A uniformly converging scheme for fractal conservation laws

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Abstract The fractal conservation law $\partial_t u + \partial_x(f(u)) + (-\Delta)^{\alpha/2} u = 0$ changes characteristics as $\alpha \to 2$ from non-local and weakly diffusive to local and strongly diffusive. In this paper we present a corrected finite difference quadrature method for $(-\Delta)^{\alpha/2}$ with $\alpha \in [0,2]$, combined with usual finite volume methods for the hyperbolic term, that automatically adjusts to this change and is uniformly convergent with respect to $\alpha \in [\eta, 2]$ for any $\eta > 0$. We provide numerical results which illustrate this asymptotic-preserving property as well as the non-uniformity of previous finite difference or finite volume type of methods.

1 Introduction

We consider the following fractional conservation law

$$\begin{array}{ll} \partial_t u_\alpha + \partial_x(f(u_\alpha)) + \mathscr{L}_\alpha[u_\alpha] = 0\,, & t > 0\,,\ x \in \mathbb{R}\,, \\ u_\alpha(0,x) = u_{\rm ini}(x)\,, & x \in \mathbb{R}\,, \end{array} \tag{1}$$

where $\alpha \in [0,2]$, $\mathscr{L}_\alpha = (-\Delta)^{\alpha/2}$,

$$u_{\rm ini} \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}) \quad \text{and} \quad f : \mathbb{R} \to \mathbb{R} \text{ is locally Lipschitz-continuous.} \tag{2}$$

Such models appear for example in mathematical finance, gas detonation or semiconductor growth [23, 26, 11, 1]. The fractional Laplacian $\mathscr{L}_\alpha = (-\Delta)^{\alpha/2}$ can be

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defined e.g. as a Fourier multiplier, but for our purpose the following equivalent definition, valid for any $\varphi \in C_c^\infty(\mathbb{R})$ (set of smooth compactly supported functions), is more useful:

$$\begin{cases} \mathscr{L}_0[\varphi](x) = \varphi(x), & \alpha = 0, \\ \mathscr{L}_\alpha[\varphi](x) = -c_\alpha \int_{\mathbb{R}} \frac{\varphi(x+z)-\varphi(x)-\varphi'(x)z\mathbf{1}_{[-1,1]}(z)}{|z|^{1+\alpha}}\,dz, & \alpha \in (0,2), \\ \mathscr{L}_2[\varphi](x) = -\Delta\varphi(x), & \alpha = 2, \end{cases} \tag{3}$$

where $\mathbf{1}_{[-1,1]}$ is the characteristic function of $[-1,1]$, $c_\alpha = (2\pi)^\alpha \frac{\alpha\Gamma(\frac{1+\alpha}{2})}{2\pi^{\frac{1}{2}+\alpha}\Gamma(1-\frac{\alpha}{2})}$ and Γ is the Euler function [15].

As $\alpha \to 2$, the operator $\mathscr{L}_\alpha$ changes nature and properties. For $\alpha \in (0,2)$, $\mathscr{L}_\alpha$ is a *non-local* pseudo-differential operator of order < 2, and it has relatively weak diffusive properties since the decay at infinity of the fundamental solution of $\partial_t u + \mathscr{L}_\alpha[u] = 0$ is polynomial. At $\alpha = 2$, $\mathscr{L}_\alpha = -\Delta$ is a *local* operator with strong diffusive properties and a fundamental solution with super-exponential decay. When α vary over $[0,2]$, the qualitative behaviour of the solution u_α of (1) also changes. In the case that $\alpha = 2$, it is well-known that u_α becomes instantly smooth for $t > 0$ even when the initial data is discontinuous. On the contrary, for $\alpha = 0$, the solution may develop shocks and uniqueness of the solution requires additional entropy conditions and the corresponding notion of entropy solution [22]. The study of the fractional case $\alpha \in (0,2)$ dates back to [6], with some restrictions on α and f. The first complete study in the case $\alpha > 1$ for any locally Lipschitz f and bounded initial data $u_{\rm ini}$ can be found in [14]. Here it is proved that the solution becomes instantly smooth even if $u_{\rm ini}$ is only bounded (see also [15]). If $\alpha < 1$, then the solution can develop shocks [4] and the weak solution need not be unique [3]. The notion of entropy solution of [2] is therefore required to obtain a well-posed formulation.

