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CONVERGENCE ANALYSIS OF A MIXED FINITE VOLUME SCHEME FOR AN ELLIPTIC-PARABOLIC SYSTEM MODELING MISCIBLE FLUID FLOWS IN POROUS MEDIA*

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Abstract. We study a finite volume discretization of a strongly coupled elliptic-parabolic PDE system describing miscible displacement in a porous medium. We discretize each equation by a finite volume scheme which allows a wide variety of unstructured grids (in any space dimension) and gives strong enough convergence for handling the nonlinear coupling of the equations. We prove the convergence of the scheme as the time and space steps go to 0. Finally, we provide numerical results to demonstrate the efficiency of the proposed numerical scheme.

Key words. finite volume methods, porous medium, miscible fluid flow, convergence analysis, numerical tests

AMS subject classifications. 65M12, 65M30, 65N12, 65N30, 76S05, 76R99

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1. Introduction.

1.1. Miscible displacement in porous media. The mathematical model for the single-phase miscible displacement of one fluid by another in a porous medium, in the case where the fluids are considered incompressible, is an elliptic-parabolic coupled system [2, 4]. Let Ω be a bounded domain of $\mathbb{R}^d$ ($d = 2$ or 3) representing the reservoir and let $(0, T)$ be the time interval. The unknowns of the problem are p the pressure in the mixture, $\mathbf{U}$ its Darcy velocity, and c the concentration of the invading fluid.

We denote by $\Phi(x)$ and $\mathbf{K}(x)$ the porosity and the absolute permeability tensor of the porous medium, $\mu(c)$ the viscosity of the fluid mixture, $\widehat{c}$ the injected concentration, and q^+ and q^- the injection and the production source terms. If we neglect gravity, the model reads

$$(1)\qquad \begin{cases} \operatorname{div}(\mathbf{U}) = q^+ - q^- & \text{in } (0,T)\times\Omega, \\ \mathbf{U} = -\dfrac{\mathbf{K}(x)}{\mu(c)}\nabla p & \text{in } (0,T)\times\Omega, \end{cases}$$

$$(2)\qquad \Phi(x)\partial_t c - \operatorname{div}(D(x,\mathbf{U})\nabla c - c\mathbf{U}) + q^- c = q^+\widehat{c} \quad \text{in } (0,T)\times\Omega,$$

where D is the diffusion-dispersion tensor including molecular diffusion and mechanical dispersion

$$(3)\qquad D(x,\mathbf{U}) = \Phi(x)\Big(d_m\mathbf{I} + |\mathbf{U}|\Big(d_l E(\mathbf{U}) + d_t(\mathbf{I} - E(\mathbf{U}))\Big)\Big)$$

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with $\mathbf{I}$ the identity matrix, d_m the molecular diffusion, d_l and d_t the longitudinal and transverse dispersion coefficients, and $E(\mathbf{U}) = (\frac{\mathbf{U}_i\mathbf{U}_j}{|\mathbf{U}|^2})_{1\le i,j\le d}$. Laboratory experiments have found that the longitudinal dispersivity d_l is much greater than the transverse dispersivity d_t and that the diffusion coefficient is very small by comparison.

In reservoir simulation, the boundary $\partial\Omega$ is typically impermeable. Therefore, if $\mathbf{n}$ denotes the exterior normal to $\partial\Omega$, the system (1)–(2) is supplemented with no flow boundary conditions:

$$\begin{cases} \mathbf{U}\cdot\mathbf{n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ D(x,\mathbf{U})\nabla c\cdot\mathbf{n} = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{4}$$

An initial condition is also prescribed:

$$c(x,0) = c_0(x) \text{ in } \Omega. \tag{5}$$

Because of the homogeneous Neumann boundary conditions on $\mathbf{U}$, the injection and production source terms have to satisfy the compatibility condition $\int_\Omega q^+(\cdot,x)\ dx = \int_\Omega q^-(\cdot,x)\ dx$ in $(0,T)$, and since the pressure is defined only up to an arbitrary constant, we normalize p by the following condition:

$$\int_\Omega p(\cdot,x)\ dx = 0 \quad \text{in } (0,T). \tag{6}$$

The viscosity μ is usually determined by the following mixing rule

$$\mu(c) = \mu(0)\Big(1 + \Big(M^{1/4}-1\Big)c\Big)^{-4} \text{ in } [0,1], \tag{7}$$

where $M = \frac{\mu(0)}{\mu(1)}$ is the mobility ratio (μ can be extended to $\mathbb{R}$ by letting $\mu = \mu(0)$ on $(-\infty,0)$ and $\mu = \mu(1)$ on $(1,\infty)$). The porosity Φ and the permeability $\mathbf{K}$ are in general assumed to be bounded from above and from below by positive constants (or positive multiples of $\mathbf{I}$ for the tensor $\mathbf{K}$).

In [15], Feng proved the existence of a weak solution to the problem (1)–(7) in the two-dimensional case and with $d_l \ge d_t > 0$ and $d_m > 0$. This result has been generalized by Chen and Ewing in [3] to the three-dimensional case and with gravity effects and various boundary conditions. At high flow velocities the effects of mechanical dispersion are much greater than those of molecular diffusion. Therefore, Amirat and Ziani studied in [1] the asymptotic behavior of the weak solution as d_m goes to 0 and proved the existence of a weak solution in the case where $d_m = 0$.

From a numerical point of view, various methods have already been developed for this problem. In general the pressure equation is discretized by a finite element method. However, the key point is that equation (2) on c is a convection-dominated equation, which is not well adapted to the discretization by finite difference or finite element methods. Douglas, Ewing, and Wheeler [6] used a mixed finite element method for the pressure equation and a Galerkin finite element method for the concentration equation. In [19], Russell introduced a modified method of characteristic for the resolution of (2), while (1) is solved by a finite element method. Then, Ewing, Russell, and Wheeler [10] combined a mixed finite element method for (1) and a modified method of characteristic for (2). In [20, 21], the authors also used a mixed finite element method for (1) but developed an Eulerian Lagrangian localized adjoint method for (2).

Convergence of numerical schemes to (1)–(7) (or connected problems) has already been studied (see, e.g., [5, 6, 11, 12, 17]). But, to the best of our knowledge, these proofs of convergence are based on a priori error estimates, which need regularity assumptions on the solution $(p, \mathbf{U}, c)$ to the continuous problem. Such regularity does not seem provable in general, such as if we take a discontinuous permeability tensor (which is expected in field applications; see [20]).

Finite volume methods are well adapted to the discretization of conservation laws; see, for instance, the reference book by Eymard, Gallouët, and Herbin [13]. They provide efficient numerical schemes for elliptic equations as well as for convection-dominated parabolic equations. However, because of the anisotropic diffusion in (1) (due to $\mathbf{K}(x)$) and of the dispersion terms in (2)–(3), the standard four-point finite volume schemes cannot be used here. Besides, as said above, (2) is convection-dominated and, therefore, a good approximation of $\mathbf{U}$ is needed in the discretization of (2) in order to obtain admissible numerical results. In [9], Droniou and Eymard recently proposed a mixed finite volume scheme which handles anisotropic heterogeneous diffusion problems on any grid and precisely provides, for equations such as (1), good approximations of $\mathbf{U}$; this scheme is therefore a natural candidate to discretize such coupled problems as (1)–(7), especially as it has been shown to behave well from a numerical point of view.

In this paper, we extend the mixed finite volume scheme of [9] to a system, presented in section 1.2, which generalizes (1)–(7). Section 2 contains the definition of the scheme and the statement of the main results: existence and uniqueness of an approximate solution and its convergence to the solution of the continuous problem as the time and space steps tend to 0. A priori estimates on the approximate solution are established in section 3, and in section 4 we prove the existence and uniqueness of the solution to our scheme. The proof of convergence is presented in section 5, under no regularity assumption on the solution to the continuous problem. Section 6 presents some numerical experiments to demonstrate the efficiency of our numerical scheme. Section 7 is an appendix containing a few technical results.

1.2. Formulation of the problem and assumptions. Let us now rewrite the problem (1)–(7) under the following synthesized and more general form (notice that, from now on, we use letters with bar accents to denote the exact solutions, and we use letters without bar accents to denote approximate solutions):

$$(8)\quad \begin{cases} \operatorname{div}(\bar{\mathbf{U}}) = q^+ - q^- \ \text{ in } (0,T)\times\Omega, & \bar{\mathbf{U}} = -A(\cdot,\bar{c})\nabla\bar{p} \ \text{ in } (0,T)\times\Omega, \\ \displaystyle\int_\Omega \bar{p}(\cdot,x)\,dx = 0 \ \text{ in } (0,T), & \bar{\mathbf{U}}\cdot\mathbf{n} = 0 \ \text{ on } (0,T)\times\partial\Omega, \end{cases}$$

$$(9)\quad \begin{cases} \Phi\partial_t\bar{c} - \operatorname{div}(D(\cdot,\bar{\mathbf{U}})\nabla\bar{c}) + \operatorname{div}(\bar{c}\bar{\mathbf{U}}) + q^-\bar{c} = q^+\hat{c} & \text{in } (0,T)\times\Omega, \\ \bar{c}(0,\cdot) = c_0 & \text{in } \Omega, \\ D(\cdot,\bar{\mathbf{U}})\nabla\bar{c}\cdot\mathbf{n} = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

In what follows, we assume that Ω is a convex polygonal bounded domain of $\mathbb{R}^d$, $T > 0$, and the following:

$$(10)\quad \begin{aligned} &(q^+, q^-) \in L^\infty(0,T;L^2(\Omega)) \text{ are nonnegative,} \\ &\int_\Omega q^+(\cdot,x)\,dx = \int_\Omega q^-(\cdot,x)\,dx \text{ a.e. in } (0,T), \end{aligned}$$

$$A:\Omega\times\mathbb{R}\to M_d(\mathbb{R}) \text{ is a Carathéodory function satisfying the following:}$$

$$\exists\alpha_A>0\,,\ \exists\Lambda_A>0 \text{ such that, for a.e. } x\in\Omega, \text{ all } s\in\mathbb{R}, \text{ and all } \xi\in\mathbb{R}^d,$$
$$A(x,s)\xi\cdot\xi\geq\alpha_A|\xi|^2 \text{ and } |A(x,s)|\leq\Lambda_A, \tag{11}$$

$$D:\Omega\times\mathbb{R}^d\to M_d(\mathbb{R}) \text{ is a Carathéodory function satisfying the following:}$$

$$\exists\alpha_D>0\,,\ \exists\Lambda_D>0 \text{ such that, for a.e. } x\in\Omega, \text{ all } \mathbf{W}\in\mathbb{R}^d, \text{ and all } \xi\in\mathbb{R}^d,$$
$$D(x,\mathbf{W})\xi\cdot\xi\geq\alpha_D(1+|\mathbf{W}|)|\xi|^2 \text{ and } |D(x,\mathbf{W})|\leq\Lambda_D(1+|\mathbf{W}|), \tag{12}$$

$$\Phi\in L^\infty(\Omega) \text{ and there exists } \Phi_*>0 \text{ such that } \Phi_*\leq\Phi\leq\Phi_*^{-1} \text{ a.e. in } \Omega, \tag{13}$$

$$\widehat{c}\in L^\infty((0,T)\times\Omega) \text{ satisfies } 0\leq\widehat{c}\leq 1 \text{ a.e. in } (0,T)\times\Omega, \tag{14}$$

$$c_0\in L^\infty(\Omega) \text{ satisfies } 0\leq c_0\leq 1 \text{ a.e. in } \Omega. \tag{15}$$

Remark 1.1. Since $E(\mathbf{U})=\left(\mathbf{U}_i\mathbf{U}_j/|\mathbf{U}|^2\right)_{1\leq i,j\leq d}$ is the orthogonal projector on $\mathbb{R}\mathbf{U}$, the model in section 1.1 satisfies this assumptions with $\alpha_D=\phi_*\inf(d_m,d_l,d_t)$ and $\Lambda_D=\phi_*^{-1}\sup(d_m,d_l,d_t)$.

As Φ does not depend on t, the following definition (similar to the one in [15]) of weak solution to (8)–(9) makes sense.

DEFINITION 1.1. *Under assumptions* (10)–(15), *a weak solution to* (8)–(9) *is* $(\bar{p},\bar{\mathbf{U}},\bar{c})$ *such that* $\bar{p}\in L^\infty(0,T;H^1(\Omega))$, $\bar{\mathbf{U}}\in L^\infty(0,T;L^2(\Omega))^d$, $\bar{c}\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$,

$$\int_\Omega\bar{p}(t,\cdot)=0 \textit{ for a.e. } t\in(0,T)\,,\quad \bar{\mathbf{U}}=-A(\cdot,\bar{c})\nabla\bar{p} \textit{ a.e. in } (0,T)\times\Omega,$$
$$\forall\varphi\in C^\infty([0,T]\times\overline{\Omega})\,,\ -\int_0^T\int_\Omega\bar{\mathbf{U}}\cdot\nabla\varphi=\int_0^T\int_\Omega(q^+-q^-)\varphi\,,$$
$$\forall\psi\in C_c^\infty([0,T)\times\overline{\Omega})\,,\ -\int_0^T\int_\Omega\Phi\bar{c}\partial_t\psi+\int_0^T\int_\Omega D(\cdot,\bar{\mathbf{U}})\nabla\bar{c}\cdot\nabla\psi-\int_0^T\int_\Omega\bar{c}\bar{\mathbf{U}}\cdot\nabla\psi$$
$$+\int_0^T\int_\Omega q^-\bar{c}\psi-\int_\Omega\Phi c_0\psi(0,\cdot)=\int_0^T\int_\Omega q^+\widehat{c}\psi.$$

2. Scheme and main results. Let us first define the notion of admissible mesh of Ω and some notation associated with it.

DEFINITION 2.1. *Let Ω be a convex polygonal bounded domain in $\mathbb{R}^d$. An admissible mesh of Ω is given by $\mathcal{D}=(\mathcal{M},\mathcal{E})$, where the following hold:*

(i) *$\mathcal{M}$ is a finite family of nonempty disjoint convex polygonal domains in Ω (the "control volumes") such that $\overline{\Omega}=\cup_{K\in\mathcal{M}}\overline{K}$.*

(ii) *$\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh), such that, for all $\sigma\in\mathcal{E}$, there exists an affine hyperplane E of $\mathbb{R}^d$ and $K\in\mathcal{M}$ verifying that $\sigma\subset\partial K\cap E$ and σ is a nonempty open convex subset of E. We assume that, for all $K\in\mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K=\cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$. We also assume that, for all $\sigma\in\mathcal{E}$, either $\sigma\subset\partial\Omega$ or $\overline{\sigma}=\overline{K}\cap\overline{L}$ for some $(K,L)\in\mathcal{M}\times\mathcal{M}$.*

The d-dimensional measure of a control volume K is denoted by $\mathrm{m}(K)$, and the $(d-1)$-dimensional measure of an edge σ by $\mathrm{m}(\sigma)$; in the integral signs, γ denotes the measure on the edges. If $\sigma \in \mathcal{E}_K$, then $\mathbf{n}_{K,\sigma}$ is the unit normal to σ outward to K. In the case where $\sigma \in \mathcal{E}$ satisfies $\overline{\sigma} = \overline{K} \cap \overline{L}$ for $(K, L) \in \mathcal{M} \times \mathcal{M}$, we denote $\sigma = K|L$ (K and L are then called "neighboring control volumes"). We define the set of interior (resp., boundary) edges as $\mathcal{E}_{\mathrm{int}} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$ (resp., $\mathcal{E}_{\mathrm{ext}} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$). For all $K \in \mathcal{M}$ and all $\sigma \in \mathcal{E}$, $\mathbf{x}_K$ and $\mathbf{x}_\sigma$ are the respective barycenters of K and σ.

The size of a mesh $\mathcal{D}$ is $\mathrm{size}(\mathcal{D}) = \sup_{K\in\mathcal{M}} \mathrm{diam}(K)$. The following quantity measures the regularity of the mesh

$$\mathrm{regul}(\mathcal{D}) = \sup\left\{\max\left(\frac{\mathrm{diam}(K)^d}{\rho_K^d}, \mathrm{Card}(\mathcal{E}_K)\right)\ ;\ K \in \mathcal{M}\right\},$$

where, for $K \in \mathcal{M}$, ρ_K is the supremum of the radius of the balls contained in K. The definition of $\mathrm{regul}(\mathcal{D})$ implies that, if ω_d is the volume of the unit ball in $\mathbb{R}^d$, for all $K \in \mathcal{M}$,

$$\mathrm{diam}(K)^d \le \mathrm{regul}(\mathcal{D})\rho_K^d \le \frac{\mathrm{regul}(\mathcal{D})}{\omega_d}\mathrm{m}(K)\,. \tag{16}$$

Remark 2.1. We ask for very few geometrical constraints on the mesh of Ω. This is particularly important since, in real-world problems, meshes used in basin and reservoir simulations can be quite irregular and not admissible in the usual finite element or finite volume senses (see [14]).

Our scheme is based on the mixed finite volume scheme introduced in [9] and, for elliptic equations, [8]. Its main goal is to handle a wide variety of grids for heterogeneous and anisotropic operators while giving strong convergence of approximate gradients. Therefore, this scheme applied to (8) provides a strong approximation of $\bar{\mathbf{U}}$, which can then be used in the discretization of the convective term $\mathrm{div}(\bar{c}\bar{\mathbf{U}})$ in the parabolic equation.

