GRADIENT SCHEMES FOR LINEAR AND NON-LINEAR ELASTICITY EQUATIONS

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Abstract. The Gradient Scheme framework provides a unified analysis setting for many different families of numerical methods for diffusion equations. We show in this paper that the Gradient Scheme framework can be adapted to elasticity equations, and provides error estimates for linear elasticity and convergence results for non-linear elasticity. We also establish that several classical and modern numerical methods for elasticity are embedded in the Gradient Scheme framework, which allows us to obtain convergence results for these methods in cases where the solution does not satisfy the full H^2-regularity or for non-linear models.

1. Introduction

We are interested in the numerical approximation of the (possibly non-linear) elasticity equation

$$\begin{aligned}
&-\mathrm{div}(\boldsymbol{\sigma}(x,\boldsymbol{\varepsilon}(\overline{\mathbf{u}})) = \mathbf{F}\,, && \text{in } \Omega,\\
&\boldsymbol{\varepsilon}(\overline{\mathbf{u}}) = \tfrac{\nabla\overline{\mathbf{u}}+(\nabla\overline{\mathbf{u}})^T}{2}\,, && \text{in } \Omega,\\
&\overline{\mathbf{u}} = 0\,, && \text{on } \Gamma_D\,,\\
&\boldsymbol{\sigma}(x,\boldsymbol{\varepsilon}(\overline{\mathbf{u}}))\mathbf{n} = \mathbf{g}\,, && \text{on } \Gamma_N,
\end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^d$ is the body submitted to the force field $\mathbf{F}$, $\mathbf{n}$ is the unit normal to $\partial\Omega$ pointing outward Ω, Γ_D and Γ_N are subsets of $\partial\Omega$ on which the body is respectively fixed and submitted to traction, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the second-order stress and strain tensors, $\overline{\mathbf{u}} = (\bar{u}_i)_{i=1,\ldots,d} : \Omega \to \mathbf{R}^d$ describes local displacements and the gradient is written in columns: $\nabla\overline{\mathbf{u}} = (\partial_j \bar{u}_i)_{i,j=1,\ldots,d}$.

Although it does not include geometrically non-linear elasticity [33], Model (1.1) covers a number of meaningful linear and non-linear elasticity models:

- the linear elasticity model with $\boldsymbol{\sigma}(x,\boldsymbol{\varepsilon}(\mathbf{u})) = \mathbb{C}(x)\boldsymbol{\varepsilon}(\mathbf{u})$, in which $\mathbb{C}$ is a 4th order stiffness tensor,
- the damage models of [9] with $\boldsymbol{\sigma}(x,\boldsymbol{\varepsilon}(\mathbf{u})) = (1 - D(\boldsymbol{\varepsilon}(\mathbf{u})))\mathbb{C}(x)\boldsymbol{\varepsilon}(\mathbf{u})$, where the damage index D is a scalar function,

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- the non-linear Hencky–von Mises elasticity model [35] in which $\boldsymbol{\sigma}(x, \boldsymbol{\varepsilon}(\mathbf{u})) = \widetilde{\lambda}(\mathrm{dev}(\boldsymbol{\varepsilon}(\mathbf{u}))) \,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} + 2\widetilde{\mu}(\mathrm{dev}(\boldsymbol{\varepsilon}(\mathbf{u})))\boldsymbol{\varepsilon}(\mathbf{u})$, where $\widetilde{\lambda}$ and $\widetilde{\mu}$ are the non-linear Lamé coefficients, tr is the trace operator and $\mathrm{dev}(\boldsymbol{\tau}) = (\boldsymbol{\tau} - \frac{1}{d}\,\mathrm{tr}(\boldsymbol{\tau})\mathbf{I}) : (\boldsymbol{\tau} - \frac{1}{d}\,\mathrm{tr}(\boldsymbol{\tau})\mathbf{I})$ is the deviatoric operator.

