Study of the mixed finite volume method for Stokes and Navier-Stokes equations

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23/09/2007

Abstract

We present finite volume schemes for Stokes and Navier-Stokes equations. These schemes are based on the mixed finite volume introduced in [6], and can be applied to any type of grid (without "orthogonality" assumptions as for classical finite volume methods) and in any space dimension. We present numerical results on some irregular grids, and we prove, for both Stokes and Navier-Stokes equations, the convergence of the scheme toward a solution of the continuous problem.

Keywords. Mixed finite volume scheme, Stokes problem, Navier-Stokes problem, general grids, numerical results.

1 Introduction

Finding an approximate solution of the Navier-Stokes equations can be done by a large range of numerical methods: finite element methods, mostly used by the mathematician community (see for example [10, 11, 12] and references therein), spectral methods and finite volume methods, largely used by the engineering and physicists community (one can first refer to [15] for finite volume methods on staggered grids, and for example to [2, 13, 8, 9] for collocated finite volume schemes). One reason for this difference of practice is that an advantage of finite volume methods on finite element ones lies on its easy physical interpretation and on simpler implementations.

However, on domains with complex shapes, it remains difficult to account for constraints provided by finite volume schemes on the meshes: the well-known Patankar scheme on staggered grids can hardly be extended to unstructured meshes, and the implementation of collocated finite volume schemes is complex on general meshes, demanding a stabilization procedure for the pressure.

These constraints on the grids are due to the simultaneous discretization of the viscous term in the momentum balance equation and of the mass conservation equation. On the other hand, a mixed finite volume scheme has recently been shown to be well suited for the resolution of diffusion problems on any type of 2D or 3D grid, structured or not, admissible or not in the sense

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of finite element or classical finite volume methods. Hence one could expect that this scheme would provide new gridding possibilities in the case of Stokes and Navier-Stokes equations. This is the point that we focus on in this paper. Let us first recall the continuous problems that are to be approximated.

We first consider the Stokes problem and we therefore search for an approximation of $\bar{\boldsymbol{u}}$ and $\bar{p}$, weak solution to

$$\begin{aligned} &-\Delta\bar{\boldsymbol{u}} + \nabla\bar{p} = \boldsymbol{f} \qquad \text{in } \Omega,\\ &\mathrm{div}\bar{\boldsymbol{u}} = 0 \qquad \text{in } \Omega,\\ &\bar{\boldsymbol{u}} = 0 \qquad \text{on } \partial\Omega,\\ &\int_\Omega \bar{p}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0, \end{aligned} \tag{1}$$

under the following assumptions:

$$\Omega \text{ is an open bounded connected polygonal subset of } \mathbb{R}^d,\ d = 2 \text{ or } 3, \tag{2}$$

and

$$\boldsymbol{f} \in L^2(\Omega)^d. \tag{3}$$

Thanks to Lax-Milgram theorem, there exists a unique weak solution to (1) in the following sense.

Definition 1.1 [Weak solution to the Stokes equation] *Assume that (2) and (3) hold. A weak solution to (1) is* $\bar{\boldsymbol{u}}$ *such that*

$$\left\{ \begin{array}{l} \bar{\boldsymbol{u}} \in E(\Omega),\\ \displaystyle\int_\Omega \nabla\bar{\boldsymbol{u}}(\boldsymbol{x}) : \nabla\boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,, \qquad \forall\boldsymbol{\varphi} \in E(\Omega), \end{array}\right. \tag{4}$$

where $E(\Omega) = \{\boldsymbol{\varphi} \in H_0^1(\Omega)^d\,,\ \mathrm{div}(\boldsymbol{\varphi}) = 0\}$.

Remark 1.1 *If* $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_d)$ *is a function* $\Omega \to \mathbb{R}^d$, *we denote by* $\nabla\boldsymbol{\varphi}$ *the second order tensor* $((\partial_j\varphi_i))_{(i,j)\in[1,d]^2}$. *If* $\mathbf{v} = ((\mathbf{v}_{i,j}))_{(i,j)\in[1,d]^2}$ *and* $\mathbf{w} = ((\mathbf{w}_{i,j}))_{(i,j)\in[1,d]^2}$ *are two second order tensors, we let* $\mathbf{v} : \mathbf{w} = \sum_{i,j=1}^d \mathbf{v}_{i,j}\mathbf{w}_{i,j}$ *(note that this definition of the doubly contracted product of tensors differs from the usual one in which one of the tensors is transposed).*

We also consider the incompressible transient Navier-Stokes problem:

$$\begin{aligned} &\partial_t\bar{\boldsymbol{u}} + (\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} - \Delta\bar{\boldsymbol{u}} + \nabla\bar{p} = \boldsymbol{f} \qquad \text{in }]0,T[\times\Omega,\\ &\mathrm{div}(\bar{\boldsymbol{u}}) = 0 \qquad \text{in }]0,T[\times\Omega,\\ &\bar{\boldsymbol{u}} = 0 \qquad \text{on }]0,T[\times\partial\Omega,\\ &\bar{\boldsymbol{u}}(0,\cdot) = \boldsymbol{u}_0 \qquad \text{in } \Omega\\ &\int_\Omega \bar{p}(\cdot,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0 \qquad \text{on }]0,T[\,, \end{aligned} \tag{5}$$

under the assumption

$$\boldsymbol{f} \in L^2(]0,T[\times\Omega)^d\,,\ \boldsymbol{u}_0 \in L^2(\Omega)^d. \tag{6}$$

Remark 1.2 [Renormalization] *If we replace the first equation of (5) by* $\partial_t \bar{\boldsymbol{u}} + (\bar{\boldsymbol{u}} \cdot \nabla)\bar{\boldsymbol{u}} - \mu\Delta\bar{\boldsymbol{u}} + \nabla\bar{p} = \boldsymbol{f}$ *for some* $\mu > 0$*, then any solution* $(\bar{\boldsymbol{u}}(t, \boldsymbol{x}), \bar{p}(t, \boldsymbol{x}))$ *of the system of equations thus obtained is such that* $(\bar{\boldsymbol{u}}(t/\mu, \boldsymbol{x})/\mu, \bar{p}(t/\mu, \boldsymbol{x})/\mu^2)$ *is a solution of (5), replacing* $\boldsymbol{u}_0(\boldsymbol{x})$ *by* $\boldsymbol{u}_0(\boldsymbol{x})/\mu$*,* $\boldsymbol{f}(t, \boldsymbol{x})$ *by* $\boldsymbol{f}(t/\mu, \boldsymbol{x})/\mu^2$ *and* T *by* μT*. We can therefore let* $\mu = 1$ *without loss of generality.*

It is known [17, 3] that there exists a weak solution to (5) in the following sense (notice however that we do not use, in the following, the existence of such a solution).

Definition 1.2 [Weak solution to the Navier-Stokes equation] *Assume that (2) and (6) hold. A weak solution to (5) is* $\bar{\boldsymbol{u}} \in L^2(0, T; H_0^1(\Omega))^d$ *such that* $\mathrm{div}(\bar{\boldsymbol{u}}) = 0$ *a.e. on* $]0, T[\times\Omega$ *and, for all* $\boldsymbol{\varphi} \in C_c^\infty([0, T[\times\Omega)^d$ *such that* $\mathrm{div}(\boldsymbol{\varphi}) = 0$*,*

$$\begin{aligned}
-\int_0^T \int_\Omega \bar{\boldsymbol{u}}(t, \boldsymbol{x}) \cdot \partial_t \boldsymbol{\varphi}(t, \boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x} + \int_0^T \int_\Omega [(\bar{\boldsymbol{u}}(t, \boldsymbol{x}) \cdot \nabla)\bar{\boldsymbol{u}}(t, \boldsymbol{x})] \cdot \boldsymbol{\varphi}(t, \boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}& \\
+ \int_0^T \int_\Omega \nabla\bar{\boldsymbol{u}}(t, \boldsymbol{x}) : \nabla\boldsymbol{\varphi}(t, \boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}& \\
= \int_\Omega \boldsymbol{u}_0(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(0, \boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_0^T \int_\Omega \boldsymbol{f}(t, \boldsymbol{x}) \cdot \boldsymbol{\varphi}(t, \boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}.& \qquad (7)
\end{aligned}$$

The principle of our scheme, described in Section 2, is the following. We simultaneously look for approximations $\boldsymbol{u}_K \in \mathbb{R}^d$, $\mathbf{v}_K \in \mathbb{R}^{d\times d}$ of $\bar{\boldsymbol{u}}$, $\nabla\bar{\boldsymbol{u}}$ in each control volume K and for approximation $\boldsymbol{F}_\sigma \in \mathbb{R}^d$ of $\int_\sigma \nabla\bar{\boldsymbol{u}}(\boldsymbol{x})\boldsymbol{n}_\sigma\,\mathrm{d}\gamma(\boldsymbol{x})$ at each edge σ of the mesh, where $\boldsymbol{n}_\sigma$ is a unit vector normal to σ. The values $\boldsymbol{F}_\sigma$ must then satisfy the balance equation in each control volume, and consistency relations are imposed on $\boldsymbol{u}_K$, $\mathbf{v}_K$ and $\boldsymbol{F}_\sigma$. We present some numerical examples in Section 3, which demonstrate the aptitude of the mixed finite volume scheme to provide accurate results on meshes including refinements, vertices inside internal edges and general quadrangular control volumes. In Sections 4 and 5, we study the mixed finite volume approximation respectively for Stokes and Navier-Stokes equations: we show that this method leads to systems (linear in the case of Stokes problem, non-linear in the case of Navier-Stokes problem) which, in general, have at least one approximate solution $\boldsymbol{u}$, $\mathbf{v}$ and $\boldsymbol{F}$ (this solution is unique in the case of Stokes problem), and we prove the convergence of these approximate solutions toward a solution of the continuous equations. An appendix (Section 6) concludes the paper by providing various lemmas involved in the analysis of the schemes.

2 The mixed finite volume schemes

2.1 Admissible discretization of Ω

We present the notion of admissible discretization of the domain Ω, which is necessary to give the expression of the finite volume scheme.

Definition 2.1 [Admissible discretization] *Let* Ω *be an open bounded polygonal subset of* $\mathbb{R}^d$ $(d \geq 1)$*, and* $\partial\Omega = \overline{\Omega} \setminus \Omega$ *its boundary. An admissible discretization of* Ω *is given by* $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$*, where:*

- *$\mathcal{M}$ is a finite family of non empty open polygonal disjoint subsets of Ω (the "control volumes") such that $\overline{\Omega} = \cup_{K\in\mathcal{M}}\overline{K}$. For all $K \in \mathcal{M}$, we denote by $K^\star$ the set of all point $\boldsymbol{x} \in K$ such that K is star-shaped with respect to $\boldsymbol{x}$ (i.e., for all $\boldsymbol{x}' \in K$, the segment between $\boldsymbol{x}$ and $\boldsymbol{x}'$ is a subset of K), and we assume that $K^\star$ has a nonempty interior.*

- *$\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh), such that, for all $\sigma \in \mathcal{E}$, there exists an affine hyperplane E of $\mathbb{R}^d$ and $K \in \mathcal{M}$ with $\sigma \subset \partial K \cap E$ and σ is a non empty open subset of E. We assume that, for all $K \in \mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$. We also assume that, for all $\sigma \in \mathcal{E}$, either $\sigma \subset \partial\Omega$ or $\sigma \subset \overline{K}\cap\overline{L}$ for some $(K, L) \in \mathcal{M}^2$ $(K \neq L)$.*

- *$\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (\boldsymbol{x}_K)_{K\in\mathcal{M}}$ and such that, for all $K \in \mathcal{M}$, $\boldsymbol{x}_K \in K$.*

Remark 2.1 *Though the elements of $\mathcal{E}_K$ may not be the real edges of the control volume K (each $\sigma \in \mathcal{E}_K$ may be only the part of a full edge, especially in the case of locally refined grids), we will in the following call "edges of K" the elements of $\mathcal{E}_K$. Notice also that the control volumes can be non-convex, so that two neighboring control volumes can share multiple edges.*

Notations. The measure of a control volume K is denoted by $\mathrm{m}(K)$ and the $(d-1)$-dimensional measure of an edge σ by $\mathrm{m}(\sigma)$. If σ is a given edge, we sometimes write it $\sigma^{K|L}$ to indicate that the control volumes on each side of σ are K and L; if σ is a boundary edge, $\sigma^{K|\partial}$ indicates that the control volume whose boundary contains σ is K. For all $\sigma \in \mathcal{E}$, $\boldsymbol{x}_\sigma$ is the barycenter of σ. The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\rm int}$ (resp. $\mathcal{E}_{\rm ext}$), that is $\mathcal{E}_{\rm int} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\rm ext} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$). For all $K \in \mathcal{M}$, we denote by $\mathcal{N}_K$ the subset of $\mathcal{M}$ of the neighboring control volumes (that is, the $L \neq K$ such that $\overline{K}\cap\overline{L}$ contains an edge of the discretization), and we denote by $\mathcal{E}_{K,{\rm ext}} = \mathcal{E}_K \cap \mathcal{E}_{\rm ext}$ and $\mathcal{E}_{K,{\rm int}} = \mathcal{E}_K \cap \mathcal{E}_{\rm int}$.

To study the convergence of the schemes, we need the following two quantities: the size of the discretization

$$\mathrm{size}(\mathcal{D}) = \sup\{\mathrm{diam}(K)\,;\ K \in \mathcal{M}\}$$

and the regularity of the discretization

$$\begin{aligned}\mathrm{regul}(\mathcal{D}) \;=\; &\sup\left\{\max\left(\frac{\mathrm{diam}(K)^d}{\rho_K^d}, \mathrm{Card}(\mathcal{E}_K)\right)\ ;\ K \in \mathcal{M}\right\}\\ &+\sup\left\{\frac{\mathrm{diam}(K)}{\mathrm{diam}(L)}\ ;\ K \in \mathcal{M}\,,\ L \in \mathcal{N}_K\right\}\end{aligned}$$

where, for $K \in \mathcal{M}$, ρ_K is defined by

$$\rho_K = \sup\{r > 0 \mid \exists \boldsymbol{x} \in K^\star, B(\boldsymbol{x}, r) \subset K^\star\} \tag{8}$$

(see the meaning of $K^\star$ in Definition 2.1). Notice in particular that, for all $K \in \mathcal{M}$, $\mathrm{diam}(K)^d \le \mathrm{regul}(\mathcal{D})\rho_K^d \le \frac{\mathrm{regul}(\mathcal{D})}{\omega_d}\mathrm{m}(K)$ (with ω_d the volume of the unit ball in $\mathbb{R}^d$); hence, since $\mathrm{Card}(\mathcal{E}_K) \le \mathrm{regul}(\mathcal{D})$ and $\mathrm{m}(\sigma) \le \omega_{d-1}\mathrm{diam}(K)^{d-1}$ if $\sigma \in \mathcal{E}_K$, we have

$$\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\mathrm{diam}(K) \le \frac{\omega_{d-1}\mathrm{regul}(\mathcal{D})^2}{\omega_d}\mathrm{m}(K). \tag{9}$$

2.2 A mixed finite volume scheme for Stokes problem

If $\mathcal{D}$ is an admissible discretization of Ω, we denote by $H_\mathcal{D}$ the set of functions $w : \Omega \to \mathbb{R}$ which are piecewise constant on each control volume $K \in \mathcal{M}$, and we identify $w \in H_\mathcal{D}$ with the family of its values $(w_K)_{K\in\mathcal{M}}$ on the control volumes. $\mathcal{F}_\mathcal{D}$ is the set of families of real numbers $(F_{K,\sigma})_{K\in\mathcal{M}\,,\,\sigma\in\mathcal{E}_K}$.
Taking $\nu > 0$, the numerical scheme for the Stokes problem is the following: find $(p, \boldsymbol{u}, \mathbf{v}, \boldsymbol{F}) \in H_\mathcal{D} \times H_\mathcal{D}^d \times H_\mathcal{D}^{d\times d} \times \mathcal{F}_\mathcal{D}^d$ which satisfies the following equations. The first relation states that the second order tensor $\mathbf{v}$ is the gradient of $\boldsymbol{u}$ (we penalize by $\boldsymbol{F}$, which plays the role of the fluxes of $\mathbf{v}$, in order to estimate these fluxes later on):

$$\begin{aligned} &\mathbf{v}_K(\boldsymbol{x}_\sigma - \boldsymbol{x}_K) + \mathbf{v}_L(\boldsymbol{x}_L - \boldsymbol{x}_\sigma) + \nu\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}\boldsymbol{F}_{K,\sigma} - \nu\frac{\mathrm{diam}(L)}{\mathrm{m}(\sigma)}\boldsymbol{F}_{L,\sigma} = \boldsymbol{u}_L - \boldsymbol{u}_K\,, \\ &\hspace{20em} \forall\sigma^{K|L} \in \mathcal{E}_{\mathrm{int}}\,, \\ &\mathbf{v}_K(\boldsymbol{x}_\sigma - \boldsymbol{x}_K) + \nu\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}\boldsymbol{F}_{K,\sigma} = -\boldsymbol{u}_K\,, \quad \forall\sigma^{K|\partial} \in \mathcal{E}_{\mathrm{ext}}\,. \end{aligned} \tag{10}$$

We then impose that the fluxes of momentum, involved in the first P.D.E. of (1), respect an exact conservation across each internal edge of the mesh:

$$(\boldsymbol{F}_{K,\sigma} - p_K\mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}) + (\boldsymbol{F}_{L,\sigma} - p_L\mathrm{m}(\sigma)\boldsymbol{n}_{L,\sigma}) = 0\,, \quad \forall\sigma^{K|L} \in \mathcal{E}_{\mathrm{int}}\,. \tag{11}$$

The tensor $\mathbf{v}$ and its fluxes $\boldsymbol{F}$ are linked through the fact that the latter allows to reconstruct the former (see Lemma 6.1):

$$\mathrm{m}(K)\mathbf{v}_K = \sum_{\sigma\in\mathcal{E}_K} \boldsymbol{F}_{K,\sigma} \otimes (\boldsymbol{x}_\sigma - \boldsymbol{x}_K)\,, \quad \forall K \in \mathcal{M}\,. \tag{12}$$

The following equation translates the incompressibility condition, taking into account the penalization introduced in (10):

$$\mathrm{m}(K)\mathrm{Tr}(\mathbf{v}_K) + \nu\mathrm{diam}(K)\sum_{\sigma\in\mathcal{E}_K} \boldsymbol{F}_{K,\sigma}\cdot\boldsymbol{n}_{K,\sigma} = 0\,, \quad \forall K \in \mathcal{M} \tag{13}$$

(where $\mathrm{Tr}(\mathbf{w}) = \mathbf{w} : \mathbf{Id} = \sum_{i=1}^d \mathbf{w}_{i,i}$ is the trace — or contraction — of a second order tensor $\mathbf{w} = ((\mathbf{w}_{i,j}))_{(i,j)\in[1,d]^2}$). We then write the balance of fluxes, that is to say the discrete counterpart of the integration of the first P.D.E. in (1) on each control volume:

$$-\sum_{\sigma\in\mathcal{E}_K}(\boldsymbol{F}_{K,\sigma} - p_K\mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}) = -\sum_{\sigma\in\mathcal{E}_K}\boldsymbol{F}_{K,\sigma} = \int_K \boldsymbol{f}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,, \quad \forall K \in \mathcal{M} \tag{14}$$

(notice that $\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma} = 0$ thanks to Stokes' formula), and we normalize the choice of the pressure:

$$\sum_{K\in\mathcal{M}} \mathrm{m}(K)p_K = 0. \tag{15}$$

Remark 2.2 [Square system and implementation] *A close examination of the preceding scheme shows that it is over-determined. Indeed, by (11) there is in fact only one unknown flux vector $\boldsymbol{F}_\sigma$ for each edge of the mesh — since the knowledge of $\boldsymbol{F}_{K,\sigma}$ gives back $\boldsymbol{F}_{L,\sigma}$ using p_K*

and p_L — and (10) precisely provides d equations per edge; (12) and (14) respectively give as many equations as there are unknowns $\mathbf{v}_K$ and $\boldsymbol{u}_K$, and (13) gives as many equations as the unknowns p_K. With (15), we therefore have written one more equation than we have unknowns. However, these equations are not independent: take the scalar product of each equation (10) with $\mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}$ and sum on $\sigma \in \mathcal{E}$. Gathering by control volumes, the general formula $\mathbf{v}_K(\boldsymbol{x}_\sigma - \boldsymbol{x}_K)\cdot \boldsymbol{n}_{K,\sigma} = \mathbf{v}_K : (\boldsymbol{n}_{K,\sigma} \otimes (\boldsymbol{x}_\sigma - \boldsymbol{x}_K))$ and Lemma 6.1 give

$$\sum_{K\in\mathcal{M}} \mathrm{m}(K)\mathrm{Tr}(\mathbf{v}_K) + \sum_{K\in\mathcal{M}} \nu \mathrm{diam}(K) \sum_{\sigma\in\mathcal{E}_K} \boldsymbol{F}_{K,\sigma}\cdot \boldsymbol{n}_{K,\sigma} = -\sum_{K\in\mathcal{M}} \boldsymbol{u}_K \cdot \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}.$$

Since $\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma} = 0$, the right-hand side of this equation is zero, and (13) shows that the left-hand side is also zero. Hence, equations (10) and (13) are linked in a non-trivial fashion, and, removing one of the equations (13), we obtain a square system equivalent to (10)—(15). Instead of imposing a zero mean value for the pressure (see (15)), we can also impose a zero value for the pressure in a particular control volume (for example the one corresponding to the removed equation (13)); the resulting system is equivalent to (10)—(15) up to the addition of a constant value to the pressure, and this is the system we implement to compute the solution of the scheme (we will see in Section 4 that these square systems are invertible).