There exists a vast literature on the numerical approximation of scalar conservation laws (i.e. (1) without $\mathscr{L}_\alpha$), see e.g. [17, 18, 19] and references therein. The study of numerical methods for fractal conservation laws is much more recent with a corresponding less extensive literature. Probabilistic methods have been studied in [21, 24], but must be applied to the equation satisfied by $\partial_x u_\alpha$ in order to avoid noisy results, and recovering from this a numerical approximation of u_α may be challenging in dimension greater than 1. Deterministic methods for (1) like finite difference, volume, and element methods (discontinuous Galerkin) are given in [13, 8, 10], while a high order spectral vanishing viscosity method is introduced in [9]. The latter method and its analysis is very different from the former three methods, with convergence and (non-optimal) error estimates that are independent of $\alpha \in (0,2)$. As opposed to the spectral method, the other methods are monotone or have low order monotone variants.

Surprisingly, *for all the non-spectral monotone methods the convergence deteriorates as* $\alpha \to 2$, and the schemes themselves are not even defined in the limit $\alpha = 2$. The purpose of this paper is to present an *asymptotic-preserving monotone* scheme for (1) defined for any $\alpha \in [0,2]$, i.e. a scheme that provides a monotone approximation of u_α which is uniform with respect to $\alpha \in [0,2]$. In particular, our scheme naturally adapts to the change of behaviour of $\mathscr{L}_\alpha$ as $\alpha \to 2$ and $\alpha \to 0$ and its convergence properties do not deteriorate in these extreme cases. The idea behind our scheme is to add a correction term in the form of a suitably chosen vanishing local viscosity term. Similar ideas have been used for other equations before, see e.g. [12] for linear equations and [20] for fully nonlinear equations. A stochastic interpretation can be found in [5].

This paper is organised as follows. The numerical method is presented in Section 2, and its asymptotic-preserving characteristics are discussed. Due to lack of space and the technical nature of the proofs, we skip them and refer instead to [16]. In Sections 3 and 4, we define precisely what asymptotic preserving means and the we give a couple numerical simulations to illustrate this property of the method.

2 The scheme

The new scheme is based on monotone convervative finite difference approximations of the local terms combined with quadrature, truncation of $\frac{1}{|z|^{1+\alpha}}$ near the singularity, and a second order correction term (vanishing viscosity) for the non-local term. Except for the correction term, the scheme is similar to the schemes of [13, 8] and of [10] with P_0-elements. It is monotone, conservative, and converges in L^1_{loc} uniformly in $\alpha \in [\eta,2]$ for all $\eta > 0$.

For given space and time steps $\delta x, \delta t > 0$, we introduce the grid $t_n := n\delta t$ and $x_i := i\delta x + \frac{\delta x}{2}$ for $n \in \mathbb{N}_0$ and $i \in \mathbb{Z}$. We identify sequences $(\varphi_i)_{i\in\mathbb{Z}}$ of numbers with piecewise constant functions $\varphi_{\delta x} : \mathbb{R} \to \mathbb{R}$ equal to φ_i on $[i\delta x,(i+1)\delta x)$ for all $i \in \mathbb{Z}$. Similarly, $(\varphi_i^n)_{n\geq 0,\, i\in\mathbb{Z}}$ is identified with $\varphi_{\delta x,\delta t} : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ equal to φ_i^n on $[n\delta t,(n+1)\delta t)\times[i\delta x,(i+1)\delta x)$ for all $n \geq 0$ and $i \in \mathbb{Z}$. The discretisation of (1) can then we written as: find $u_{\alpha,\delta x,\delta t} = (u_i^n)_{n\geq 0,\, i\in\mathbb{Z}}$ such that

$$u_i^0 = \frac{1}{\delta x}\int_{[i\delta x,(i+1)\delta x)} u_0(x)\,dx \quad \text{for all } i \in \mathbb{Z}, \tag{4}$$