Convergence of conforming Finite Element methods for the linear elasticity problem can be obtained by using standard techniques [6,10]. This convergence analysis covers the case when the solution does not possess a full H^2-regularity. However, convergence analysis of non-conforming Finite Element methods for linear elasticity is most often done using the full H^2-regularity of the solution [4, 6–8, 31]. The situation is even more restricted when it comes to non-linear models. Convergence analysis of numerical methods for non-linear elasticity has been mostly carried out for conforming approximations, and assuming the full H^2-regularity of the solution [5, 25]. Conforming approximations are formulated by simply writting the weak formulation of the problem, in which the infinite-dimensional spaces are replaced with finite-dimensional subspaces. A number of schemes for elasticity are however non-conforming, either because the spaces of approximate solutions are not subspaces of the spaces of continuous solutions, or because discrete projection/averaging operators are introduced in the weak formulation of the scheme. This is in particular the case for the stabilised nodal strain formulation [24, 30, 38] and the modified Hu-Washizu scheme [31], both designed to efficiently deal with the nearly incompressible limit. The convergence of some non-conforming methods for non-linear elasticity has been carried out in [37], using the minimisation formulation of non-linear elasticity (see also [36] for the convergence of Crouzeix-Raviart non-conforming finite elements for an elliptic minimisation problem). This analysis is however conducted under strong regularity assumptions on the solution (it must belong to H^{1+m} for some $m > d/2$) which prevents its generalisation to cases where $\boldsymbol{\sigma}$ is discontinuous with respect to x, a case which occurs in the modelling of composite materials for example. Moreover, [37] does not cover methods, such as the stabilised nodal strain formulation or the modified Hu-Washizu scheme mentioned above, that are of particular importance in the numerical approximation of elasticity. For such methods, convergence analyses therefore seem to be limited to linear models and H^2 solutions. In practice, the full H^2-regularity is not achieved due to the non-convexity of the domain, corner singularities, discontinuities of the stiffness tensor (e.g. in composite materials) and mixed boundary conditions.

The Gradient Scheme framework is a setting, based on a few discrete elements and properties, which has been recently developed to analyse numerical methods for a vast number of diffusion models: linear or non-linear, local or non-local, stationary or transient models, etc. [15–17, 20, 21]. This framework is also currently being extended to the linear poroelasticity equation, see [1]. It has been shown that a number of well-known methods for diffusion equations are Gradient Schemes [15, 16, 19, 22]: Galerkin methods (including conforming Finite Element methods), Mixed Finite Element methods, Hybrid Mimetic Mixed methods (including Hybrid Finite Volumes, Mimetic Finite Differences and Mixed Finite Volumes), Discrete Duality Finite Volume methods, etc. The Gradient Scheme framework provides a unified convergence analysis of conforming and non-conforming schemes, for linear and non-linear diffusion equations, under assumptions on the data and solution which are compatible with field applications (in particular, no restrictive regularity assumption is required).

The aim of this paper is to extend the Gradient Scheme framework to linear and non-linear elasticity models, thus showing that all the advantages of this analysis framework can be applied to conforming and non-conforming numerical methods developed for linear elasticity equations. Since Gradient Schemes are seamlessly applicable to linear and non-linear models, interpreting numerical methods as Gradient Schemes also allows us to adapt these methods to non-linear elasticity equations, and to establish their convergence for such models.

The paper is organised as follows. In the next section, we introduce the notion of Gradient Discretisations, used to define Gradient Schemes for (1.1). We also state the three properties, *consistency*, *limit-conformity* and *coercivity*, that a Gradient Discretisation must satisfy in order to lead to a stable and convergent numerical scheme. In Section 3.1, we first analyse the convergence of Gradient Schemes for linear elasticity equations, providing an error estimate under very weak regularity assumptions on the data and solution. We then carry out the convergence analysis for fully non-linear models, proving the convergence of the approximate solution under the same unrestrictive assumptions. Section 4 is devoted to the study of some examples of Gradient Scheme. We show in particular that several conforming and non-conforming schemes for elasticity equations, including methods developed to handle the nearly incompressible limit and acute bending, do fall in the framework of Gradient Schemes and that our convergence analysis – for both linear and non-linear models – therefore applies to them. Some conclusions of the paper are summarised in the final section.