The idea is to consider, besides unknowns which approximate the functions $(\bar{p}, \bar{c})$, unknowns which approximate the gradients of these functions, as well as unknowns which stand for the fluxes associated with the differential operators. Thus, if $\mathcal{D}$ is an admissible mesh of Ω and $k > 0$ is a time step (we always choose time steps such that $N_k = T/k$ is an integer), we consider, for all $n = 1, \ldots, N_k$ and all $K \in \mathcal{M}$, unknowns $(p_K^n, \mathbf{v}_K^n)$ which stand for approximate values of $(\bar{p}, \nabla\bar{p})$ on $[(n-1)k, nk) \times K$ and numbers $F_{K,\sigma}^n$ (for $\sigma \in \mathcal{E}_K$) which stand for approximate values of $-\int_\sigma \bar{\mathbf{U}} \cdot \mathbf{n}_{K,\sigma}\, d\gamma$ on $[(n-1)k, nk)$. Similarly, the unknowns $(c_K^n, \mathbf{w}_K^n)$ approximate $(\bar{c}, \nabla\bar{c})$ on $[(n-1)k, nk) \times K$ and the numbers $G_{K,\sigma}^n$ (for $\sigma \in \mathcal{E}_K$) approximate $\int_\sigma D(\cdot, \bar{\mathbf{U}})\nabla\bar{c} \cdot \mathbf{n}_{K,\sigma}\, d\gamma$ on $[(n-1)k, nk)$.

The quantities $q_K^{+,n}$, $q_K^{-,n}$, and $\widehat{c}_K^n$ denote the mean values of q^+, q^-, and $\widehat{c}$ on $[(n-1)k, nk) \times K$, and Φ_K, c_K^0, $A_K(s)$, and $D_K(\xi)$ are the mean values of Φ, c_0, $A(\cdot, s)$, and $D(\cdot, \xi)$ on K. We also take positive numbers $(\nu_K)_{K\in\mathcal{M}}$. The scheme for (8) reads as follows: for all $n = 1, \ldots, N_k$,

$$\begin{aligned} &\mathbf{v}_K^n \cdot (\mathbf{x}_\sigma - \mathbf{x}_K) + \mathbf{v}_L^n \cdot (\mathbf{x}_L - \mathbf{x}_\sigma) + \nu_K \mathrm{m}(K) F_{K,\sigma}^n - \nu_L \mathrm{m}(L) F_{L,\sigma}^n \\ &\qquad = p_L^n - p_K^n \quad \forall \sigma = K|L \in \mathcal{E}_{\mathrm{int}}, \end{aligned} \tag{17}$$

$$F_{K,\sigma}^n + F_{L,\sigma}^n = 0 \quad \forall \sigma = K|L \in \mathcal{E}_{\mathrm{int}}\,, \tag{18}$$

$$\mathbf{U}_K^n = -A_K(c_K^{n-1})\mathbf{v}_K^n \quad \forall K \in \mathcal{M}\,, \tag{19}$$

$$\mathrm{m}(K)\mathbf{U}_K^n = -\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^n(\mathbf{x}_\sigma - \mathbf{x}_K) \quad \forall K \in \mathcal{M}\,, \tag{20}$$

$$-\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^n = \mathrm{m}(K)q_K^{+,n} - \mathrm{m}(K)q_K^{-,n} \quad \forall K \in \mathcal{M}\,, \tag{21}$$

$$\sum_{K\in\mathcal{M}} \mathrm{m}(K)p_K^n = 0\,, \tag{22}$$

$$F_{K,\sigma}^n = 0 \quad \forall K \in \mathcal{M}\,,\ \forall \sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}}\,. \tag{23}$$

Denoting by $(-F_{K,\sigma}^n)^+$ and $(-F_{K,\sigma}^n)^-$ the positive and negative parts of $-F_{K,\sigma}^n$, the scheme for (9) reads as follows: for all $n = 1, \ldots, N_k$,

$$\begin{aligned} &\mathbf{w}_K^n \cdot (\mathbf{x}_\sigma - \mathbf{x}_K) + \mathbf{w}_L^n \cdot (\mathbf{x}_L - \mathbf{x}_\sigma) + \nu_K \mathrm{m}(K) G_{K,\sigma}^n - \nu_L \mathrm{m}(L) G_{L,\sigma}^n \\ &\qquad = c_L^n - c_K^n \quad \forall \sigma = K|L \in \mathcal{E}_{\mathrm{int}}\,, \end{aligned} \tag{24}$$

$$G_{K,\sigma}^n + G_{L,\sigma}^n = 0 \quad \forall \sigma = K|L \in \mathcal{E}_{\mathrm{int}}\,, \tag{25}$$

$$\mathrm{m}(K) D_K(\mathbf{U}_K^n)\mathbf{w}_K^n = \sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}^n(\mathbf{x}_\sigma - \mathbf{x}_K) \quad \forall K \in \mathcal{M}\,, \tag{26}$$

$$\begin{aligned} &\mathrm{m}(K)\Phi_K \frac{c_K^n - c_K^{n-1}}{k} - \sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}^n + \sum_{\substack{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int}}\\ \sigma=K|L}} \left[(-F_{K,\sigma}^n)^+ c_K^n - (-F_{K,\sigma}^n)^- c_L^n\right] \\ &\qquad + \mathrm{m}(K) q_K^{-,n} c_K^n = \mathrm{m}(K) q_K^{+,n} \hat{c}_K^n \quad \forall K \in \mathcal{M}\,, \end{aligned} \tag{27}$$

$$G_{K,\sigma}^n = 0 \quad \forall K \in \mathcal{M}\,,\ \forall \sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}}\,. \tag{28}$$

Let us explain why each equation of this scheme is quite natural.

- If we take $\nu_K = 0$, (17) and (24) state that $\mathbf{v}_K^n$ ($\approx \nabla\bar{p}$) and $\mathbf{w}_K^n$ ($\approx \nabla\bar{c}$) are "discrete gradients" of p_K^n ($\approx \bar{p}$) and c_K^n ($\approx \bar{c}$). The penalization using the fluxes (i.e., with $\nu_K > 0$) is added to ensure the stability of the scheme.
- Equations (18) and (25) state the conservation of the fluxes, and (23) and (28) translate the no flow boundary conditions.
- Equations (21) and (27) come from the integration on a control volume and on a time step of the PDEs in (8) and (9). Notice that, as usual, we have chosen a time-implicit scheme for the convection-diffusion equation with an upwind discretization of the convective term.
- Equations (19) and (22) are expressions of $\bar{\mathbf{U}} = -A(\cdot,\bar{c})\nabla\bar{p}$ and $\int_\Omega \bar{p}(t,\cdot) = 0$.
- Equations (20) and (26) come from the reconstruction formula given in Lemma 7.1, since $F_{K,\sigma}^n$ and $G_{K,\sigma}^n$ are approximations of the fluxes of $-\bar{\mathbf{U}}$ and $D(\cdot,\bar{\mathbf{U}})\nabla\bar{c}$.

In the following, if $a = (a_K^n)_{n=1,\dots,N_k\,,\,K\in\mathcal{M}}$ is a family of numbers (or vectors), we use a to denote the piecewise constant function on $[0,T)\times\Omega$ which is equal to a_K^n on $[(n-1)k, nk)\times K$. Similarly, for a fixed n, $a^n = (a_K^n)_{K\in\mathcal{M}}$ is identified with the function on Ω which takes the constant value a_K^n on the control volume K. Hence p denotes both the family $(p_K^n)_{n=1,\dots,N_k\,,\,K\in\mathcal{M}}$ and the corresponding function on $[0,T)\times\Omega$. We also denote by F and G the families $(F_{K,\sigma}^n)_{n=1,\dots,N_k\,,\,K\in\mathcal{M}\,,\,\sigma\in\mathcal{E}_K}$ and $(G_{K,\sigma}^n)_{n=1,\dots,N_k\,,\,K\in\mathcal{M}\,,\,\sigma\in\mathcal{E}_K}$.

THEOREM 2.1. *Let Ω be a convex polygonal bounded domain in $\mathbb{R}^d$ and let $T>0$. Assume* (10)–(15) *hold. Let $\mathcal{D}$ be an admissible mesh of Ω and $k>0$ such that T/k is an integer. Then there exists a unique solution $(p,\mathbf{v},\mathbf{U},F,c,\mathbf{w},G)$ to* (17)–(28).

THEOREM 2.2. *Let Ω be a convex polygonal bounded domain in $\mathbb{R}^d$ and let $T>0$. Assume* (10)–(15) *hold. Let $\nu_0>0$ and $\beta\in(2-2d, 4-2d)$. Let $(\mathcal{D}_m)_{m\geq 1}$ be a sequence of admissible meshes of Ω such that $\mathrm{size}(\mathcal{D}_m)\to 0$ as $m\to\infty$ and $(\mathrm{regul}(\mathcal{D}_m))_{m\geq 1}$ is bounded; assume that there exists C_1 such that, for all $m\geq 1$,*

$$(29)\quad \forall K,L\in\mathcal{M}_m \text{ neighboring control volumes, } \mathrm{diam}(K)^{2-\beta-d}\leq C_1\mathrm{diam}(L)^{d-2}.$$

For all $K\in\mathcal{M}_m$, we take $\nu_K=\nu_0\mathrm{diam}(K)^\beta$. Let $k_m>0$ be such that $N_{k_m}=T/k_m$ is an integer and $k_m\to 0$ as $m\to\infty$, and denote by $(p^m,\mathbf{v}^m,\mathbf{U}^m,F^m,c^m,\mathbf{w}^m,G^m)$ the solution to (17)–(28) *with $\mathcal{D}=\mathcal{D}_m$ and $k=k_m$. Then, up to a subsequence, as $m\to\infty$,*

$$\begin{aligned}
&p^m\to\bar p && \text{weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ and strongly in } L^p(0,T;L^q(\Omega))\\
& && \text{for all } p<\infty \text{ and all } q<2;\\
&\mathbf{v}^m\to\nabla\bar p && \text{weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega))^d \text{ and strongly in } L^2((0,T)\times\Omega)^d;\\
&\mathbf{U}^m\to\bar{\mathbf{U}} && \text{weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega))^d \text{ and strongly in } L^2((0,T)\times\Omega)^d;\\
&c^m\to\bar c && \text{weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ and strongly in } L^p(0,T;L^q(\Omega))\\
& && \text{for all } p<\infty \text{ and all } q<2;\\
&\mathbf{w}^m\to\nabla\bar c && \text{weakly in } L^2((0,T)\times\Omega)^d,
\end{aligned}$$

where $(\bar p,\bar{\mathbf{U}},\bar c)$ is a weak solution to (8)–(9).

Remark 2.2. As usual in finite volume schemes, we do not assume the existence of a solution to the continuous problem; this existence is obtained as a byproduct of the proof of convergence. In particular, this means that, contrary to [5] or [11], the convergence of the mixed finite volume scheme is proved here under no regularity assumption on the solution to (8)–(9). The convergence occurs only up to a subsequence because, with such a lack of regularity, the uniqueness of the solution is not known (see [15]); in the case where the solution is unique (for instance, under suitable regularity assumptions), then the whole sequence converges.

Remark 2.3. Note that, since $4-\beta-2d\geq 0$, one way to satisfy (29) is to ask that $\mathrm{diam}(K)\leq C_2\mathrm{diam}(L)$ for all neighboring control volumes K and L of a mesh. But (29) allows more freedom on the meshes (for example, if $d=1$ and $\beta\in(0,1]$ or if $d=2$ and $\beta\in(-2,0)$, then (29) is always satisfied).

3. The a priori estimates. We prove a priori estimates on the solution to the scheme.

PROPOSITION 3.1. *Let Ω be a convex polygonal bounded domain in $\mathbb{R}^d$ and let $T>0$. Assume* (10)–(11) *hold. Let $\mathcal{D}$ be an admissible mesh of Ω such that $\mathrm{regul}(\mathcal{D})\leq\theta$ for some $\theta>0$, and let $k>0$ be such that $N_k=T/k$ is an integer. Let*

$(\nu_K)_{K\in\mathcal{M}}$ be a family of positive numbers such that, for some $\nu_0>0$ and $\beta\geq 2-2d$, $\nu_K\leq \nu_0 \mathrm{diam}(K)^\beta$ for all $K\in\mathcal{M}$. Then there exists C_3 only depending on d, Ω, θ, β, ν_0, α_A, and Λ_A such that, for any numbers $(c_K^{n-1})_{n=1,\dots,N_k\,,\,K\in\mathcal{M}}$, any solution $(p,\mathbf{v},\mathbf{U},F)$ to (17)–(23) satisfies

$$\|p\|^2_{L^\infty(0,T;L^2(\Omega))}+\|\mathbf{v}\|^2_{L^\infty(0,T;L^2(\Omega))^d}+\|\mathbf{U}\|^2_{L^\infty(0,T;L^2(\Omega))^d}$$
$$+\sup_{n=1,\dots,N_k}\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu_K \mathrm{m}(K)|F^n_{K,\sigma}|^2\leq C_3\|q^+-q^-\|^2_{L^\infty(0,T;L^2(\Omega))}.$$

Proof. Let $n\in[1,N_k]$. Multiply (21) by p_K^n, sum over all control volumes, and gather by edges using (18). Thanks to (23), the terms involving boundary edges disappear, and this leads to

$$\sum_{\sigma=K|L\in\mathcal{E}_{\rm int}}F^n_{K,\sigma}(p^n_L-p^n_K)=\sum_{K\in\mathcal{M}}\mathrm{m}(K)(q_K^{+,n}-q_K^{-,n})p_K^n=\int_\Omega(q^{+,n}-q^{-,n})p^n,$$

where $q^{+,n}(\cdot)-q^{-,n}(\cdot)=\frac{1}{k}\int_{(n-1)k}^{nk}q^+(t,\cdot)-q^-(t,\cdot)\,dt$. Substituting (17) into this equality and gathering by control volumes (still using (18) and (23)), we deduce

$$\int_\Omega(q^{+,n}-q^{-,n})p^n=\sum_{\sigma=K|L\in\mathcal{E}_{\rm int}}F^n_{K,\sigma}\left(\mathbf{v}^n_K\cdot(\mathbf{x}_\sigma-\mathbf{x}_K)+\mathbf{v}^n_L\cdot(\mathbf{x}_L-\mathbf{x}_\sigma)\right)$$
$$+\sum_{\sigma=K|L\in\mathcal{E}_{\rm int}}F^n_{K,\sigma}\left(\nu_K\mathrm{m}(K)F^n_{K,\sigma}-\nu_L\mathrm{m}(L)F^n_{L,\sigma}\right)$$
$$=\sum_{K\in\mathcal{M}}\mathbf{v}^n_K\cdot\sum_{\sigma\in\mathcal{E}_K}F^n_{K,\sigma}(\mathbf{x}_\sigma-\mathbf{x}_K)+\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu_K\mathrm{m}(K)|F^n_{K,\sigma}|^2. \tag{30}$$

Thanks to (20), (19), and hypothesis (11), we find

$$\|q^{+,n}-q^{-,n}\|_{L^2(\Omega)}\|p^n\|_{L^2(\Omega)}\geq\alpha_A\|\mathbf{v}^n\|^2_{L^2(\Omega)^d}+\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu_K\mathrm{m}(K)|F^n_{K,\sigma}|^2. \tag{31}$$

We notice that (17) is exactly (61) for $(p^n,\mathbf{v}^n,F^n)$. Hence, since p^n satisfies (22), we can apply the discrete Poincaré–Wirtinger inequality given in Lemma 7.2 to get

$$\|p^n\|_{L^2(\Omega)}\leq C_4\left(\|\mathbf{v}^n\|_{L^2(\Omega)^d}+\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\mathrm{diam}(K)^{2d-2}\nu_K^2\mathrm{m}(K)|F^n_{K,\sigma}|^2\right)^{\frac12}\right),$$

where C_4 depends only on d, Ω, and θ. By choice of ν_K, we have $\mathrm{diam}(K)^{2d-2}\nu_K\leq\nu_0\mathrm{diam}(K)^{2d-2+\beta}$; but $2d-2+\beta\geq0$, and thus $\mathrm{diam}(K)^{2d-2}\nu_K\leq\nu_0\mathrm{diam}(\Omega)^{2d-2+\beta}$. Hence

$$\|p^n\|_{L^2(\Omega)}\leq C_5\left(\|\mathbf{v}^n\|_{L^2(\Omega)^d}+\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu_K\mathrm{m}(K)|F^n_{K,\sigma}|^2\right)^{\frac12}\right), \tag{32}$$

where C_5 depends only on d, Ω, θ, β, and ν_0. Substituting this into (31), we obtain

$$\alpha_A\|\mathbf{v}^n\|^2_{L^2(\Omega)^d}+\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu_K\mathrm{m}(K)|F^n_{K,\sigma}|^2\leq C_5\|q^{+,n}-q^{-,n}\|_{L^2(\Omega)}\|\mathbf{v}^n\|_{L^2(\Omega)^d}$$
$$+C_5\|q^{+,n}-q^{-,n}\|_{L^2(\Omega)}\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu_K\mathrm{m}(K)|F^n_{K,\sigma}|^2\right)^{\frac12}.$$

Using Young's inequality, this gives the desired bound on $\mathbf{v}$ and F and, coming back to (32), the bound on p. The bound on $\mathbf{U}$ derives from the one on $\mathbf{v}$, since A is bounded (see (11)). ∎