$$\frac{u_i^{n+1}-u_i^n}{\delta t} + \mathscr{F}_{\delta x}(u^n)_i + \mathscr{L}_{\alpha,\delta x}[u^{n+1}]_i = 0 \quad \text{for all } n \geq 0 \text{ and all } i \in \mathbb{Z}. \tag{5}$$

where $\mathscr{F}_{\delta x}$ is any monotone consistent and consevative discretization of $\partial_x(f(u))$ (see e.g [17, 18, 19]), and $\mathscr{L}_{\alpha,\delta x}$ is a monotone discretisation of $\mathscr{L}_\alpha$ to be defined. Note that the scheme has explicit convection and implicit diffusion terms.

The first and simplest idea to obtain a monotone discretization of $\mathscr{L}_\alpha$ for $\alpha \in (0,2)$ is to discretize the integral in (3) using a simple (weighted) midpoint type quadrature rule, see e.g. [13, 10, 8]. For $\varphi \in C_c^\infty(\mathbb{R})$ and letting $\varphi_l = \varphi(x_l)$ if $l \in \mathbb{Z}$, this leads to

$$\mathscr{L}_\alpha[\varphi](x_i) \approx \tilde{\mathscr{L}}_{\alpha,\delta x}[\varphi]_i := -\sum_{j\in\mathbb{Z}\setminus\{0\}}\Big(\varphi_{i+j}-\varphi_i\Big)\int_{(j\delta x-\frac{\delta x}{2},\,j\delta x+\frac{\delta x}{2})}\frac{c_\alpha}{|z|^{1+\alpha}}\,dz. \tag{6}$$

However, as $\alpha \to 2$ we have $c_\alpha \to 0$ and therefore $\tilde{\mathscr{L}}_{\alpha,\delta x} \to 0$ for fixed δx. In the limit $\alpha \to 2$ the scheme then converges to

$$\frac{u_i^{n+1} - u_i^n}{\delta t} + \mathscr{F}_{\delta x}(u^n)_i = 0 \quad \text{for all } n \geq 0 \text{ and all } i \in \mathbb{Z},$$

which is a discretisation of $\partial_t u + \partial_x(f(u)) = 0$ and not $\partial_t u + \partial_x(f(u)) - \Delta u = 0$. Hence the limits $\alpha \to 2$ and $\delta x \to 0$ do not commute and the scheme is not asymptotic-preserving.

Note that $\tilde{\mathscr{L}}_{\alpha,\delta x}$ vanishes in the limit because the measure $\frac{c_\alpha dz}{|z|^{1+\alpha}}$ concentrates around 0 as $\alpha \to 2$, while in the above midpoint rule the integral in (3) over $(-\frac{\delta x}{2}, \frac{\delta x}{2})$ will always be zero by symmetry. We therefore need to replace the midpoint rule on this interval by a more accurate rule based on the second order interpolation polynomial P_i of φ around the node x_i. We find that this polynomial satisfies $P_i(x_i + z) - P_i(x_i) - P_i'(x_i)z = \frac{1}{2\delta x^2}\left(z^2\varphi_{i-1} - 2z^2\varphi_i + z^2\varphi_{i+1}\right)$ and the new discretization therefore becomes

$$\begin{aligned}\hat{\mathscr{L}}_{\alpha,\delta x}[\varphi]_i &:= -c_\alpha \int_{-\frac{\delta x}{2}}^{\frac{\delta x}{2}} \frac{P(x_i + z) - P(x_i) - P'(x_i)z}{|z|^{1+\alpha}}\, dz + \tilde{\mathscr{L}}_{\alpha,\delta x}[\varphi]_i \\ &= \frac{\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}}{\delta x^2} \int_{(-\frac{\delta x}{2}, \frac{\delta x}{2})} \frac{c_\alpha |z|^{1-\alpha}}{2}\, dz + \tilde{\mathscr{L}}_{\alpha,\delta x}[\varphi]_i.\end{aligned}$$