Remark 2.3 [Exact incompressibility] *The equations (10) allow to define $\boldsymbol{u}_\sigma$ by*

$$\mathbf{v}_K(\boldsymbol{x}_\sigma - \boldsymbol{x}_K) + \nu\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}\boldsymbol{F}_{K,\sigma} = \boldsymbol{u}_\sigma - \boldsymbol{u}_K, \quad \forall K\in\mathcal{M},\ \forall \sigma \in \mathcal{E}_K$$

(and $\boldsymbol{u}_\sigma$ indeed only depends on σ, not on K such that $\sigma\in\mathcal{E}_K$). Taking the scalar product of these equations by $\mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}$ and summing on $\sigma\in\mathcal{E}_K$, we obtain, thanks to Lemma 6.1 and since $\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma} = 0$,

$$\mathrm{m}(K)\mathrm{Tr}(\mathbf{v}_K) + \nu\mathrm{diam}(K)\sum_{\sigma\in\mathcal{E}_K} \boldsymbol{F}_{K,\sigma}\cdot\boldsymbol{n}_{K,\sigma} = \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}.$$

Using (13) leads to

$$\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma} = 0,$$

which provides the conservation property expected from the finite volume methods.

2.3 A mixed finite volume scheme for transient Navier-Stokes problem

Let $T > 0$, $N \geq 1$ be an integer, and define $\delta t = T/N$. If $\mathcal{D}$ is an admissible discretization of Ω in the sense of Definition 2.1, we denote by $H_{\mathcal{D},\delta t}$ the families of real numbers $w = (w_K^{n+1/2})_{K\in\mathcal{M}\,,\,n=0,\ldots,N-1}$, and we identify $w\in H_{\mathcal{D},\delta t}$ with the piecewise constant function $w :]\,0,T\,[\,\times\Omega\to\mathbb{R}$ which is equal to $w_K^{n+1/2}$ on $]n\delta t,(n+1)\delta t[\times K$ (for $n = 0,\ldots,N-1$ and $K\in\mathcal{M}$). If $w\in H_{\mathcal{D},\delta t}$, we let $w^{n+1/2} = (w_K^{n+1/2})_{K\in\mathcal{M}}\in H_{\mathcal{D}}$.

Defining $\mathcal{F}_{\mathcal{D},\delta t} = \{(G^{n+1/2})_{n=0,\ldots,N-1}\,;\ \forall n = 0,\ldots,N-1\,,\ G^{n+1/2}\in\mathcal{F}_{\mathcal{D}}\}$, the mixed finite volume scheme for the transient Navier-Stokes problem is a natural generalization of the scheme for the Stokes problem, using a Crank-Nicolson discretization of the time derivative (hence the natural exponent $n+1/2$, since this time discretization involves quantities at half time steps). We search for $(p,\boldsymbol{u},\mathbf{v},\boldsymbol{F})\in H_{\mathcal{D},\delta t}\times H^d_{\mathcal{D},\delta t}\times H^{d\times d}_{\mathcal{D},\delta t}\times\mathcal{F}^d_{\mathcal{D},\delta t}$ such that, for all $n = 0,\ldots,N-1$,

- $\mathbf{v}$ plays the role of a gradient of $\boldsymbol{u}$:

$$\mathbf{v}_K^{n+1/2}(\boldsymbol{x}_\sigma-\boldsymbol{x}_K)+\mathbf{v}_L^{n+1/2}(\boldsymbol{x}_L-\boldsymbol{x}_\sigma)+\nu\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}\boldsymbol{F}_{K,\sigma}^{n+1/2}-\nu\frac{\mathrm{diam}(L)}{\mathrm{m}(\sigma)}\boldsymbol{F}_{L,\sigma}^{n+1/2} = \boldsymbol{u}_L^{n+1/2}-\boldsymbol{u}_K^{n+1/2}\,,\quad \forall\sigma^{K|L}\in\mathcal{E}_{\mathrm{int}}\,, \tag{16}$$

$$\mathbf{v}_K^{n+1/2}(\boldsymbol{x}_\sigma-\boldsymbol{x}_K)+\nu\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}\boldsymbol{F}_{K,\sigma}^{n+1/2}=-\boldsymbol{u}_K^{n+1/2}\,,\quad \forall\sigma^{K|\partial}\in\mathcal{E}_{\mathrm{ext}}\,,$$

- the fluxes of momentum, involving the pressure, are conservative:

$$\left(\boldsymbol{F}_{K,\sigma}^{n+1/2}-p_K^{n+1/2}\mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}\right)+\left(\boldsymbol{F}_{L,\sigma}^{n+1/2}-p_L^{n+1/2}\mathrm{m}(\sigma)\boldsymbol{n}_{L,\sigma}\right)=0\,,\quad \forall\sigma^{K|L}\in\mathcal{E}_{\mathrm{int}}\,, \tag{17}$$

- $\mathbf{v}$ can be reconstructed from its fluxes:

$$\mathrm{m}(K)\mathbf{v}_K^{n+1/2}=\sum_{\sigma\in\mathcal{E}_K}\boldsymbol{F}_{K,\sigma}^{n+1/2}\otimes(\boldsymbol{x}_\sigma-\boldsymbol{x}_K)\,,\quad \forall K\in\mathcal{M}\,, \tag{18}$$

- the incompressibility condition holds:

$$\mathrm{m}(K)\mathrm{Tr}(\mathbf{v}_K^{n+1/2})+\nu\mathrm{diam}(K)\sum_{\sigma\in\mathcal{E}_K}\boldsymbol{F}_{K,\sigma}^{n+1/2}\cdot\boldsymbol{n}_{K,\sigma}=0\,,\quad \forall K\in\mathcal{M}\,, \tag{19}$$

- the PDE is satisfied on the discrete level ([1]):

$$\mathrm{m}(K)\frac{\boldsymbol{u}_K^{n+1}-\boldsymbol{u}_K^n}{\delta t}+\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}(\sigma)\boldsymbol{u}_\sigma^{n+1/2}\cdot\boldsymbol{n}_{K,\sigma}\left(\frac{\boldsymbol{u}_K^{n+1/2}+\boldsymbol{u}_L^{n+1/2}}{2}\right) -\sum_{\sigma\in\mathcal{E}_K}(\boldsymbol{F}_{K,\sigma}^{n+1/2}-p_K^{n+1/2}\mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma})=\frac{1}{\delta t}\int_{n\delta t}^{(n+1)\delta t}\int_K\boldsymbol{f}(t,\boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}\,,\quad \forall K\in\mathcal{M}\,, \tag{20}$$

  where the values at full time steps and half time steps are linked together by the following relation:

$$\boldsymbol{u}_K^{n+1/2}=\frac{\boldsymbol{u}_K^{n+1}+\boldsymbol{u}_K^n}{2}\,,\quad \forall K\in\mathcal{M}\,, \tag{21}$$

  and where, as in Remark 2.3, we define $\boldsymbol{u}_\sigma^{n+1/2}$ by ([2])

$$\mathbf{v}_K^{n+1/2}(\boldsymbol{x}_\sigma-\boldsymbol{x}_K)+\nu\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}\boldsymbol{F}_{K,\sigma}^{n+1/2}=\boldsymbol{u}_\sigma^{n+1/2}-\boldsymbol{u}_K^{n+1/2}\,,\quad \forall K\in\mathcal{M}\,,\ \forall\sigma\in\mathcal{E}_K\,, \tag{22}$$

- the discretization of the initial condition is defined by:

$$\boldsymbol{u}_K^0=\frac{1}{\mathrm{m}(K)}\int_K\boldsymbol{u}_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,,\quad \forall K\in\mathcal{M}\,, \tag{23}$$

[1] In the first sum on the edges, we let L be the neighboring control volume of K on the other side of σ, if $\sigma\in\mathcal{E}_{K,\mathrm{int}}$, or we let $\boldsymbol{u}_L^{n+1/2}=0$, if $\sigma\in\mathcal{E}_{K,\mathrm{ext}}$.

[2] This definition makes sense thanks to (16).

- and the choice of the pressure is normalized:

$$\sum_{K\in\mathcal{M}} \mathrm{m}(K) p_K^{n+1/2} = 0. \tag{24}$$

It will be useful to notice that, thanks to (22), (19) and as in Remark 2.3, we have

$$\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) \boldsymbol{u}_\sigma^{n+1/2} \cdot \boldsymbol{n}_{K,\sigma} = 0\,, \quad \forall K \in \mathcal{M}\,,\ \forall n = 0, \dots, N-1. \tag{25}$$

Remark 2.4 [Scheme for the steady problem] *A scheme for the steady problem can be obtained by suppressing all the time indices $n + 1/2$ in equations (16), (17), (18), (19), (22), (24), and replacing (20) with*

$$\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma} \left(\frac{\boldsymbol{u}_K + \boldsymbol{u}_L}{2}\right) - \sum_{\sigma\in\mathcal{E}_K} (\boldsymbol{F}_{K,\sigma} - p_K \mathrm{m}(\sigma) \boldsymbol{n}_{K,\sigma}) = \int_K \boldsymbol{f}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\,, \quad \forall K \in \mathcal{M}\,.$$

The scheme thus obtained can be studied in the same way as the transient scheme (see Section 5), which leads to similar convergence results.

Remark 2.5 [Implicit discretization] *All the mathematical results presented in this paper hold for the θ-scheme, which consists in replacing (21) by $\boldsymbol{u}_K^{n+1/2} = \theta \boldsymbol{u}_K^{n+1} + (1-\theta)\boldsymbol{u}_K^n$ with $\theta \in [1/2, 1]$ (the implicit discretization is obtained with $\theta = 1$, the Crank-Nicolson discretization with $\theta = 1/2$). The crucial point is that, in the course of the proof of Proposition 5.1, the new term $(\theta - \frac{1}{2}) \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} \mathrm{m}(K) \left|\boldsymbol{u}_K^{n+1} - \boldsymbol{u}_K^n\right|^2$ appearing in T_2 is non-negative for $\theta \in [1/2, 1]$.*

3 Numerical results

Since this paper is focused on the presentation of the scheme and on the proof of its convergence, we have no room to develop here a thorough comparison between its results and the ones of other schemes. We therefore limit the presentation of numerical results to the illustration of the aptitude of the scheme for handling various types of grids, in the case of steady and transient Navier-Stokes problems, while preserving good qualitative properties on the solution. The resolution of equations (16)—(24) has been implemented in a prototype code written in FORTRAN, and the resolution procedure at each time step is based on under-relaxed Newtonian iterations coupling all the equations (after eliminating $\mathbf{v}$ thanks to (18)). The resulting linear systems are solved by a direct method (Gaussian elimination) or an iterative method (BICGSTAB solver with an ILU preconditioner, see for example [16]). The implementation of the steady problem is done with the modifications presented in Remark 2.4, and the steady solution is therefore directly obtained (there is no need to approximate this solution by a transient one).

3.1 Lid driven cavity

We first focus on the classical lid driven cavity example with Re = 1000. Figure 1 shows the results obtained thanks to the scheme (16)—(24) (using nonhomogeneous boundary conditions instead of homogeneous ones) on different grids which are not admissible for classical finite

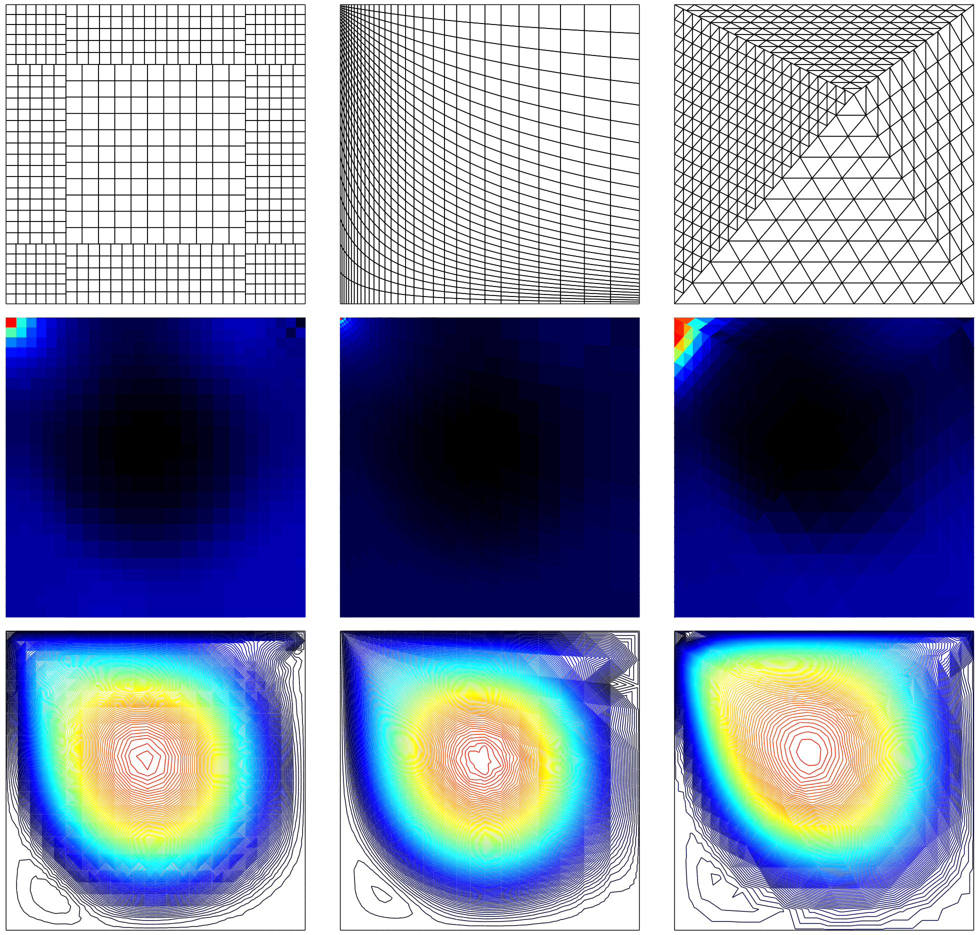

Figure 1: Lid driven cavity on unstructured and irregular grids: grids (top pictures), pressure field (middle pictures) and streamlines (bottom pictures).

element or finite volume schemes. The accuracy of these results on those coarse grids appear to be acceptable. We also notice that the quality of the numerical streamlines is mainly linked with the size of the control volumes (the streamlines are deformed in regions with large control volumes, and good in regions with small control volumes), and not with the fact that different regions are discretized with grids which are connected in "non-admissible" ways (in the sense of finite element methods); such a situation can occur, for instance, during a refinement procedure. We present in Figure 2 the effect of the value of the stabilization parameter ν, in the case of the lid driven cavity with $\mathrm{Re} = 1000$ on a 30×30 square grid; these results show that, in order to obtain a good approximate pressure field in this case, the stabilization parameter must be chosen not too small. We however want to emphasize that the choice of ν has no perceptible influence on the quality of the velocity: we have noticed that, on the same 30×30 grid, the results for the streamlines and the velocity field with ν in the range $[10^{-3}, 10^{-7}]$ are indistinguishable from the case $\nu = 10^{-7}$ presented in Figure 3.

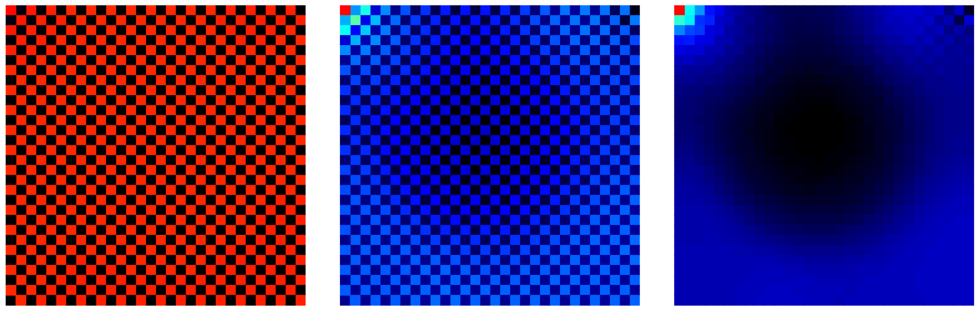

Figure 2: Lid driven cavity on a 30×30 square grid, pressure fields for: $\nu = 10^{-7}$ (left), $\nu = 10^{-5}$ (middle), $\nu = 10^{-3}$ (right).

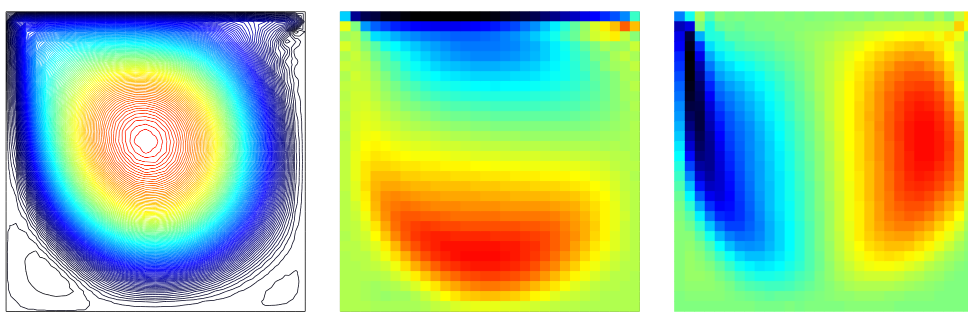

Figure 3: Lid driven cavity on a 30×30 square grid for $\nu = 10^{-7}$: streamlines (left), horizontal velocity (middle) and vertical velocity (right).

3.2 Backward facing step

We then study the behavior of the scheme in the case of the flow into a pipe whose dimensions vary discontinuously (the backward facing step problem, included in the domain $]-2,30[\times]0,1.94[$, the step being at $x=0$; see for example [1]). We let Re $=800$ and we use a quite coarse mesh, made of 5625 rectangles and triangles and deliberately chosen to be non admissible in the sense of classical finite element or finite volume schemes (some edges are cut in two, see Figure 4). The results we obtain show a good accuracy: the reattachment length for the bottom vortex is obtained at $x=10.5$, the detachment position for the top vortex is obtained at $x=9.0$ and its reattachment position is given by $x=17.5$, which is in the order of magnitude of the values supplied in literature, to within 10% (see Figure 5). Let us also observe that in this case, where we impose the pressure at the right vertical boundary, nearly no stabilization is necessary: we chose $\nu=10^{-7}$ for this calculation and we obtained a good pressure field (see Figure 6). Notice finally that, as for the lid driven cavity, the quality of the numerical results is not deteriorated in the region where the grid is not admissible (in the sense of classical finite element or finite volume schemes).