PROPOSITION 3.2. *Let Ω be a convex polygonal bounded domain in $\mathbb{R}^d$ and let $T>0$. Assume* (10) *and* (12)–(15) *hold. Let $\mathcal{D}$ be an admissible mesh of Ω, and let $k>0$ be such that $N_k = T/k$ is an integer. Let $(\nu_K)_{K\in\mathcal{M}}$ be a family of positive numbers. Assume that $F=(F^n_{K,\sigma})_{n=1,\dots,N_k\,,\,K\in\mathcal{M}\,,\,\sigma\in\mathcal{E}_K}$ satisfies* (18), (21), *and* (23), *and let $\mathbf{U}=(\mathbf{U}^n_K)_{n=1,\dots,N_k\,,\,K\in\mathcal{M}}$ be a family of vectors in $\mathbb{R}^d$. Then there exists C_6 depending only on d, Ω, T, α_D, and Φ_* such that any solution $(c,\mathbf{w},G)$ to* (24)–(28) *satisfies*

$$\|c\|^2_{L^\infty(0,T;L^2(\Omega))}+\|\mathbf{w}\|^2_{L^2((0,T)\times\Omega)^d}+\|\,|\mathbf{U}|^{1/2}\,|\mathbf{w}|\,\|^2_{L^2((0,T)\times\Omega)}$$
$$+\sum_{n=1}^{N_k}k\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu_K\mathrm{m}(K)|G^n_{K,\sigma}|^2\le C_6\|c_0\|^2_{L^2(\Omega)}+C_6\|q^+\|^2_{L^\infty(0,T;L^2(\Omega))}.$$

Proof. Multiply (27) by c^n_K and sum over all control volumes. Noting that $(c^n_K - c^{n-1}_K)c^n_K\ge\frac12\left((c^n_K)^2-(c^{n-1}_K)^2\right)$ and using (25) to gather by edges (no boundary term remains thanks to (28)), we obtain, since $\Phi_K\ge 0$,

$$\frac{1}{2k}\sum_{K\in\mathcal{M}}\mathrm{m}(K)\Phi_K\left((c^n_K)^2-(c^{n-1}_K)^2\right)+\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}}G^n_{K,\sigma}(c^n_L-c^n_K)+\sum_{K\in\mathcal{M}}\mathrm{m}(K)q^{-,n}_K(c^n_K)^2$$
$$(33)\quad +\sum_{K\in\mathcal{M}}\sum_{\substack{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\rm int}\\ \sigma=K|L}}\left[(-F^n_{K,\sigma})^+c^n_K-(-F^n_{K,\sigma})^-c^n_L\right]c^n_K\le\sum_{K\in\mathcal{M}}\mathrm{m}(K)|q^{+,n}_K\widehat{c}^n_K|\,|c^n_K|.$$

Let us denote by $\mathcal{T}$ the fourth term of the inequality. Gathering by edges and using (18), which implies $(-F^n_{L,\sigma})^+=(-F^n_{K,\sigma})^-$ and $(-F^n_{L,\sigma})^-=(-F^n_{K,\sigma})^+$, yields

$$\mathcal{T}\;=\sum_{\sigma=K|L\in\mathcal{E}_{\rm int}}\left[(-F^n_{K,\sigma})^+\left(c^n_K(c^n_K-c^n_L)\right)+(-F^n_{K,\sigma})^-\left(c^n_L(c^n_L-c^n_K)\right)\right].$$

But $c^n_K(c^n_K-c^n_L)\ge\frac12\left((c^n_K)^2-(c^n_L)^2\right)$ and $c^n_L(c^n_L-c^n_K)\ge\frac12\left((c^n_L)^2-(c^n_K)^2\right)$, hence

$$\mathcal{T}\ge\frac12\sum_{\sigma=K|L\in\mathcal{E}_{\rm int}}\left[(-F^n_{K,\sigma})^+-(-F^n_{K,\sigma})^-\right]\left((c^n_K)^2-(c^n_L)^2\right)$$
$$\ge\frac12\sum_{\sigma=K|L\in\mathcal{E}_{\rm int}}-F^n_{K,\sigma}\left((c^n_K)^2-(c^n_L)^2\right),$$

which gives, gathering by control volumes and using (18), (23), and (21),

$$\mathcal{T}\ge\frac12\sum_{K\in\mathcal{M}}(c^n_K)^2\left(-\sum_{\sigma\in\mathcal{E}_K}F^n_{K,\sigma}\right)\ge\frac12\sum_{K\in\mathcal{M}}\mathrm{m}(K)(c^n_K)^2(q^{+,n}_K-q^{-,n}_K).$$

Since

$$\frac12\sum_{K\in\mathcal{M}}\mathrm{m}(K)(c^n_K)^2(q^{+,n}_K-q^{-,n}_K)+\sum_{K\in\mathcal{M}}\mathrm{m}(K)q^{-,n}_K(c^n_K)^2$$
$$=\frac12\sum_{K\in\mathcal{M}}\mathrm{m}(K)(q^{+,n}_K+q^{-,n}_K)(c^n_K)^2\ge 0$$

(because q^+ and q^- are nonnegative), we deduce from (33) that

$$\frac{1}{2k}\sum_{K\in\mathcal{M}} \mathrm{m}(K)\Phi_K\left((c_K^n)^2-(c_K^{n-1})^2\right)+\sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{int}}} G_{K,\sigma}^n(c_L^n-c_K^n)$$
$$\le \sum_{K\in\mathcal{M}} \mathrm{m}(K)|q_K^{+,n}\widehat{c}_K^n|\,|c_K^n|. \tag{34}$$

Using (24) and gathering by control volumes, we get, thanks to (25), (28), and (26),

$$\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\\ \sigma=K|L}} G_{K,\sigma}^n(c_L^n-c_K^n)=\sum_{K\in\mathcal{M}}\mathbf{w}_K^n\cdot\sum_{\sigma\in\mathcal{E}_K}G_{K,\sigma}^n(\mathbf{x}_\sigma-\mathbf{x}_K)+\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu_K\mathrm{m}(K)|G_{K,\sigma}^n|^2$$
$$=\sum_{K\in\mathcal{M}}\mathrm{m}(K)D_K(\mathbf{U}_K^n)\mathbf{w}_K^n\cdot\mathbf{w}_K^n+\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu_K\mathrm{m}(K)|G_{K,\sigma}^n|^2. \tag{35}$$

We then use (12) and plug the corresponding lower bound into (34), which we multiply by k and sum over $n=1,\dots,N$ (for some $N\in[1,N_k]$); since $|\widehat{c}|\le 1$, this leads to

$$\frac{1}{2}\sum_{K\in\mathcal{M}}\mathrm{m}(K)\Phi_K\left((c_K^N)^2-(c_K^0)^2\right)+\alpha_D\sum_{n=1}^N k\sum_{K\in\mathcal{M}}\mathrm{m}(K)(1+|\mathbf{U}_K^n|)|\mathbf{w}_K^n|^2$$
$$+\sum_{n=1}^N k\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu_K\mathrm{m}(K)|G_{K,\sigma}^n|^2\le T\|q^+\|_{L^\infty(0,T;L^2(\Omega))}\|c\|_{L^\infty(0,T;L^2(\Omega))}. \tag{36}$$

This gives in particular, by (13) and the definition of $(c_K^0)_{K\in\mathcal{M}}$,

$$\frac{\Phi_*}{2}\sum_{K\in\mathcal{M}}\mathrm{m}(K)(c_K^N)^2\le\frac{\Phi_*^{-1}}{2}\|c_0\|_{L^2(\Omega)}^2+\frac{T^2}{\Phi_*}\|q^+\|_{L^\infty(0,T;L^2(\Omega))}^2+\frac{\Phi_*}{4}\|c\|_{L^\infty(0,T;L^2(\Omega))}^2.$$

Since $\|c\|_{L^\infty(0,T;L^2(\Omega))}^2=\sup_{r=1,\dots,N_k}\sum_{K\in\mathcal{M}}\mathrm{m}(K)(c_K^r)^2$, this inequality, valid for all $1\le N\le N_k$, gives the estimate on $\|c\|_{L^\infty(0,T;L^2(\Omega))}$. Plugged into (36), it gives the desired bounds on $\mathbf{w}$, $|\mathbf{U}|^{1/2}\,|\mathbf{w}|$ and G. ◻

4. Existence and uniqueness of numerical solutions. In this section, we prove Theorem 2.1. Note first that (17)–(23) and (24)–(28) are decoupled systems: at time step n, the knowledge of c_K^{n-1} (or of c_K^0 if $n=1$) shows that (17)–(23) is a linear system for $(p^n,\mathbf{v}^n,\mathbf{U}^n,(F_{K,\sigma}^n)_{K\in\mathcal{M},\sigma\in\mathcal{E}_K})$; once this system is solved, $\mathbf{U}^n$ is known and (24)–(28) becomes a linear system for $(c^n,\mathbf{w}^n,(G_{K,\sigma}^n)_{K\in\mathcal{M},\sigma\in\mathcal{E}_K})$. Hence, to prove Theorem 2.1 we only need to show that these linear systems are solvable.

Let us first consider the system on $(c^n,\mathbf{w}^n,(G_{K,\sigma}^n)_{K\in\mathcal{M},\sigma\in\mathcal{E}_K})$. By (25) and (28), we can consider that there is only one flux by interior edge and this system therefore has $(d+1)\mathrm{Card}(\mathcal{M})+\mathrm{Card}(\mathcal{E}_{\mathrm{int}})$ unknowns, with as many remaining equations ((26) gives $d\mathrm{Card}(\mathcal{M})$ equations, (27) another $\mathrm{Card}(\mathcal{M})$ equations, and (24) the last $\mathrm{Card}(\mathcal{E}_{\mathrm{int}})$ equations). Hence, this first system is a square system. Assume that $(c^n,\mathbf{w}^n,(G_{K,\sigma}^n)_{K\in\mathcal{M},\sigma\in\mathcal{E}_K})$ is a solution with a null right-hand side, i.e., with $c^{n-1}=\widehat{c}^n=0$; then (34) and (35) show that this solution is null, and therefore that this system is invertible.

Without the relation (22) and since we can eliminate $\mathbf{U}^n$ by (19), the system on $(p^n,\mathbf{v}^n,\mathbf{U}^n,(F_{K,\sigma}^n)_{K\in\mathcal{M},\sigma\in\mathcal{E}_K})$ also has $(d+1)\mathrm{Card}(\mathcal{M})+\mathrm{Card}(\mathcal{E}_{\mathrm{int}})$ unknowns and the same number of equations. However, it is not invertible since its kernel clearly contains $(\mathcal{C}, 0, 0, 0)$, where $\mathcal{C} \in \mathbb{R}^{\mathrm{Card}(\mathcal{M})}$ is any constant vector; in fact, the estimates in the preceding section show that these vectors fully describe the kernel of ((17)–(21), (23)): if $(p^n, \mathbf{v}^n, \mathbf{U}^n, (F^n_{K,\sigma})_{K\in\mathcal{M},\sigma\in\mathcal{E}_K})$ belongs to this kernel, then $(p^n - \mathcal{C}, \mathbf{v}^n, \mathbf{U}^n, (F^n_{K,\sigma})_{K\in\mathcal{M},\sigma\in\mathcal{E}_K})$, where $\mathcal{C}$ is a constant vector such that (22) holds with $p^n - \mathcal{C}$, satisfies (17)–(23) with $q^{+,n} - q^{-,n} = 0$, and is therefore null by Proposition 3.1, which shows that $(p^n, \mathbf{v}^n, \mathbf{U}^n, (F^n_{K,\sigma})_{K\in\mathcal{M},\sigma\in\mathcal{E}_K}) = (\mathcal{C}, 0, 0, 0)$.

Summing (21) over K and using (18) and (23), we obtain that a necessary condition for ((17)–(21), (23)) to have a solution is $\sum_{K\in\mathcal{M}} \mathrm{m}(K) q_K^{n,+} - \mathrm{m}(K) q_K^{n,-} = 0$. Since the kernel of the square system ((17)–(21), (23)) has dimension 1, this condition is also sufficient, and is clearly satisfied by the data we consider thanks to (10). We can therefore always find a solution to ((17)–(21), (23)) and, in view of the kernel of this system, (22) then selects one and only one solution.

Remark 4.1. As said above, at each time step the scheme (17)–(28) can be decoupled in two successive linear systems, (17)–(23) and then (24)–(28), each one with size $(d+1)\mathrm{Card}(\mathcal{M}) + \mathrm{Card}(\mathcal{E}_{\mathrm{int}})$. However, it is possible to proceed to an algebraic elimination which leads to smaller sparse linear systems, following [18] for the mixed finite element method and [9] for the mixed finite volume method for anisotropic diffusion problems.

The computation of $(p, \mathbf{v}, \mathbf{U}, F)$ at each time step reduces to the resolution of a linear system of size $\mathrm{Card}(\mathcal{E}_{\mathrm{int}})$, while the computation of $(c, \mathbf{w}, G)$ demands the resolution of a linear system of size $\mathrm{Card}(\mathcal{M}) + \mathrm{Card}(\mathcal{E}_{\mathrm{int}})$ (the size of this last system cannot be reduced to $\mathrm{Card}(\mathcal{E}_{\mathrm{int}})$ because of the upwind and implicit discretization of the convective term $\mathrm{div}(c\mathbf{U})$).

5. Proof of the convergence of the scheme. In this section, we prove Theorem 2.2. To simplify the notation, we drop the index m and thus prove the desired convergence as $\mathrm{size}(\mathcal{D}) \to 0$ and $k \to 0$, with $\mathrm{regul}(\mathcal{D})$ bounded and (29) uniformly satisfied for all considered meshes. Under these assumptions, Propositions 3.1 and 3.2 give estimates which are uniform with respect to the meshes and time steps.

5.1. Compactness of the concentration. We prove the strong compactness of the concentration.

LEMMA 5.1. *Under the assumptions of Theorem* 2.2, *c is relatively compact in* $L^1(0, T; L^1_{\mathrm{loc}}(\Omega))$.

Proof. We first construct an affine interpolant $\widetilde{c}$ of c and prove, thanks to Aubin's theorem, the relative compactness of this interpolant in a weaker space. We then deduce the compactness of c in $L^1(0, T; L^1_{\mathrm{loc}}(\Omega))$. ∎

Step 1. An affine interpolant of c.

We define $\widetilde{c} : [0, T) \times \Omega \to \mathbb{R}$ as, for all $n = 1, \ldots, N_k$ and all $t \in [(n-1)k, nk)$,

$$\widetilde{c}(t, \cdot) = \frac{t - (n-1)k}{k} c_K^n + \frac{nk - t}{k} c_K^{n-1} \quad \text{on } K.$$

The estimates of Proposition 3.2 and the definition of $(c_K^0)_{K\in\mathcal{M}}$ ensure the bound of $\|\widetilde{c}\|_{L^\infty(0,T;L^2(\Omega))}$. For all $n = 1, \ldots, N_k$ and all $t \in [(n-1)k, nk)$, we have $\partial_t \widetilde{c}(t, \cdot) = \frac{c_K^n - c_K^{n-1}}{k}$ on K. Hence, denoting by $\Phi_{\mathcal{D}}$ the piecewise constant function on Ω equal to Φ_K on K and taking $\varphi \in C_c^2(\Omega)$, we deduce from (27) that if φ_K is the mean value of φ on K,

$$\int_\Omega \Phi_{\mathcal{D}}(x)\partial_t\widetilde{c}(t,x)\varphi(x)\,dx = \sum_{K\in\mathcal{M}} \mathrm{m}(K)\Phi_K \frac{c_K^n - c_K^{n-1}}{k}\varphi_K$$

$$= \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}^n \varphi_K - \sum_{K\in\mathcal{M}} \mathrm{m}(K) q_K^{-,n} c_K^n \varphi_K + \sum_{K\in\mathcal{M}} \mathrm{m}(K) q_K^{+,n}\widehat{c}_K^n\varphi_K$$

$$- \sum_{K\in\mathcal{M}}\sum_{\sigma=K|L\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int}}} \left[(-F_{K,\sigma}^n)^+ c_K^n - (-F_{K,\sigma}^n)^- c_L^n\right]\varphi_K. \tag{37}$$

Let us denote by T_1, T_3, T_4, and T_2 the four terms on the right-hand side of this equality. In the following, C_i denote constants which do not depend on k, $\mathcal{D}$, n, K, or φ; we induce $C_c^2(\Omega)$ with the norm $\|\varphi\| = \sup_{x\in\Omega}\left(|\varphi(x)| + |\nabla\varphi(x)| + |D^2\varphi(x)|\right)$.