We can check that the new approximation has the following truncation error [16]:

$$\begin{aligned}&|\mathscr{L}_\alpha[\varphi](x_i) - \hat{\mathscr{L}}_{\alpha,\delta x}[\varphi]_i| \\ &\quad \leq C\Big(\|\varphi^{(4)}\|_{L^\infty} \delta x^{4-\alpha} + \|\varphi''\|_{L^\infty} c_\alpha \big(\tfrac{1}{\alpha} + \tfrac{1}{|1-\alpha|}\big)\delta x^{\min(1,2-\alpha)} + \|\varphi'\|_{L^\infty} \delta x\Big),\end{aligned}$$

which is $O(\delta x^2) + o(1)$ as $\alpha \to 2$ and therefore does not deteriorate in this limit. Note that if $\alpha = 1$, then $\frac{1}{|1-\alpha|}\delta x^{\min(1,2-\alpha)}$ must be replaced with $\delta x|\ln(\delta x)|$.

In order to obtain an approximation which uses only a finite number of discrete values, we also truncate the sum in (6) as in [13] at some index $J_{\delta x} > 0$ (which may depend upon α) where $J_{\delta x}\delta x \to \infty$ as $\delta x \to 0$. The final approximate operator $\mathscr{L}_{\alpha,\delta x}$ is therefore

$$\begin{aligned}\mathscr{L}_{\alpha,\delta x}[\varphi]_i = &-\sum_{0<|j|\leq J_{\delta x}} W^j_{\alpha,\delta x}(\varphi_{i+j} - \varphi_i) - W^{J_{\delta x}+1}_{\alpha,\delta x}\left(\varphi_{i-J_{\delta x}-1} - \varphi_i\right) \\ &- W^{J_{\delta x}+1}_{\alpha,\delta x}\left(\varphi_{i+J_{\delta x}+1} - \varphi_i\right) - W^0_{\alpha,\delta x}\frac{\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}}{\delta x^2}, \qquad (7)\end{aligned}$$

with weights

$$
\begin{aligned}
W^0_{\alpha,\delta x} &= \int_{(-\frac{\delta x}{2},\frac{\delta x}{2})} \frac{c_\alpha |z|^{1-\alpha}}{2}\,dz\,,\\
W^j_{\alpha,\delta x} &= \int_{(j\delta x-\frac{\delta x}{2},\,j\delta x+\frac{\delta x}{2})} \frac{c_\alpha}{|z|^{1+\alpha}}\,dz \qquad \text{for} \qquad 0<|j|\le J_{\delta x},\\
W^{J_{\delta x}+1}_{\alpha,\delta x} &= \int_{z>J_{\delta x}\delta x+\frac{\delta x}{2}} \frac{c_\alpha}{|z|^{1+\alpha}}\,dz = \int_{z<-J_{\delta x}\delta x-\frac{\delta x}{2}} \frac{c_\alpha}{|z|^{1+\alpha}}\,dz.
\end{aligned}
\tag{8}
$$

The last term in (7) contains the classical discretization of $\varphi''(x_i)$ and is the new correction term compared with the discretisations of [13, 10, 8]. This discretisations fit with the generic framework of [13] from which we can conclude:

Theorem 1 ([16]). *Under a standard CFL condition for the convection term,*

1. *There is a unique solution $u_{\alpha,\delta x,\delta t}$ of the scheme defined by* (4), (5), (7) *and* (8), *satisfying $\|u_{\alpha,\delta x,\delta t}\|_{L^\infty} \le \|u_{ini}\|_{L^\infty}$ and $|u_{\alpha,\delta x,\delta t}(t,\cdot)|_{BV} \le |u_{ini}|_{BV}$ for all $t>0$.*
2. *For fixed α, $u_{\alpha,\delta x,\delta t}$ converges in $C([0,\infty);L^1_{loc})$ as $(\delta x,\delta t)\to 0$ to the unique entropy solution u_α of* (1).

Remark 1. We set $\mathscr{L}_{2,\delta x}[\varphi]_i = -(\varphi_{i+1}-2\varphi_i+\varphi_{i-1})/\delta x^2$ and $\mathscr{L}_{0,\delta x}[\varphi]_i = \varphi_i$. This consists in fixing δx and sending $\alpha\to 2$ or $\alpha\to 0$ in (7). Taking the limits in the scheme (5), we obtain the classical implicit scheme for the (1) with $\alpha=2$ or $\alpha=0$.