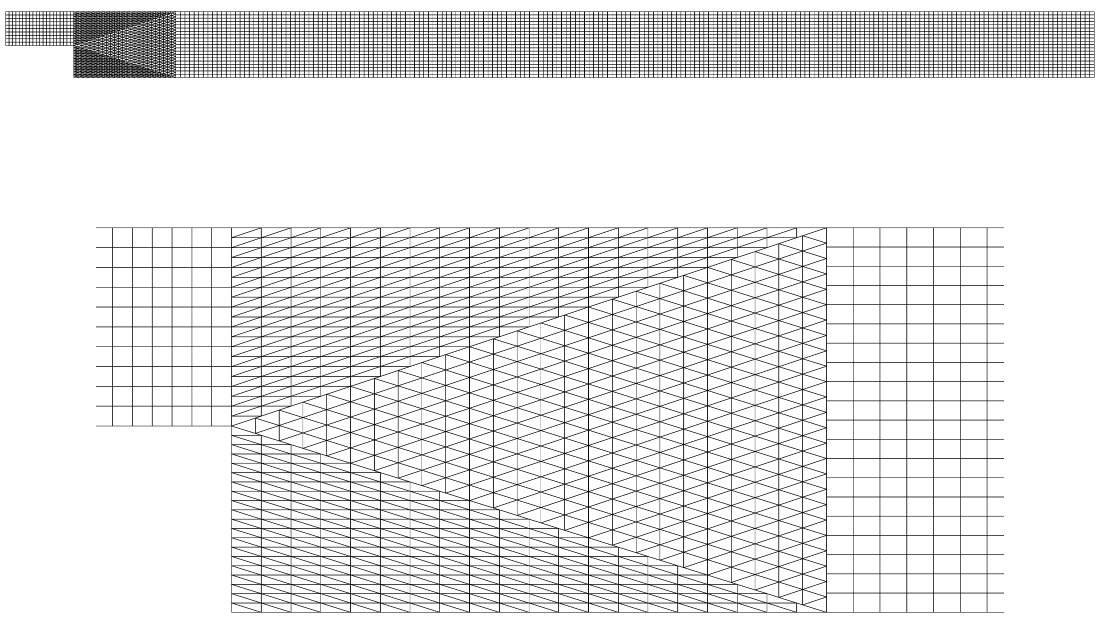

Figure 4: Backward facing step, mesh: in the full pipe (top), and zoom on a neighborhood of the step (bottom).

3.3 Green-Taylor analytical example

We take $\Omega=]0,1[\times]0,1[$ and $T=0.02$. Let $\mu>0$ be given and let the pair of functions $(\bar{\boldsymbol{u}},\bar{p})$ be defined on $]0,T[\times\Omega$ by

$$\begin{aligned}
\bar{u}_1(t,\boldsymbol{x}) &= -\frac{1}{\mu}\cos(2\pi(x_1+\tfrac{1}{4}))\ \sin(2\pi(x_2+\tfrac{1}{2}))\ \exp(-8\pi^2 t)\\
\bar{u}_2(t,\boldsymbol{x}) &= \frac{1}{\mu}\sin(2\pi(x_1+\tfrac{1}{4}))\ \cos(2\pi(x_2+\tfrac{1}{2}))\ \exp(-8\pi^2 t)\\
\bar{p}(t,\boldsymbol{x}) &= -\frac{1}{4\,\mu^2}\left(\cos(4\pi(x_1+\tfrac{1}{4}))+\cos(4\pi(x_2+\tfrac{1}{2}))\right)\ \exp(-16\,\pi^2 t).
\end{aligned}$$

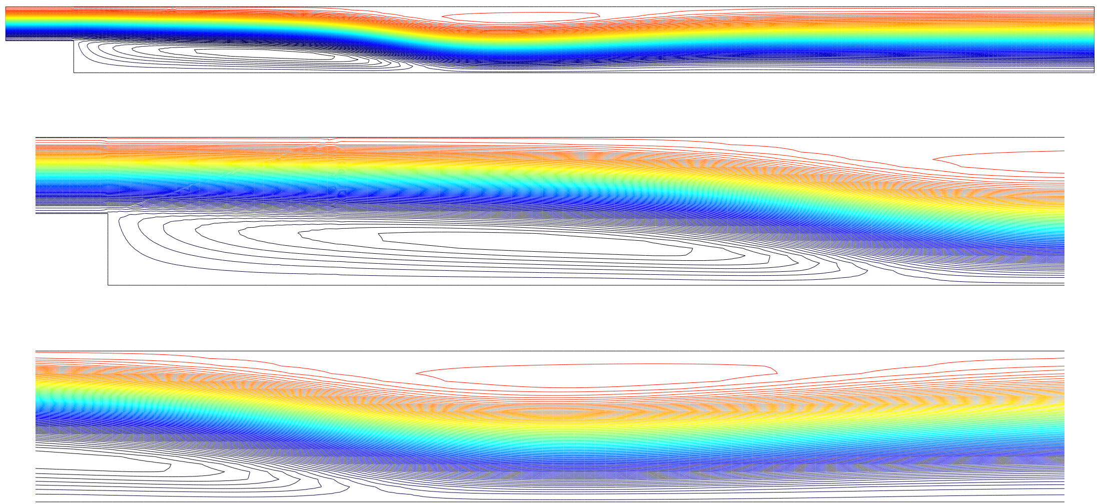

Figure 5: Backward facing step, streamlines: full pipe (top) and zooms on the first (middle) and second (bottom) vortices.

Figure 6: Backward facing step, pressure field.

Then $(\bar{\boldsymbol{u}}, \bar{p})$ is the unique solution of the transient Navier-Stokes equations with $\boldsymbol{f} = 0$, the initial condition and nonhomogeneous boundary conditions being respectively given by $\bar{\boldsymbol{u}}(0, \cdot)$ and $\bar{\boldsymbol{u}}(t, \boldsymbol{x})$ for all $(t, \boldsymbol{x}) \in]0, T[\times \partial\Omega$ (a small time $T = 0.02$ has been selected in order to take into account the exponential decay of the solution: for larger times, the solution nearly vanishes).
We denote by $(\boldsymbol{u}, p)$ the approximate velocity and pressure fields resulting from the time implicit version of (16)—(24) (see Remark 2.5; the results given by the Crank-Nicolson scheme have led in this case to lower convergence properties) with $\boldsymbol{f} = 0$ and the initial condition and the nonhomogeneous boundary conditions satisfied by the continuous solution. The obtained results are given in Table 1 (in which the regular grids and the time steps we used are precised), assuming $\mu = 0.01$ and setting $\nu = 10^{-7}$ for all calculations. These results show that the convergence

grid	δt	$\frac{\|\bar{u}_1(T,\cdot)-u_1(T,\cdot)\|_{L^2}}{\|\bar{u}_1(T,\cdot)\|_{L^2}}$	$\frac{\|\bar{u}_2(T,\cdot)-u_2(T,\cdot)\|_{L^2}}{\|\bar{u}_2(T,\cdot)\|_{L^2}}$	$\frac{\|\bar{p}(T,\cdot)-p(T,\cdot)\|_{L^2}}{\|\bar{p}(T,\cdot)\|_{L^2}}$
10×10	0.004	0.14	0.15	0.38
20×20	0.001	0.038	0.043	0.086
40×40	0.00025	0.011	0.012	0.023
80×80	0.0000625	0.0029	0.0035	0.0064

Table 1: Green-Taylor analytical example, relative errors of the different fields of unknowns at time $T = 0.02$.

properties of the method are compatible with space order not far from 2 and time order not far from 1 for the velocity and the pressure, although ν remains constant.

3.4 Conservation of kinetic energy

In order to obtain a stable and dissipation-free numerical method, one of the important behaviors of the scheme must be the conservation, at very high Reynolds numbers and without source terms, of the kinetic energy (see [14]). In other words, for Reynolds number equal to 1 and small times (see the renormalization in Remark 1.2), the decay of kinetic energy should only come from the viscous term, not the convective nonlinear term. Let us check the behavior of the mixed finite volume method on the kinetic energy.
We consider (5) with $\Omega =]0,1[\times]0,1[$, $T = 10^{-5}$, $\boldsymbol{f} = 0$ and $\boldsymbol{u}_0$ given by

$$\boldsymbol{u}_0(\boldsymbol{x}) = \begin{pmatrix} -\partial_2\Psi(\boldsymbol{x}) \\ \partial_1\Psi(\boldsymbol{x}) \end{pmatrix} \quad \text{with} \quad \Psi(x_1,x_2) = 0.0001 \times (x_1(1-x_1)x_2(1-x_2))^2 .$$

Denoting by $\bar{\boldsymbol{u}}$ the solution to (5) and by $\boldsymbol{u}$ the solution to (16)—(24), we define the exact kinetic energy by $\bar{\mathcal{E}}_c(t) = \frac{1}{2}\int_\Omega |\bar{\boldsymbol{u}}(t,\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}$ and the approximate one by $\mathcal{E}_c(t) = \frac{1}{2}\int_\Omega |\boldsymbol{u}(t,\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}$. The exact kinetic energy follows the equation

$$\bar{\mathcal{E}}_c(t) = \bar{\mathcal{E}}_c(0) - t\left(\int_\Omega \nabla\boldsymbol{u}_0(\boldsymbol{x}) : \nabla\boldsymbol{u}_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \varepsilon(t)\right), \quad \forall t \in]0,T[,$$

with $\varepsilon(t) \to 0$ as $t \to 0$ (this relation states that the infinitesimal decay of $\bar{\mathcal{E}}_c$ only comes from the viscous term). Let us check that this is approximately verified by the discrete solution, using a mesh with 20×20 control volumes and a time step equal to $\delta t = 10^{-7}$ (hence T corresponds to a hundred time steps) and letting $\nu = 10^{-7}$. We have $\bar{\mathcal{E}}_c(0) = \frac{4}{1323}\times 10^6 \simeq 3023.43$ and $\int_\Omega \nabla\boldsymbol{u}_0(\boldsymbol{x}) : \nabla\boldsymbol{u}_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \frac{16}{49}\times 10^6 \simeq 3.27\times 10^5$, and the computation of the numerical solution gives $\mathcal{E}_c(0) \simeq 3023.74$ and $\mathcal{E}_c(T) \simeq 3020.44$; this shows a decrease rate $(\mathcal{E}_c(0) - \mathcal{E}_c(T))/T$ equal to 3.3×10^5, very close to the theoretical value 3.27×10^5. This example shows that the mixed finite volume scheme does not introduce any significant artificial energy decay: the only decay is due to the diffusion term, and the discretization of the convective nonlinear term induces no additional diffusion phenomenon (as we will prove during the analysis of the scheme — see (35)).

4 Mathematical study of the scheme for Stokes problem

Here are the results we prove on the scheme for Stokes problem.

Theorem 4.1 [Existence of a unique solution to the scheme for Stokes problem] *Assume that (2) and (3) hold. Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 and let $\nu > 0$. Then there exists a unique $(p,\boldsymbol{u},\mathrm{v},\boldsymbol{F}) \in H_\mathcal{D}\times H_\mathcal{D}^d \times H_\mathcal{D}^{d\times d}\times \mathcal{F}_\mathcal{D}^d$ solution to (10)—(15).*

Theorem 4.2 [Convergence of the scheme for Stokes problem] *Assume that (2) and (3) hold. Let $(\mathcal{D}_m)_{m\geq 1}$ be a sequence of admissible discretizations of Ω in the sense of Definition 2.1, such that $\mathrm{size}(\mathcal{D}_m) \to 0$ as $m\to\infty$ and $(\mathrm{regul}(\mathcal{D}_m))_{m\geq 1}$ is bounded. Let $\lambda > 0$ and $\alpha \in]0,2[$, and define $\nu_m = \lambda \mathrm{size}(\mathcal{D}_m)^\alpha$. Let $(p_m,\boldsymbol{u}_m,\mathrm{v}_m,\boldsymbol{F}_m)$ be the solution to (10)—(15) with $\mathcal{D} = \mathcal{D}_m$ and $\nu = \nu_m$. Let $\bar{\boldsymbol{u}}$ be the unique solution to (4).*
Then, as $m\to\infty$, $\boldsymbol{u}_m \to \bar{\boldsymbol{u}}$ strongly in $L^q(\Omega)^d$ for all $q < \frac{2d}{d-2}$ (and weakly in $L^6(\Omega)^3$ if $d = 3$) and $\mathrm{v}_m \to \nabla\bar{\boldsymbol{u}}$ strongly in $L^2(\Omega)^{d\times d}$.

4.1 A priori estimates

As is usual in finite volume schemes, the proof of convergence relies on a priori estimates on the solution to the scheme.

Proposition 4.1 [A priori estimates on v and $\boldsymbol{F}$] *Assume that (2) and (3) hold. Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 and $\theta \geq \mathrm{regul}(\mathcal{D})$. Let $\nu_0 > 0$ and assume that $0 \leq \nu \leq \nu_0$. If $(p, \boldsymbol{u}, \mathbf{v}, \boldsymbol{F}) \in H_{\mathcal{D}} \times H_{\mathcal{D}}^d \times H_{\mathcal{D}}^{d\times d} \times \mathcal{F}_{\mathcal{D}}^d$ is a solution to (10)—(15), then*

$$\|\mathbf{v}\|^2_{L^2(\Omega)^{d\times d}} + \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} |\boldsymbol{F}_{K,\sigma}|^2 \leq C_1 \|\boldsymbol{f}\|^2_{L^2(\Omega)^d},$$

where $\|\mathbf{v}\|^2_{L^2(\Omega)^{d\times d}} = \int_\Omega |\mathbf{v}(\boldsymbol{x})|^2\, dx = \int_\Omega \mathbf{v}(\boldsymbol{x}) : \mathbf{v}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$ and C_1 only depends on d, Ω, θ and ν_0.

Proof of Proposition 4.1
Take the scalar product of (14) and $\boldsymbol{u}_K$ and sum on the control volumes K. Using the conservation property (11), we can gather by edges to find

$$\sum_{\sigma^{K|L}\in\mathcal{E}_{\mathrm{int}}} (\boldsymbol{F}_{K,\sigma} - p_K \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}) \cdot (\boldsymbol{u}_L - \boldsymbol{u}_K)$$
$$+ \sum_{\sigma^{K|\partial}\in\mathcal{E}_{\mathrm{ext}}} (\boldsymbol{F}_{K,\sigma} - p_K \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}) \cdot (-\boldsymbol{u}_K) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

Thanks to (10) and using again (11), we deduce, gathering by control volumes,

$$\begin{aligned}&\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} (\mathbf{v}_K(\boldsymbol{x}_\sigma - \boldsymbol{x}_K)) \cdot (\boldsymbol{F}_{K,\sigma} - p_K \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma})\\ &+ \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} (\boldsymbol{F}_{K,\sigma} - p_K \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}) \cdot \boldsymbol{F}_{K,\sigma} = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.\end{aligned} \tag{26}$$

We then use

$$(\mathbf{v}_K(\boldsymbol{x}_\sigma - \boldsymbol{x}_K)) \cdot (\boldsymbol{F}_{K,\sigma} - p_K \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}) = \mathbf{v}_K : ((\boldsymbol{F}_{K,\sigma} - p_K \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}) \otimes (\boldsymbol{x}_\sigma - \boldsymbol{x}_K))$$

and Lemma 6.1 with $\mathbf{a} = \mathbf{Id}$ to obtain, thanks to (12) and (13),

$$\begin{aligned}\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} (\mathbf{v}_K(\boldsymbol{x}_\sigma - \boldsymbol{x}_K)) \cdot (\boldsymbol{F}_{K,\sigma} - p_K \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma})&\\ = \|\mathbf{v}\|^2_{L^2(\Omega)^{d\times d}} - \sum_{K\in\mathcal{M}} \mathrm{m}(K) p_K \mathrm{Tr}(\mathbf{v}_K)&\\ = \|\mathbf{v}\|^2_{L^2(\Omega)^{d\times d}} + \sum_{K\in\mathcal{M}} \nu \mathrm{diam}(K) p_K \sum_{\sigma\in\mathcal{E}_K} \boldsymbol{F}_{K,\sigma} \cdot \boldsymbol{n}_{K,\sigma}.&\end{aligned} \tag{27}$$

Plugging (27) into (26), we notice that the pressure terms disappear, which leads to

$$\|\mathbf{v}\|^2_{L^2(\Omega)^{d\times d}} + \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} |\boldsymbol{F}_{K,\sigma}|^2 = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}. \tag{28}$$

Using Young's inequality and Lemma 6.2, we have, for all $\varepsilon > 0$,

$$\begin{aligned}\int_\Omega \boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &\leq \frac{1}{2\varepsilon}\|\boldsymbol{f}\|^2_{L^2(\Omega)^d} + \frac{\varepsilon}{2}\|\boldsymbol{u}\|^2_{L^2(\Omega)^d}\\ &\leq \frac{1}{2\varepsilon}\|\boldsymbol{f}\|^2_{L^2(\Omega)^d} + \varepsilon C_2\|\mathbf{v}\|^2_{L^2(\Omega)^{d\times d}} + \varepsilon C_2\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu^2\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}|\boldsymbol{F}_{K,\sigma}|^2\\ &\leq \frac{1}{2\varepsilon}\|\boldsymbol{f}\|^2_{L^2(\Omega)^d} + \varepsilon C_2\|\mathbf{v}\|^2_{L^2(\Omega)^{d\times d}} + \varepsilon C_2\nu_0\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}|\boldsymbol{F}_{K,\sigma}|^2\,,\end{aligned}$$

where C_2 only depends on d, Ω and θ. The proof is concluded by taking $\varepsilon = \inf(\frac{1}{2C_2}, \frac{1}{2\nu_0 C_2})$ and plugging the result into (28). □

4.2 Proof of the theorems

Using the preceding estimates, we can now prove the existence of a unique solution to (10)—(15).

Proof of Theorem 4.1
As explained in Remark 2.2, (10)—(15) can in fact be considered as a square system (as many equations as unknowns). Since this system is linear, Proposition 4.1 shows that if the terms $\int_K \boldsymbol{f}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ in the right-hand side of (10)—(15) are equal to zero, then so is $(\mathbf{v}, \boldsymbol{F})$, which in turn implies (thanks to (10)) that $\boldsymbol{u}$ vanishes. From (11), we deduce $p_K = p_L$ for all neighboring control volumes K and L; since Ω is connected this means that p is constant and, by (15), that it vanishes. Hence, the square system (10)—(15) is well-posed and has a unique solution. □

Let us now see that the approximate solution converges to the weak solution of (1).

Proof of Theorem 4.2
To simplify the notations, we drop the index m in $\mathcal{D}_m$, p_m, $\boldsymbol{u}_m$, $\mathbf{v}_m$ and $\boldsymbol{F}_m$. As is usual, since the solution to (4) is unique, it is enough to prove the convergence of a subsequence of $(\boldsymbol{u}, \mathbf{v})$ toward the solution of this problem.
Proposition 4.1 gives estimates on $(\mathbf{v}, \boldsymbol{F})$ which are uniform with respect to $\mathcal{D}$, since $\mathrm{regul}(\mathcal{D})$ and $\nu = \lambda\mathrm{size}(\mathcal{D})^\alpha$ are bounded. We can therefore write,

$$\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu^2\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}|\boldsymbol{F}_{K,\sigma}|^2 \leq C_3\nu = C_3\lambda\mathrm{size}(\mathcal{D})^\alpha \tag{29}$$

with C_3 not depending on $\mathcal{D}$. This last quantity tends to 0 as $\mathrm{size}(\mathcal{D}) \to 0$ and by Lemma 6.4 we deduce that there exists $\bar{\boldsymbol{u}} \in H^1_0(\Omega)^d$ such that, up to a subsequence as $\mathrm{size}(\mathcal{D}) \to 0$, $\boldsymbol{u} \to \bar{\boldsymbol{u}}$ strongly in $L^q(\Omega)^d$ for all $q < \frac{2d}{d-2}$ (and weakly in $L^6(\Omega)^d$ if $d = 3$) and $\mathbf{v} \to \nabla\bar{\boldsymbol{u}}$ weakly in $L^2(\Omega)^{d\times d}$.