Since $\mathbf{x}_K$ is the barycenter of K and φ is regular we have $\varphi(\mathbf{x}_\sigma) - \varphi_K = \nabla\varphi(\mathbf{x}_K)\cdot(\mathbf{x}_\sigma - \mathbf{x}_K) + R_{K,\sigma}$ for all $\sigma\in\mathcal{E}_K$, with $|R_{K,\sigma}| \le C_7\|\varphi\|\mathrm{diam}(K)^2$. Hence,

$$\varphi_L - \varphi_K = \nabla\varphi(\mathbf{x}_K)\cdot(\mathbf{x}_\sigma - \mathbf{x}_K) + \nabla\varphi(\mathbf{x}_L)\cdot(\mathbf{x}_L - \mathbf{x}_\sigma) + R_{K,\sigma} - R_{L,\sigma}. \tag{38}$$

Using this equality and gathering by control volumes, we get

$$-T_1 = \sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{int}}} G_{K,\sigma}^n(\varphi_L - \varphi_K)$$

$$= \sum_{K\in\mathcal{M}} \nabla\varphi(\mathbf{x}_K)\cdot\sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}^n(\mathbf{x}_\sigma - \mathbf{x}_K) + \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}^n R_{K,\sigma}$$

$$= \sum_{K\in\mathcal{M}} \mathrm{m}(K)\nabla\varphi(\mathbf{x}_K)\cdot D_K(\mathbf{U}_K^n)\mathbf{w}_K^n + \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}^n R_{K,\sigma}. \tag{39}$$

On one hand, thanks to (12) and to the estimate on $\mathbf{U}$ in $L^\infty(0,T;L^2(\Omega))^d$ (which gives in particular an estimate in $L^\infty(0,T;L^1(\Omega))^d$), we have

$$\left|\sum_{K\in\mathcal{M}} \mathrm{m}(K)\nabla\varphi(\mathbf{x}_K)\cdot D_K(\mathbf{U}_K^n)\mathbf{w}_K^n\right| \le C_8\|\varphi\|\sum_{K\in\mathcal{M}} \mathrm{m}(K)(1+|\mathbf{U}_K^n|)\,|\mathbf{w}_K^n|$$

$$\le C_9\|\varphi\|\left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)(1+|\mathbf{U}_K^n|)|\mathbf{w}_K^n|^2\right)^{\frac{1}{2}}. \tag{40}$$

On the other hand, using $|R_{K,\sigma}| \le C_7\|\varphi\|\mathrm{diam}(K)^2$ and the Cauchy–Schwarz inequality, we get

$$\left|\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}^n R_{K,\sigma}\right|$$

$$\le C_7\|\varphi\|\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K)|G_{K,\sigma}^n|^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \frac{\mathrm{diam}(K)^4}{\nu_K\mathrm{m}(K)}\right)^{\frac{1}{2}}$$

$$\le C_{10}\|\varphi\|\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K)|G_{K,\sigma}^n|^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{M}} \mathrm{diam}(K)^{4-2d-\beta}\mathrm{m}(K)\right)^{\frac{1}{2}} \tag{41}$$

because (16) and the definition of ν_K imply

$$\frac{\mathrm{diam}(K)^4}{\nu_K \mathrm{m}(K)} = \mathrm{m}(K)\frac{\mathrm{diam}(K)^{4-\beta}}{\nu_0 \mathrm{m}(K)^2} \le \frac{1}{\nu_0}\left(\frac{\mathrm{regul}(\mathcal{D})}{\omega_d}\right)^2 \mathrm{m}(K)\mathrm{diam}(K)^{4-2d-\beta}. \tag{42}$$

But $4-2d-\beta \ge 0$ and thus $\mathrm{diam}(K)^{4-2d-\beta} \le \mathrm{diam}(\Omega)^{4-2d-\beta}$. Using this in (41) and substituting the result along with (40) into (39), we deduce the final estimate:

$$\begin{aligned}|T_1| \le {}& C_{11}\|\varphi\| \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)(1+|\mathbf{U}_K^n|)|\mathbf{w}_K^n|^2\right)^{\frac12}\\ &+ C_{11}\|\varphi\|\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K)|G_{K,\sigma}^n|^2\right)^{\frac12}.\end{aligned} \tag{43}$$

For $\sigma = K|L$, set $b_{K,\sigma}^n = (-F_{K,\sigma}^n)^+ c_K^n - (-F_{K,\sigma}^n)^- c_L^n$. By (18), we have $b_{K,\sigma}^n = -b_{L,\sigma}^n$. Hence, using (38) and gathering by control volumes, we get

$$\begin{aligned}T_2 &= \sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{int}}} b_{K,\sigma}^n(\varphi_L - \varphi_K)\\ &= \sum_{K\in\mathcal{M}} \nabla\varphi(\mathbf{x}_K)\cdot \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int}}} b_{K,\sigma}^n(\mathbf{x}_\sigma - \mathbf{x}_K) + \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int}}} b_{K,\sigma}^n R_{K,\sigma}.\end{aligned}$$

But $b_{K,\sigma}^n = -F_{K,\sigma}^n c_K^n + (-F_{K,\sigma}^n)^-(c_K^n - c_L^n)$ and thus, by (23) and (20),

$$T_2 = -\sum_{K\in\mathcal{M}} c_K^n \nabla\varphi(\mathbf{x}_K)\cdot \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^n(\mathbf{x}_\sigma - \mathbf{x}_K) + T_5 = \sum_{K\in\mathcal{M}} \mathrm{m}(K) c_K^n \nabla\varphi(\mathbf{x}_K)\cdot \mathbf{U}_K^n + T_5$$

with

$$T_5 = \sum_{K\in\mathcal{M}} \nabla\varphi(\mathbf{x}_K)\cdot \sum_{\substack{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int}}\\ \sigma=K|L}} (-F_{K,\sigma}^n)^-(c_K^n - c_L^n)(\mathbf{x}_\sigma - \mathbf{x}_K) + \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int}}} b_{K,\sigma}^n R_{K,\sigma}.$$

Let us estimate T_5. The corresponding calculations will be useful later in the proof of the convergence of the concentration. We have

$$\begin{aligned}|T_5| \le {}& \|\varphi\| \sum_{K\in\mathcal{M}}\sum_{\sigma=K|L\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int}}} |F_{K,\sigma}^n|\,|c_K^n - c_L^n|\mathrm{diam}(K)\\ &+ C_7\|\varphi\|\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int}}} |b_{K,\sigma}^n|\mathrm{diam}(K)^2.\end{aligned} \tag{44}$$

But (24) entails

$$|c_K^n - c_L^n| \le |\mathbf{w}_K^n|\mathrm{diam}(K) + |\mathbf{w}_L^n|\mathrm{diam}(L) + \nu_K\mathrm{m}(K)|G_{K,\sigma}^n| + \nu_L \mathrm{m}(L)|G_{L,\sigma}^n|$$

and thus, using $|F_{K,\sigma}^n| = |F_{L,\sigma}^n|$ whenever $\sigma = K|L$,

$$\begin{aligned}|b_{K,\sigma}^n| \le {}& |F_{K,\sigma}^n|\,|c_K^n| + |F_{K,\sigma}^n|\,|\mathbf{w}_K^n|\mathrm{diam}(K) + |F_{K,\sigma}^n|\,|\mathbf{w}_L^n|\mathrm{diam}(L)\\ &+ \nu_K\mathrm{m}(K)|F_{K,\sigma}^n|\,|G_{K,\sigma}^n| + \nu_L\mathrm{m}(L)|F_{L,\sigma}^n|\,|G_{L,\sigma}^n|.\end{aligned}$$

Substituting these two estimates into (44) and bounding $\operatorname{diam}(K)$ either by $\operatorname{diam}(\Omega)$ or $\operatorname{size}(\mathcal{D})$, we get

$$
\begin{aligned}
|T_5| \le{}& C_{12}\|\varphi\| \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} |F^n_{K,\sigma}|\, (|\mathbf{w}^n_K| + |c^n_K|)\operatorname{diam}(K)^2 \\
&+ C_{12}\|\varphi\| \sum_{K\in\mathcal{M}} \sum_{\sigma=K|L\in\mathcal{E}_K\cap\mathcal{E}_{\rm int}} |F^n_{K,\sigma}|\, |\mathbf{w}^n_L| \operatorname{diam}(K)\operatorname{diam}(L) \\
&+ C_{12}\|\varphi\|\operatorname{size}(\mathcal{D}) \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K)|F^n_{K,\sigma}|\, |G^n_{K,\sigma}| \\
&+ C_{12}\|\varphi\|\operatorname{size}(\mathcal{D}) \sum_{K\in\mathcal{M}} \sum_{\sigma=K|L\in\mathcal{E}_K\cap\mathcal{E}_{\rm int}} \nu_L \mathrm{m}(L)|F^n_{L,\sigma}|\, |G^n_{L,\sigma}|. \qquad (45)
\end{aligned}
$$

We successively apply the Cauchy–Schwarz inequality, the fact that $\operatorname{regul}(\mathcal{D})$ is bounded, inequality (42), and the estimates on F from Proposition 3.1. This yields

$$
\begin{aligned}
&\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} |F^n_{K,\sigma}|\, (|\mathbf{w}^n_K| + |c^n_K|)\operatorname{diam}(K)^2 \\
&\le C_{13}\left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K)|F^n_{K,\sigma}|^2\right)^{\frac12} \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)\operatorname{diam}(K)^{4-2d-\beta}(|\mathbf{w}^n_K| + |c^n_K|)^2\right)^{\frac12} \\
&\le C_{14}\operatorname{size}(\mathcal{D})^{\frac{4-2d-\beta}{2}} \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)(|\mathbf{w}^n_K| + |c^n_K|)^2\right)^{\frac12}. \qquad (46)
\end{aligned}
$$

Then we note that (thanks to (23))

$$
\sum_{K\in\mathcal{M}} \sum_{\sigma=K|L\in\mathcal{E}_K\cap\mathcal{E}_{\rm int}} \nu_L \mathrm{m}(L)|F^n_{L,\sigma}|\, |G^n_{L,\sigma}| = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K)|F^n_{K,\sigma}|\, |G^n_{K,\sigma}| \qquad (47)
$$

and, with the estimates on F from Proposition 3.1, we get

$$
\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K)|F^n_{K,\sigma}|\, |G^n_{K,\sigma}| \le C_{15}\left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K)|G^n_{K,\sigma}|^2\right)^{\frac12}. \qquad (48)
$$

Using the fact that $\nu_K = \nu_0\operatorname{diam}(K)^\beta$ and inequalities (16) and (29),we get

$$
\begin{aligned}
&\sum_{K\in\mathcal{M}} \sum_{\sigma=K|L\in\mathcal{E}_K\cap\mathcal{E}_{\rm int}} |F^n_{K,\sigma}|\, |\mathbf{w}^n_L|\operatorname{diam}(K)\operatorname{diam}(L) \\
&\le \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K)|F^n_{K,\sigma}|^2\right)^{\frac12} \left(\sum_{K\in\mathcal{M}} \sum_{\substack{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\rm int}\\ \sigma=K|L}} \frac{\operatorname{diam}(K)^2\operatorname{diam}(L)^2}{\nu_K \mathrm{m}(K)}|\mathbf{w}^n_L|^2\right)^{\frac12} \\
&\le C_{16}\left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K)|F^n_{K,\sigma}|^2\right)^{\frac12} \left(\sum_{L\in\mathcal{M}} |\mathbf{w}^n_L|^2 \sum_{\substack{\sigma\in\mathcal{E}_L\cap\mathcal{E}_{\rm int}\\ \sigma=L|K}} \operatorname{diam}(K)^{2-\beta-d}\operatorname{diam}(L)^2\right)^{\frac12} \\
&\le C_{17}\left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K)|F^n_{K,\sigma}|^2\right)^{\frac12} \left(\sum_{L\in\mathcal{M}} \mathrm{m}(L)|\mathbf{w}^n_L|^2\right)^{\frac12}.
\end{aligned}
$$

Finally, gathering (45), (46), (47), (48), and this last inequality, it yields

$$
\begin{aligned}
|T_5| &= \left| T_2 - \sum_{K\in\mathcal{M}} \mathrm{m}(K) c_K^n \nabla\varphi(\mathbf{x}_K)\cdot \mathbf{U}_K^n \right| \\
&\le C_{18}\|\varphi\| \mathrm{size}(\mathcal{D})^{\frac{4-2d-\beta}{2}} \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)(|\mathbf{w}_K^n| + |c_K^n|)^2 \right)^{\frac12} \\
&\quad + C_{18}\|\varphi\| \left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K) |F_{K,\sigma}^n|^2 \right)^{\frac12} \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)|\mathbf{w}_K^n|^2 \right)^{\frac12} \\
&\quad + C_{18}\|\varphi\| \mathrm{size}(\mathcal{D}) \left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K) |G_{K,\sigma}^n|^2 \right)^{\frac12}. \qquad (49)
\end{aligned}
$$

Thanks to the $L^\infty(0,T;L^2(\Omega))$ estimates on c and $\mathbf{U}$, we also have

$$
\begin{aligned}
\left| \sum_{K\in\mathcal{M}} \mathrm{m}(K) c_K^n \nabla\varphi(\mathbf{x}_K)\cdot \mathbf{U}_K^n \right| &\le \|\varphi\| \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)|c_K^n|^2 \right)^{\frac12} \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)|\mathbf{U}_K^n|^2 \right)^{\frac12} \\
&\le C_{19}\|\varphi\|,
\end{aligned}
$$

and, using the bound on the fluxes $F_{K,\sigma}^n$ from Proposition 3.1, the final estimate on T_2 reads

$$
\begin{aligned}
|T_2| &\le C_{20}\|\varphi\| + C_{20}\|\varphi\| \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)(|\mathbf{w}_K^n| + |c_K^n|)^2 \right)^{\frac12} \qquad (50) \\
&\quad + C_{20}\|\varphi\| \left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K) |G_{K,\sigma}^n|^2 \right)^{\frac12}.
\end{aligned}
$$

The estimates on T_3 and T_4 are straightforward, thanks to the $L^\infty(0,T;L^2(\Omega))$-bound on c; plugging (43) and (50) into (37), we obtain, for all $n=1,\dots,N_k$ and all $t\in[(n-1)k,nk)$,

$$
\begin{aligned}
&\left| \int_\Omega \Phi_{\mathcal{D}}(x)\partial_t \widetilde{c}(t,x)\varphi(x)\,dx \right| \\
&\le C_{21}\|\varphi\| \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)(1+|\mathbf{U}_K^n|)|\mathbf{w}_K^n|^2 \right)^{\frac12} + C_{21}\|\varphi\| \left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K) |G_{K,\sigma}^n|^2 \right)^{\frac12} \\
&\quad + C_{21}\|\varphi\| + C_{21}\|\varphi\| \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)(|\mathbf{w}_K^n| + |c_K^n|)^2 \right)^{\frac12}.
\end{aligned}
$$

Since this inequality is satisfied for all $\varphi\in C_c^2(\Omega)$ and $\Phi_{\mathcal{D}}$ does not depend on t, this gives an estimate on $\|\partial_t(\Phi_{\mathcal{D}}\widetilde{c})(t,\cdot)\|_{(C_c^2(\Omega))'}$ which, squared, leads to

$$
\begin{aligned}
\|\partial_t(\Phi_{\mathcal{D}}\widetilde{c})(t,\cdot)\|_{(C_c^2(\Omega))'}^2 &\le C_{22} \sum_{K\in\mathcal{M}} \mathrm{m}(K)(1+|\mathbf{U}_K^n|)|\mathbf{w}_K^n|^2 + C_{22} \\
&\quad + C_{22} \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K) |G_{K,\sigma}^n|^2 + C_{22} \sum_{K\in\mathcal{M}} \mathrm{m}(K)(|\mathbf{w}_K^n| + |c_K^n|)^2
\end{aligned}
$$

for all $n = 1, \dots, N_k$ and all $t \in [(n-1)k, nk)$. Integrating this last inequality on $t \in [(n-1)k, nk)$ and summing over $n = 1, \dots, N_k$, we prove, thanks to the estimates of Proposition 3.2, that $\partial_t(\Phi_{\mathcal{D}}\widetilde{c})$ is bounded in $L^2(0,T;(C_c^2(\Omega))')$.

Noting that $\Phi_{\mathcal{D}}\widetilde{c}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ (because $\widetilde{c}$ is bounded in this space and $\Phi_{\mathcal{D}}$ is bounded in $L^\infty(\Omega)$), and since $L^2(\Omega)$ is continuously embedded in $(C_c^2(\Omega))'$ (via the natural embedding $f \to (\varphi \to \int_\Omega f\varphi)$), this shows that $\Phi_{\mathcal{D}}\widetilde{c}$ is bounded in $H^1(0,T;(C_c^2(\Omega))')$. But $C_c^2(\Omega)$ is compactly and densely embedded in $C_0(\Omega)$, and, by duality, $(C_0(\Omega))'$ (the space of bounded measures on Ω) is compactly embedded in $(C_c^2(\Omega))'$. Since $L^2(\Omega)$ is continuously embedded in $(C_0(\Omega))'$ (via an embedding which is compatible with the preceding one), the embedding of $L^2(\Omega)$ in $(C_c^2(\Omega))'$ is in fact compact. Hence, by Aubin's compactness theorem we deduce that $\Phi_{\mathcal{D}}\widetilde{c}$ is relatively compact in $C([0,T];(C_c^2(\Omega))')$.

Step 2. Conclusion.

For all $n = 1, \dots, N_k$ and $t \in [(n-1)k, nk)$, we have $\Phi_{\mathcal{D}}c(t,\cdot) = \Phi_{\mathcal{D}}\widetilde{c}(nk,\cdot)$ on Ω (these functions are both equal to $\Phi_K c_K^n$ on each $K \in \mathcal{M}$). We also know (see, e.g., [7]) that $H^1(0,T;(C_c^2(\Omega))')$ is continuously embedded in $C^{1/2}([0,T];(C_c^2(\Omega))')$ (the space of 1/2-Hölder continuous functions $[0,T] \to (C_c^2(\Omega))'$). Hence, $\Phi_{\mathcal{D}}\widetilde{c}$ is also bounded in $C^{1/2}([0,T];(C_c^2(\Omega))')$ and there exists C_{23} not depending on k or $\mathcal{D}$ such that, for all $n = 1, \dots, N_k$ and all $t \in [(n-1)k, nk)$,

$$\|\Phi_{\mathcal{D}}c(t,\cdot) - \Phi_{\mathcal{D}}\widetilde{c}(t,\cdot)\|_{(C_c^2(\Omega))'} = \|\Phi_{\mathcal{D}}\widetilde{c}(nk,\cdot) - \Phi_{\mathcal{D}}\widetilde{c}(t,\cdot)\|_{(C_c^2(\Omega))'} \le C_{23}\sqrt{k}.$$

This means that, as $k \to 0$, $\Phi_{\mathcal{D}}c - \Phi_{\mathcal{D}}\widetilde{c} \to 0$ in $L^\infty(0,T;(C_c^2(\Omega))')$; since $\Phi_{\mathcal{D}}\widetilde{c}$ is relatively compact in this space, we deduce that $\Phi_{\mathcal{D}}c$ is also relatively compact in this same space, and thus in particular in $L^1(0,T;(C_c^2(\Omega))')$.