3 The asymptotic-preserving property

The scheme is asymptotic-preserving if its solution $u_{\alpha,\delta x,\delta t}$ satisfies the following uniform approximation result away from $\alpha=0$ (see [16] for the case $\alpha=0$):

$$
\forall\eta>0\,,\ \sup_{\alpha\in[\eta,2]} d_{L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R})}(u_{\alpha,\delta x,\delta t},u_\alpha)\to 0 \text{ as } (\delta x,\delta t)\to 0 \tag{9}
$$

where $d_{L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R})}(u,v)=\sum_{n=1}^{\infty}2^{-n}\min(1,||u-v||_{L^1([0,n)\times(-n,n))})$ is the usual distance defining the topology of $L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R})$. Here and elsewhere, the convergence $(\delta x,\delta t)\to 0$ is always taken under a standard CFL condition depending on the definition of the convective flux $\mathscr{F}$ in (5) (see e.g. [13, 10, 8]). That this formulation of the asymptotic-preserving property is very general and does not require an explicit error estimate independent on α. Such an estimate seems particularly challenging to obtain in the absence of regularity of the solution as $t\to 0$.

Theorem 2 ([16]). *Under a standard CFL for the convection part, the numerical scheme defined by* (4) (5), (7) *and* (8) *is asymptotic-preserving.*

Next we want to illustrate this property numerically. As it is formulated now, this would require us to have access to the exact solution u_α, which is not the case. We overcome this difficulty by using instead the following equivalent reformulation of (9) (see [16]), which can be checked by computing approximate solutions only:

$$\forall \alpha_0 \in (0,2]\,,\ \text{for any sequence } (\delta x_k, \delta t_k)_{k\in\mathbb{N}} \text{ converging to 0:}$$
$$\sup_{k\geq 1} d_{L^1_{\rm loc}([0,\infty)\times\mathbb{R})}(u_{\alpha,\delta x_k,\delta t_k}, u_{\alpha_0,\delta x_k,\delta t_k}) \to 0 \text{ as } \alpha\to\alpha_0. \quad (10)$$

Remark 2. The matrix of $\mathscr{L}_{\alpha,\delta x}$ defined by (7) is a semi-definite Toepliz matrix as in [13, 10, 8]. Implementation of the scheme thus takes advantage of super-fast multiplication and inversion algorithms for these matrices [7, 25]. Computing several approximate solutions, as required in (10), is therefore not very expensive.

4 Numerical results

In all these tests, we take the Burgers flux $f(u) = \frac{u^2}{2}$ and $\mathscr{F}_{\delta x}$ given by a MUSCL method. The final time is $T = 1$ and the spatial computational domain is $[-1,1]$. We use the same truncation parameters (in particular $J_{\delta x}$) as in [13, Section 4.1.2].

For each test, we choose the discretisation steps $(\delta x_k, \delta t_k) = (\frac{1}{2^k\times 50}, \frac{1}{2^k\times 100})$ for $k = 1,\dots,4$, which all satisfy the CFL for (5). We also select four values $(\alpha_m)_{m=1,\dots,4} = (1.8, 1.9, 1.99, 1.999)$ which are near $\alpha_0 = 2$, the difficult case in assessing the uniformity of the convergence in (10) and the reason why we introduced the correction term in (7). We then indicate, for $m = 1,\dots,4$, the value of

$$E_m = \sup_{t\in[0,1]} ||u_{\alpha_m,\delta x,\delta t}(\cdot,t) - u_{\alpha_0,\delta x,\delta t}(\cdot,t)||_{L^1([-1,1])},$$

that is the $L^\infty(L^1)$ norm of $u_{\alpha_m,\delta x,\delta t} - u_{\alpha_0,\delta x,\delta t}$ on the computational domain. This is a stronger norm that the $L^1(L^1)$ norm used in (10). Hence, E_m approaching 0 as m increases is an even better indication that the scheme is asymptotic-preserving.