Step 1: $\bar{\boldsymbol{u}}$ belongs to $E(\Omega)$.
Let $\Gamma : \Omega \to \mathbb{R}$ be the piecewise function equal to $\frac{\nu\mathrm{diam}(K)}{\mathrm{m}(K)}\sum_{\sigma\in\mathcal{E}_K}\boldsymbol{F}_{K,\sigma}\cdot\boldsymbol{n}_{K,\sigma}$ on $K \in \mathcal{M}$. From Cauchy-Schwarz inequality, we have

$$\|\Gamma\|_{L^1(\Omega)} = \sum_{K\in\mathcal{M}}\left|\sum_{\sigma\in\mathcal{E}_K}\nu\mathrm{diam}(K)\boldsymbol{F}_{K,\sigma}\cdot\boldsymbol{n}_{K,\sigma}\right|$$

$$\leq \quad \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu^2 \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} |\boldsymbol{F}_{K,\sigma}|^2 \right)^{1/2} \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{diam}(K)\mathrm{m}(\sigma) \right)^{1/2}.$$

Hence, using (9) and (29), we have $\Gamma \to 0$ in $L^1(\Omega)$ as $\mathrm{size}(\mathcal{D}) \to 0$. Since $\mathrm{Tr}(\mathbf{v}) + \Gamma = 0$ on Ω (this is (13) divided by $\mathrm{m}(K)$), we deduce from the weak convergence of $\mathbf{v}$ to $\nabla\bar{\boldsymbol{u}}$ that $\mathrm{Tr}(\nabla\bar{\boldsymbol{u}}) = \mathrm{div}(\bar{\boldsymbol{u}}) = 0$ and thus that $\bar{\boldsymbol{u}} \in E(\Omega)$.

Step 2: $\bar{\boldsymbol{u}}$ satisfies (4).
By the density of $\{\boldsymbol{\varphi} \in C_c^\infty(\Omega)^d\,,\ \mathrm{div}(\boldsymbol{\varphi}) = 0\}$ in $E(\Omega)$ (see [17]), it is sufficient to prove (4) for $\boldsymbol{\varphi}$ regular with compact support. Let $\boldsymbol{\varphi}$ be such a function; we take the scalar product of (14) and $\boldsymbol{\varphi}(\boldsymbol{x}_K)$ and we sum on K:

$$-\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} (\boldsymbol{F}_{K,\sigma} - p_K \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}_K) = \sum_{K\in\mathcal{M}} \int_K \boldsymbol{\varphi}(\boldsymbol{x}_K) \cdot \boldsymbol{f}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{30}$$

Let $\boldsymbol{\varphi}_\sigma = \frac{1}{\mathrm{m}(\sigma)} \int_\sigma \boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})$. By (11) and since $\boldsymbol{\varphi}_\sigma = 0$ for $\sigma \in \mathcal{E}_{\mathrm{ext}}$, we have

$$\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} (\boldsymbol{F}_{K,\sigma} - p_K \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}) \cdot \boldsymbol{\varphi}_\sigma = \sum_{\sigma^{K|L}\in\mathcal{E}_{\mathrm{int}}} [(\boldsymbol{F}_{K,\sigma} - p_K \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}) + (\boldsymbol{F}_{L,\sigma} - p_L \mathrm{m}(\sigma)\boldsymbol{n}_{L,\sigma})] \cdot \boldsymbol{\varphi}_\sigma = 0.$$

Equation (30) can therefore be written, with $\boldsymbol{\varphi}_\mathcal{D}$ equal to $\boldsymbol{\varphi}(\mathbf{x}_K)$ on each mesh K,

$$\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} (\boldsymbol{F}_{K,\sigma} - p_K \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}) \cdot (\boldsymbol{\varphi}_\sigma - \boldsymbol{\varphi}(\boldsymbol{x}_K)) = \int_\Omega \boldsymbol{\varphi}_\mathcal{D}(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{31}$$

We have, since $\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma} = 0$ and $\mathrm{div}(\boldsymbol{\varphi}) = 0$,

$$\begin{aligned} &\sum_{K\in\mathcal{M}} p_K \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma} \cdot (\boldsymbol{\varphi}_\sigma - \boldsymbol{\varphi}(\boldsymbol{x}_K)) \\ &\qquad = \sum_{K\in\mathcal{M}} p_K \sum_{\sigma\in\mathcal{E}_K} \int_\sigma \boldsymbol{\varphi}(\boldsymbol{x}) \cdot \boldsymbol{n}_{K,\sigma}\,\mathrm{d}\gamma(\boldsymbol{x}) = \sum_{K\in\mathcal{M}} p_K \int_K \mathrm{div}(\boldsymbol{\varphi})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0 \end{aligned}$$

and therefore (31) leads to

$$\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \boldsymbol{F}_{K,\sigma} \cdot (\boldsymbol{\varphi}_\sigma - \boldsymbol{\varphi}(\boldsymbol{x}_K)) = \int_\Omega \boldsymbol{\varphi}_\mathcal{D}(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{32}$$

Since $\boldsymbol{x}_\sigma$ is the barycenter of σ and $\boldsymbol{\varphi}_\sigma$ is the mean value on σ of the regular function $\boldsymbol{\varphi}$, we have $\boldsymbol{\varphi}_\sigma - \boldsymbol{\varphi}(\boldsymbol{x}_K) = \frac{1}{\mathrm{m}(K)} \int_K \nabla\boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,(\boldsymbol{x}_\sigma - \boldsymbol{x}_K) + \boldsymbol{R}_{K,\sigma}$ with $|\boldsymbol{R}_{K,\sigma}| \leq C_{\boldsymbol{\varphi}}\mathrm{diam}(K)^2$ (where $C_{\boldsymbol{\varphi}}$ only depends on $\boldsymbol{\varphi}$). From (32), we deduce

$$\sum_{K\in\mathcal{M}} \frac{1}{\mathrm{m}(K)} \int_K \nabla\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} : \sum_{\sigma\in\mathcal{E}_K} \boldsymbol{F}_{K,\sigma} \otimes (\boldsymbol{x}_\sigma - \boldsymbol{x}_K) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{\varphi}_\mathcal{D}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + T_1\,,$$

where $|T_1| \leq C_{\boldsymbol{\varphi}} \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} |\boldsymbol{F}_{K,\sigma}| \mathrm{diam}(K)^2$. Using then (12), this gives

$$\int_\Omega \mathbf{v}(\boldsymbol{x}) : \nabla\boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{\varphi}_{\mathcal{D}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + T_1. \tag{33}$$

By the weak convergence of $\mathbf{v}$ to $\nabla\bar{\boldsymbol{u}}$ and the regularity of $\boldsymbol{\varphi}$, the first two terms of this equality respectively converge to $\int_\Omega \nabla\bar{\boldsymbol{u}}(\boldsymbol{x}) : \nabla\boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ and $\int_\Omega \boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ as $\mathrm{size}(\mathcal{D}) \to 0$. Hence, it remains to prove that $T_1 \to 0$ to conclude the proof that $\bar{\boldsymbol{u}}$ satisfies (4).
The convergence of T_1 is quite easy to establish thanks to Proposition 4.1. Indeed, from the estimates in this proposition and using (9), we have

$$\begin{aligned}
|T_1| &\leq C_4 \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{diam}(K)^4 \frac{\mathrm{m}(\sigma)}{\nu\,\mathrm{diam}(K)} \right)^{1/2} \\
&\leq C_4 \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \frac{\mathrm{size}(\mathcal{D})^2}{\lambda\,\mathrm{size}(\mathcal{D})^\alpha} \mathrm{m}(\sigma)\mathrm{diam}(K) \right)^{1/2} \leq C_5 \mathrm{size}(\mathcal{D})^{\frac{2-\alpha}{2}}
\end{aligned}$$

where C_4 and C_5 do not depend on $\mathcal{D}$. Since $\alpha < 2$, this last term tends to 0 as $\mathrm{size}(\mathcal{D}) \to 0$, which concludes the proof that $\bar{\boldsymbol{u}}$ is the weak solution to the Stokes equation.

Step 3: it remains to prove that the convergence of $\mathbf{v}$ to $\nabla\bar{\boldsymbol{u}}$ is strong.
In order to do so, we recall (28), which implies

$$\|\mathbf{v}\|^2_{L^2(\Omega)^{d\times d}} \leq \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

By convergence of $\boldsymbol{u}$ to $\bar{\boldsymbol{u}}$, and since $\bar{\boldsymbol{u}}$ is a solution to (4), we deduce

$$\limsup_{\mathrm{size}(\mathcal{D})\to 0} \|\mathbf{v}\|^2_{L^2(\Omega)^{d\times d}} \leq \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \bar{\boldsymbol{u}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega |\nabla\bar{\boldsymbol{u}}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}.$$

On the other hand, since $\mathbf{v} \to \nabla\bar{\boldsymbol{u}}$ weakly in $L^2(\Omega)^{d\times d}$,

$$\int_\Omega |\nabla\bar{\boldsymbol{u}}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} = \|\nabla\bar{\boldsymbol{u}}\|^2_{L^2(\Omega)^{d\times d}} \leq \liminf_{\mathrm{size}(\mathcal{D})\to 0} \|\mathbf{v}\|^2_{L^2(\Omega)^{d\times d}}$$

and therefore $\|\mathbf{v}\|^2_{L^2(\Omega)^{d\times d}} \to \|\nabla\bar{\boldsymbol{u}}\|^2_{L^2(\Omega)^{d\times d}}$ as $\mathrm{size}(\mathcal{D}) \to 0$. The weak convergence of $\mathbf{v}$ to $\nabla\bar{\boldsymbol{u}}$ and the convergence of the norm of $\mathbf{v}$ toward the norm of $\nabla\bar{\boldsymbol{u}}$ imply the strong convergence of $\mathbf{v}$ in $L^2(\Omega)^{d\times d}$. $\square$

5 Mathematical study of the scheme for Navier-Stokes problem

Here are the two results we prove on the scheme (16)—(24).

Theorem 5.1 [Existence of a solution to the scheme for Navier-Stokes problem] *Let $T > 0$, $N \geq 1$ be an integer and $\delta t = T/N$. Assume that (2) and (6) hold. Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 and let $\nu > 0$. Then there exists at least one $(p, \boldsymbol{u}, \mathbf{v}, \boldsymbol{F}) \in H_{\mathcal{D},\delta t} \times H^d_{\mathcal{D},\delta t} \times H^{d\times d}_{\mathcal{D},\delta t} \times \mathcal{F}^d_{\mathcal{D},\delta t}$ solution to (16)—(24).*

Theorem 5.2 **[Convergence of the scheme for Navier-Stokes problem]** *Let $T > 0$ and assume that (2) and (6) hold. Let $N_m \to +\infty$ be a sequence of integers and define $\delta t_m = T/N_m$. Let $(\mathcal{D}_m)_{m\geq 1}$ be a sequence of admissible discretizations of Ω in the sense of Definition 2.1, such that $\mathrm{size}(\mathcal{D}_m) \to 0$ as $m \to \infty$ and $(\mathrm{regul}(\mathcal{D}_m))_{m\geq 1}$ is bounded. Let $\lambda > 0$ and $\alpha \in]0, 2[$, and define $\nu_m = \lambda \mathrm{size}(\mathcal{D}_m)^\alpha$. Let $(p_m, \boldsymbol{u}_m, \mathbf{v}_m, \boldsymbol{F}_m)$ be the solution to (16)—(24) with $\delta t = \delta t_m$, $\mathcal{D} = \mathcal{D}_m$ and $\nu = \nu_m$.*
Then there exists a weak solution $\bar{\boldsymbol{u}}$ to (5) such that, up to a subsequence as $m \to \infty$, $\boldsymbol{u}_m \to \bar{\boldsymbol{u}}$ strongly in $L^2(]0, T[\times\Omega)^d$ and $\mathbf{v}_m \to \nabla\bar{\boldsymbol{u}}$ weakly in $L^2(]0, T[\times\Omega)^{d\times d}$.

Remark 5.1 *In dimension $d = 2$, the solution $\bar{\boldsymbol{u}}$ is unique and is regular enough to be used as a test function in (7) (see [17]). Hence, in this case, the whole sequence of approximate solutions converges toward the weak solution and we can mimic the method used in [6], [4] or the proof of Theorem 4.2 to see that the convergence of $\mathbf{v}_m$ is in fact strong in $L^2(]0, T[\times\Omega)^{d\times d}$.*

5.1 A priori estimates and existence of an approximate solution

We begin with *a priori* estimates, similar to the ones obtained for the scheme on Stokes problem.

Proposition 5.1 *Let $T > 0$, $N \geq 1$ be an integer and $\delta t = T/N$. Assume that (2) and (6) hold. Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 and take $\theta \geq \mathrm{regul}(\mathcal{D})$. Let $\nu_0 > 0$ and assume that $0 \leq \nu \leq \nu_0$. If $(p, \boldsymbol{u}, \mathbf{v}, \boldsymbol{F}) \in H_{\mathcal{D},\delta t} \times H^d_{\mathcal{D},\delta t} \times H^{d\times d}_{\mathcal{D},\delta t} \times \mathcal{F}^d_{\mathcal{D},\delta t}$ is a solution to (16)—(24) then*

$$\|\boldsymbol{u}\|^2_{L^\infty(0,T;L^2(\Omega)^d)} + \|\mathbf{v}\|^2_{L^2(]0,T[\times\Omega)^{d\times d}} \leq C_6(\|\boldsymbol{f}\|^2_{L^2(]0,T[\times\Omega)^d} + \|\boldsymbol{u}_0\|^2_{L^2(\Omega)^d})$$

and

$$\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} \left|\boldsymbol{F}^{n+1/2}_{K,\sigma}\right|^2 \leq C_6(\|\boldsymbol{f}\|^2_{L^2(]0,T[\times\Omega)^d} + \|\boldsymbol{u}_0\|^2_{L^2(\Omega)^d})$$

where C_6 only depends on d, Ω, T, θ and ν_0.

Proof of Proposition 5.1
Take the scalar product of (20) with $\delta t \boldsymbol{u}^{n+1/2}_K = \delta t \frac{\boldsymbol{u}^{n+1}_K + \boldsymbol{u}^n_K}{2}$, sum on $K \in \mathcal{M}$ and $n = 0, \ldots, I-1$ (with $1 \leq I \leq N$). This gives $T_2 + T_3 + T_4 = T_5$ with

$$T_2 = \frac{1}{2}\sum_{n=0}^{I-1} \sum_{K\in\mathcal{M}} \mathrm{m}(K) \left(\left|\boldsymbol{u}^{n+1}_K\right|^2 - |\boldsymbol{u}^n_K|^2\right),$$

$$T_3 = \sum_{n=0}^{I-1} \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) \boldsymbol{u}^{n+1/2}_\sigma \cdot \boldsymbol{n}_{K,\sigma} \left(\frac{\boldsymbol{u}^{n+1/2}_K + \boldsymbol{u}^{n+1/2}_L}{2}\right) \cdot \boldsymbol{u}^{n+1/2}_K,$$

$$T_4 = -\sum_{n=0}^{I-1} \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} (\boldsymbol{F}^{n+1/2}_{K,\sigma} - p^{n+1/2}_K \mathrm{m}(\sigma) \boldsymbol{n}_{K,\sigma}) \cdot \boldsymbol{u}^{n+1/2}_K,$$

$$T_5 = \int_0^{I\delta t} \int_\Omega \boldsymbol{f}(t, \boldsymbol{x}) \cdot \boldsymbol{u}(t, \boldsymbol{x}) \,\mathrm{d}t \,\mathrm{d}\boldsymbol{x}$$

(in T_3, recall that L is the neighboring control volume of K on the other side of σ, if $\sigma \in \mathcal{E}_{K,\text{int}}$, or that $\boldsymbol{u}_L^{n+1/2} = 0$ if $\sigma \in \mathcal{E}_{K,\text{ext}}$). We clearly have, denoting $\boldsymbol{u}^I \in H_{\mathcal{D}}^d$ the function equal to $\boldsymbol{u}_K^I$ on $K \in \mathcal{M}$,

$$T_2 = \frac{1}{2}\left(\|\boldsymbol{u}^I\|_{L^2(\Omega)^d}^2 - \|\boldsymbol{u}^0\|_{L^2(\Omega)^d}^2\right). \tag{34}$$

Gathering by edges and denoting $\sigma = \sigma^{K|L}$ if $\sigma \in \mathcal{E}_{\text{int}}$, or $\boldsymbol{u}_L^{n+1/2} = 0$ if $\sigma = \sigma^{K|\partial} \in \mathcal{E}_{\text{ext}}$, we can write

$$\begin{aligned} T_3 &= \sum_{n=0}^{I-1} \delta t \sum_{\sigma\in\mathcal{E}} \mathrm{m}(\sigma)\boldsymbol{u}_\sigma^{n+1/2}\cdot\boldsymbol{n}_{K,\sigma}\left(\frac{\boldsymbol{u}_K^{n+1/2}+\boldsymbol{u}_L^{n+1/2}}{2}\right)\cdot\left(\boldsymbol{u}_K^{n+1/2}-\boldsymbol{u}_L^{n+1/2}\right) \\ &= \frac{1}{2}\sum_{n=0}^{I-1} \delta t \sum_{\sigma\in\mathcal{E}} \mathrm{m}(\sigma)\boldsymbol{u}_\sigma^{n+1/2}\cdot\boldsymbol{n}_{K,\sigma}\left(\left|\boldsymbol{u}_K^{n+1/2}\right|^2-\left|\boldsymbol{u}_L^{n+1/2}\right|^2\right). \end{aligned}$$

We now gather back by control volumes and we find, thanks to (25),

$$T_3 = \frac{1}{2}\sum_{n=0}^{I-1}\delta t \sum_{K\in\mathcal{M}}\left|\boldsymbol{u}_K^{n+1/2}\right|^2 \sum_{\sigma\in\mathcal{E}_K}\mathrm{m}(\sigma)\boldsymbol{u}_\sigma^{n+1/2}\cdot\boldsymbol{n}_{K,\sigma} = 0. \tag{35}$$

The term T_4 is handled exactly as in the Stokes equation (see the proof of Proposition 4.1) and gives the transient equivalent of the left-hand side (28)

$$\begin{aligned} T_4 &= \sum_{n=0}^{I-1}\delta t\|\mathbf{v}^{n+1/2}\|_{L^2(\Omega)^{d\times d}}^2 + \sum_{n=0}^{I-1}\delta t\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}\left|\boldsymbol{F}_{K,\sigma}^{n+1/2}\right|^2 \\ &= \|\mathbf{v}\|_{L^2(]0,I\delta t[\times\Omega)^{d\times d}}^2 + \sum_{n=0}^{I-1}\delta t\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}\left|\boldsymbol{F}_{K,\sigma}^{n+1/2}\right|^2. \end{aligned} \tag{36}$$

We also have the following bound, independent on I:

$$T_5 \le \|\boldsymbol{f}\|_{L^2(]0,T[\times\Omega)^d}\|\boldsymbol{u}\|_{L^2(]0,T[\times\Omega)^d}. \tag{37}$$

We now gather (34), (35), (36) and (37) in $T_2+T_3+T_4 = T_5$; since this relation is valid for any $I = 1,\dots,N$ and since $\|\boldsymbol{u}^0\|_{L^2(\Omega)^d} \le \|\boldsymbol{u}_0\|_{L^2(\Omega)^d}$ (see (23)), we deduce that

$$\begin{aligned} \frac{1}{2}\sup_{I=0,\dots,N}\left(\|\boldsymbol{u}^I\|_{L^2(\Omega)^d}^2\right) + \|\mathbf{v}\|_{L^2(]0,T[\times\Omega)^{d\times d}}^2 + \sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}\left|\boldsymbol{F}_{K,\sigma}^{n+1/2}\right|^2 \\ \le \frac{1}{2}\|\boldsymbol{u}_0\|_{L^2(\Omega)^d}^2 + \|\boldsymbol{f}\|_{L^2(]0,T[\times\Omega)^d}\|\boldsymbol{u}\|_{L^2(]0,T[\times\Omega)^d}. \end{aligned} \tag{38}$$

For all $t \in]0,T[$, $\boldsymbol{u}(t,\cdot)$ is equal, for some $n = 0,\dots,N-1$, to $\boldsymbol{u}^{n+1/2} = \frac{\boldsymbol{u}^{n+1}+\boldsymbol{u}^n}{2}$. Hence, $\|\boldsymbol{u}\|_{L^2(]0,T[\times\Omega)^d}^2 \le T\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega)^d)}^2 \le T\sup_{I=0,\dots,N}\|\boldsymbol{u}^I\|_{L^2(\Omega)^d}^2$ and Young's inequality concludes the proof. □

We can now prove the existence of at least one solution to the scheme for Navier-Stokes problem.