Let $n = 1, \dots, N_k$ and $t \in [(n-1)k, nk)$. By (24), Lemma 7.3 gives, for all ω relatively compact in Ω and all $|\xi| < \mathrm{dist}(\omega, \mathbb{R}^d \backslash \Omega)$,

$$\begin{aligned}\|c(t,\cdot+\xi) - c(t,\cdot)\|_{L^1(\omega)} \le C_{24}|\xi| \sum_{K\in\mathcal{M}} \mathrm{m}(K)|\mathbf{w}_K^n| \\ + C_{24}|\xi| \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{diam}(K)^{d-1}\nu_K \mathrm{m}(K)|G_{K,\sigma}^n|.\end{aligned}$$

Integrating on $t \in [(n-1)k, nk)$ and summing over $n = 1, \dots, N_k$, this implies that

$$\begin{aligned}&\|c(\cdot,\cdot+\xi) - c\|_{L^1((0,T)\times\omega)} \\ &\le C_{24}|\xi|\,\|\mathbf{w}\|_{L^1((0,T)\times\Omega)^d} + C_{24}|\xi| \left(\sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{diam}(K)^{2d-2}\nu_K \mathrm{m}(K)\right)^{\frac{1}{2}} \\ &\qquad \times \left(\sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K)|G_{K,\sigma}^n|^2\right)^{\frac{1}{2}} \\ &\le C_{25}|\xi| + C_{25}|\xi| \left(\sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \mathrm{diam}(K)^{2d-2+\beta}\mathrm{m}(K)\right)^{\frac{1}{2}}\end{aligned}$$

thanks to the estimates of Proposition 3.2. But $2d-2+\beta \ge 0$ and $\mathrm{diam}(K)^{2d-2+\beta} \le \mathrm{diam}(\Omega)^{2d-2+\beta}$. Hence, we see that $\|c(\cdot,\cdot+\xi) - c\|_{L^1((0,T)\times\omega)} \to 0$ as $\xi \to 0$, independent of k or $\mathcal{D}$.

Since $\Phi_{\mathcal{D}}$ is bounded in $L^\infty(\Omega)$ and c is bounded in $L^\infty(0,T;L^2(\Omega))$, we have

$$\begin{aligned}\|(\Phi_{\mathcal{D}}c)(\cdot,\cdot+\xi)-\Phi_{\mathcal{D}}c\|_{L^1((0,T)\times\omega)}\\ &=\|\Phi_{\mathcal{D}}(\cdot+\xi)(c(\cdot,\cdot+\xi)-c)+(\Phi_{\mathcal{D}}(\cdot+\xi)-\Phi_{\mathcal{D}})c\|_{L^1((0,T)\times\omega)}\\ &\le C_{26}\|c(\cdot,\cdot+\xi)-c\|_{L^1((0,T)\times\omega)}+C_{27}\|\Phi_{\mathcal{D}}(\cdot+\xi)-\Phi_{\mathcal{D}}\|_{L^2(\omega)},\end{aligned}$$

where C_{26} and C_{27} do not depend on $\mathcal{D}$ or k. But it is classical that $\Phi_{\mathcal{D}}\to\Phi$ in $L^2(\Omega)$ as size$(\mathcal{D})\to 0$ and thus $\|\Phi_{\mathcal{D}}(\cdot+\xi)-\Phi_{\mathcal{D}}\|_{L^2(\omega)}\to 0$ as $\xi\to 0$, independent of $\mathcal{D}$. We therefore obtain $\|(\Phi_{\mathcal{D}}c)(\cdot,\cdot+\xi)-\Phi_{\mathcal{D}}c\|_{L^1((0,T)\times\omega)}\to 0$ as $\xi\to 0$, independent of k or $\mathcal{D}$. Since $\Phi_{\mathcal{D}}c$ is relatively compact in $L^1(0,T;(C_c^2(\Omega))')$, Lemma 7.5 then shows that $\Phi_{\mathcal{D}}c$ is relatively compact in $L^1(0,T;L^1_{\rm loc}(\Omega))$.

Up to a subsequence as $k\to 0$ and size$(\mathcal{D})\to 0$, $\Phi_{\mathcal{D}}c\to f$ in $L^1(0,T;L^1_{\rm loc}(\Omega))$. Using again the fact that $\Phi_{\mathcal{D}}\to\Phi$ in $L^2(\Omega)$ we also have, up to another subsequence, $\Phi_{\mathcal{D}}\to\Phi$ a.e. on Ω; moreover, $\Phi_{\mathcal{D}}\ge\Phi_*>0$ and thus $\frac{1}{\Phi_{\mathcal{D}}}$ stays bounded on Ω (independent of $\mathcal{D}$) and converges a.e. to $\frac{1}{\Phi}$. The Lebesgue dominated convergence theorem then shows that $c=\frac{1}{\Phi_{\mathcal{D}}}\Phi_{\mathcal{D}}c\to\frac{1}{\Phi}f$ in $L^1(0,T;L^1_{\rm loc}(\Omega))$, which concludes the proof. ∎

In what follows, we extract a sequence such that c converges in $L^1(0,T;L^1_{\rm loc}(\Omega))$ to some $\bar{c}$.

5.2. Convergence of the pressure. Let us now turn to the convergence of $(p,\mathbf{v},\mathbf{U})$. By Proposition 3.1, we can assume, up to a subsequence, that $p\to\bar{p}$ weakly-$*$ in $L^\infty(0,T;L^2(\Omega))$ and that $\mathbf{v}\to\bar{\mathbf{v}}$ weakly-$*$ in $L^\infty(0,T;L^2(\Omega))^d$. Since $\int_\Omega p(t,\cdot)=0$ for all $t\in(0,T)$, it is quite clear that $\int_\Omega\bar{p}(t,\cdot)=0$ for a.e. $t\in(0,T)$. By choice of ν_K and thanks to the estimate on F in Proposition 3.1 and the fact that $2d-2+\beta>0$, we have

$$\begin{aligned}&\sum_{n=1}^{N_k}k\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\mathrm{diam}(K)^{d-1}\nu_K\mathrm{m}(K)|F^n_{K,\sigma}|\\ &\le\left(\sum_{n=1}^{N_k}k\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}(K)\right)^{\frac12}\left(\sum_{n=1}^{N_k}k\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\mathrm{diam}(K)^{2d-2}\nu_K^2\mathrm{m}(K)|F^n_{K,\sigma}|^2\right)^{\frac12}\\ &\le C_{28}\left(\sup_{n=1,\dots,N_k}\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\mathrm{diam}(K)^{2d-2+\beta}\nu_K\mathrm{m}(K)|F^n_{K,\sigma}|^2\right)^{\frac12}\\ &\le C_{29}\,\mathrm{size}(\mathcal{D})^{\frac{2d-2+\beta}{2}}.\end{aligned}\tag{51}$$

Hence, Lemma 7.4 shows that $\bar{p}\in L^2(0,T;H^1(\Omega))$ and that $\nabla\bar{p}=\bar{\mathbf{v}}$, so that $\bar{p}\in L^\infty(0,T;H^1(\Omega))$. Let $A_{\mathcal{D}}:\Omega\times\mathbb{R}\to M_d(\mathbb{R})$ be the function defined by $A_{\mathcal{D}}(x,s)=A_K(s)$ whenever $s\in\mathbb{R}$ and x belongs to $K\in\mathcal{M}$. We also define $\breve{c}:(0,T)\times\Omega\to\mathbb{R}$ by $\breve{c}=c_K^{n-1}$ on $[(n-1)k,nk)\times K$ ($n=1,\dots,N_k$ and $K\in\mathcal{M}$); noticing that $\breve{c}=c_K^0\in[0,1]$ on $[0,k[\times K$ and that $\breve{c}=c(\cdot-k,\cdot)$ on $[k,T[\times\Omega$, it is clear that $\breve{c}\to\bar{c}$ in $L^1(0,T;L^1_{\rm loc}(\Omega))$ as $k\to 0$ and size$(\mathcal{D})\to 0$. We have $\mathbf{U}=-A_{\mathcal{D}}(\cdot,\breve{c})\mathbf{v}$ and thus, for all $\mathbf{Z}\in L^2((0,T)\times\Omega)^d$, $\int_0^T\int_\Omega\mathbf{Z}\cdot\mathbf{U}=\int_0^T\int_\Omega -A_{\mathcal{D}}(\cdot,\breve{c})^T\mathbf{Z}\cdot\mathbf{v}$. Applying Lemma 7.6 (with $-A^T$ instead of A, $u^m=\breve{c}$, and $\mathbf{Z}^m$ constant equal to $\mathbf{Z}$), and since $\mathbf{v}$ converges to $\nabla\bar{p}$ weakly in $L^2((0,T)\times\Omega)^d$, we obtain that $\int_0^T\int_\Omega\mathbf{Z}\cdot\mathbf{U}\to\int_0^T\int_\Omega -A(\cdot,\bar{c})^T\mathbf{Z}\cdot\nabla\bar{p}$, which proves that $\mathbf{U}\to\bar{\mathbf{U}}=-A(\cdot,\bar{c})\nabla\bar{p}$ weakly in $L^2((0,T)\times\Omega)^d$ (since $\mathbf{U}$ is bounded in $L^\infty(0,T;L^2(\Omega))^d$, the convergence also holds weakly-$*$ in this space).

Let us now prove that $\bar{p}$ is the weak solution to (8) with $\bar{c}$ fixed as given above. Let $\varphi \in C^\infty([0,T]\times\overline{\Omega})$ and define $\varphi^n(x) = \frac{1}{k}\int_{(n-1)k}^{nk}\varphi(t,x)\,dt$ for $n = 1,\ldots,N_k$. Multiply (21) by $\varphi^n(\mathbf{x}_K)$, sum over all control volumes, and, using (18) and (23), gather by edges; this gives

$$\sum_{K\in\mathcal{M}} \mathrm{m}(K)(q_K^{+,n} - q_K^{-,n})\varphi^n(\mathbf{x}_K) = \sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{int}}} F_{K,\sigma}^n(\varphi^n(\mathbf{x}_L) - \varphi^n(\mathbf{x}_K)).$$

However, since φ is regular, we have

$$\begin{aligned}\varphi^n(\mathbf{x}_L) - \varphi^n(\mathbf{x}_K) &= \nabla\varphi^n(\mathbf{x}_K)\cdot(\mathbf{x}_\sigma - \mathbf{x}_K) + \nabla\varphi^n(\mathbf{x}_L)\cdot(\mathbf{x}_L - \mathbf{x}_\sigma)\\ &\quad + R_{K,\sigma}^n - R_{L,\sigma}^n\\ &\quad \text{with } |R_{K,\sigma}^n| \le C_{30}\mathrm{diam}(K)^2\,,\end{aligned}\tag{52}$$

where C_{30} does not depend on n, $\sigma = K|L$, k, or $\mathcal{D}$. Therefore,

$$\begin{aligned}\sum_{K\in\mathcal{M}} \mathrm{m}(K)(q_K^{+,n} - q_K^{-,n})\varphi^n(\mathbf{x}_K) &= \sum_{K\in\mathcal{M}} \nabla\varphi^n(\mathbf{x}_K)\cdot\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^n(\mathbf{x}_\sigma - \mathbf{x}_K)\\ &\quad + \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^n R_{K,\sigma}^n\\ &= -\sum_{K\in\mathcal{M}} \mathrm{m}(K)\nabla\varphi^n(\mathbf{x}_K)\cdot\mathbf{U}_K^n + \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^n R_{K,\sigma}^n.\end{aligned}\tag{53}$$

If $\varphi_{k,\mathcal{D}}$ and $\Psi_{k,\mathcal{D}}$ denote the functions on $[0,T)\times\Omega$ which are equal to $\varphi^n(\mathbf{x}_K)$ and to $\nabla\varphi^n(\mathbf{x}_K)$ on $[(n-1)k, nk)\times K$, it is clear that $\varphi_{k,\mathcal{D}} \to \varphi$ and $\Psi_{k,\mathcal{D}} \to \nabla\varphi$ uniformly on $(0,T)\times\Omega$ as $k\to 0$ and $\mathrm{size}(\mathcal{D})\to 0$; multiplying (53) by k and summing over $n = 1,\ldots,N_k$, we obtain

$$\int_0^T\int_\Omega (q^+ - q^-)\varphi_{k,\mathcal{D}} = -\int_0^T\int_\Omega \Psi_{k,\mathcal{D}}\cdot\mathbf{U} + \sum_{n=1}^{N_k} k\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^n R_{K,\sigma}^n.\tag{54}$$

Adapting the proof of (41) to F by using Proposition 3.1, we get

$$\begin{aligned}\left|\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^n R_{K,\sigma}^n\right| &\le C_{31}\left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)\mathrm{diam}(K)^{4-2d-\beta}\right)^{\frac{1}{2}}\\ &\le C_{32}\ \mathrm{size}(\mathcal{D})^{\frac{4-2d-\beta}{2}}.\end{aligned}\tag{55}$$

Hence, by the weak convergence of $\mathbf{U}$, we can pass to the limit in (54) and find $\int_0^T\int_\Omega (q^+ - q^-)\varphi = -\int_0^T\int_\Omega \nabla\varphi\cdot\bar{\mathbf{U}}$; since this equation is satisfied for all $\varphi \in C^\infty([0,T]\times\overline{\Omega})$, this concludes the proof that $\bar{p}$ is the weak solution to (8) for the given $\bar{c}$ (limit of c).

We now want to prove the strong convergence of $\mathbf{v}$ to $\nabla\bar{p}$ in $L^2((0,T)\times\Omega)^d$. To do so, we use (20) and (19) in (30), which we then multiply by k and sum over $n = 1,\ldots,N_k$; this leads to

$$\int_0^T\int_\Omega (q^+ - q^-)p = \int_0^T\int_\Omega A(\cdot,\check{c})\mathbf{v}\cdot\mathbf{v} + \sum_{n=1}^{N_k} k\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \nu_K\mathrm{m}(K)|F_{K,\sigma}^n|^2.\tag{56}$$

Dropping the last term (which is nonnegative), the weak convergence of p gives, since $\bar{p}$ is a solution to (8),

$$\limsup_{k\to 0\,,\ \mathrm{size}(\mathcal{D})\to 0}\int_0^T\int_\Omega A(\cdot,\check{c})\mathbf{v}\cdot\mathbf{v}\le\int_0^T\int_\Omega (q^+-q^-)\bar{p}=\int_0^T\int_\Omega A(\cdot,\bar{c})\nabla\bar{p}\cdot\nabla\bar{p} \tag{57}$$

(the last equality is obtained using $\bar{p}$ as a test function in (8), which is possible since the weak formulation of (8) is in fact valid with test functions in $L^1(0,T;H^1(\Omega))$). We now write, thanks to (11),

$$\begin{aligned}\alpha_A\int_0^T\int_\Omega|\mathbf{v}-\nabla\bar{p}|^2&\le\int_0^T\int_\Omega A(\cdot,\check{c})(\mathbf{v}-\nabla\bar{p})\cdot(\mathbf{v}-\nabla\bar{p})\\&=\int_0^T\int_\Omega A(\cdot,\check{c})\mathbf{v}\cdot\mathbf{v}-\int_0^T\int_\Omega A(\cdot,\check{c})\mathbf{v}\cdot\nabla\bar{p}-\int_0^T\int_\Omega A(\cdot,\check{c})\nabla\bar{p}\cdot\mathbf{v}\\&\quad+\int_0^T\int_\Omega A(\cdot,\check{c})\nabla\bar{p}\cdot\nabla\bar{p}.\end{aligned} \tag{58}$$

Up to a subsequence, we can assume that $\check{c}\to\bar{c}$ a.e. on $(0,T)\times\Omega$, and (11) then gives $A(\cdot,\check{c})\nabla\bar{p}\to A(\cdot,\bar{c})\nabla\bar{p}$ and $A(\cdot,\check{c})^T\nabla\bar{p}\to A(\cdot,\bar{c})^T\nabla\bar{p}$ strongly in $L^2((0,T)\times\Omega)^d$. Hence, the weak convergence of $\mathbf{v}$ to $\nabla\bar{p}$ allows us to pass to the limit in the second and third terms on the right-hand side of (58); the last term on this right-hand side obviously converges and (57) therefore gives

$$\begin{aligned}\limsup_{k\to 0\,,\ \mathrm{size}(\mathcal{D})\to 0}\alpha_A\int_0^T\int_\Omega|\mathbf{v}-\nabla\bar{p}|^2&\le\limsup_{k\to 0\,,\ \mathrm{size}(\mathcal{D})\to 0}\int_0^T\int_\Omega A(\cdot,\check{c})\mathbf{v}\cdot\mathbf{v}\\&\quad-\int_0^T\int_\Omega A(\cdot,\bar{c})\nabla\bar{p}\cdot\nabla\bar{p}\le 0\,,\end{aligned}$$

which concludes the proof of the strong convergence of $\mathbf{v}$ to $\nabla\bar{p}$ in $L^2((0,T)\times\Omega)^d$. The strong convergence of $\mathbf{U}$ in the same space is then a consequence of Lemma 7.6, of the equality $\mathbf{U}=-A_{\mathcal{D}}(\cdot,\check{c})\mathbf{v}$, and of the strong convergence of $\mathbf{v}$.

We conclude by proving that, up to subsequence and as $k\to 0$ and $\mathrm{size}(\mathcal{D})\to 0$, $p(t)\to\bar{p}(t)$ in $L^1_{\mathrm{loc}}(\Omega)$ for a.e. $t\in(0,T)$. Since p is bounded in $L^\infty(0,T;L^2(\Omega))$, and thus in $L^\infty(0,T;L^1_{\mathrm{loc}}(\Omega))$, this a.e. convergence and Vitali's theorem imply the convergence in $L^1(0,T;L^1_{\mathrm{loc}}(\Omega))$, and, using once again the bound on p in $L^\infty(0,T;L^2(\Omega))$, we deduce the strong convergences stated in Theorem 2.2.

