03	3.93E-03	1.03E-03	1.04E-02	2.89E+00	9.74E-04	1.79E+00
4	1.44E-03	9.79E-04	7.82E-04	3.06E-03	2.80E+00	2.44E-04	1.82E+00
5	5.92E-04	2.44E-04	4.65E-04	8.89E-04	2.75E+00	6.10E-05	1.83E+00

• **Test 2 Numerical locking,** $u(x, y) = \sin(2\pi x)e^{-2\pi\sqrt{\frac{1}{\delta}}y}$, $\delta = 10^5$**,** $\min = -1,\ \max = 1$**, regular triangular mesh,** `mesh1`

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	92	424	6.25E-13	2.61E-01	3.54E-01		
2	352	1692	-4.14E-10	3.57E+00	1.61E+01	-3.90	-5.69
3	1376	6748	2.05E-10	6.51E-01	5.75E+00	2.50	1.51
4	5440	26940	3.36E-10	1.00E-01	1.97E+00	2.73	1.56
5	21632	107644	5.29E-10	1.62E-02	6.83E-01	2.64	1.53
6	86272	430332	2.72E-12	2.76E-03	2.40E-01	2.56	1.51
7	344576	1720828	-2.91E-10	5.21E-04	8.44E-02	2.41	1.51

ocvl2$= 2.41$, **ocvgradl2**$= 1.51$.

i	erflx0	erflx1	fluy0	fluy1	erflm	umin	umax
1	0.00E+00	0.00E+00	-1.92E-11	1.98E-11	5.69E+02	-1.06E+00	1.07E+00
2	2.86E-16	1.43E-15	-4.17E-10	2.96E-12	5.06E+02	-6.50E+00	5.75E+00
3	1.43E-16	8.57E-16	6.29E-11	1.42E-10	4.27E+02	-2.46E+00	2.29E+00
4	0.00E+00	5.71E-16	2.51E-10	8.50E-11	2.73E+02	-1.10E+00	1.09E+00
5	0.00E+00	1.43E-16	3.86E-10	1.43E-10	1.42E+02	-1.01E+00	1.01E+00
6	0.00E+00	0.00E+00	5.03E-10	-5.00E-10	7.15E+01	-1.00E+00	1.00E+00
7	0.00E+00	2.86E-16	-8.76E-10	5.84E-10	3.58E+01	-1.00E+00	1.00E+00

• **Test 2 Numerical locking,** $u(x,y) = \sin(2\pi x)e^{-2\pi\sqrt{\frac{1}{\delta}}y}$, $\delta = 10^6$**,** $\min = -1$, $\max = 1$**, regular triangular mesh,** `mesh1`

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	92	424	3.17E-10	2.61E-01	3.54E-01		
2	352	1692	-1.14E-08	1.13E+01	5.10E+01	-5.62	-7.41
3	1376	6748	-2.62E-09	2.06E+00	1.82E+01	2.50	1.51
4	5440	26940	7.76E-10	3.16E-01	6.21E+00	2.73	1.56
5	21632	107644	-2.73E-10	5.12E-02	2.16E+00	1.64	1.53
6	86272	430332	-6.86E-09	8.69E-03	7.57E-01	2.56	1.52
7	344576	1720828	-1.14E-08	1.64E-03	2.67E-01	2.41	1.51

ocvl2$= 2.41$, **ocvgradl2**$= 1.51$.

i	erflx0	erflx1	fluy0	fluy1	erflm	umin	umax
1	2.84E-16	1.42E-16	6.22E-10	-3.05E-10	1.80E+03	-1.07E+00	1.07E+00
2	5.67E-16	1.42E-15	-1.47E-09	-9.94E-09	1.61E+03	-1.86E+01	1.63E+01
3	1.84E-15	2.13E-15	-1.62E-09	-9.98E-10	1.36E+03	-6.59E+00	6.08E+00
4	9.93E-16	1.42E-16	-3.02E-10	1.08E-09	8.70E+02	-1.84E+00	1.75E+00
5	5.67E-16	0.00E+00	-1.52E-09	1.25E-09	4.51E+02	-1.06E+00	1.06E+00
6	0.00E+00	1.42E-16	-7.04E-09	1.77E-10	2.27E+02	-1.00E+00	1.00E+00
7	5.67E-16	0.00E+00	-1.29E-08	1.48E-09	1.14E+02	-1.00E+00	1.00E+00

• **Test 3 Oblique flow,** $\min = 0$, $\max = 1$**, uniform rectangular mesh,** `mesh2`

i	nunkw	nnmat	sumflux	umin	umax
1	24	128	-6.50E-09	1.22E-01	8.78E-01
2	112	696	-2.87E-08	1.76E-07	1.00E+00
3	480	3176	-1.24E-07	1.20E-02	9.88E-01
4	1984	13512	-5.14E-07	1.02E-02	9.90E-01
5	8064	55688	-2.09E-06	4.32E-03	9.96E-01
ref	204160	1425288	3.27E-06	7.92E-04	9.99E-01