Proof of Theorem 5.1

Notice first that the *a priori* estimates of Proposition 5.1 still hold (with exactly the same C_6) if we multiply the second term of (20) (the only non-linear term of the scheme) by some $\beta \in [0,1]$. Moreover, from the estimates on $\boldsymbol{F}$ and (17), we have, for all K and L neighboring control volumes, $|p_K^{n+1/2} - p_L^{n+1/2}| \leq \frac{1}{\mathrm{m}(\sigma)} |\boldsymbol{F}_{K,\sigma}^{n+1/2} + \boldsymbol{F}_{L,\sigma}^{n+1/2}| \leq C_7$ where C_7 depends on the mesh and time step (but not on the aforementioned β); thus, for any control volumes K and M (not necessarily neighbors), we have $|p_K^{n+1/2} - p_M^{n+1/2}| \leq \mathrm{Card}(\mathcal{M}) C_7$ and therefore, by (24), $|p_M^{n+1/2}| = \frac{1}{\mathrm{m}(\Omega)} |\sum_{K \in \mathcal{M}} \mathrm{m}(K)(p_M^{n+1/2} - p_K^{n+1/2})| \leq \mathrm{Card}(\mathcal{M}) C_7$ for any control volume M; this gives a rough estimate, not depending on β, on the pressure (this estimate however strongly depends on the mesh and the time step).
By the same reasoning as in Remark 2.2, the non-linear system (16)—(24) can be considered square. The properties of the topological degree (see [5]) and the preceding estimates then imply that the degree of the function defining this system is equal to the degree of the same function without the non-linear term in (20). The resulting system is square and linear and the estimates above, which imply that any solution to this system is bounded, show that it is invertible. Hence, the topological degree of the linear function defining this system differs from zero, and so does the topological degree of the function defining (16)—(24). This shows that there exists at least one solution to the scheme. □

5.2 Translations estimates

In order to pass to the limit in the nonlinear term of the equation, we need to obtain enough compactness on the approximate solution $\boldsymbol{u}$, which demands some estimates on its translations in time (the translations in space are estimated thanks to Lemma 6.3). To prove those estimates, we introduce, for $\mathcal{D}$ an admissible discretization of Ω and $\nu \geq 0$, the space $L_{\mathcal{D},\nu}$ of the functions $\widehat{\boldsymbol{u}} \in H_{\mathcal{D}}^d$ for which there exists $(\widehat{\mathbf{v}}, \widehat{\boldsymbol{F}}) \in H_{\mathcal{D}}^{d\times d} \times \mathcal{F}_{\mathcal{D}}^d$ such that $(\widehat{\boldsymbol{u}}, \widehat{\mathbf{v}}, \widehat{\boldsymbol{F}})$ satisfies (10) and (13). We call such $(\widehat{\mathbf{v}}, \widehat{\boldsymbol{F}})$ "compatible" with $\widehat{\boldsymbol{u}}$ and we endow $L_{\mathcal{D},\nu}$ with the norm

$$\|\widehat{\boldsymbol{u}}\|_{L_{\mathcal{D},\nu}}^2 = \inf \left\{ \|\widehat{\mathbf{v}}\|_{L^2(\Omega)^{d\times d}}^2 + \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \nu \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} |\widehat{\boldsymbol{F}}_{K,\sigma}|^2 \,;\, (\widehat{\mathbf{v}}, \widehat{\boldsymbol{F}}) \text{ is compatible with } \widehat{\boldsymbol{u}} \right\}$$

(notice that this infimum is in fact a minimum). Defining, for $\widehat{\boldsymbol{w}} \in H_{\mathcal{D}}^d$, the semi-norm

$$\|\widehat{\boldsymbol{w}}\|_{L_{\mathcal{D},\nu}^*} = \sup \left\{ \sum_{K \in \mathcal{M}} \mathrm{m}(K) \widehat{\boldsymbol{w}}_K \cdot \widehat{\boldsymbol{u}}_K \,;\, \widehat{\boldsymbol{u}} \in L_{\mathcal{D},\nu} \,,\, \|\widehat{\boldsymbol{u}}\|_{L_{\mathcal{D},\nu}} = 1 \right\},$$

we notice that, if $\widehat{\boldsymbol{u}} \in L_{\mathcal{D},\nu}$,

$$\|\widehat{\boldsymbol{u}}\|_{L^2(\Omega)^d}^2 \leq \|\widehat{\boldsymbol{u}}\|_{L_{\mathcal{D},\nu}} \|\widehat{\boldsymbol{u}}\|_{L_{\mathcal{D},\nu}^*}. \tag{39}$$

These tools will allow us to prove the following estimate.

Proposition 5.2 *Let $T > 0$, $N \geq 1$ be an integer and $\delta t = T/N$. Assume that (2) and (6) hold. Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 and take $\theta \geq \mathrm{regul}(\mathcal{D})$. Let $\nu_0 >$ and assume that $0 \leq \nu \leq \nu_0$. If $(p, \boldsymbol{u}, \mathbf{v}, \boldsymbol{F}) \in H_{\mathcal{D},\delta t} \times H_{\mathcal{D},\delta t}^d \times H_{\mathcal{D},\delta t}^{d\times d} \times \mathcal{F}_{\mathcal{D},\delta t}^d$ is a solution to (16)—(24) then, for all $\tau \in]0,T[$,*

$$\int_0^{T-\tau} \|\boldsymbol{u}(t+\tau,\cdot) - \boldsymbol{u}(t,\cdot)\|_{L^2(\Omega)^d} \,\mathrm{d}t \leq C_8 \sqrt{\tau}$$

where C_8 only depends on d, Ω, T, θ and ν_0.

PROOF OF PROPOSITION 5.2
In this proof, C_i are constants which only depend on d, Ω, T, θ and ν_0.

Step 1: we estimate the $L^*_{\mathcal{D},\nu}$ norm of $\boldsymbol{u}^{n+1} - \boldsymbol{u}^n$.
Let $\widehat{\boldsymbol{u}} \in L_{\mathcal{D},\nu}$ such that $\|\widehat{\boldsymbol{u}}\|_{L_{\mathcal{D},\nu}} = 1$, and take $(\widehat{\mathbf{v}}, \widehat{\boldsymbol{F}})$ compatible with $\widehat{\boldsymbol{u}}$ such that

$$\|\widehat{\mathbf{v}}\|^2_{L^2(\Omega)^{d\times d}} + \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \nu \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} |\widehat{\boldsymbol{F}}_{K,\sigma}|^2 = 1. \tag{40}$$

Define $\widehat{\boldsymbol{u}}_\sigma$ as in Remark 2.3 with $(\boldsymbol{u}, \mathbf{v}, \boldsymbol{F}) = (\widehat{\boldsymbol{u}}, \widehat{\mathbf{v}}, \widehat{\boldsymbol{F}})$ (which satisfies (10) and (13)); we have $\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\widehat{\boldsymbol{u}}_\sigma \cdot \boldsymbol{n}_{K,\sigma} = 0$.
By (25), we can write

$$\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\boldsymbol{u}_\sigma^{n+1/2} \cdot \boldsymbol{n}_{K,\sigma} \left(\frac{\boldsymbol{u}_K^{n+1/2} + \boldsymbol{u}_L^{n+1/2}}{2}\right)$$
$$= \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\boldsymbol{u}_\sigma^{n+1/2} \cdot \boldsymbol{n}_{K,\sigma} \left(\frac{\boldsymbol{u}_L^{n+1/2} - \boldsymbol{u}_K^{n+1/2}}{2}\right). \tag{41}$$

Plugging this into (20), taking the scalar product of the resulting equation with $\delta t\widehat{\boldsymbol{u}}_K$ and summing on $K \in \mathcal{M}$, we obtain $T_6 = -T_7 + T_8 + T_9$ with

$$T_6 = \sum_{K\in\mathcal{M}} \mathrm{m}(K) \left(\boldsymbol{u}_K^{n+1} - \boldsymbol{u}_K^n\right) \cdot \widehat{\boldsymbol{u}}_K$$
$$T_7 = \delta t \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\boldsymbol{u}_\sigma^{n+1/2} \cdot \boldsymbol{n}_{K,\sigma} \left(\frac{\boldsymbol{u}_L^{n+1/2} - \boldsymbol{u}_K^{n+1/2}}{2}\right) \cdot \widehat{\boldsymbol{u}}_K$$
$$T_8 = \delta t \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} (\boldsymbol{F}_{K,\sigma}^{n+1/2} - p_K^{n+1/2}\mathrm{m}(\sigma)\boldsymbol{n}_{K,\sigma}) \cdot \widehat{\boldsymbol{u}}_K$$
$$T_9 = \int_{n\delta t}^{(n+1)\delta t} \int_\Omega \boldsymbol{f}(t,\boldsymbol{x}) \cdot \widehat{\boldsymbol{u}}(\boldsymbol{x}) \,\mathrm{d}t \,\mathrm{d}\boldsymbol{x}.$$

Let us estimate T_7. We have

$$\begin{aligned} 2T_7 &= \delta t \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) \left(\boldsymbol{u}_\sigma^{n+1/2} - \boldsymbol{u}_K^{n+1/2}\right) \cdot \boldsymbol{n}_{K,\sigma} \left(\boldsymbol{u}_L^{n+1/2} - \boldsymbol{u}_K^{n+1/2}\right) \cdot \widehat{\boldsymbol{u}}_K \\ &\quad + \delta t \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\boldsymbol{u}_K^{n+1/2} \cdot \boldsymbol{n}_{K,\sigma} \left(\boldsymbol{u}_L^{n+1/2} - \boldsymbol{u}_K^{n+1/2}\right) \cdot \widehat{\boldsymbol{u}}_K \\ &= \delta t\, T_7^a + \delta t\, T_7^b. \end{aligned}$$

Using Hölder's inequality with exponents 2, 4 and 4, we have

$$|T_7^a| \leq \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\mathrm{diam}(K) \frac{\left|\boldsymbol{u}_\sigma^{n+1/2} - \boldsymbol{u}_K^{n+1/2}\right|}{\mathrm{diam}(K)} \left|\boldsymbol{u}_L^{n+1/2} - \boldsymbol{u}_K^{n+1/2}\right| |\widehat{\boldsymbol{u}}_K|$$

$$\leq \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\mathrm{diam}(K) \frac{\left|\boldsymbol{u}_\sigma^{n+1/2} - \boldsymbol{u}_K^{n+1/2}\right|^2}{\mathrm{diam}(K)^2} \right)^{1/2}$$
$$\times \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\mathrm{diam}(K) \left(\left|\boldsymbol{u}_L^{n+1/2}\right| + \left|\boldsymbol{u}_K^{n+1/2}\right| \right)^4 \right)^{1/4}$$
$$\times \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\mathrm{diam}(K) \left|\widehat{\boldsymbol{u}}_K\right|^4 \right)^{1/4}.$$

The same way we estimate the discrete H^1-norm in Lemma 6.2, it is easy to see from the definition of $\boldsymbol{u}_\sigma^{n+1/2}$ that

$$\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{\mathrm{diam}(K)} \left|\boldsymbol{u}_\sigma^{n+1/2} - \boldsymbol{u}_K^{n+1/2}\right|^2$$
$$\leq C_9 \left(\|\mathbf{v}^{n+1/2}\|^2_{L^2(\Omega)^{d\times d}} + \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu^2 \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} \left|\boldsymbol{F}_{K,\sigma}^{n+1/2}\right|^2 \right) \tag{42}$$

and therefore, thanks to (9) and to the definition of regul($\mathcal{D}$),

$$|T_7^a| \leq C_{10} \left(\|\mathbf{v}^{n+1/2}\|^2_{L^2(\Omega)^{d\times d}} + \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu^2 \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} \left|\boldsymbol{F}_{K,\sigma}^{n+1/2}\right|^2 \right)^{\frac{1}{2}} \|\boldsymbol{u}^{n+1/2}\|_{L^4(\Omega)^d} \|\widehat{\boldsymbol{u}}\|_{L^4(\Omega)^d}.$$

Using Lemma 6.2 for $\boldsymbol{u}^{n+1/2}$ and $\widehat{\boldsymbol{u}}$ with $q = 4$, and recalling (40), we obtain

$$|T_7^a| \leq C_{11} \left(\|\mathbf{v}^{n+1/2}\|^2_{L^2(\Omega)^{d\times d}} + \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu^2 \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} \left|\boldsymbol{F}_{K,\sigma}^{n+1/2}\right|^2 \right).$$

For T_7^b, we write

$$|T_7^b| \leq \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\mathrm{diam}(K) \left|\boldsymbol{u}_K^{n+1/2}\right| \left| \frac{\boldsymbol{u}_L^{n+1/2} - \boldsymbol{u}_K^{n+1/2}}{\mathrm{diam}(K)} \right| \left|\widehat{\boldsymbol{u}}_K\right|$$
$$\leq \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\mathrm{diam}(K) \left|\boldsymbol{u}_K^{n+1/2}\right|^4 \right)^{1/4}$$
$$\times \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\mathrm{diam}(K) \frac{\left|\boldsymbol{u}_L^{n+1/2} - \boldsymbol{u}_K^{n+1/2}\right|^2}{\mathrm{diam}(K)^2} \right)^{1/2}$$
$$\times \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\mathrm{diam}(K) \left|\widehat{\boldsymbol{u}}_K\right|^4 \right)^{1/4}$$

and, thanks to the estimates on the discrete H^1-norm and the L^4 norm in Lemma 6.2, T_7^b is estimated the same way as T_7^a. This finally gives, since $\nu \le \nu_0$,

$$|T_7| \le C_{12}\delta t \left(\|\mathbf{v}^{n+1/2}\|^2_{L^2(\Omega)^{d\times d}} + \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} \left| \boldsymbol{F}^{n+1/2}_{K,\sigma} \right|^2 \right). \tag{43}$$

We now turn to T_8. Noting that $\widehat{\boldsymbol{u}}_\sigma = 0$ if $\sigma \in \mathcal{E}_{\mathrm{ext}}$, we have, thanks to (17),

$$\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} (\boldsymbol{F}^{n+1/2}_{K,\sigma} - p^{n+1/2}_K \mathrm{m}(\sigma) \boldsymbol{n}_{K,\sigma}) \cdot \widehat{\boldsymbol{u}}_\sigma =$$
$$\sum_{\sigma^{K|L}\in\mathcal{E}_{\mathrm{int}}} \left[(\boldsymbol{F}^{n+1/2}_{K,\sigma} - p^{n+1/2}_K \mathrm{m}(\sigma) \boldsymbol{n}_{K,\sigma}) + (\boldsymbol{F}^{n+1/2}_{L,\sigma} - p^{n+1/2}_L \mathrm{m}(\sigma) \boldsymbol{n}_{L,\sigma}) \right] \cdot \widehat{\boldsymbol{u}}_\sigma = 0$$

and therefore

$$-T_8 = \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \left(\boldsymbol{F}^{n+1/2}_{K,\sigma} - p^{n+1/2}_K \mathrm{m}(\sigma) \boldsymbol{n}_{K,\sigma} \right) \cdot (\widehat{\boldsymbol{u}}_\sigma - \widehat{\boldsymbol{u}}_K).$$

Since $\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) \widehat{\boldsymbol{u}}_\sigma \cdot \boldsymbol{n}_{K,\sigma}$ and $\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) \boldsymbol{n}_{K,\sigma}$ are zero, we infer from the definition of $\widehat{\boldsymbol{u}}_\sigma$ that

$$\begin{aligned} -T_8 &= \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \boldsymbol{F}^{n+1/2}_{K,\sigma} \cdot (\widehat{\boldsymbol{u}}_\sigma - \widehat{\boldsymbol{u}}_K) \\ &= \delta t \sum_{K\in\mathcal{M}} \widehat{\mathbf{v}}_K : \sum_{\sigma\in\mathcal{E}_K} \boldsymbol{F}^{n+1/2}_{K,\sigma} \otimes (\boldsymbol{x}_\sigma - \boldsymbol{x}_K) + \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} \boldsymbol{F}^{n+1/2}_{K,\sigma} \cdot \widehat{\boldsymbol{F}}_{K,\sigma} \\ &= \delta t \sum_{K\in\mathcal{M}} \mathrm{m}(K) \widehat{\mathbf{v}}_K : \mathbf{v}^{n+1/2}_K + \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} \boldsymbol{F}^{n+1/2}_{K,\sigma} \cdot \widehat{\boldsymbol{F}}_{K,\sigma}. \end{aligned}$$

The choice (40) then implies

$$|T_8| \le \delta t \|\mathbf{v}^{n+1/2}\|_{L^2(\Omega)^{d\times d}} + \delta t \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} \left| \boldsymbol{F}^{n+1/2}_{K,\sigma} \right|^2 \right)^{1/2}. \tag{44}$$

We apply Lemma 6.2 to $(\widehat{\boldsymbol{u}}, \widehat{\mathbf{v}}, \widehat{\boldsymbol{F}})$ to estimate $\|\widehat{\boldsymbol{u}}\|_{L^2(\Omega)^d}$ and, since $\nu \le \nu_0$, the equation (40) gives

$$|T_9| \le C_{13} \int_{n\delta t}^{(n+1)\delta t} \|\boldsymbol{f}(t,\cdot)\|_{L^2(\Omega)^d} \,\mathrm{d}t. \tag{45}$$

Gathering (43), (44) and (45) in $T_6 = -T_7 + T_8 + T_9$, and since the resulting estimate is valid for all $\widehat{\boldsymbol{u}} \in L_{\mathcal{D},\nu}$ with norm 1, we conclude, from Young's inequality, that

$$\begin{aligned} \left|\left| \boldsymbol{u}^{n+1} - \boldsymbol{u}^n \right|\right|_{L^*_{\mathcal{D},\nu}} \le\ & C_{14} \int_{n\delta t}^{(n+1)\delta t} \|\boldsymbol{f}(t,\cdot)\|_{L^2(\Omega)^d} \,\mathrm{d}t + C_{14}\delta t \|\mathbf{v}^{n+1/2}\|^2_{L^2(\Omega)^{d\times d}} \\ & + C_{14}\delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} \left| \boldsymbol{F}^{n+1/2}_{K,\sigma} \right|^2 + C_{14}\delta t. \end{aligned} \tag{46}$$

Step 2: conclusion.
For all $t \in]0,T[$, $\boldsymbol{u}(t,\cdot)$ belongs to $L_{\mathcal{D},\nu}$ (since $\boldsymbol{u}(t,\cdot) = \boldsymbol{u}^{n+1/2}$ for some $n = 0,\ldots,N-1$). Hence, we can apply (39) and we have, by Young's inequality,

$$\begin{aligned}\int_0^{T-\tau} \|\boldsymbol{u}(t+\tau,\cdot) - \boldsymbol{u}(t,\cdot)\|_{L^2(\Omega)^d}\,\mathrm{d}t \quad &\leq \quad \frac{1}{2\sqrt{\tau}} \int_0^{T-\tau} \|\boldsymbol{u}(t+\tau,\cdot) - \boldsymbol{u}(t,\cdot)\|_{L^*_{\mathcal{D},\nu}}\,\mathrm{d}t \\ &\quad + \frac{\sqrt{\tau}}{2} \int_0^{T-\tau} \|\boldsymbol{u}(t+\tau,\cdot) - \boldsymbol{u}(t,\cdot)\|_{L_{\mathcal{D},\nu}}\,\mathrm{d}t. \qquad (47)\end{aligned}$$

But, for a.e. $t \in]0,T[$, denoting by n the integer such that $t \in]n\delta t,(n+1)\delta t[$ (so that $\boldsymbol{u}(t,\cdot) = \boldsymbol{u}^{n+1/2}$), we have by definition of the norm in $L_{\mathcal{D},\nu}$,

$$\|\boldsymbol{u}(t,\cdot)\|^2_{L_{\mathcal{D},\nu}} \leq \|\mathbf{v}^{n+1/2}\|^2_{L^2(\Omega)^{d\times d}} + \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} \left|\boldsymbol{F}^{n+1/2}_{K,\sigma}\right|^2.$$