2. Definition of Gradient Schemes for elasticity equations

Our general assumptions on the data are as follows.

$$\begin{aligned}&\Omega \text{ is a connected open subset of } \mathbf{R}^d\ (d\ge 1) \text{ with Lipschitz boundary,}\\&\Gamma_D \text{ and } \Gamma_N \text{ are disjoint subsets of } \partial\Omega \text{ such that } \partial\Omega=\Gamma_D\cup\Gamma_N,\\&\Gamma_D \text{ has a non-zero } (d-1)\text{-dimensional measure and } \Gamma_N \text{ is open in } \partial\Omega,\end{aligned}\tag{2.1}$$

$$\mathbf{F}\in\mathbf{L}^2(\Omega)\,,\quad \mathbf{g}\in\mathbf{L}^2(\Gamma_N)\tag{2.2}$$

(where $\mathbf{L}^2(X)=(L^2(X))^d$) and, denoting by $\mathcal{S}_{d\times d}$ the set of symmetric $d\times d$ tensors,

$$\begin{aligned}&\boldsymbol{\sigma}:(x,\boldsymbol{\tau})\in\Omega\times\mathcal{S}_{d\times d}\mapsto\boldsymbol{\sigma}(x,\boldsymbol{\tau})\in\mathcal{S}_{d\times d} \text{ is a Caratheodory}\\&\text{function (i.e. measurable w.r.t. } x \text{ and continuous w.r.t. } \boldsymbol{\tau}) \text{ and}\\&\exists\sigma^*,\sigma_*>0 \text{ such that, for a.e. } x\in\Omega\,,\ \forall\boldsymbol{\tau},\boldsymbol{\omega}\in\mathcal{S}_{d\times d}\,,\\&\quad|\boldsymbol{\sigma}(x,\boldsymbol{\tau})|\le\sigma^*|\boldsymbol{\tau}|+\sigma^* \qquad \text{(growth)},\\&\quad\boldsymbol{\sigma}(x,\boldsymbol{\tau}):\boldsymbol{\tau}\ge\sigma_*|\boldsymbol{\tau}|^2 \qquad \text{(coercivity)},\\&\quad(\boldsymbol{\sigma}(x,\boldsymbol{\tau})-\boldsymbol{\sigma}(x,\boldsymbol{\omega})):(\boldsymbol{\tau}-\boldsymbol{\omega})\ge 0 \qquad \text{(monotonicity)},\end{aligned}\tag{2.3}$$

where, for $\boldsymbol{\tau},\boldsymbol{\omega}\in\mathbf{R}^{d\times d}$, $\boldsymbol{\tau}:\boldsymbol{\omega}=\sum_{i,j=1}^d\boldsymbol{\tau}_{ij}\boldsymbol{\omega}_{ij}$ and $|\boldsymbol{\tau}|^2=\boldsymbol{\tau}:\boldsymbol{\tau}$. In the following, we also denote by $\cdot$ and $|\cdot|$ the Euclidean product and norm on $\mathbf{R}^d$.

Remark 2.1. *Note that the linear elasticity and the Hencky–von Mises models both satisfy these assumptions (see [2, Lemma 4.1] for a proof of the monotonicity of the Hencky–von Mises model). One can also see that the damage model* $\boldsymbol{\sigma}(x,\boldsymbol{\varepsilon}(\mathbf{u}))=(1-D(\boldsymbol{\varepsilon}(\mathbf{u})))\mathbb{C}(x)\boldsymbol{\varepsilon}(\mathbf{u})$ *satisfies* (2.3) *if* $1-D(\xi)=f(|\xi|)$ *where, for some* $0<\underline{d}\le\overline{d}$, f *is continuous* $[0,\infty)\to[\underline{d},\overline{d}]$ *and such that* $s\in[0,\infty)\to sf(s)$ *is non-decreasing. However, this does not include geometrically non-linear elasticity [33].*