Reference mesh: `mesh5_ref`

i	flux0	flux1	fluy0	fluy1	ener1	ener2	eren
1	-8.52E-02	8.52E-02	8.52E-02	-8.52E-02	4.85E-01	4.85E-01	8.23E-07
2	-2.90E-01	2.90E-01	-2.90E-01	2.90E-01	5.38E-01	5.38E-01	2.98E-06
3	-2.16E-01	2.16E-01	-1.48E-01	1.48E-01	2.66E-01	2.66E-01	1.90E-06
4	-1.68E-01	1.68E-01	-5.07E-02	5.07E-02	2.58E-01	2.58E-01	6.84E-06
5	-1.87E-01	1.87E-01	-8.67E-02	8.67E-02	2.43E-01	2.43E-01	6.59E-06
ref	-1.93E-01	1.93E-01	-9.87E-02	9.87E-02	2.42E-01	2.42E-01	9.74E-06

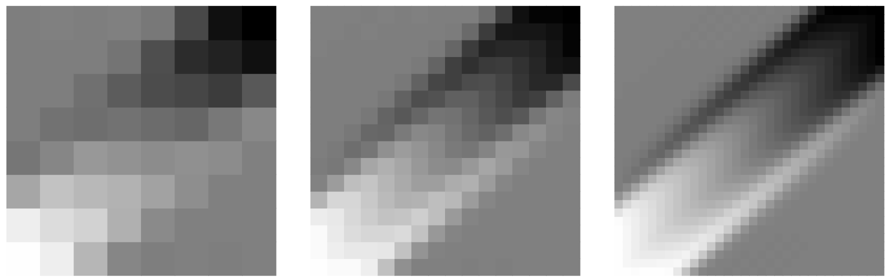

Figure 1. *Test 3, solutions for* `mesh2_2` *(left),* `mesh2_3` *(center) and* `mesh2_4` *(right). Black=min, white= max.*

• **Test 4 Vertical fault,** $\min = 0., \max = 1.$**, non conforming rectangular mesh,** `mesh5`

i	nunkw	nnmat	sumflux	umin	umax
1	199	1361	6.49E-09	4.93E-02	9.54E-01
reg	760	5080	1.03E-08	2.15E-02	9.80E-01
ref	204160	1425280	4.65E-06	1.32E-03	9.99E-01

Reference mesh: `mesh5_ref`

i	flux0	flux1	fluy0	fluy1	ener1	ener2	eren
1	-4.40E+01	5.03E+01	-8.03E+00	1.72E+00	4.99E+01	4.99E+01	4.21E-05
reg	-4.25E+01	4.47E+01	-2.18E+00	1.34E-03	4.36E+01	4.36E+01	1.92E-05
ref	-4.21E+01	4.44E+01	-2.33E+00	8.33E-04	4.32E+01	4.32E+01	1.88E-05

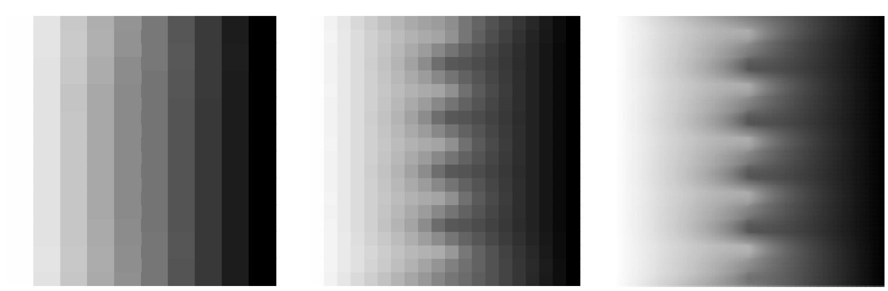

Figure 2. *Test 4, solutions for* `mesh5` *(left),* `mesh5_reg` *(center) and* `mesh5_ref` *(right). Black=min, white= max.*

• **Test 5 Heterogeneous rotating anisotropy,** $\min = 0, \max = 1$ **, uniform rectangular mesh,** `mesh2`

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	24	128	-2.33E-09	7.86E-02	2.21E-01		
2	112	696	4.29E-09	2.07E-02	8.82E-02	1.73	1.19
3	480	3176	1.75E-09	5.23E-03	3.02E-02	1.89	1.47
4	1984	13512	-3.46E-08	1.32E-03	8.85E-03	1.94	1.73
5	8064	55688	-2.97E-08	3.29E-04	2.32E-03	1.98	1.91

ocvl2$= 2.00$, **ocvgradl2**$= 1.93$.

i	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1	1.12E-01	6.02E-02	1.12E-01	6.02E-02	9.11E-01	1.38E-01	8.50E-01
2	3.55E-02	4.28E-03	3.55E-02	4.28E-03	4.81E-01	3.71E-02	9.53E-01
3	1.08E-02	5.36E-04	1.08E-02	5.36E-04	2.45E-01	9.54E-03	9.87E-01
4	3.22E-03	3.49E-04	3.22E-03	3.49E-04	1.23E-01	2.40E-03	9.97E-01
5	9.40E-04	1.11E-04	9.40E-04	1.11E-04	6.15E-02	6.02E-04	9.99E-01