Hence, from the estimates in Proposition 5.1,

$$\int_0^T \|\boldsymbol{u}(t,\cdot)\|^2_{L_{\mathcal{D},\nu}} \leq \sum_{n=0}^{N-1} \delta t \|\mathbf{v}^{n+1/2}\|^2_{L^2(\Omega)^{d\times d}} + \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} \left|\boldsymbol{F}^{n+1/2}_{K,\sigma}\right|^2 \leq C_{15}$$

and (47) leads to

$$\int_0^{T-\tau} \|\boldsymbol{u}(t+\tau,\cdot) - \boldsymbol{u}(t,\cdot)\|_{L^2(\Omega)^d}\,\mathrm{d}t \leq \frac{1}{2\sqrt{\tau}} \int_0^{T-\tau} \|\boldsymbol{u}(t+\tau,\cdot) - \boldsymbol{u}(t,\cdot)\|_{L^*_{\mathcal{D},\nu}} + C_{16}\sqrt{\tau}. \qquad (48)$$

For $t \in]0,T-\tau[$, let $n_0(t)$ and $n_1(t)$ be the integer parts of $t/\delta t$ and $(t+\tau)/\delta t$ (these integers belong to $\{0,\ldots,N-1\}$). We have $\|\boldsymbol{u}(t+\tau,\cdot) - \boldsymbol{u}(t,\cdot)\|_{L^*_{\mathcal{D},\nu}} \leq \sum_{n=n_0(t)}^{n_1(t)-1} \|\boldsymbol{u}^{n+3/2} - \boldsymbol{u}^{n+1/2}\|_{L^*_{\mathcal{D},\nu}}$. By (21), $\boldsymbol{u}^{n+3/2} - \boldsymbol{u}^{n+1/2} = \frac{\boldsymbol{u}^{n+2}+\boldsymbol{u}^{n+1}}{2} - \frac{\boldsymbol{u}^{n+1}+\boldsymbol{u}^{n}}{2} = \frac{\boldsymbol{u}^{n+2}-\boldsymbol{u}^{n+1}}{2} + \frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n}}{2}$ so that, using Fubini's theorem,

$$\begin{aligned}&\int_0^{T-\tau} \|\boldsymbol{u}(t+\tau,\cdot) - \boldsymbol{u}(t,\cdot)\|_{L^*_{\mathcal{D},\nu}}\,\mathrm{d}t \\ &\qquad \leq \int_0^{T-\tau} \sum_{n=n_0(t)}^{n_1(t)-1} \frac{\|\boldsymbol{u}^{n+2} - \boldsymbol{u}^{n+1}\|_{L^*_{\mathcal{D},\nu}}}{2} + \frac{\|\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n}\|_{L^*_{\mathcal{D},\nu}}}{2}\,\mathrm{d}t \\ &\qquad \leq \sum_{n=0}^{N-2} \left(\frac{\|\boldsymbol{u}^{n+2} - \boldsymbol{u}^{n+1}\|_{L^*_{\mathcal{D},\nu}}}{2} + \frac{\|\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n}\|_{L^*_{\mathcal{D},\nu}}}{2} \right) \int_0^{T-\tau} \chi(n,t)\,\mathrm{d}t\end{aligned}$$

where $\chi(n,t) = 1$ if $n_0(t) \leq n \leq n_1(t)-1$, and $\chi(n,t) = 0$ otherwise. We have $\chi(n,t) = 1$ if and only if $n > (t/\delta t) - 1$ and $n+1 \leq (t+\tau)/\delta t$, i.e. $t \in [(n+1)\delta t - \tau, (n+1)\delta t[$, so that $\int_0^{T-\tau} \chi(n,t)\,\mathrm{d}t \leq \tau$. By (46) and the estimates of Proposition 5.1, we obtain

$$\begin{aligned}&\int_0^{T-\tau} \|\boldsymbol{u}(t+\tau,\cdot) - \boldsymbol{u}(t,\cdot)\|_{L^*_{\mathcal{D},\nu}}\,\mathrm{d}t \\ &\leq C_{17}\tau \left(\|\boldsymbol{f}\|_{L^1(0,T;L^2(\Omega))^d} + \|\mathbf{v}\|^2_{L^2(]0,T[\times\Omega)^{d\times d}} + \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} \left|\boldsymbol{F}^{n+1/2}_{K,\sigma}\right|^2 + T \right) \\ &\leq C_{18}\tau\end{aligned}$$

and the proof is concluded by using this estimate in (48). □

5.3 Proof of the convergence

We now prove Theorem 5.2. To simplify the notations, we drop the index m and we study the convergence of $\boldsymbol{u}$ and $\mathbf{v}$ as $\mathrm{size}(\mathcal{D}) \to 0$ and $\delta t \to 0$ while $\mathrm{regul}(\mathcal{D})$ remains bounded.
Proposition 5.1 gives estimates, independent on $\mathcal{D}$ or δt, on $(\boldsymbol{u}, \mathbf{v}, \boldsymbol{F})$. Using the definition of N_2 from Lemma 6.2, these estimates show that

$$\sum_{n=0}^{N-1} \delta t N_2(\mathcal{D}, \nu, \boldsymbol{F}^{n+1/2})^2 = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu^2 \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} \left|\boldsymbol{F}_{K,\sigma}^{n+1/2}\right|^2 \to 0 \tag{49}$$

as $\mathrm{size}(\mathcal{D}) \to 0$ (recall that $\nu = \lambda \mathrm{size}(\mathcal{D})^\alpha$ with $\alpha \in]0,2[$). In particular, applying Lemma 6.2 to $(\boldsymbol{u}^{n+1/2}, \mathbf{v}^{n+1/2}, \boldsymbol{F}^{n+1/2})$, taking the square of the resulting L^q estimate, multiplying by δt and summing on $n = 0, \ldots, N-1$, we see that $\boldsymbol{u}$ is bounded in $L^2(0,T; L^q(\Omega)^d)$ for all $q < +\infty$ if $d = 2$ and all $q \le 6$ if $d = 3$. Since $\boldsymbol{u}$ is also bounded in $L^\infty(0,T; L^2(\Omega)^d)$ (Proposition 5.1), we deduce by interpolation that it is bounded in $L^{2+\epsilon}(]0,T[\times\Omega)^d$ for some $\epsilon > 0$.
By Cauchy-Schwarz inequality, (9) and (49), we have

$$\begin{aligned} &\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu \mathrm{diam}(K) \left|\boldsymbol{F}_{K,\sigma}^{n+1/2}\right| \\ &\quad\le C_{19} \left(\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu^2 \frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)} \left|\boldsymbol{F}_{K,\sigma}^{n+1/2}\right|^2 \right)^{1/2} \to 0 \quad \text{as } \mathrm{size}(\mathcal{D}) \to 0 \end{aligned} \tag{50}$$

(C_{19} does not depend on $\mathcal{D}$ or δt). Applying Lemma 6.3 to $(\boldsymbol{u}^{n+1/2}, \mathbf{v}^{n+1/2}, \boldsymbol{F}^{n+1/2})$, multiplying the resulting estimate by δt and summing on $n = 0, \ldots, N-1$, we deduce that $\|\boldsymbol{u}(\cdot, \cdot + \xi) - \boldsymbol{u}\|_{L^1(]0,T[\times\mathbb{R}^d)^d} \to 0$ as $|\xi| \to 0$, independently on $\mathcal{D}$ or δt (we have extended $\boldsymbol{u}$ in space by 0 outside Ω). In conjunction with the estimates on the time translates from Proposition 5.2, this proves that $\boldsymbol{u}$ is relatively compact in $L^1_{\mathrm{loc}}(]0,T[\times\Omega)^d$. Since it is bounded in $L^{2+\epsilon}(]0,T[\times\Omega)^d$ for some $\epsilon > 0$, $\boldsymbol{u}$ is also relatively compact in $L^2(]0,T[\times\Omega)^d$. Up to a subsequence, we can thus assume that $\boldsymbol{u} \to \bar{\boldsymbol{u}}$ strongly in $L^2(]0,T[\times\Omega)^d$ and that $\mathbf{v}$ weakly converges in $L^2(]0,T[\times\Omega)^{d\times d}$. It is then easy to deduce from (49) that the weak limit of $\mathbf{v}$ is $\nabla\bar{\boldsymbol{u}}$ and that $\bar{\boldsymbol{u}} \in L^2(0,T; H^1_0(\Omega))^d$ (this is similar to Lemma 6.4; see the proof of [6, Lemma 3.3] or the proof of [4, Lemma 7.4] for an example in a transient case).

Step 1: we prove that $\mathrm{div}(\bar{\boldsymbol{u}}) = 0$.
Let $\Gamma :]\,0,T\,[\,\times\Omega \to \mathbb{R}$ be the piecewise constant function equal to $\frac{\nu\mathrm{diam}(K)}{\mathrm{m}(K)} \sum_{\sigma\in\mathcal{E}_K} \boldsymbol{F}_{K,\sigma}^{n+1/2} \cdot \boldsymbol{n}_{K,\sigma}$ on $]n\delta t, (n+1)\delta t[\times K$. Thanks to (50), we have $\Gamma \to 0$ in $L^1(]0,T[\times\Omega)$ as $\mathrm{size}(\mathcal{D}) \to 0$. Since $\mathrm{Tr}(\mathbf{v}) + \Gamma = 0$ (this is (19) divided by $\mathrm{m}(K)$), we deduce from the weak convergence of $\mathbf{v}$ toward $\nabla\bar{\boldsymbol{u}}$ that $\mathrm{Tr}(\nabla\bar{\boldsymbol{u}}) = \mathrm{div}(\bar{\boldsymbol{u}}) = 0$.

Step 2: we prove that $\bar{\boldsymbol{u}}$ satisfies (7).
Let $\boldsymbol{\varphi} \in C_c^\infty([0,T[\times\Omega)^d$ such that $\mathrm{div}(\boldsymbol{\varphi}) = 0$. We use (41) to transform the second term of (20), take the scalar product of the resulting equation and $\delta t \boldsymbol{\varphi}(n\delta t, \boldsymbol{x}_K)$ and sum on K and n. This gives $T_{10} + T_{11} + T_{12} = T_{13}$ with

$$T_{10} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} \mathrm{m}(K) \left(\boldsymbol{u}_K^{n+1} - \boldsymbol{u}_K^n\right) \cdot \boldsymbol{\varphi}(n\delta t, \boldsymbol{x}_K),$$

$$T_{11} = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) \boldsymbol{u}_\sigma^{n+1/2} \cdot \boldsymbol{n}_{K,\sigma} \left(\frac{\boldsymbol{u}_L^{n+1/2} - \boldsymbol{u}_K^{n+1/2}}{2} \right) \cdot \boldsymbol{\varphi}(n\delta t, \boldsymbol{x}_K)\,,$$

$$T_{12} = -\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} (\boldsymbol{F}_{K,\sigma}^{n+1/2} - p_K^{n+1/2} \mathrm{m}(\sigma) \boldsymbol{n}_{K,\sigma}) \cdot \boldsymbol{\varphi}(n\delta t, \boldsymbol{x}_K)\,,$$

$$T_{13} = \sum_{n=0}^{N-1} \int_{n\delta t}^{(n+1)\delta t} \int_K \boldsymbol{f}(t,\boldsymbol{x}) \,\mathrm{d}t \,\mathrm{d}\boldsymbol{x} \ \cdot \boldsymbol{\varphi}(n\delta t, \boldsymbol{x}_K).$$

We now study the convergence of each of these terms as $\mathrm{size}(\mathcal{D}) \to 0$ and $\delta t \to 0$.

Since $\boldsymbol{\varphi} = 0$ on a neighborhood of $t = T$, for δt small enough we have

$$T_{10} = \sum_{n=1}^{N-1} \delta t \sum_{K\in\mathcal{M}} \mathrm{m}(K) \boldsymbol{u}_K^n \cdot \frac{\boldsymbol{\varphi}((n-1)\delta t, \boldsymbol{x}_K) - \boldsymbol{\varphi}(n\delta t, \boldsymbol{x}_K)}{\delta t} - \sum_{K\in\mathcal{M}} \mathrm{m}(K) \boldsymbol{u}_K^0 \cdot \boldsymbol{\varphi}(0, \boldsymbol{x}_K).$$

By (23) and regularity of $\boldsymbol{\varphi}$, the second term of this right-hand side tends to $-\int_\Omega \boldsymbol{u}_0(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(0,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$ as $\mathrm{size}(\mathcal{D}) \to 0$. Let $\boldsymbol{\Pi} :]\,0,T\,[\times \Omega \to \mathbb{R}^d$ and $\boldsymbol{U} :]\,0,T\,[\times \Omega \to \mathbb{R}^d$ be the piecewise functions respectively equal to $\frac{\boldsymbol{\varphi}((n-1)\delta t, \boldsymbol{x}_K) - \boldsymbol{\varphi}(n\delta t, \boldsymbol{x}_K)}{\delta t}$ and $\boldsymbol{u}_K^n$ on $]n\delta t, (n+1)\delta t[\times K$ for all $n = 0, \ldots, N-1$ and all $K \in \mathcal{M}$. We have

$$T_{10} = \int_0^T \int_\Omega \boldsymbol{U}(t,\boldsymbol{x}) \cdot \boldsymbol{\Pi}(t,\boldsymbol{x}) \,\mathrm{d}t \,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{u}_0(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(0,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} + \zeta(\mathcal{D}) \tag{51}$$

where $\zeta(\mathcal{D}) \to 0$ as $\mathrm{size}(\mathcal{D}) \to 0$. The regularity of $\boldsymbol{\varphi}$ ensures that $\boldsymbol{\Pi} \to -\partial_t \boldsymbol{\varphi}$ uniformly on $]0,T[\times\Omega$. The estimate (38) shows that $\boldsymbol{U}$ is bounded in $L^\infty(0,T;L^2(\Omega)^d)$ and therefore that, up to a subsequence, it converges to some $\bar{\boldsymbol{U}}$ weakly-$*$ in this space. But, on $]0,T-\delta t[\times\Omega$, (21) states that $\boldsymbol{u}(t,\boldsymbol{x}) = \frac{\boldsymbol{U}(t+\delta t,\boldsymbol{x}) + \boldsymbol{U}(t,\boldsymbol{x})}{2}$ and $\boldsymbol{u}$ therefore also converges weakly-$*$ in $L^\infty(0,T;L^2(\Omega)^d)$ to $\bar{\boldsymbol{U}}$; hence, $\bar{\boldsymbol{U}} = \bar{\boldsymbol{u}}$ and we can pass to the limit in (51) to find

$$T_{10} \to -\int_0^T \int_\Omega \bar{\boldsymbol{u}}(t,\boldsymbol{x}) \cdot \partial_t \boldsymbol{\varphi}(t,\boldsymbol{x}) \,\mathrm{d}t \,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{u}_0(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(0,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \quad \text{as } \mathrm{size}(\mathcal{D}) \to 0 \text{ and } \delta t \to 0. \tag{52}$$

Gathering T_{11} by edges, we have, for $\mathrm{size}(\mathcal{D})$ small enough (so that $\varphi = 0$ on the boundary control volumes),

$$T_{11} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma^{K|L} \in \mathcal{E}_{\mathrm{int}}} \mathrm{m}(\sigma) \boldsymbol{u}_\sigma^{n+1/2} \cdot \boldsymbol{n}_{K,\sigma} \left(\boldsymbol{u}_L^{n+1/2} - \boldsymbol{u}_K^{n+1/2} \right) \cdot \frac{\boldsymbol{\varphi}(n\delta t, \boldsymbol{x}_K) + \boldsymbol{\varphi}(n\delta t, \boldsymbol{x}_L)}{2}.$$

Let $\diamondsuit$ be a $\mathcal{D}$-adapted diamond partition of Ω according to Definition 6.1. We have

$$\begin{aligned} T_{11} \;\; = \;\; & \sum_{n=0}^{N-1} \delta t \sum_{\sigma^{K|L} \in \mathcal{E}_{\mathrm{int}}} \mathrm{m}(\diamondsuit_\sigma) \boldsymbol{u}_\sigma^{n+1/2} \\ & \cdot \left(\frac{\mathrm{m}(\sigma)}{\mathrm{m}(\diamondsuit_\sigma)} \left[\boldsymbol{n}_{K,\sigma} \otimes \left(\boldsymbol{u}_L^{n+1/2} - \boldsymbol{u}_K^{n+1/2} \right) \right] \frac{\boldsymbol{\varphi}(n\delta t, \boldsymbol{x}_K) + \boldsymbol{\varphi}(n\delta t, \boldsymbol{x}_L)}{2} \right). \end{aligned}$$

Denoting by $\mathbf{\Psi}$, $\boldsymbol{u}_\diamondsuit$ and $\mathbf{V}_\diamondsuit$ the functions respectively equal to $\frac{\boldsymbol{\varphi}(n\delta t,\boldsymbol{x}_K)+\boldsymbol{\varphi}(n\delta t,\boldsymbol{x}_L)}{2}$, $\boldsymbol{u}_\sigma^{n+1/2}$ and $\frac{\mathrm{m}(\sigma)}{\mathrm{m}(\diamondsuit_\sigma)}\boldsymbol{n}_{K,\sigma}\otimes\left(\boldsymbol{u}_L^{n+1/2}-\boldsymbol{u}_K^{n+1/2}\right)$ on $]n\delta t,(n+1)\delta t[\times\diamondsuit_\sigma$, we can write

$$T_{11}=\int_0^T\int_\Omega \boldsymbol{u}_\diamondsuit(t,\boldsymbol{x})\cdot(\mathbf{V}_\diamondsuit(t,\boldsymbol{x})\mathbf{\Psi}(t,\boldsymbol{x}))\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}. \tag{53}$$

We clearly have, by regularity of $\boldsymbol{\varphi}$, $\mathbf{\Psi}\to\boldsymbol{\varphi}$ uniformly on $]0,T[\times\Omega$ as size$(\mathcal{D})\to 0$ and $\delta t\to 0$. From the estimates in Proposition 5.1 and Lemma 6.2, we see that $\sum_{n=0}^{N-1}\delta t\|\boldsymbol{u}^{n+1/2}\|_{1,\mathcal{D}}^2$ remains bounded and Lemma 6.5 therefore shows that $\mathbf{V}_\diamondsuit\to(\nabla\bar{\boldsymbol{u}})^*$ weakly in $L^2(]0,T[\times\Omega)^{d\times d}$ (we denote by $(\nabla\bar{\boldsymbol{u}})^*$ the transpose of $\nabla\bar{\boldsymbol{u}}$). It is not very difficult to see that $\boldsymbol{u}_\diamondsuit\to\bar{\boldsymbol{u}}$ strongly in $L^2(]0,T[\times\Omega)^d$; indeed, we have

$$\|\boldsymbol{u}_\diamondsuit-\boldsymbol{u}\|^2_{L^2(]0,T[\times\Omega)^d}=\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}(\triangle_{K,\sigma})\left|\boldsymbol{u}_\sigma^{n+1/2}-\boldsymbol{u}_K^{n+1/2}\right|^2$$

where $\triangle_{K,\sigma}=\mathrm{co}(\{\boldsymbol{p}_K\}\cup\sigma)$ (co denotes the convex hull and $\boldsymbol{p}_K$ is the point chosen during the construction of $\diamondsuit$; see Definition 6.1). But $\mathrm{m}(\triangle_{K,\sigma})\le\mathrm{m}(\sigma)\mathrm{diam}(K)$ so that, by (42) and the estimates in Proposition 5.1, $\|\boldsymbol{u}_\diamondsuit-\boldsymbol{u}\|^2_{L^2(]0,T[\times\Omega)^d}$ is bounded by $C_{20}\mathrm{size}(\mathcal{D})^2$ (with C_{20} not depending on $\mathcal{D}$ or δt), and tends to 0 as size$(\mathcal{D})\to 0$. Since $\boldsymbol{u}\to\bar{\boldsymbol{u}}$ strongly in $L^2(]0,T[\times\Omega)^d$, so does $\boldsymbol{u}_\diamondsuit$. We can thus pass to the limit in (53) and, since $\bar{\boldsymbol{u}}(t,\boldsymbol{x})\cdot((\nabla\bar{\boldsymbol{u}}(t,\boldsymbol{x}))^*\boldsymbol{\varphi}(t,\boldsymbol{x}))=[(\bar{\boldsymbol{u}}(t,\boldsymbol{x})\cdot\nabla)\bar{\boldsymbol{u}}(t,\boldsymbol{x})]\cdot\boldsymbol{\varphi}(t,\boldsymbol{x})$, we find

$$T_{11}\to\int_0^T\int_\Omega[(\bar{\boldsymbol{u}}(t,\boldsymbol{x})\cdot\nabla)\bar{\boldsymbol{u}}(t,\boldsymbol{x})]\cdot\boldsymbol{\varphi}(t,\boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}\quad\text{as size}(\mathcal{D})\to 0\text{ and }\delta t\to 0. \tag{54}$$

To handle T_{12}, we use the same technique as for the Stokes equation. Introducing $(\boldsymbol{\varphi}_\sigma)^n=\frac{1}{\mathrm{m}(\sigma)}\int_\sigma\boldsymbol{\varphi}(n\delta t,\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})$ and using the fact that $\mathrm{div}(\boldsymbol{\varphi})=0$, we see (the same way we arrived at (33)) that

$$T_{12}=\int_0^T\int_\Omega\mathbf{v}(t,\boldsymbol{x}):(\nabla\boldsymbol{\varphi})_{\delta t}(t,\boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}+T_{14}$$

where $(\nabla\boldsymbol{\varphi})_{\delta t}(\cdot,\boldsymbol{x})=\nabla\boldsymbol{\varphi}(n\delta t,\boldsymbol{x})$ on $]n\delta t,(n+1)\delta t[$ and

$$|T_{14}|\le C_{\boldsymbol{\varphi}}\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}|\boldsymbol{F}_{K,\sigma}^{n+1/2}|\mathrm{diam}(K)^2.$$

Using Proposition 5.1, we can show (as we did for T_1) that $T_{14}\to 0$ and therefore, by weak convergence of $\mathbf{v}$ to $\nabla\bar{\boldsymbol{u}}$ and uniform convergence of $(\nabla\boldsymbol{\varphi})_{\delta t}$ to $\nabla\boldsymbol{\varphi}$,

$$T_{12}\to\int_0^T\int_\Omega\nabla\bar{\boldsymbol{u}}(t,\boldsymbol{x}):\nabla\boldsymbol{\varphi}(t,\boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}\quad\text{as size}(\mathcal{D})\to 0\text{ and }\delta t\to 0. \tag{55}$$

The convergence $T_{13}\to\int_0^T\int_\Omega\boldsymbol{f}(t,\boldsymbol{x})\cdot\boldsymbol{\varphi}(t,\boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}$ is obvious and we conclude, from $T_{10}+T_{11}+T_{12}=T_{13}$ and using (52), (54) and (55), that $\bar{\boldsymbol{u}}$ satisfies (7).