Test 1 (rarefaction): we select a Riemann initial condition, $u_{\rm ini} = -1$ if $x < 0$ and $u_{\rm ini} = 1$ if $x > 0$. In this case both convection and diffusion work to smooth out the intial data. Table 1 shows the values of $(E_m)_{m=1,\dots,4}$ for both the uncorrected scheme from [13] based on (6) and our corrected scheme based on (7).

Table 1 Comparison between the uncorrected scheme of [13] and our corrected scheme, $u_{\rm ini} = -1$ on $(-\infty,0)$, $u_{\rm ini} = 1$ on $(0,\infty)$.

	E_1	E_2	E_3	E_4
Uncorrected scheme	1.8E-1	3E-1	8.8E-1	9.1E-1
Corrected scheme	5.1E-2	2.2E-2	1.7E-4	1.7E-5

Test 2 (smooth shock): the initial condition is $u_{\rm ini}(x) = 1$ if $x < 0$ and $u_{\rm ini}(x) = -1$ if $x > 0$. Here the hyperbolic and non-local terms in (1) compete to maintain or diffuse the initial shock. Since α_m is near 2 however, any solution is instantly smooth, but has much larger gradients near $x = 0$ than the solution in Test 1.

Both tests confirm that the scheme defined by (4), (5), (7) and (8) is asymptotic-preserving. They also confirm that, without the order 2 correction in (7), the scheme

Table 2 Comparison between the uncorrected scheme of [13] and our corrected scheme, $u_{\text{ini}} = 1$ on $(-\infty, 0)$, $u_{\text{ini}} = -1$ on $(0, \infty)$.

	E_1	E_2	E_3	E_4
Uncorrected scheme	2.1E-1	3.9E-1	1.3	1.3
Corrected scheme	5.3E-2	2.3E-2	3.2E-4	4.2E-5

deteriorates as $\alpha \to 2$ and does not provide a correct numerical solution at any reasonable resolution. This is also illustrated in Figure 1, where we plot the solutions of both schemes for $\alpha = 1.99$ for the initial condition of Test 2 and $(\delta x, \delta t) = (\frac{1}{2^4 \times 50}, \frac{1}{2^4 \times 100})$. Even at this very high resolution, the uncorrected scheme provides an incorrect approximate solution which, as expected, is closer to the solution of $\partial_t u + \partial_x(f(u))$ than to the solution of (1).

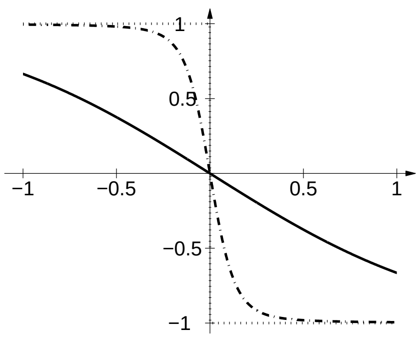

Fig. 1 Approximate solutions provided at $T = 1$ by the corrected (continuous) and uncorrected (dashed) schemes for (1) with $\alpha = 1.99$. The dotted line is both the initial condition and the solution to $\partial_t u + \partial_x(f(u)) = 0$.

5 Conclusion

We have presented a monotone numerical method for fractional conservation laws which is asymptotic-preserving with respect to the fractional power of the Laplacian. The scheme automatically adjusts to the change of nature of the equation as the power of the Laplacian goes to 1 (i.e. $\alpha \to 2$ in (1)) and therefore provides accurate approximate solutions for any power of the fractional Laplacian. We have given numerical results to illustrate the asymptotic-preserving property of our method, as well as the necessity of modifying previously studied monotone methods to obtain this property.

The complete theoretical study of such monotone asymptotic-preserving schemes will be presented in the forthcomming paper [16]. Here a general class of fractional degenerate parabolic equations are considered that include (1) as a special case.