• **Test 6 Oblique drain,** $\min = -1.2, \max = 0$**, coarse (C) and fine (F) oblique meshes,** `mesh6` **and** `mesh7`

grid	nunkw	nnmat	sumflux	erl2	ergrad
C	389	2545	6.93E-09	2.06E-08	2.17E-06
F	449	3189	1.29E-08	4.14E-08	6.57E-06

grid	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
C	8.10E-06	8.10E-06	9.65E-08	9.58E-08	9.62E-04	-1.15E+00	-5.38E-02
F	2.43E-05	2.43E-05	2.31E-09	2.64E-09	2.91E-03	-1.15E+00	-5.38E-02

• **Test 7 Oblique barrier,** $\min = -5.575, \max = 0.575$**, coarse oblique mesh** `mesh6`

nunkw	nnmat	sumflux	erl2	ergrad
389	2545	1.04E-08	1.40E-08	4.69E-08

erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
2.03E-07	2.42E-07	2.95E-07	2.92E-07	1.38E-06	-5.54E+00	5.37E-01

• **Test 8 Perturbed parallelograms,** $\min = 0$**, perturbed parallelograms mesh** `mesh8`

nunkw	nnmat	sumflux	umin	umax
220	1416	8.46E-11	-8.08E-03	5.81E-02

flux0	flux1	fluy0	fluy1
-2.30E-02	4.95E-02	2.74E-01	6.99E-01

• **Test 9 Anisotropy with wells,** $\min = 0$, $\max = 1.$**, square uniform grid** `mesh9`

nunkw	nnmat	sumflux	umin	umax
264	1718	1.11E-08	-1.22E-01	1.07E+00

3. Comments on the results

3.1. *Implementation*

The mixed finite volume method has been implemented in two different computing frameworks: the first one is a code written in FORTRAN, the second one in MATLAB. The results obtained within both implementations are identical, up to the precision provided by the 8-bytes storage for a real value.

In the FORTRAN implementation, we have taken advantage from the "hybrid" structure of the linear system to be solved. Indeed, in such a case, it is easy to apply an efficient recursive elimination of the unknowns. This method provides dramatic time and memory spares in a Gaussian elimination, compared to a classical band elimination (such a property is not so easily expected from multi-point flux schemes). For example, in the case of the 160×160 mesh of the `mesh2` family, the storage for a band elimination is equal to $22.1\ 10^6$ real numbers in double precision, whereas it is only equal to $1.13\ 10^6$ using the recursive elimination (the number of unknowns is equal to 50880). In the 320×320 mesh, the corresponding values are $176.\ 10^6$ and $5.3\ 10^6$ for 204160 unknowns.

The same remarkable property can be used with the `mesh1` family. For `mesh1_6`, the storage needed for 85760 unknowns is decreased from $44.4\ 10^6$ to $9.56\ 10^5$, and for `mesh1_7`, the storage needed for 343552 unknowns is decreased from $353.\ 10^6$ to $4.30\ 10^6$ (and the computing time is less than 30 seconds on a standard laptop).

3.2. *Selection of a value for* ν_0

In the case of triangular meshes (or tetrahedral meshes in 3D), it is possible to take $\nu_0 h^\alpha = 0$ (see [DE06]): we made such a choice in the test 1.2 on the meshes `mesh1`, in order to show a numerical result of the scheme without penalization. However, even on triangular meshes, it is sometimes better to take $\nu_0 h^\alpha \neq 0$ in order to numerically stabilize the scheme. One can then fix $\nu_0 > 0$ and let the penalization $\nu_0 h^\alpha$ vary with the size of the mesh, which is what we did in the test 1.2 on meshes `mesh3` (with $\nu_0 = 6.\ 10^{-3}$). However, in conditions close to the ones encountered in the course of engineering study, there is no time to explore the sensitivity of the results with respect to the numerical parameters. Hence, in order to stay close to these situations, we made the choice in all the other tests to fix $\nu_0 h^\alpha = 10^{-7}$, as a kind of optimal compromise between the condition number of the linear systems, and the precision of real values stored in 8 bytes. This is the strategy that we already successfully adopted in the works given in the bibliography.

4. References

[CD07] CHAINAIS-HILLAIRET C. AND DRONIOU J., «Convergence analysis of a mixed finite volume scheme for an elliptic-parabolic system modeling miscible fluid flows in porous media», *SIAM J. Numer. Anal.*, vol 45, n^{o} 5, 2007, p. 2228–2258.

[DRO06] DRONIOU J., «Finite volume schemes for fully non-linear elliptic equations in divergence form», *M2AN Math. Model. Numer. Anal.*, vol 40, n^{o} 6, 2006, p. 1069-1100.

[DE06] DRONIOU J. AND EYMARD R., «A mixed finite volume scheme for anisotropic diffusion problems on any grid», *Num. Math.*, vol 105, n^{o} 1, 2006, p. 35–71.

[DE08] DRONIOU J. AND EYMARD R., «Study of the mixed finite volume method for Stokes and Navier-Stokes equations», to be published in *Numerical Methods for Partial Differential Equations*, 2008.