6 Appendix

6.1 A key relation

The following lemma is a simple application of Stokes' formula. Its proof in the vector case can be found in [6] (in the case of convex control volumes and convex edges, but the proof does not use these convexity assumptions), and the tensor case follows from the vector case.

Lemma 6.1 *Let K be a non empty open polygonal set in $\mathbb{R}^d$. For $\sigma \in \mathcal{E}_K$ (the edges of K, in the sense given in Definition 2.1), we let $\boldsymbol{x}_\sigma$ be the barycenter of σ; we also denote by $\boldsymbol{n}_{K,\sigma}$ the unit normal to σ outward to K.*
Then, for any vector $\boldsymbol{e} \in \mathbb{R}^d$ and any $\boldsymbol{x}_K \in \mathbb{R}^d$, we have

$$\mathrm{m}(K)\boldsymbol{e} = \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\boldsymbol{e}\cdot\boldsymbol{n}_{K,\sigma}(\boldsymbol{x}_\sigma - \boldsymbol{x}_K).$$

Similarly, for any second order tensor $\mathbf{a} \in \mathbb{R}^{d\times d}$ and any $\boldsymbol{x}_K \in \mathbb{R}^d$, we have

$$\mathrm{m}(K)\mathbf{a} = \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\,(\mathbf{a}\boldsymbol{n}_{K,\sigma})\otimes(\boldsymbol{x}_\sigma - \boldsymbol{x}_K).$$

6.2 The discretization space

Assume (2) and let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1. Let $\nu \geq 0$. We give here some properties on the $(u, \boldsymbol{v}, F) \in H_\mathcal{D} \times H_\mathcal{D}^d \times \mathcal{F}_\mathcal{D}$ which satisfy

$$\begin{aligned}
&\boldsymbol{v}_K\cdot(\boldsymbol{x}_\sigma - \boldsymbol{x}_K) + \boldsymbol{v}_L\cdot(\boldsymbol{x}_L - \boldsymbol{x}_\sigma) + \nu\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}F_{K,\sigma} - \nu\frac{\mathrm{diam}(L)}{\mathrm{m}(\sigma)}F_{L,\sigma} = u_L - u_K\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\sigma^{K|L}\in\mathcal{E}_{\mathrm{int}}\,, \qquad (56)\\
&\boldsymbol{v}_K\cdot(\boldsymbol{x}_\sigma - \boldsymbol{x}_K) + \nu\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}F_{K,\sigma} = -u_K\,, \quad \forall\sigma^{K|\partial}\in\mathcal{E}_{\mathrm{ext}}.
\end{aligned}$$

In order to state these properties, we need to introduce (and in fact estimate) the following discrete H^1-norm, defined for all $u \in H_\mathcal{D}$ by

$$\|u\|_{1,\mathcal{D}} = \left(\sum_{\sigma^{K|L}\in\mathcal{E}_{\mathrm{int}}}\left(\frac{\mathrm{m}(\sigma)}{\mathrm{diam}(K)} + \frac{\mathrm{m}(\sigma)}{\mathrm{diam}(L)}\right)|u_K - u_L|^2 + \sum_{\sigma^{K|\partial}\in\mathcal{E}_{\mathrm{ext}}}\frac{\mathrm{m}(\sigma)}{\mathrm{diam}(K)}|u_K|^2\right)^{1/2}. \quad (57)$$

Lemma 6.2 [Estimate on the discrete H^1 norm and Sobolev inequalities] *Assume that (2) holds. Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 and let $\theta \geq \mathrm{regul}(\mathcal{D})$. Then there exists C_{21} only depending on d, Ω and θ such that, for all $\nu \geq 0$ and all $(u, \boldsymbol{v}, F)$ satisfying (56), denoting $N_2(\mathcal{D},\nu,F) = \left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu^2\,\frac{\mathrm{diam}(K)}{\mathrm{m}(\sigma)}|F_{K,\sigma}|^2\right)^{1/2}$, we have*

$$\|u\|_{1,\mathcal{D}} \leq C_{21}\left(\|\boldsymbol{v}\|_{L^2(\Omega)^d} + N_2(\mathcal{D},\nu,F)\right) \quad (58)$$

and, for all $q < \frac{2d}{d-2} = +\infty$ if $d = 2$ and all $q \leq \frac{2d}{d-2} = 6$ if $d = 3$,

$$\|u\|_{L^q(\Omega)} \leq C_{21}q\left(\|\boldsymbol{v}\|_{L^2(\Omega)^d} + N_2(\mathcal{D},\nu,F)\right). \quad (59)$$

Proof of Lemma 6.2

The relations (56) show that

$$\begin{aligned}\|u\|_{1,\mathcal{D}}^2 &\le 4\sum_{\sigma^{K|L}\in\mathcal{E}_{\rm int}}\left(\frac{{\rm m}(\sigma)}{{\rm diam}(K)}+\frac{{\rm m}(\sigma)}{{\rm diam}(L)}\right)\left({\rm diam}(K)^2|\boldsymbol{v}_K|^2+{\rm diam}(L)^2|\boldsymbol{v}_L|^2\right)\\ &+4\sum_{\sigma^{K|L}\in\mathcal{E}_{\rm int}}\left(\frac{{\rm m}(\sigma)}{{\rm diam}(K)}+\frac{{\rm m}(\sigma)}{{\rm diam}(L)}\right)\nu^2\left(\frac{{\rm diam}(K)^2}{{\rm m}(\sigma)^2}|F_{K,\sigma}|^2+\frac{{\rm diam}(L)^2}{{\rm m}(\sigma)^2}|F_{L,\sigma}|^2\right)\\ &+2\sum_{\sigma^{K|\partial}\in\mathcal{E}_{\rm ext}}\frac{{\rm m}(\sigma)}{{\rm diam}(K)}{\rm diam}(K)^2|\boldsymbol{v}_K|^2+2\sum_{\sigma^{K|\partial}\in\mathcal{E}_{\rm ext}}\frac{{\rm m}(\sigma)}{{\rm diam}(K)}\nu^2\frac{{\rm diam}(K)^2}{{\rm m}(\sigma)^2}|F_{K,\sigma}|^2.\end{aligned}$$

The definition of regul($\mathcal{D}$) and its bound by θ ensure that, if K and L are neighboring control volumes, ${\rm diam}(K)\le\theta{\rm diam}(L)$ (and *vice-versa*). Hence, gathering by control volumes,

$$\|u\|_{1,\mathcal{D}}^2 \le 4\sum_{K\in\mathcal{M}}|\boldsymbol{v}_K|^2\sum_{\sigma\in\mathcal{E}_K}{\rm m}(\sigma){\rm diam}(K)(1+\theta)+4\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\nu^2|F_{K,\sigma}|^2\frac{{\rm diam}(K)}{{\rm m}(\sigma)}(1+\theta).$$

By (9), the proof of (58) is complete.

Let us prove (59) in the case $d=2$. It is shown, in the proof of Lemma 9.5 in [7, p. 792] (using no assumption on the discretization of Ω), that, for all $\alpha>1$,

$$\left(\int_\Omega|u(\boldsymbol{x})|^{2\alpha}\,{\rm d}\boldsymbol{x}\right)^{\frac12}\le\alpha\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}{\rm m}(\sigma)|u_K|^{\alpha-1}D_\sigma u$$

where $D_\sigma u=|u_K-u_L|$ if $\sigma^{K|L}\in\mathcal{E}_{\rm int}$ and $D_\sigma u=|u_K|$ if $\sigma^{K|\partial}\in\mathcal{E}_{\rm ext}$. Hölder's inequality with $p=\frac{2\alpha}{\alpha-1}>2$ and $p'=\frac{p}{p-1}$ then gives

$$\begin{aligned}\left(\int_\Omega|u(\boldsymbol{x})|^{2\alpha}\,{\rm d}\boldsymbol{x}\right)^{\frac12} &\le \alpha\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}{\rm m}(\sigma){\rm diam}(K)|u_K|^{p(\alpha-1)}\right)^{\frac1p}\\ &\times\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}{\rm m}(\sigma){\rm diam}(K)\frac{|D_\sigma u|^{p'}}{{\rm diam}(K)^{p'}}\right)^{\frac{1}{p'}}.\end{aligned}$$

Since $p(\alpha-1)=2\alpha$, (9) shows that

$$\left(\int_\Omega|u(\boldsymbol{x})|^{2\alpha}\,{\rm d}\boldsymbol{x}\right)^{\frac12}\le C_{22}\alpha\left(\int_\Omega|u(\boldsymbol{x})|^{2\alpha}\,{\rm d}\boldsymbol{x}\right)^{\frac12-\frac{1}{2\alpha}}\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}{\rm m}(\sigma){\rm diam}(K)\frac{|D_\sigma u|^{p'}}{{\rm diam}(K)^{p'}}\right)^{\frac{1}{p'}}$$

where C_{22} only depends on d and θ. Since $p'<2$, we can apply Hölder's inequality with exponent $2/p'$ to find, thanks again to (9),

$$\begin{aligned}\|u\|_{L^{2\alpha}(\Omega)} &\le C_{22}\alpha\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}{\rm m}(\sigma){\rm diam}(K)\right)^{\frac{2-p'}{2p'}}\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}{\rm m}(\sigma){\rm diam}(K)\frac{|D_\sigma u|^2}{{\rm diam}(K)^2}\right)^{\frac12}\\ &\le C_{22}\alpha\left(\frac{\omega_{d-1}\theta^2}{\omega_d}{\rm m}(\Omega)\right)^{\frac{2-p'}{2p'}}\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\frac{{\rm m}(\sigma)}{{\rm diam}(K)}|D_\sigma u|^2\right)^{\frac12}.\end{aligned}$$

Gathering the last sum by edges, we recognize the discrete H^1-norm $\|u\|_{1,\mathcal{D}}$ of u and (59) for $q = 2\alpha > 2$ is therefore a consequence of (58); the case $1 \le q \le 2$ is immediate from the case $q > 2$ using Hölder's inequality.

To prove (59) in the case $d = 3$, we still use an inequality from the proof of Lemma 9.5 in [7, p. 792]:

$$\int_\Omega |u(\boldsymbol{x})|^6 \,\mathrm{d}\boldsymbol{x} \le \left(4 \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)|u_K|^3 D_\sigma u\right)^{3/2}.$$

Using Cauchy-Schwarz inequality and (9), we deduce

$$\int_\Omega |u(\boldsymbol{x})|^6 \,\mathrm{d}\boldsymbol{x} \le C_{23} \left(\int_\Omega |u(\boldsymbol{x})|^6 \,\mathrm{d}\boldsymbol{x}\right)^{3/4} \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\mathrm{diam}(K)\frac{|D_\sigma u|^2}{\mathrm{diam}(K)^2}\right)^{3/4}$$

with C_{23} only depending on d and θ. Since the last term (involving $D_\sigma u$) is $\|u\|_{1,\mathcal{D}}^{3/2}$, (58) concludes the proof of (59) for $d = 3$ and $q = 6$; the case $1 \le q \le 6$ can be deduced from the case $q = 6$ thanks to Hölder's inequality.

Although the cases $d = 1$ and $d \ge 4$ are not useful to us, we can notice that the Sobolev injections (59) are also valid for $d = 1$ (with $q = +\infty$) and $d \ge 4$ (with $q \le \frac{2d}{d-2}$); the proof is done by induction on d (see the technique in [7, Lemma 9.5] for $d = 2$ and $d = 3$). □

The two following lemmas are similar to [6, Lemmas 3.2 and 3.3], the only differences being that, in [6], the fluxes are not penalized the same way as in (56) and that we use the result of Lemma 6.2 to improve the convergence of u_m in Lemma 6.4. We let the reader check that the proofs of these lemmas are straightforward adaptations of the proofs in [6].

Lemma 6.3 [Equicontinuity of the translations] *Assume that (2) holds. Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1 and let $\theta \ge \mathrm{regul}(\mathcal{D})$. Let $\nu \ge 0$. Then there exists C_{24} only depending on d, Ω and θ such that, for all $(u, \boldsymbol{v}, F)$ satisfying (56) and all $\xi \in \mathbb{R}^d$,*

$$\|u(\cdot + \xi) - u\|_{L^1(\mathbb{R}^d)} \le C_{24} \left(\|\boldsymbol{v}\|_{L^1(\Omega)^d} + N_1(\mathcal{D}, \nu, F)\right) |\xi| \,, \tag{60}$$

where $N_1(\mathcal{D}, \nu, F) = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \nu\mathrm{diam}(K)|F_{K,\sigma}|$ and u has been extended by 0 outside Ω.

Lemma 6.4 [Compactness property] *Assume that (2) holds. Let $(\mathcal{D}_m)_{m\ge1}$ be admissible discretizations of Ω in the sense of Definition 2.1, such that $\mathrm{size}(\mathcal{D}_m) \to 0$ as $m \to \infty$ and $(\mathrm{regul}(\mathcal{D}_m))_{m\ge1}$ is bounded. Let $(\nu_m)_{m\ge1}$ be a sequence of nonnegative real numbers and, for all $m \ge 1$, let $(u_m, \boldsymbol{v}_m, F_m)_{m\ge1}$ satisfy (56) with $\mathcal{D} = \mathcal{D}_m$ and $\nu = \nu_m$. Assume that the sequence $(\boldsymbol{v}_m)_{m\ge1}$ is bounded in $L^2(\Omega)^d$ and that $N_2(\mathcal{D}_m, \nu_m, F_m) \to 0$ as $m \to \infty$ (N_2 has been defined in Lemma 6.2).*
Then there exists a subsequence of $(\mathcal{D}_m)_{m\ge1}$ (still denoted by $(\mathcal{D}_m)_{m\ge1}$) and $\bar{u} \in H^1_0(\Omega)$ such that the corresponding sequence $(u_m)_{m\ge1}$ converges to $\bar{u}$ strongly in $L^q(\Omega)$ for all $q < \frac{2d}{d-2}$ (and weakly in $L^6(\Omega)$ if $d = 3$), and such that $(\boldsymbol{v}_m)_{m\ge1}$ converges to $\nabla\bar{u}$ weakly in $L^2(\Omega)^d$.

6.3 Tools for the convergence of the nonlinear term

To prove the convergence of the nonlinear term in the scheme for Navier-Stokes problem, it is convenient to introduce partitions of Ω adapted to the edges of the discretization, and to study the convergence of some special functions defined on such partitions.

Definition 6.1 *Assume that (2) holds and let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1. An $\mathcal{D}$-adapted diamond partition of Ω is any partition (up to sets of zero measure) $\diamondsuit$ of Ω defined in the following way: for all $K \in \mathcal{M}$, take $\boldsymbol{p}_K \in K$ such that $B(\boldsymbol{p}_K, \rho_K) \subset K^\star$ (see Definition 2.1 and (8) for the meaning of $K^\star$ and ρ_K) and define $\diamondsuit = (\diamondsuit_\sigma)_{\sigma\in\mathcal{E}}$ by $\diamondsuit_\sigma = \mathrm{co}(\{\boldsymbol{p}_K\} \cup \sigma) \cup \mathrm{co}(\{\boldsymbol{p}_L\} \cup \sigma)$ if $\sigma^{K|L} \in \mathcal{E}_{\rm int}$ and $\diamondsuit_\sigma = \mathrm{co}(\{\boldsymbol{p}_K\} \cup \sigma)$ if $\sigma^{K|\partial} \in \mathcal{E}_{\rm ext}$ (where $\mathrm{co}(A)$ is the convex hull of A).*
If $w = (w_\sigma)_{\sigma\in\mathcal{E}}$ is a given family of values and $\diamondsuit$ is a $\mathcal{D}$-adapted diamond partition of Ω, the diamond-adapted function defined by w is the piecewise function $w_\diamondsuit : \Omega \to \mathbb{R}$ which is equal, on each $\diamondsuit_\sigma$, to w_σ.

Lemma 6.5 *Let $T > 0$ and assume that (2) holds. Let $(\mathcal{D}_m)_{m\geq 1}$ be a sequence of admissible discretizations of Ω in the sense of Definition 2.1 such that $(\mathrm{regul}(\mathcal{D}_m))_{m\geq1}$ is bounded and $\mathrm{size}(\mathcal{D}_m) \to 0$ as $m \to \infty$. We assume that, for all $m \geq 1$ and all integer $N \geq 1$, defining $\delta t = T/N$, we have $u_{m,\delta t} \in H_{\mathcal{D}_m,\delta t}$ (see Section 2.3) such that $\sum_{n=0}^{N-1} \delta t \|(u_{m,\delta t})^{n+1/2}\|^2_{1,\mathcal{D}_m}$ remains bounded as $m \to \infty$ and $\delta t \to 0$ ($\|\cdot\|_{1,\mathcal{D}_m}$ is given by (57) with $\mathcal{D} = \mathcal{D}_m$).*
For each $m \geq 1$, we choose a $\mathcal{D}_m$-adapted diamond partition $\diamondsuit_m$. We let, for $n \in \{0, \dots, N-1\}$ and $t \in]n\delta t, (n+1)\delta t[$, $(\boldsymbol{v}_{m,\delta t})_\diamondsuit(t, \cdot)$ be the (vector-valued) diamond-adapted function defined by the family of (vector) values

$$\begin{cases} \dfrac{\mathrm{m}(\sigma)}{\mathrm{m}(\diamondsuit_\sigma)}((u_{m,\delta t})_L^{n+1/2} - (u_{m,\delta t})_K^{n+1/2})\boldsymbol{n}_{K,\sigma} & \text{if } \sigma^{K|L} \in \mathcal{E}_{m,\rm int} \\ \dfrac{\mathrm{m}(\sigma)}{\mathrm{m}(\diamondsuit_\sigma)}(0 - (u_{m,\delta t})_K^{n+1/2})\boldsymbol{n}_{K,\sigma} & \text{if } \sigma^{K|\partial} \in \mathcal{E}_{m,\rm ext}. \end{cases}$$

We also assume that $u_{m,\delta t}$ converges to some $\bar u$ weakly in $L^2(]0,T[\times\Omega)$ as $m \to \infty$ and $\delta t \to 0$. Then $(\boldsymbol{v}_{m,\delta t})_\diamondsuit$ converges to $\nabla \bar u$ weakly in $L^2(]0,T[\times\Omega)^d$ as $m \to \infty$ and $\delta t \to 0$.