As $\mathbf{v}$ converges in $L^2(0,T;L^2(\Omega))^d$, we can assume that, up to a subsequence, $\mathbf{v}(t)\to\nabla\bar{p}(t)$ in $L^2(\Omega)^d$ for a.e. $t\in(0,T)$. Take a t_0 for which this convergence holds, and such that $\int_\Omega\bar{p}(t_0)=0$; we now prove, using the method of proof by contradiction, that $p(t_0)\to\bar{p}(t_0)$ in $L^1_{\mathrm{loc}}(\Omega)$ (along the same subsequence as the one chosen for $\mathbf{v}$, which thus does not depend on t_0). If this convergence does not hold, then we can assume, up to a new subsequence, that, for some $\eta>0$, $\mathrm{d}_1(p(t_0),\bar{p}(t_0))\ge\eta$, where d_1 is the distance in $L^1_{\mathrm{loc}}(\Omega)$. By (17), $(p(t_0),\mathbf{v}(t_0),F^{n(t_0,k)})\in L_\nu(\mathcal{D})$ (where $n(t_0,k)$ is such that $(n(t_0,k)-1)k\le t_0<n(t_0,k)k$) and Proposition 3.1 proves, with the help of the Cauchy–Schwarz inequality, that $M_1(\mathcal{D},\nu,F^{n(t_0,k)})$ (defined in Lemma 7.3) stays bounded; hence, since $p(t_0)$ is bounded in $L^2(\Omega)$ (see again Proposition 3.1), Lemma 7.3 and Kolmogorov's compactness theorem show that, up to a subsequence, $p(t_0)$ converges to some P strongly in $L^1_{\mathrm{loc}}(\Omega)$ and weakly in $L^2(\Omega)$. By (22), it is

clear that $\int_\Omega P = 0$ (use the weak convergence in $L^2(\Omega)$). Applying Lemma 7.4 to the functions constant in time $(u, \mathbf{r}) = (p(t_0), \mathbf{v}(t_0))$ and to the fluxes $H = F^{n(t_0,k)}$, the estimates in Proposition 3.1 allow us to see that (64) is satisfied and thus that $\nabla P = \nabla \bar{p}(t_0)$ (because $\mathbf{v}(t_0) \to \nabla \bar{p}(t_0)$); hence, since $\int_\Omega \bar{p}(t_0) = 0$, we deduce that $P = \bar{p}(t_0)$, and therefore that $p(t_0) \to \bar{p}(t_0)$ in $L^1_{\mathrm{loc}}(\Omega)$. Since the subsequence along which this convergence holds has been extracted from a sequence which satisfies $\mathrm{d}_1(p(t_0), \bar{p}(t_0)) \geq \eta$, this gives the contradiction we sought.

Remark 5.1. From the strong convergence of $\mathbf{v}$ and the a.e. convergence of $\check{c}$, we have $\int_0^T \int_\Omega A(\cdot, \check{c})\mathbf{v} \cdot \mathbf{v} \to \int_0^T \int_\Omega A(\cdot, \bar{c})\nabla \bar{p} \cdot \nabla \bar{p} = \int_0^T \int_\Omega (q^+ - q^-)\bar{p}$. Hence, (56) implies

$$\lim_{k \to 0\,,\ \mathrm{size}(\mathcal{D}) \to 0} \sum_{n=1}^{N_k} k \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \nu_K \mathrm{m}(K) |F^n_{K,\sigma}|^2 = 0. \tag{59}$$

5.3. Convergence of the concentration. Let us now turn to the convergence of $(c, \mathbf{w})$. By the estimates of Proposition 3.2, the convergence of c to $\bar{c}$ holds not only in $L^1(0, T; L^1_{\mathrm{loc}}(\Omega))$, but also in $L^\infty(0, T; L^2(\Omega))$ weak-$*$ and strongly in $L^p(0, T; L^q(\Omega))$ for all $p < \infty$ and $q < 2$. Up to a subsequence, we can assume that $\mathbf{w} \to \bar{\mathbf{w}}$ weakly in $L^2((0, T) \times \Omega)^d$. Thanks to the estimates on G from Proposition 3.2, the analogue of (51) reads

$$\sum_{k=1}^{N_k} k \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \mathrm{diam}(K)^{d-1} \nu_K \mathrm{m}(K) |G^n_{K,\sigma}| \leq C_{33} \mathrm{size}(\mathcal{D})^{\frac{2d-2+\beta}{2}}$$
$$\to 0 \quad \text{as } \mathrm{size}(\mathcal{D}) \to 0.$$

Hence, by (24) and Lemma 7.4, we have $\bar{c} \in L^2(0, T; H^1(\Omega))$ and $\bar{\mathbf{w}} = \nabla \bar{c}$. We now prove that $\bar{c}$ is a solution to (9), with $\bar{\mathbf{U}}$ the strong limit of $\mathbf{U}$ found in section 5.2. Let $\psi \in C_c^\infty([0, T) \times \overline{\Omega})$ and, for $n = 1, \ldots, N_k$, $\psi^n(x) = \frac{1}{k} \int_{(n-1)k}^{nk} \psi(t, x)\, dt$. We multiply (27) by $k\psi^n(\mathbf{x}_K)$ and sum over all $K \in \mathcal{M}$ and over $n = 1, \ldots, N_k$; this gives $T_6 + T_7 + T_8 + T_9 = T_{10}$. Let us study the limit of each of these terms as $k \to 0$ and $\mathrm{size}(\mathcal{D}) \to 0$.

5.3.1. Limit of T_6. We have, since $\psi^{N_k} = \psi^{N_k+1} = 0$ for k small enough (the support of ψ does not touch $t = T$),

$$\begin{aligned} T_6 &= \sum_{n=1}^{N_k} k \sum_{K \in \mathcal{M}} \mathrm{m}(K) \Phi_K \frac{c^n_K - c^{n-1}_K}{k} \psi^n(\mathbf{x}_K) \\ &= \sum_{n=1}^{N_k} k \sum_{K \in \mathcal{M}} \mathrm{m}(K) \Phi_K c^n_K \frac{\psi^n(\mathbf{x}_K) - \psi^{n+1}(\mathbf{x}_K)}{k} - \sum_{K \in \mathcal{M}} \mathrm{m}(K) \Phi_K c^0_K \psi^1(\mathbf{x}_K) \\ &= \int_0^T \int_\Omega \Phi c \zeta_{k,\mathcal{D}} - \int_\Omega \Phi_{\mathcal{D}} c_0 \pi_{k,\mathcal{D}}, \end{aligned}$$

where $\Phi_{\mathcal{D}} = \Phi_K$ on K (as before), $\zeta_{k,\mathcal{D}} = \frac{\psi^n(\mathbf{x}_K) - \psi^{n+1}(\mathbf{x}_K)}{k}$ on $[(n-1)k, nk) \times K$, and $\pi_{k,\mathcal{D}} = \psi^1_K$ on K ($n = 1, \ldots, N_k$ and $K \in \mathcal{M}$). By regularity of ψ, it is clear that $\zeta_{k,\mathcal{D}} \to -\partial_t \psi$ uniformly on $(0, T) \times \Omega$ and $\pi_{k,\mathcal{D}} \to \psi(0, \cdot)$ uniformly on Ω; we also recall that $\Phi_{\mathcal{D}} \to \Phi$ strongly in $L^2(\Omega)$. The weak-$*$ convergence of c in $L^\infty(0, T; L^2(\Omega))$ then implies $T_6 \to -\int_0^T \int_\Omega \Phi \bar{c} \partial_t \psi - \int_\Omega \Phi c_0 \psi(0, \cdot)$.

5.3.2. Limit of T_7. Making use of manipulations which should be, at this stage, familiar to the reader, we get, using (52) with $\varphi = \psi$ and letting $\Psi_{k,\mathcal{D}}$ be the function on $[0,T)\times\Omega$ equal to $\nabla\psi^n(\mathbf{x}_K)$ on $[(n-1)k, nk)\times K$,

$$\begin{aligned}
T_7 &= -\sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}^n \psi^n(\mathbf{x}_K)\\
&= \sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \nabla\psi^n(\mathbf{x}_K)\cdot \sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}^n(\mathbf{x}_\sigma - \mathbf{x}_K) + \sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}^n R_{K,\sigma}^n\\
&= \sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \mathrm{m}(K) D_K(\mathbf{U}_K^n)\mathbf{w}_K^n\cdot\nabla\psi^n(\mathbf{x}_K) + \sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}^n R_{K,\sigma}^n\\
&= \int_0^T \int_\Omega \mathbf{w}\cdot D(\cdot,\mathbf{U})^T\Psi_{k,\mathcal{D}} + \sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}^n R_{K,\sigma}^n. \qquad (60)
\end{aligned}$$

However, thanks to the estimates on G from Proposition 3.2, the analogue of (55) reads

$$\left|\sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma}^n R_{K,\sigma}^n\right| \le C_{34}\ \mathrm{size}(\mathcal{D})^{\frac{4-2d-\beta}{2}} \to 0 \quad \text{ as } \mathrm{size}(\mathcal{D})\to 0.$$

Since $\mathbf{U}\to\bar{\mathbf{U}}$ strongly in $L^2((0,T)\times\Omega)^d$, hypothesis (12) classically implies that $D(\cdot,\mathbf{U})\to D(\cdot,\bar{\mathbf{U}})$ strongly in $L^2((0,T)\times\Omega)^{d\times d}$. Since $\Psi_{k,\mathcal{D}}\to\nabla\psi$ uniformly on $(0,T)\times\Omega$, we deduce that $D(\cdot,\mathbf{U})^T\Psi_{k,\mathcal{D}}\to D(\cdot,\bar{\mathbf{U}})^T\nabla\psi$ in $L^2((0,T)\times\Omega)^d$ and the weak convergence of $\mathbf{w}$ to $\nabla\bar{c}$ allows us to pass to the limit in (60), and we get $T_7\to\int_0^T\int_\Omega D(\cdot,\bar{\mathbf{U}})\nabla\bar{c}\cdot\nabla\psi$.

5.3.3. Limit of T_8. The term T_8 is built by writing $-kT_2$ (introduced in the proof of Lemma 5.1) with $\psi^n(\mathbf{x}_K)$ instead of ψ_K and summing over n, that is,

$$T_8 = \sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \sum_{\sigma=K|L\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{int}}} \left[(-F_{K,\sigma}^n)^+ c_K^n - (-F_{L,\sigma}^n)^- c_L^n\right]\psi^n(\mathbf{x}_K).$$

In the proof of Lemma 5.1, the estimate (49) on T_2 has been proved for test functions φ in $C_c^2(\Omega)$, but it is also valid for test functions in $C^2(\overline{\Omega})$; in the same way, it is still valid if we use, in the definition of T_2, $\varphi(\mathbf{x}_K)$ rather than the mean value of φ on K (because (52) is similar to (38) without requiring $\mathbf{x}_K$ to be the barycenter of K). Therefore,

$$\begin{aligned}
&\left|T_8 + \sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \mathrm{m}(K) c_K^n \mathbf{U}_K^n\cdot\nabla\psi^n(\mathbf{x}_K)\right|\\
&\quad\le C_{35}\ \mathrm{size}(\mathcal{D})^{\frac{4-2d-\beta}{2}} \sum_{n=1}^{N_k} k \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)(|\mathbf{w}_K^n| + |c_K^n|)^2\right)^{\frac12}\\
&\qquad + C_{35}\sum_{n=1}^{N_k} k \left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K)|F_{K,\sigma}^n|^2\right)^{\frac12} \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)|\mathbf{w}_K^n|^2\right)^{\frac12}\\
&\qquad + C_{35}\ \mathrm{size}(\mathcal{D})\sum_{n=1}^{N_k} k \left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K)|G_{K,\sigma}^n|^2\right)^{\frac12}
\end{aligned}$$

and, using the Cauchy–Schwarz inequality, the estimates of Proposition 3.2, and (59), this right-hand side tends to 0 as $k \to 0$ and $\mathrm{size}(\mathcal{D}) \to 0$. With the same $\Psi_{k,\mathcal{D}}$ as before, we have $\sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \mathrm{m}(K) c_K^n \mathbf{U}_K^n \cdot \nabla\psi^n(\mathbf{x}_K) = \int_0^T \int_\Omega c\mathbf{U}\cdot\Psi_{k,\mathcal{D}}$, and we therefore can pass to the limit (using the weak convergence of c in $L^2((0,T)\times\Omega)$, the strong convergence of $\mathbf{U}$ in $L^2((0,T)\times\Omega)^d$, and the uniform convergence of $\Psi_{k,\mathcal{D}}$ on $(0,T)\times\Omega$) to obtain $T_8 \to -\int_0^T\int_\Omega \bar{c}\bar{\mathbf{U}}\cdot\nabla\psi$.

5.3.4. Limits of T_9 and T_{10}. We have, with $\psi_{k,\mathcal{D}}$ equal to $\psi^n(\mathbf{x}_K)$ on $[(n-1)k, nk)\times K$,

$$T_9 = \sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \mathrm{m}(K) q_K^{-,n} c_K^n \psi^n(\mathbf{x}_K) = \int_0^T \int_\Omega q^- c\psi_{k,\mathcal{D}} \to \int_0^T\int_\Omega q^-\bar{c}\psi.$$

It is also easy to pass to the limit in

$$T_{10} = \sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \mathrm{m}(K) q_K^{+,n} \widehat{c}_K^n \psi^n(\mathbf{x}_K) = \int_0^T \int_\Omega q^+ \widehat{c}_{k,\mathcal{D}}\psi_{k,\mathcal{D}}$$

once we notice that, as for $\Phi_{\mathcal{D}}$, the function $\widehat{c}_{k,\mathcal{D}}$ equal to $\widehat{c}_K^n$ on $[(n-1)k, nk)\times K$ converges to $\widehat{c}$ in $L^2((0,T)\times\Omega)$. Hence, $T_{10} \to \int_0^T\int_\Omega q^+\widehat{c}\psi$.

Gathering the preceding convergences in $T_6 + T_7 + T_8 + T_9 = T_{10}$, we deduce that $\bar{c}$ is a weak solution to (9) with the function $\bar{\mathbf{U}}$ being the limit of $\mathbf{U}$.

6. Numerical results. In this section, we illustrate the behavior of the mixed finite volume scheme by applying it to the system (1)–(7), which describes the miscible displacement of one fluid by another in a porous medium. Some of the tests cases come from [20], where an ELLAM-MFEM scheme is used, and our results compare very well to the ones in this reference. In practice, for the implementation of the numerical scheme we have used the hybrid method mentioned in Remark 4.1.

In all the test cases, the spatial domain is $\Omega = (0, 1000)\times(0, 1000)$ ft^2 and the time period is $[0, 3600]$ days. The injection well is located at the upper-right corner $(1000, 1000)$ with an injection rate $q^+ = 30$ ft^2/day and an injection concentration $\hat{c} = 1.0$. The production well is located at the lower-left corner $(0, 0)$ with a production rate $q^- = 30$ ft^2/day. The viscosity of the oil is $\mu(0) = 1.0$ cp, the porosity of the medium is specified as $\Phi(x) = 0.1$, and the initial concentration is $c_0(x) = 0$.

Remark 6.1. Although this does not entirely satisfy the assumptions of our theoretical study, the wells can be considered as Dirac masses; from the point of view of numerical tests, we saw no difference between using Dirac masses for q^+ and q^- or approximations of such masses by functions with small support (which would be admissible in the theoretical study).

The mesh of the domain is partitioned into 928 triangles of maximal edge length 50 ft. We take as time step $k = 36$ days, but the scheme still works with greater time steps (indeed, the discretization is implicit in time and does not require any stability condition). In fact, if we use the same time step $k = 360$ days as in [20], we obtain numerical results close to the ones in this reference but, since the computational times are in any case very short (less than 3 seconds per time step on a personal computer), we choose the smaller time step $k = 36$ days to show more accurate results with respect to the exact solution. As noticed in [9], the choice of ν_K has very little impact on the numerical outcomes and any small value for the penalization gives good results; we therefore take $\nu_K \mathrm{m}(K) = 10^{-6}$ for all K. Note that for $10^{-10} \le \nu_K\mathrm{m}(K) \le 10^{-2}$, the numerical results are similar. For each test case, we present the surface plot and/or the contour plot of the concentration c, the interesting physical quantity, at $t = 3$ years (≈ 30 time steps) and $t = 10$ years (≈ 100 time steps).

Remark 6.2. Notice that our scheme preserves the discrete mass, that is, for $n = 1, \dots, N_k$,

$$\int_\Omega \phi(x)c^n(x)\,dx + \int_{(n-1)k}^{nk}\int_\Omega q^-(t,x)c^n(x)\,dxdt = \int_\Omega \phi(x)c^{n-1}(x)\,dx + \int_{(n-1)k}^{nk}\int_\Omega q^+(t,x)\widehat{c}^n(x)\,dxdt$$

(this is obtained by summing (27) over all $K \in \mathcal{M}$ and using (25) and (28) to cancel the terms involving $G^n_{K,\sigma}$ and (18) to cancel the terms involving $F^n_{K,\sigma}$). This is of essential importance in the applications.

Remark 6.3. We also notice that, in all the following numerical tests, the computed values of the concentration remain in $[0, 1]$. This is, however, only a numerical verification, not a proof (but, thanks to assumption (11), these bounds are not needed to prove the convergence of the mixed finite volume scheme—and in fact, since the computed c remains in $[0, 1]$, the implementation of the scheme does not require extending μ outside of $[0, 1]$). The mixed finite volume method has many advantages: it works on very general meshes (which can be useful in petroleum engineering; see [14]); it ensures strong convergence of the discrete gradients (and therefore convergence of the scheme for the fully coupled system with minimal regularity assumptions on the data); it can be easily implemented. But the counterpart is that, though the *continuous* concentration remains in $[0, 1]$ (see [3] or [15]), we did not prove such bounds for the *approximate* concentration; they are just verified in numerical experiments (such is also the case for other numerical methods; see, e.g., [12, 17, 20]).