References

1. Alfaro, M., Droniou, J.: General fractal conservation laws arising from a model of detonations in gases. Appl. Math. Res. Express **2012**, 127–151 (2012)
2. Alibaud, N.: Entropy formulation for fractal conservation laws. J. Evol. Equ. **7**(1), 145–175 (2007)
3. Alibaud, N., Andreianov, B.: Non-uniqueness of weak solutions for the fractal Burgers equation. Ann. Inst. H. Poincaré Anal. Non Linéaire **27**(4), 997–1016 (2010)
4. Alibaud, N., Droniou, J., Vovelle, J.: Occurrence and non-appearance of shocks in fractal burgers equations. J. Hyperbolic Differ. Equ. **4**(3), 479–499 (2007)
5. Asmussen, S., Rosiński, J.: Approximations of small jumps of Lévy processes with a view towards simulation. J. Appl. Probab. **38**(2), 482–493 (2001)
6. Biler, P., Karch, G., Woyczynski, W.: Fractal burgers equations. J. Diff. Eq. **148**, 9–46 (1998)
7. Chan, R., Ng, M.: Conjugate gradient methods for toeplitz systems. SIAM Review **38**(3), 427–482 (1996)
8. Cifani, S., Jakobsen, E.R.: On numerical methods and error estimates for degenerate fractional convection-diffusion equations. To appear in Numer. Math. DOI 10.1007/s00211-013-0590-0
9. Cifani, S., Jakobsen, E.R.: On the spectral vanishing viscosity method for periodic fractional conservation laws. Math. Comp. **82**(283), 1489–1514 (2013)
10. Cifani, S., Jakobsen, E.R., Karlsen, K.H.: The discontinuous galerkin method for fractal conservation laws. IMA J. Numer. Anal. **31**(3), 1090–1122 (2011)
11. Clavin, P.: Instabilities and nonlinear patterns of overdriven detonations in gases. Kluwer (2002)
12. Cont, R., Tankov, P.: Financial modelling with jump processes. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton (FL) (2004)
13. Droniou, J.: A numerical method for fractal conservation laws. Math. Comp. **79**(269), 95–124 (2010)
14. Droniou, J., Gallouët, T., Vovelle, J.: Global solution and smoothing effect for a non-local regularization of an hyperbolic equation. J. Evol. Equ. **3**(3), 499–521 (2003)
15. Droniou, J., Imbert, C.: Fractal first order partial differential equations. Arch. Ration. Mech. Anal. **182**(2), 299–331 (2006)
16. Droniou, J., Jakobsen, E.R.: An asymptotic-preserving scheme for fractal conservation laws and frational degenerate parabolic equations. In preparation
17. Eymard, R., Gallouët, T., Herbin, R.: Finite volume methods. In: P.G. Ciarlet, J.L. Lions (eds.) Techniques of Scientific Computing, Part III, Handbook of Numerical Analysis, VII, pp. 713–1020. North-Holland, Amsterdam (2000)
18. Godlewski, E., Raviart, P.A.: Numerical approximation of hyperbolic systems of conservation laws, *Applied Mathematematical Sciences*, vol. 118. Springer, New-York (1996)
19. Holden, H., H., R.N.: Front tracking for Hyperbolic Conservation Laws. Springer (2002)
20. Jakobsen, E.R., Karlsen, K.H., La Chioma, C.: Error estimates for approximate solutions to Bellman equations associated with controlled jump-diffusions. Numer. Math. **110**(2), 221–255 (2008)
21. Jourdain, B., Méléard, S., Woyczynski, W.: Probabilistic approximation and inviscid limits for one-dimensional fractional conservation laws. Bernoulli **11**(4), 689–714 (2005)
22. Kruzhkov, S.N.: First order quasilinear equations with several independent variables. Math. Sb. (N.S.) **81**(123), 228–255 (1970)
23. Soner, H.: Optimal control with state-space constraint ii. SIAM J. Control Optim. **24**(6) (1986)
24. Stanescu, D., Kim, D., Woyczynski, W.: Numerical study of interacting particles approximation for integro-differential equations. Journal of Computational Physics **206**, 706–726 (2005)
25. Van Loan, C.: Computational frameworks for the fast Fourier transform, *Frontiers in Applied Mathematics*, vol. 10. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (1992)
26. Woyczynski, W.: Lévy processes in the physical sciences. Birkhäuser, Boston (2001)