Proof of Lemma 6.5
To simplify the notations, we drop the indices m and δt, and we study the convergence of $\boldsymbol{v}_\diamondsuit$ as $\mathrm{size}(\mathcal{D}) \to 0$ and $\delta t \to 0$ while $\mathrm{regul}(\mathcal{D})$ stays bounded. Let us first show that $\boldsymbol{v}_\diamondsuit$ is bounded in $L^2(]0,T[\times\Omega)^d$. By definition of $\boldsymbol{v}_\diamondsuit$, we have

$$\begin{aligned} \|\boldsymbol{v}_\diamondsuit\|^2_{L^2(]0,T[\times\Omega)^d} &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}} \mathrm{m}(\diamondsuit_\sigma) \left| \frac{\mathrm{m}(\sigma)}{\mathrm{m}(\diamondsuit_\sigma)} (u_L^{n+1/2} - u_K^{n+1/2})\boldsymbol{n}_{K,\sigma} \right|^2 \\ &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}} \frac{\mathrm{m}(\sigma)}{\mathrm{m}(\diamondsuit_\sigma)} \mathrm{m}(\sigma) \left| u_L^{n+1/2} - u_K^{n+1/2} \right|^2 \end{aligned} \tag{61}$$

(we have written $\sigma = \sigma^{K|L}$ if $\sigma \in \mathcal{E}_{\rm int}$ and $u_L^{n+1/2} = 0$ if $\sigma = \sigma^{K|\partial} \in \mathcal{E}_{\rm ext}$). If $d_{K,\sigma}$ is the distance between the $\boldsymbol{p}_K$ chosen to define $\diamondsuit$ and the hyperplane containing $\sigma \in \mathcal{E}_K$, we have $\mathrm{m}(\mathrm{co}(\{\boldsymbol{p}_K\} \cup \sigma)) = \mathrm{m}(\sigma)d_{K,\sigma}/d$ and therefore $\mathrm{m}(\diamondsuit_\sigma) = \mathrm{m}(\sigma)(d_{K,\sigma} + d_{L,\sigma})/d$ (with $d_{L,\sigma} = 0$ if

$\sigma = \sigma^{K|\partial} \in \mathcal{E}_{\rm ext}$); hence, $\frac{\mathrm{m}(\sigma)}{\mathrm{m}(\diamondsuit_\sigma)} = \frac{d}{d_{K,\sigma}+d_{L,\sigma}}$. However, since K is star-shaped with respect to all the points in $B(\boldsymbol{p}_K, \rho_K)$, it is possible to show that $d_{K,\sigma} \geq \rho_K$ (see the proof of (67) in the second step of the proof of Lemma 6.6); since $\mathrm{regul}(\mathcal{D})$ is bounded, we deduce that there exists C_{25} not depending on $\mathcal{D}$ or δt such that

$$\frac{\mathrm{m}(\sigma)}{\mathrm{m}(\diamondsuit_\sigma)} \leq \frac{C_{25}}{\mathrm{diam}(K) + \mathrm{diam}(L)}$$

(where $\mathrm{diam}(L) = 0$ if $\sigma = \sigma^{K|\partial} \in \mathcal{E}_{\rm ext}$) and, coming back to (61),

$$\|\boldsymbol{v}_\diamondsuit\|^2_{L^2(]0,T[\times\Omega)^d} \leq C_{25} \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}} \frac{\mathrm{m}(\sigma)}{\mathrm{diam}(K) + \mathrm{diam}(L)} \left| u_L^{n+1/2} - u_K^{n+1/2} \right|^2.$$

As $\frac{1}{\mathrm{diam}(K)+\mathrm{diam}(L)} \leq \frac{1}{\mathrm{diam}(K)} + \frac{1}{\mathrm{diam}(L)}$, we obtain $\|\boldsymbol{v}_\diamondsuit\|^2_{L^2(]0,T[\times\Omega)^d} \leq C_{25} \sum_{n=0}^{N-1} \delta t \|u^{n+1/2}\|^2_{1,\mathcal{D}}$ and, by assumption, $\boldsymbol{v}_\diamondsuit$ is therefore bounded in $L^2(]0,T[\times\Omega)^d$.

We now prove that $\boldsymbol{v}_\diamondsuit \to \nabla \bar{u}$ in the distribution sense on $]0,T[\times\Omega$ as $\mathrm{size}(\mathcal{D}) \to 0$ and $\delta t \to 0$, which will conclude the proof of the lemma. Let $\boldsymbol{\varphi} \in C_c^\infty(]0,T[\times\Omega)^d$ and $\boldsymbol{\psi}(t,\cdot)$ be the function equal to $\frac{1}{\mathrm{m}(\sigma)} \int_\sigma \boldsymbol{\varphi}(t,\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})$ on $\diamondsuit_\sigma$; since $\boldsymbol{\varphi}$ is regular, we have $\|\boldsymbol{\varphi} - \boldsymbol{\psi}\|_\infty \leq C_{\boldsymbol{\varphi}} \mathrm{size}(\mathcal{D})$. Hence, as $\mathrm{size}(\mathcal{D}) \to 0$ and $\delta t \to 0$, $\boldsymbol{v}_\diamondsuit$ being bounded in $L^2(]0,T[\times\Omega)^d$,

$$\left| \int_0^T \int_\Omega \boldsymbol{v}_\diamondsuit(t,\boldsymbol{x}) \cdot \boldsymbol{\varphi}(t,\boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x} - \int_0^T \int_\Omega \boldsymbol{v}_\diamondsuit(t,\boldsymbol{x}) \cdot \boldsymbol{\psi}(t,\boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x} \right| \to 0. \tag{62}$$

On the other hand, gathering by control volumes,

$$\begin{aligned}
&\int_0^T \int_\Omega \boldsymbol{v}_\diamondsuit(t,\boldsymbol{x}) \cdot \boldsymbol{\psi}(t,\boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x} \\
&\qquad = \sum_{n=0}^{N-1} \sum_{\sigma^{K|L}\in\mathcal{E}_{\rm int}} (u_L^{n+1/2} - u_K^{n+1/2}) \boldsymbol{n}_{K,\sigma} \cdot \int_{n\delta t}^{(n+1)\delta t} \int_\sigma \boldsymbol{\varphi}(t,\boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\gamma(\boldsymbol{x}) \\
&\qquad = -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} u_K^{n+1/2} \int_{n\delta t}^{(n+1)\delta t} \sum_{\sigma\in\mathcal{E}_K} \int_\sigma \boldsymbol{\varphi}(t,\boldsymbol{x}) \cdot \boldsymbol{n}_{K,\sigma}\,\mathrm{d}t\,\mathrm{d}\gamma(\boldsymbol{x}) \\
&\qquad = -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} u_K^{n+1/2} \int_{n\delta t}^{(n+1)\delta t} \int_K \mathrm{div}(\boldsymbol{\varphi})(t,\boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x} \\
&\qquad = -\int_0^T \int_\Omega u(t,\boldsymbol{x}) \mathrm{div}(\boldsymbol{\varphi})(t,\boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}
\end{aligned}$$

which converges, by assumption on u, to $-\int_0^T \int_\Omega \bar{u}(t,\boldsymbol{x}) \mathrm{div}(\boldsymbol{\varphi})(t,\boldsymbol{x})\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}$. Together with (62), this shows that $\boldsymbol{v}_\diamondsuit \to \nabla \bar{u}$ in the distribution sense, and the proof is concluded. □

6.4 A technical result

The following lemma is the generalization of Lemma 6.3 in [6] to the case of non-convex control volumes.

Lemma 6.6 *Let $\alpha > 0$ and $d \geq 1$. Assume that K is a polygonal open subset of $\mathbb{R}^d$ such that K is star-shaped with respect to all the points in a ball of radius $\alpha\mathrm{diam}(K)$. Let E be an affine hyperplane of $\mathbb{R}^d$ and σ be a non-empty open subset of $E \cap \partial K$. Then there exists C_{26} only depending on d and α such that, for all $v \in H^1(K)$,*

$$\left(\frac{1}{\mathrm{m}(\sigma)}\int_\sigma v(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) - \frac{1}{\mathrm{m}(K)}\int_K v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^2 \leq \frac{C_{26}\mathrm{diam}(K)}{\mathrm{m}(\sigma)}\int_K |\nabla v(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}.$$

Proof of Lemma 6.6

In the special case $d = 1$, K is convex and the result is a consequence of Lemma 6.3 in [6]. We therefore assume hereafter that $d \geq 2$.

Step 1: a preliminary inequality.

Let v be a regular function and U, V, A be sets in $\mathbb{R}^d$ of non-zero Lebesgue measure such that, for all $\boldsymbol{x} \in U$ and all $\boldsymbol{y} \in V$, $[\boldsymbol{x}, \boldsymbol{y}] \subset A$. We prove in this step that there exists C_{27} only depending on d such that

$$\left(\frac{1}{\mathrm{m}(U)}\int_U v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{\mathrm{m}(V)}\int_V v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^2 \leq \frac{C_{27}\mathrm{diam}(A)^{d+2}}{\mathrm{m}(U)\mathrm{m}(V)}\int_A |\nabla v(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}. \tag{63}$$

Since v is regular we can write, for all $\boldsymbol{x} \in U$ and all $\boldsymbol{y} \in V$, $v(\boldsymbol{x}) - v(\boldsymbol{y}) = \int_0^1 \nabla v(t\boldsymbol{x} + (1 - t)\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})\,\mathrm{d}t$. As $[\boldsymbol{x}, \boldsymbol{y}] \subset A$, we have $|\boldsymbol{x} - \boldsymbol{y}| \leq \mathrm{diam}(A)$ and Jensen's inequality thus implies

$$\left(\frac{1}{\mathrm{m}(U)}\int_U v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{\mathrm{m}(V)}\int_V v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^2 \\ \leq \frac{\mathrm{diam}(A)^2}{\mathrm{m}(U)\mathrm{m}(V)}\int_U\int_V\int_0^1 |\nabla v(t\boldsymbol{x} + (1-t)\boldsymbol{y})|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}. \tag{64}$$

Let $\boldsymbol{y} \in V$. Using the change of variable $\boldsymbol{x} \in U \to \boldsymbol{z} = t\boldsymbol{x} + (1 - t)\boldsymbol{y} \in A$ and Fubini's theorem, we find

$$\int_U\int_V\int_0^1 |\nabla v(t\boldsymbol{x} + (1-t)\boldsymbol{y})|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \leq \int_A |\nabla v(\boldsymbol{z})|^2 \int_V\int_{I(\boldsymbol{z},\boldsymbol{y})} t^{-d}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{z} \tag{65}$$

where $I(\boldsymbol{z}, \boldsymbol{y}) = \{t \in [0, 1] \mid \exists \boldsymbol{x} \in U\,,\ t\boldsymbol{x} + (1 - t)\boldsymbol{y} = \boldsymbol{z}\}$. If $\boldsymbol{z} \in A$, $\boldsymbol{y} \in V$ and $t \in I(\boldsymbol{z}, \boldsymbol{y})$, then $t(\boldsymbol{x} - \boldsymbol{y}) = \boldsymbol{z} - \boldsymbol{y}$ for some $\boldsymbol{x} \in U$, and therefore $\mathrm{diam}(A)t \geq t|\boldsymbol{x} - \boldsymbol{y}| = |\boldsymbol{z} - \boldsymbol{y}|$. Hence $I(\boldsymbol{z}, \boldsymbol{y}) \subset [\frac{|\boldsymbol{z}-\boldsymbol{y}|}{\mathrm{diam}(A)}, 1]$ and we deduce that

$$\int_{I(\boldsymbol{z},\boldsymbol{y})} t^{-d}\,\mathrm{d}t \leq \int_{\frac{|\boldsymbol{z}-\boldsymbol{y}|}{\mathrm{diam}(A)}}^1 t^{-d}\,\mathrm{d}t \leq \frac{1}{d-1}\frac{\mathrm{diam}(A)^{d-1}}{|\boldsymbol{z} - \boldsymbol{y}|^{d-1}}.$$

Thus, for all $\boldsymbol{z} \in A$,

$$\int_V\int_{I(\boldsymbol{z},\boldsymbol{y})} t^{-d}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{y} \leq \frac{\mathrm{diam}(A)^{d-1}}{d-1}\int_V \frac{1}{|\boldsymbol{z} - \boldsymbol{y}|^{d-1}}\,\mathrm{d}\boldsymbol{y} = \frac{\mathrm{diam}(A)^{d-1}}{d-1}\int_{\boldsymbol{z}-V} \frac{1}{|\boldsymbol{\xi}|^{d-1}}\,\mathrm{d}\boldsymbol{\xi}.$$

Since $V \subset A$, for all $\boldsymbol{z} \in A$ the set $\boldsymbol{z} - V$ is contained in $B(0, \mathrm{diam}(A))$ and, using polar coordinates, we deduce

$$\int_V \int_{I(\boldsymbol{z},\boldsymbol{y})} t^{-d} \,\mathrm{d}t \,\mathrm{d}\boldsymbol{y} \leq \frac{\mathrm{diam}(A)^{d-1}}{d-1} C_{28} \int_0^{\mathrm{diam}(A)} \frac{1}{\rho^{d-1}} \rho^{d-1} \,\mathrm{d}\rho = \frac{C_{28}}{d-1} \mathrm{diam}(A)^d$$

where C_{28} is the surface of the unit sphere in $\mathbb{R}^d$. Substituting this last inequality into (65) and coming back to (64), we conclude the proof of (63).

Step 2: proof of the lemma.
Since the regular functions are dense in $H^1(K)$ (because K is star-shaped), it is sufficient to prove the lemma for $v \in C^1(\overline{K})$. Let $\boldsymbol{p} \in K$ be such that K is star-shaped with respect to all the points in $B(\boldsymbol{p}, \alpha \mathrm{diam}(K))$.

Let $\triangle$ be the convex hull of $\boldsymbol{p}$ and σ (notice that $\triangle \subset \overline{K}$). Under the assumption that K is convex and that $B(\boldsymbol{p}, \alpha \mathrm{diam}(K)) \subset K$, Lemma 6.2 in [6] states that

$$\left(\frac{1}{\mathrm{m}(\sigma)} \int_\sigma v(\boldsymbol{x}) \,\mathrm{d}\gamma(\boldsymbol{x}) - \frac{1}{\mathrm{m}(\triangle)} \int_\triangle v(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \right)^2 \leq \frac{C_{29} \mathrm{dist}(\boldsymbol{p}, E)^2}{\mathrm{m}(\triangle)} \int_\triangle |\nabla v(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} \qquad (66)$$

with C_{29} only depending on d and α. In fact, in the proof of [6, Lemma 6.2], the convexity assumption on K is only used to ensure that

$$\mathrm{dist}(\boldsymbol{p}, E) \geq \alpha \mathrm{diam}(K)\,, \qquad (67)$$

which is a consequence of the fact that $B(\boldsymbol{p}, \alpha \mathrm{diam}(K))$ entirely lies on one side of E; it is quite easy to see that this property still holds if K is only star-shaped with respect to all the points in $B(\boldsymbol{p}, \alpha \mathrm{diam}(K))$.
Indeed, by translation we can assume that 0 is in the relative interior of σ. The definition of "polygonal subset" implies that K is, on a neighborhood of 0, on one side of its edge σ; denoting by $\boldsymbol{n}$ the outer normal to K on σ, this means that $\boldsymbol{z} \cdot \boldsymbol{n} < 0$ for all $\boldsymbol{z} \in K$ in a neighborhood of 0. Assume now that $B(\boldsymbol{p}, \alpha \mathrm{diam}(K))$ has points on the two sides of E: we can then find $\boldsymbol{y}$ in this ball such that $\boldsymbol{y} \cdot \boldsymbol{n} > 0$. Since K is star-shaped with respect to $\boldsymbol{y}$ we have, for all $\boldsymbol{x} \in K$ and all $\lambda \in]0, 1[$, $\boldsymbol{z}(\lambda, \boldsymbol{x}) = \boldsymbol{x} + \lambda(\boldsymbol{y} - \boldsymbol{x}) \in K$. If $\boldsymbol{x}$ is close to 0 (it is possible to take such a $\boldsymbol{x}$ in K since $0 \in \sigma \subset \partial K$) and λ is close to 0, we see that $\boldsymbol{z}(\lambda, \boldsymbol{x})$ is close to 0; moreover, $\boldsymbol{z}(\lambda, \boldsymbol{x}) \cdot \boldsymbol{n} = (1 - \lambda)\boldsymbol{x} \cdot \boldsymbol{n} + \lambda \boldsymbol{y} \cdot \boldsymbol{n}$ and, since $\boldsymbol{y} \cdot \boldsymbol{n} > 0$, it is possible to choose $\boldsymbol{x}$ close to 0 and then λ close to 0 such that $\boldsymbol{z}(\lambda, \boldsymbol{x}) \cdot \boldsymbol{n} > 0$, which is a contradiction since $\boldsymbol{z}(\lambda, \boldsymbol{x})$ is a point of K close to 0. Hence, $B(\boldsymbol{p}, \alpha \mathrm{diam}(K))$ lies on only one side of E, and (67) and (66) are therefore valid under our assumptions.

We apply (63) with $U = \triangle \backslash \sigma \subset K$, $V = B(\boldsymbol{p}, \alpha \mathrm{diam}(K))$ and $A = K$ (since K is star-shaped with respect to all the points in V and since $U \subset K$, we indeed have $[\boldsymbol{x}, \boldsymbol{y}] \in A$ for all $\boldsymbol{x} \in U$ and all $\boldsymbol{y} \in V$); the set $\sigma = \triangle \backslash U$ having a zero d-dimensional Lebesgue measure, we obtain

$$\begin{aligned} &\left(\frac{1}{\mathrm{m}(\triangle)} \int_\triangle v(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} - \frac{1}{\mathrm{m}(B(\boldsymbol{p}, \alpha \mathrm{diam}(K)))} \int_{B(\boldsymbol{p}, \alpha \mathrm{diam}(K))} v(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \right)^2 \\ &\qquad \leq \frac{C_{27} \mathrm{diam}(K)^{d+2}}{\mathrm{m}(\triangle) \mathrm{m}(B(\boldsymbol{p}, \alpha \mathrm{diam}(K)))} \int_K |\nabla v(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}. \end{aligned} \qquad (68)$$

Applying once again (63), with $U = B(\boldsymbol{p}, \alpha\mathrm{diam}(K))$, $V = K$ and $A = K$, we also have

$$\left(\frac{1}{\mathrm{m}(B(\boldsymbol{p},\alpha\mathrm{diam}(K)))}\int_{B(\boldsymbol{p},\alpha\mathrm{diam}(K))} v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{\mathrm{m}(K)}\int_K v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^2 \leq \frac{C_{27}\mathrm{diam}(K)^{d+2}}{\mathrm{m}(B(\boldsymbol{p},\alpha\mathrm{diam}(K)))\mathrm{m}(K)}\int_K |\nabla v(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}. \quad (69)$$

Gathering (66), (68) and (69), we get C_{30} only depending on d and α such that

$$\left(\frac{1}{\mathrm{m}(\sigma)}\int_\sigma v(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) - \frac{1}{\mathrm{m}(K)}\int_K v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^2 \leq C_{30}\left(\frac{\mathrm{dist}(\boldsymbol{p},E)^2}{\mathrm{m}(\triangle)} + \frac{\mathrm{diam}(K)^2}{\mathrm{m}(\triangle)} + \frac{\mathrm{diam}(K)^2}{\mathrm{m}(K)}\right)\int_K |\nabla v(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}.$$

But $\mathrm{m}(K) \geq \mathrm{m}(\triangle) = \frac{\mathrm{dist}(\boldsymbol{p},E)\mathrm{m}(\sigma)}{d}$ and $\mathrm{dist}(\boldsymbol{p}, E) \leq \mathrm{dist}(\boldsymbol{p}, \sigma) \leq \mathrm{diam}(K)$, so that (67) concludes the proof. □

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