Test 1. For this test case, we assume that the porous medium is homogeneous and isotropic: the permeability tensor is diagonal and constant, $\mathbf{K} = 80\mathbf{I}$. The mobility ratio between the resident and the injected fluids is $M = 1$, so that the viscosity is constant, $\mu(c) = 1.0$ cp.

We assume that $\Phi d_m = 1.0$ ft^2/day, $\Phi d_l = 5.0$ ft, and $\Phi d_t = 0.5$ ft. This means that the diffusion effects will be considerably greater than the dispersion effects, which is in fact unrealistic.

The surface plot and the contour plot of the concentration c at $t = 3$ years and $t = 10$ years are shown in Figure 1. As expected, the Darcy velocity is radial and the contour plots are circular until the invading fluid reaches the production well (see at $t = 3$ years). When the production well is reached, the invading fluid continues to fill the whole domain until $c = 1$.

Test 2. The permeability tensor is still diagonal and constant, $\mathbf{K} = 80\mathbf{I}$. The adverse mobility ratio is $M = 41$ and the viscosity $\mu(c)$ now really depends on c.

We assume that there is no molecular diffusion $\Phi d_m = 0.0$ ft^2/day and that $\Phi d_l = 5.0$ ft and $\Phi d_t = 0.5$ ft. This means that we take into account dispersion effects, which is realistic.

This test case is presented in [20] and permits us to see the macroscopic fingering phenomenon. Indeed, the viscosity $\mu(c)$ rapidly changes across the fluid interface. It induces rapid changes of the Darcy velocity $\mathbf{U}$, and the difference between the longitudinal and the transverse dispersivity coefficients implies that the fluid flow is much faster along the diagonal direction. Such effects can be seen on the surface and contour plots in Figure 2.

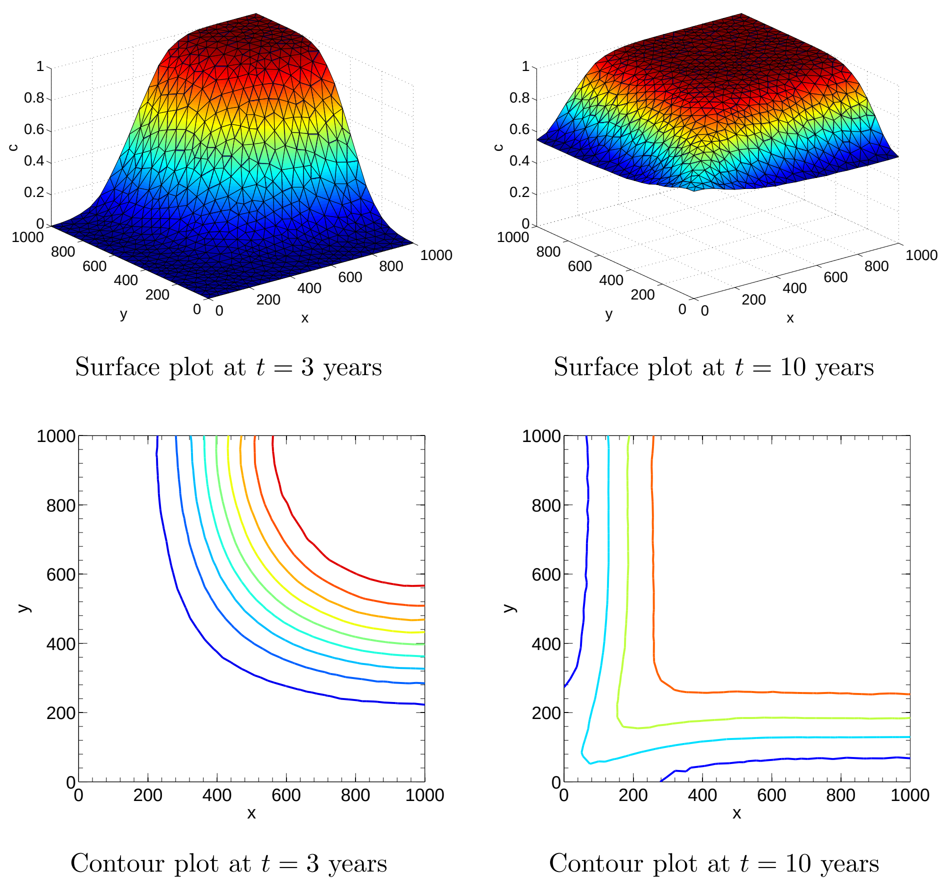


FIG. 1. *Concentration of the invading component in Test* 1.

Remark 6.4. Although this test (as well as Tests 3 and 4) does not satisfy our theoretical assumptions (because $d_m = 0$), we present its results to show that the mixed finite volume scheme is robust and can numerically handle more general cases than the ones admitted in the theoretical study, and also to compare it with other existing schemes for the same equations (note that there is no theoretical study of convergence whatsoever in [20] or [21]).

Test 3. In this test case, we consider that the permeability tensor is still diagonal but discontinuous: $\mathbf{K} = 80\mathbf{I}$ on the subdomain $(0, 1000) \times (0, 500)$ and $\mathbf{K} = 20\mathbf{I}$ on the subdomain $(0, 1000) \times (500, 1000)$. The adverse mobility ratio, the molecular diffusion, the longitudinal and the transverse dispersivities are the same as in Test 2.

The lower half domain has a larger permeability than the upper half domain. Therefore, when the invading fluid reaches the lower half domain, it "prefers" to pass through this domain rather than through the domain with lower permeability. As expected, we also notice that the upper half domain is, overall, less invaded than in Test 2. These effects are illustrated by the contour plots of c in Figure 3.

Test 4. In this last test case, the permeability tensor has the form $\mathbf{K} = \kappa(x)\mathbf{I}$ with $\kappa(x) = 80$ except on the four square subdomains $(200, 400) \times (200, 400)$, $(600, 800) \times (200, 400)$, $(200, 400) \times (600, 800)$, and $(600, 800) \times (600, 800)$, where $\kappa(x) = 20$. The

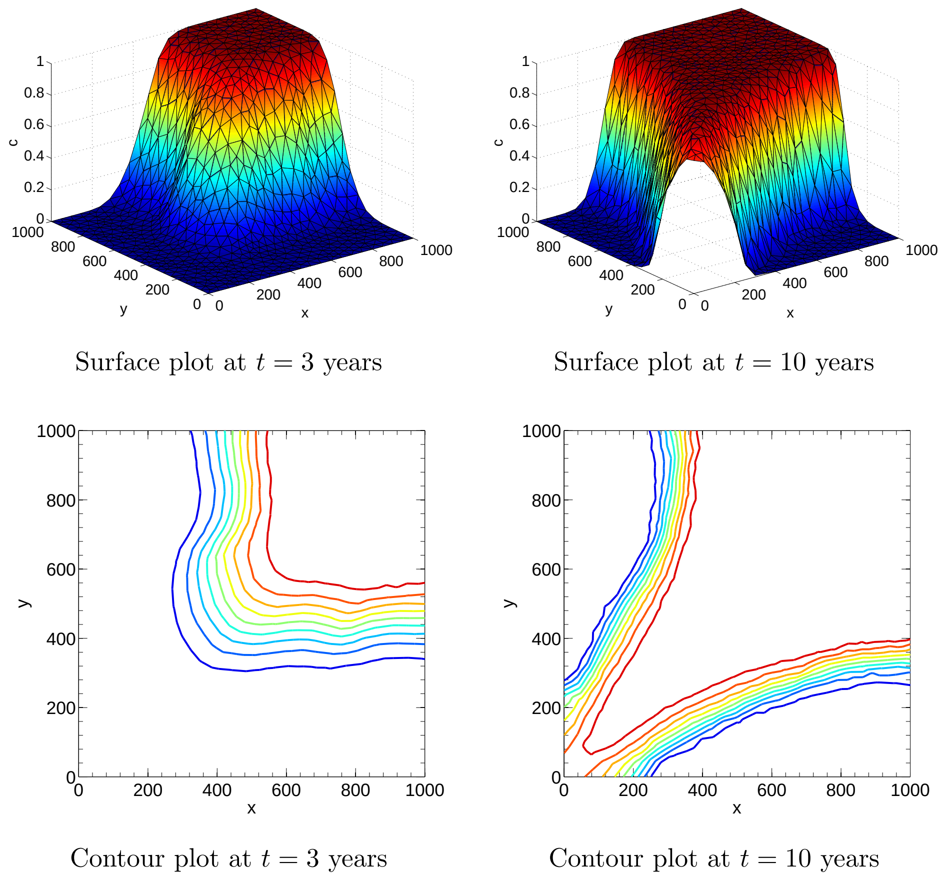


FIG. 2. *Concentration of the invading component in Test* 2.

adverse mobility ratio is $M = 41$, and we take $\Phi d_m = 0.0$ ft^2/day, $\Phi d_l = 5.0$ ft, and $\Phi d_t = 0.5$ ft.

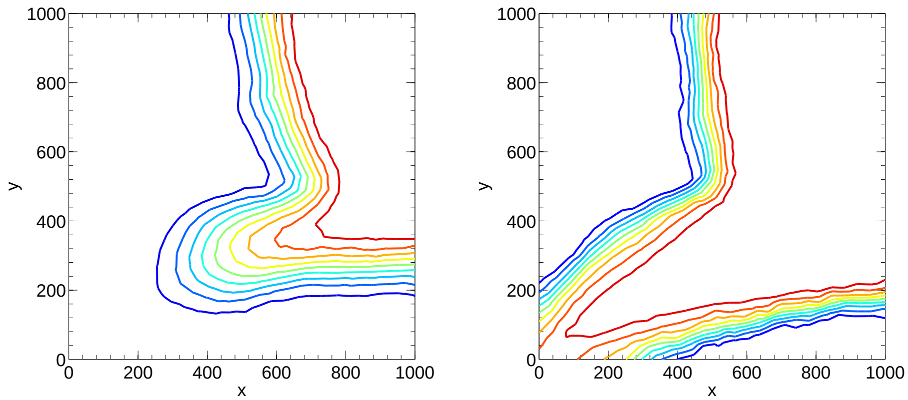
Figure 4 shows the contour plot of the concentration at $t = 3$ years and $t = 10$ years. The subdomains where the permeability is lower can easily be seen in the figures. We note that the area occupied by the invading fluid at $t = 10$ years is in this case larger than in Test 2, where the permeability was homogeneous.

7. Appendix.

7.1. A magical lemma. The proof of the following lemma (a very simple application of Stokes's formula) can be found in [9].

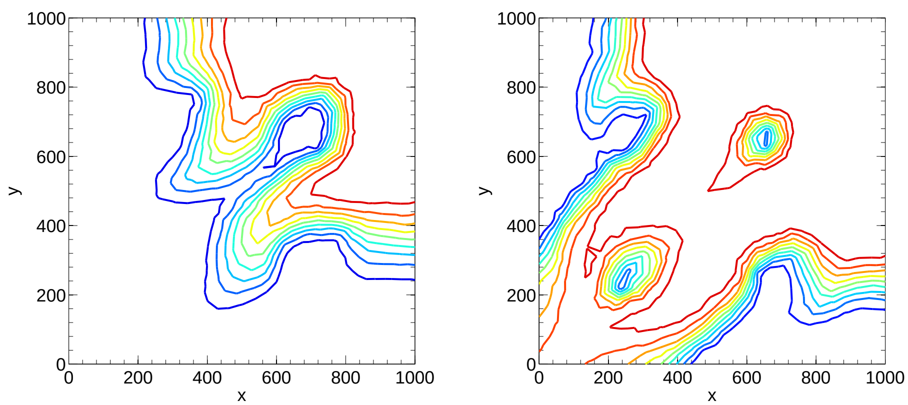
LEMMA 7.1. *Let K be a nonempty polygonal convex domain in $\mathbb{R}^d$. For $\sigma \in \mathcal{E}_K$, we define $\mathbf{x}_\sigma$ as the center of gravity of σ, and $\mathbf{n}_{K,\sigma}$ as the unit normal to σ outward to K. Then, for all vector $\mathbf{e} \in \mathbb{R}^d$ and for all point $\mathbf{x}_K \in \mathbb{R}^d$, we have $\mathrm{m}(K)\mathbf{e} = \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\mathbf{e}\cdot\mathbf{n}_{K,\sigma}(\mathbf{x}_\sigma - \mathbf{x}_K)$, where $\mathrm{m}(K)$ is the d-dimensional measure of K and $\mathrm{m}(\sigma)$ is the $(d-1)$-dimensional measure of σ.*

7.2. Lemmas on discrete gradients. For $\mathcal{D}$ an admissible mesh of Ω and $\nu = (\nu_K)_{K\in\mathcal{M}}$ a family of positive numbers, we denote by $L_\nu(\mathcal{D})$ the space of $(u, \mathbf{r}, H)$, with $u = (u_K)_{K\in\mathcal{M}}$ a family of numbers, $\mathbf{r} = (\mathbf{r}_K)_{K\in\mathcal{M}}$ a family of vectors, and



Contour plot at $t = 3$ years Contour plot at $t = 10$ years

FIG. 3. *Concentration of the invading component in Test* 3.



Contour plot at $t = 3$ years Contour plot at $t = 10$ years

FIG. 4. *Concentration of the invading component in Test* 4.

$H = (H_{K,\sigma})_{K\in\mathcal{M}\,,\,\sigma\in\mathcal{E}_K}$ a family of numbers, such that, for all $\sigma = K|L \in \mathcal{E}_{\text{int}}$,

$$\mathbf{r}_K \cdot (\mathbf{x}_\sigma - \mathbf{x}_K) + \mathbf{r}_L \cdot (\mathbf{x}_L - \mathbf{x}_\sigma) + \nu_K \mathrm{m}(K) H_{K,\sigma} - \nu_L \mathrm{m}(L) H_{L,\sigma} = u_L - u_K \tag{61}$$

(note that u and $\mathbf{r}$ are also identified with the corresponding functions on Ω constant on each control volume K). The following lemmas are the counterparts for Neumann boundary conditions of lemmas stated in [9] or [8] in the case of Dirichlet boundary conditions.

LEMMA 7.2. *Let Ω be a convex polygonal bounded domain in $\mathbb{R}^d$, $\mathcal{D}$ an admissible mesh of Ω such that* $\mathrm{regul}(\mathcal{D}) \le \theta$ *for some $\theta > 0$, and $\nu = (\nu_K)_{K\in\mathcal{M}}$ a family of positive numbers. Then there exists C_{36} depending only on d, Ω, and θ such that, for all $(u, \mathbf{r}, H) \in L_\nu(\mathcal{D})$ satisfying $\int_\Omega u = 0$,*

$$\|u\|_{L^2(\Omega)} \le C_{36} \left(\|\mathbf{r}\|_{L^2(\Omega)^d} + M_2(\mathcal{D}, \nu, H) \right)$$

with $M_2(\mathcal{D}, \nu, H) = \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{diam}(K)^{2d-2} \nu_K^2 \mathrm{m}(K) |H_{K,\sigma}|^2 \right)^{\frac{1}{2}}$.

Proof. Let w be the weak solution of $-\Delta w = u$ on Ω with homogeneous Neumann boundary conditions on $\partial\Omega$ (such a w exists thanks to the fact that $\int_\Omega u = 0$) and null mean value. Since Ω is convex, it is well known (see [16]) that $w \in H^2(\Omega)$ and that there exists C_{37} depending only on d and Ω such that $\|w\|_{H^2(\Omega)} \le C_{37}\|u\|_{L^2(\Omega)}$.

We multiply each equation of (61) by $\int_\sigma \nabla w\cdot \mathbf{n}_{K,\sigma}\,d\gamma$, sum over the interior edges, gather by control volumes, and use that $\int_\sigma \nabla w\cdot \mathbf{n}_{K,\sigma}\,d\gamma = 0$ whenever $\sigma\in\mathcal{E}_{\rm ext}$; this gives

$$\begin{aligned}\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \mathbf{r}_K\cdot(\mathbf{x}_\sigma-\mathbf{x}_K)\int_\sigma \nabla w\cdot\mathbf{n}_{K,\sigma}\,d\gamma\\ +\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu_K \mathrm{m}(K)H_{K,\sigma}\int_\sigma \nabla w\cdot\mathbf{n}_{K,\sigma}\,d\gamma = \|u\|^2_{L^2(\Omega)}.\end{aligned}\tag{62}$$

Since $\mathrm{regul}(\mathcal{D})\le\theta$, [8, Lemma 8.1] gives C_{38} depending only on d, Ω, and θ such that

$$\left|\int_\sigma \nabla w\,d\gamma\cdot\mathbf{n}_{K,\sigma}\right|^2 \le \left|\int_\sigma\nabla w\,d\gamma\right|^2\le \frac{C_{38}\mathrm{m}(\sigma)}{\mathrm{diam}(K)}\|w\|^2_{H^2(K)}.$$

Using the Cauchy–Schwarz inequality, we deduce, since $\mathrm{Card}(\mathcal{E}_K)\le\mathrm{regul}(\mathcal{D})\le\theta$ for all $K\in\mathcal{M}$,

$$\begin{aligned}&\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\mathbf{r}_K\cdot(\mathbf{x}_\sigma-\mathbf{x}_K)\int_\sigma\nabla w\cdot\mathbf{n}_{K,\sigma}\,d\gamma\\ &\le\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}(K)|\mathbf{r}_K|^2\right)^{\frac12}\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\frac{\mathrm{diam}(K)^2}{\mathrm{m}(K)}\left|\int_\sigma\nabla w\,d\gamma\cdot\mathbf{n}_{K,\sigma}\right|^2\right)^{\frac12}\\ &\le (C_{38}\theta)^{\frac12}\left(\sum_{K\in\mathcal{M}}\mathrm{m}(K)|\mathbf{r}_K|^2\right)^{\frac12}\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\frac{\mathrm{diam}(K)\mathrm{m}(\sigma)}{\mathrm{m}(K)}\|w\|^2_{H^2(K)}\right)^{\frac12}.\end{aligned}$$

We have, if $\sigma\in\mathcal{E}_K$, $\mathrm{m}(\sigma)\le\omega_{d-1}\mathrm{diam}(K)^{d-1}$ (where ω_{d-1} is the volume of the unit ball in $\mathbb{R}^{d-1}$); thus, by (16), $\frac{\mathrm{diam}(K)\mathrm{m}(\sigma)}{\mathrm{m}(K)}\le\frac{\mathrm{regul}(\mathcal{D})\omega_{d-1}}{\omega_d}$ and we obtain

$$\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\mathbf{r}_K\cdot(\mathbf{x}_\sigma-\mathbf{x}_K)\int_\sigma\nabla w\cdot\mathbf{n}_{K,\sigma}\,d\gamma\le\frac{\theta^{\frac32}\sqrt{C_{38}\omega_{d-1}}}{\sqrt{\omega_d}}\|\mathbf{r}\|_{L^2(\Omega)^d}\|w\|_{H^2(\Omega)}.\tag{63}$$

The Cauchy–Schwarz inequality also gives

$$\begin{aligned}&\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu_K\mathrm{m}(K)H_{K,\sigma}\int_\sigma\nabla w\cdot\mathbf{n}_{K,\sigma}\,d\gamma\\ &\le\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\mathrm{diam}(K)^{2d-2}\nu_K^2\mathrm{m}(K)|H_{K,\sigma}|^2\right)^{\frac12}\\ &\qquad\times\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\frac{\mathrm{m}(K)}{\mathrm{diam}(K)^{2d-2}}\left|\int_\sigma\nabla w\cdot\mathbf{n}_{K,\sigma}\,d\gamma\right|^2\right)^{\frac12}\\ &\le\sqrt{C_{38}}\,M_2(\mathcal{D},\nu,H)\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\frac{\mathrm{m}(\sigma)\mathrm{m}(K)}{\mathrm{diam}(K)^{2d-1}}\|w\|^2_{H^2(K)}\right)^{\frac12}.\end{aligned}$$

Since $\frac{\mathrm{m}(\sigma)\mathrm{m}(K)}{\mathrm{diam}(K)^{2d-1}} \le \omega_{d-1}\omega_d$, this inequality and (63) plugged in (62) conclude the proof, since $\|w\|_{H^2(\Omega)} \le C_{37}\|u\|_{L^2(\Omega)}$. □

LEMMA 7.3. *Let Ω be a convex polygonal bounded domain in $\mathbb{R}^d$, $\mathcal{D}$ an admissible mesh of Ω such that* $\mathrm{regul}(\mathcal{D}) \le \theta$ *for some $\theta > 0$, and $\nu = (\nu_K)_{K\in\mathcal{M}}$ a family of positive numbers. Let ω be relatively compact in Ω. Then there exists C_{39} depending only on d, Ω, ω, and θ such that, for all $(u,\mathbf{r},H) \in L_\nu(\mathcal{D})$ and all $|\xi| < \mathrm{dist}(\omega, \mathbb{R}^d\backslash\Omega)$,*

$$\|u(\cdot+\xi) - u\|_{L^1(\omega)} \le C_{39}\left(\|\mathbf{r}\|_{L^1(\Omega)^d} + M_1(\mathcal{D},\nu,H)\right)|\xi|,$$

where $M_1(\mathcal{D},\nu,H) = \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \mathrm{diam}(K)^{d-1}\nu_K \mathrm{m}(K)|H_{K,\sigma}|$.

We leave to the reader the proof of Lemma 7.3, counterpart of Lemma 3.2 in [9].

LEMMA 7.4. *Let Ω be a convex polygonal bounded domain in $\mathbb{R}^d$ and let $T > 0$. Let $(\mathcal{D}_m)_{m\ge1}$ be a sequence of admissible meshes of Ω such that* $\mathrm{size}(\mathcal{D}_m) \to 0$ *as $m \to \infty$ and* $(\mathrm{regul}(\mathcal{D}_m))_{m\ge1}$ *is bounded. We also take, for all $m \ge 1$, $k_m > 0$ such that $N_{k_m} = T/k_m$ is an integer and $k_m \to 0$ as $m \to \infty$, and $\nu_m = (\nu_{m,K})_{K\in\mathcal{M}_m}$ a family of positive numbers.*

For all $m \ge 1$ and all $n = 1,\dots,N_{k_m}$, we take $(u^{m,n},\mathbf{r}^{m,n}) = (u_K^{m,n},\mathbf{r}_K^{m,n})_{K\in\mathcal{M}_m}$ and a family $H^{m,n} = (H_{K,\sigma}^{m,n})_{K\in\mathcal{M}_m,\,\sigma\in\mathcal{E}_K}$ such that $(u^{m,n},\mathbf{r}^{m,n},H^{m,n}) \in L_{\nu_m}(\mathcal{D}_m)$. We let $(u^m,\mathbf{r}^m)$ be the functions on $[0,T)\times\Omega$ equal to $(u_K^{m,n},\mathbf{r}_K^{m,n})$ on $[(n-1)k,nk)\times K$ (for $n = 1,\dots,N_{k_m}$ and $K \in \mathcal{M}_m$).

Assume that, as $m \to \infty$, $u^m \to \bar{u}$ weakly in $L^2((0,T)\times\Omega)$, $\mathbf{r}^m \to \bar{\mathbf{r}}$ weakly in $L^2((0,T)\times\Omega)^d$, and

$$\sum_{n=1}^{N_{k_m}} k_m \sum_{K\in\mathcal{M}_m}\sum_{\sigma\in\mathcal{E}_K} \mathrm{diam}(K)^{d-1}\nu_{m,K}\mathrm{m}(K)|H_{K,\sigma}^{m,n}| \to 0. \tag{64}$$

Then $\bar{u} \in L^2(0,T;H^1(\Omega))$ and $\nabla\bar{u} = \bar{\mathbf{r}}$.

Proof. We first simplify the notation by dropping the index m; hence, we denote $\mathcal{D} = \mathcal{D}_m$, $k = k_m$, $u = u^m$, $\mathbf{r} = \mathbf{r}^m$, $H_{K,\sigma}^n = H_{K,\sigma}^{m,n}$, and we are interested in the convergence of quantities as $k \to 0$ and $\mathrm{size}(\mathcal{D}) \to 0$.

To prove the lemma, we just need to show that $\nabla\bar{u} = \bar{\mathbf{r}}$ in the sense of the distributions on $(0,T)\times\Omega$. Let $\varphi \in C_c^\infty((0,T)\times\Omega)$ and $\mathbf{e} \in \mathbb{R}^d$; we multiply each equation (61) on $(u^n,\mathbf{r}^n,H^n)$ by $\int_{(n-1)k}^{nk}\int_\sigma \varphi\mathbf{e}\cdot\mathbf{n}_{K,\sigma}\,d\gamma$. We then sum over all the edges and, using $\mathbf{n}_{K,\sigma} = -\mathbf{n}_{L,\sigma}$ if $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$, we gather by control volumes. Thanks to the fact that $\int_{(n-1)k}^{nk}\int_\sigma \varphi\mathbf{e}\cdot\mathbf{n}_{K,\sigma}\,d\gamma = 0$ if $\sigma \in \mathcal{E}_{\mathrm{ext}}$, we can freely introduce the terms corresponding to boundary edges (which are otherwise not present). Finally summing over $n = 1,\dots,N_k$, we obtain

$$\begin{aligned}&\sum_{n=1}^{N_k}\sum_{K\in\mathcal{M}}\mathbf{r}_K^n\cdot\sum_{\sigma\in\mathcal{E}_K}\int_{(n-1)k}^{nk}\int_\sigma \varphi\mathbf{e}\cdot\mathbf{n}_{K,\sigma}\,d\gamma(\mathbf{x}_\sigma - \mathbf{x}_K)\\ &+\sum_{n=1}^{N_k}\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu_K\mathrm{m}(K)H_{K,\sigma}^n\int_{(n-1)k}^{nk}\int_\sigma \varphi\mathbf{e}\cdot\mathbf{n}_{K,\sigma}\,d\gamma = -\int_0^T\int_\Omega u\mathrm{div}(\varphi\mathbf{e}).\end{aligned} \tag{65}$$

By convergence of u, this right-hand side tends, as $k \to 0$ and $\mathrm{size}(\mathcal{D}) \to 0$, to $-\int_0^T\int_\Omega \bar{u}\mathrm{div}(\varphi\mathbf{e})$. Let us denote by T_{11} and T_{12} the two terms on the left-hand side of this equality.

We have, since φ is bounded and $\mathrm{m}(\sigma) \le \omega_{d-1}\mathrm{diam}(K)^{d-1}$ if $\sigma \in \mathcal{E}_K$,

$$|T_{12}| \le \|\varphi\|_\infty \omega_{d-1} \sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu_K \mathrm{m}(K) |H^n_{K,\sigma}| \mathrm{diam}(K)^{d-1}$$

and thus, by assumption, $T_{12} \to 0$ as $k \to 0$ and $\mathrm{size}(\mathcal{D}) \to 0$. We now compare T_{11} with

$$T_{13} = \sum_{n=1}^{N_k} \sum_{K\in\mathcal{M}} \mathbf{r}^n_K \cdot \int_{(n-1)k}^{nk} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) \left(\frac{1}{\mathrm{m}(K)} \int_K \varphi\, \mathbf{e} \right) \cdot \mathbf{n}_{K,\sigma} (\mathbf{x}_\sigma - \mathbf{x}_K).$$

Since φ is regular, we have C_{40} depending only on φ such that

$$|T_{11} - T_{13}| \le C_{40} \mathrm{size}(\mathcal{D}) \sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} |\mathbf{r}^n_K| \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) \mathrm{diam}(K).$$

Using the fact that $\mathrm{regul}(\mathcal{D})$ stays bounded and that $\mathrm{m}(\sigma) \le \omega_{d-1}\mathrm{diam}(K)^{d-1}$, we get

$$|T_{11} - T_{13}| \le C_{41} \mathrm{size}(\mathcal{D}) \sum_{n=1}^{N_k} k \sum_{K\in\mathcal{M}} \mathrm{m}(K) |\mathbf{r}^n_K| = C_{41} \mathrm{size}(\mathcal{D}) \|\mathbf{r}\|_{L^1((0,T)\times\Omega)^d}.$$

Since $\mathbf{r}$ is bounded in $L^2((0,T)\times\Omega)^d$, this shows that $T_{11} - T_{13} \to 0$ as $\mathrm{size}(\mathcal{D}) \to 0$. Using Lemma 7.1 with $\frac{1}{\mathrm{m}(K)} \int_K \varphi(t,\cdot)\, \mathbf{e}$ instead of $\mathbf{e}$, we get

$$T_{13} = \sum_{n=1}^{N_k} \sum_{K\in\mathcal{M}} \mathbf{r}^n_K \cdot \int_{(n-1)k}^{nk} \int_K \varphi\, \mathbf{e} = \int_0^T \int_\Omega \mathbf{r} \cdot \varphi\, \mathbf{e} \longrightarrow \int_0^T \int_\Omega \bar{\mathbf{r}} \cdot \varphi\, \mathbf{e}$$

as $k \to 0$ and $\mathrm{size}(\mathcal{D}) \to 0$. Hence, the limit of (65) as $k \to 0$ and $\mathrm{size}(\mathcal{D}) \to 0$ gives $\int_0^T \int_\Omega \bar{\mathbf{r}} \cdot \varphi\, \mathbf{e} = -\int_0^T \int_\Omega \bar{u} \mathrm{div}(\varphi \mathbf{e})$, which concludes the proof. ∎

7.3. A compactness lemma. The following lemma, whose proof is inspired by classical proofs of Kolmogorov's or Aubin's compactness theorems, mixes a weak time-compactness and a space-equicontinuity property to obtain a strong time-space compactness.

LEMMA 7.5. *Let Ω be a bounded domain in $\mathbb{R}^d$, let $T > 0$, and let $A \subset L^1(0,T;L^1_{\mathrm{loc}}(\Omega))$. If A is relatively compact in $L^1(0,T;(C^2_c(\Omega))')$ and if, for all ω relatively compact in Ω,*

$$\sup_{u\in A} \|u(\cdot,\cdot+\xi) - u\|_{L^1((0,T)\times\omega)} \to 0 \quad \text{as } |\xi| \to 0,$$

then A is relatively compact in $L^1(0,T;L^1_{\mathrm{loc}}(\Omega))$.

Proof. Let ω be relatively compact in Ω and take $(\rho_\mu)_{0<\mu<\mathrm{dist}(\omega,\mathbb{R}^d\setminus\Omega)}$ smoothing kernels on $\mathbb{R}^d$ such that $\mathrm{supp}(\rho_\mu)$ is included in the ball of center 0 and radius μ. For $u \in A$, let $u_\mu = u * \rho_\mu$ (the convolution being only on the space variable), which is defined on $(0,T)\times\omega$.

We first prove that, for all μ, $A_\mu = \{u_\mu,\ u \in A\}$ is relatively compact in $L^1((0,T)\times\omega)$. Let $(u^n_\mu)_{n\ge1}$ be a sequence in A_μ. Since $(u^n)_{n\ge1}$ lies in A, it is relatively compact in $L^1(0,T;(C_c^2(\Omega))')$ and we can assume, up to a subsequence, that it converges in this space. We then have, for all $(t,x)\in(0,T)\times\omega$, since $\mathrm{supp}(\rho_\mu(x-\cdot))\subset\Omega$ by choice of μ,

$$|u_\mu^n(t,x)-u_\mu^m(t,x)| = \left|\int_\Omega (u^n(t,y)-u^m(t,y))\rho_\mu(x-y)\,dx\right|$$
$$\le \|u^n(t,\cdot)-u^m(t,\cdot)\|_{(C_c^2(\Omega))'}\|\rho_\mu(x-\cdot)\|_{C_c^2(\Omega)}.$$

Hence, integrating on $x\in\omega$ and $t\in(0,T)$, we find C_μ depending on μ but not on n or m such that $\|u_\mu^n-u_\mu^m\|_{L^1((0,T)\times\omega)}\le C_\mu\|u^n-u^m\|_{L^1(0,T;(C_c^2(\Omega))')}$, which shows that $(u_\mu^n)_{n\ge1}$ converges in $L^1((0,T)\times\omega)$ since $(u^n)_{n\ge1}$ converges in $L^1(0,T;(C_c^2(\Omega))')$. Hence, for all $\mu\in(0,\mathrm{dist}(\omega,\mathbb{R}^d\backslash\Omega))$, A_μ is relatively compact in $L^1((0,T)\times\omega)$.

Let us now conclude. It is sufficient to show that $\sup_{u\in A}\|u-u_\mu\|_{L^1((0,T)\times\omega)}$ goes to 0 as $\mu\to0$. Indeed, once this is done, we get $A\subset A_\mu+B_{L^1((0,T)\times\omega)}(0,\delta(\mu))$ with $\delta(\mu)\to0$ as $\mu\to0$, which clearly shows, since A_μ is precompact in $L^1((0,T)\times\omega)$, that A is also precompact (and thus relatively compact) in this space. Let $u\in A$, $t\in(0,T)$, and $x\in\omega$; we have $|u(t,x)-u_\mu(t,x)|\le\int_{B(0,\mu)}|u(t,x)-u(t,x-y)|\rho_\mu(y)\,dy$ and thus, integrating on $x\in\omega$ and $t\in(0,T)$,

$$\|u-u_\mu\|_{L^1((0,T)\times\omega)}\le\int_{B(0,\mu)}\int_0^T\int_\omega|u(t,x)-u(t,x-y)|\,dtdx\,\rho_\mu(y)\,dy$$
$$\le\sup_{|y|\le\mu}\int_0^T\int_\omega|u(t,x)-u(t,x-y)|\,dtdx,$$

and the proof is concluded. ∎

7.4. A technical lemma. The proof of the following technical lemma is left to the reader.

LEMMA 7.6. *Let Ω be a convex polygonal bounded domain in $\mathbb{R}^d$, let $T>0$, and let $A:\Omega\times\mathbb{R}\to M_d(\mathbb{R})$ be a Carathéodory bounded matrix-valued function. Let $(\mathcal{D}_m)_{m\ge1}$ be a sequence of admissible meshes of Ω such that* $\mathrm{size}(\mathcal{D}_m)\to0$ *as $m\to\infty$, and let $k_m>0$ be such that $N_{k_m}=T/k_m$ is an integer and $k_m\to0$ as $m\to\infty$.*

Let $u^m=(u_K^{m,n})_{n=1,\dots,N_{k_m},\,K\in\mathcal{M}}$ be a function on $(0,T)\times\Omega$, constant on each $[(n-1)k,nk)\times K$ ($n=1,\dots,N_{k_m}$, $K\in\mathcal{M}_m$). We assume that $u^m\to\bar u$ in $L^1(0,T;L^1_{\mathrm{loc}}(\Omega))$ as $m\to\infty$. Let $\mathbf{Z}^m\in L^2((0,T)\times\Omega)^d$, which converges to $\bar{\mathbf{Z}}$ in $L^2((0,T)\times\Omega)^d$ as $m\to\infty$. Define $A_{\mathcal{D}_m}:\Omega\times\mathbb{R}\to M_d(\mathbb{R})$ by $A_{\mathcal{D}_m}(x,s)=\frac{1}{\mathrm{m}(K)}\int_K A(y,s)\,dy$ whenever x belongs to $K\in\mathcal{M}_m$.

Then $A_{\mathcal{D}_m}(\cdot,u^m)\mathbf{Z}^m\to A(\cdot,\bar u)\bar{\mathbf{Z}}$ in $L^2((0,T)\times\Omega)^d$ as $m\to\infty$.

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