Under these assumptions, and defining $\mathbf{H}^1(\Omega)=H^1(\Omega)^d$, $\gamma:\mathbf{H}^1(\Omega)\to\mathbf{L}^2(\partial\Omega)$ the trace operator and $\mathbf{H}^1_{\Gamma_D}(\Omega)=\{\mathbf{v}\in\mathbf{H}^1(\Omega)\ :\ \gamma(\mathbf{v})=0 \text{ on } \Gamma_D\}$, the weak formulation of (1.1) is

$$\begin{aligned}&\text{Find } \overline{\mathbf{u}}\in\mathbf{H}^1_{\Gamma_D}(\Omega) \text{ such that, for any } \mathbf{v}\in\mathbf{H}^1_{\Gamma_D}(\Omega),\\&\int_\Omega\boldsymbol{\sigma}(x,\boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x)):\boldsymbol{\varepsilon}(\mathbf{v})(x)\mathrm{d}x = \int_\Omega\mathbf{F}(x)\cdot\mathbf{v}(x)\mathrm{d}x\\&\qquad\qquad+\int_{\Gamma_N}\mathbf{g}(x)\cdot\gamma(\mathbf{v})(x)\mathrm{d}S(x)\end{aligned}\tag{2.4}$$

where $dS(x)$ denotes the $(d-1)$-dimensional integral over $\partial\Omega$.

Gradient Schemes for such equations are based on Gradient Discretisations, which consist in introducing a discrete space, gradient, trace and reconstructed function, and using those to approximate (2.4). The following definitions are adapted to elasticity equations, and to non-homogeneous mixed boundary conditions, from the theory developed in [16,21] for diffusion equations with homogeneous Dirichlet boundary conditions.

Definition 2.2 (Gradient Discretisation for elasticity equations)**.**
A Gradient Discretisation $\mathcal{D}$ for Problem (1.1) *is* $\mathcal{D} = (\mathbf{X}_{\mathcal{D},\Gamma_D}, \Pi_{\mathcal{D}}, \mathcal{T}_{\mathcal{D}}, \nabla_{\mathcal{D}})$*, where:*

(1) *the set of discrete unknowns $\mathbf{X}_{\mathcal{D},\Gamma_D}$ is a finite dimensional vector space on $\mathbf{R}$, whose definition includes the null trace condition on Γ_D,*
(2) *the linear mapping $\Pi_{\mathcal{D}} : \mathbf{X}_{\mathcal{D},\Gamma_D} \to \mathbf{L}^2(\Omega)$ is the reconstruction of the approximate function,*
(3) *the linear mapping $\mathcal{T}_{\mathcal{D}} : \mathbf{X}_{\mathcal{D},\Gamma_D} \to \mathbf{L}^2(\Gamma_N)$ is a discrete trace operator,*
(4) *the linear mapping $\nabla_{\mathcal{D}} : \mathbf{X}_{\mathcal{D},\Gamma_D} \to \mathbf{L}^2(\Omega)^d$ is the discrete gradient operator. It must be chosen such that $\| \cdot \|_{\mathcal{D}} := \|\nabla_{\mathcal{D}} \cdot \|_{\mathbf{L}^2(\Omega)^d}$ is a norm on $\mathbf{X}_{\mathcal{D},\Gamma_D}$.*

Once a Gradient Discretisation is available, the related Gradient Scheme consists in writing the weak formulation (2.4) with the continuous spaces and operators replaced with their discrete counterparts.

Definition 2.3 (Gradient Scheme for elasticity equations)**.**
If $\mathcal{D} = (\mathbf{X}_{\mathcal{D},\Gamma_D}, \Pi_{\mathcal{D}}, \mathcal{T}_{\mathcal{D}}, \nabla_{\mathcal{D}})$ is a Gradient Discretisation in the sense of Definition 2.2 then we define the related Gradient Scheme *for* (1.1) *by*

Find $\mathbf{u} \in \mathbf{X}_{\mathcal{D},\Gamma_D}$ *such that,* $\forall \mathbf{v} \in \mathbf{X}_{\mathcal{D},\Gamma_D}$,

$$\int_\Omega \boldsymbol{\sigma}(x, \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{u})(x)) : \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v})(x) \mathrm{d}x = \int_\Omega \mathbf{F}(x) \cdot \Pi_{\mathcal{D}}\mathbf{v}(x) \mathrm{d}x + \int_{\Gamma_N} \mathbf{g}(x) \cdot \mathcal{T}_{\mathcal{D}}(\mathbf{v})(x) \mathrm{d}S(x) \tag{2.5}$$

where $\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v}) = \frac{\nabla_{\mathcal{D}}\mathbf{v} + (\nabla_{\mathcal{D}}\mathbf{v})^T}{2}$.

The definitions of consistency, limit-conformity and compactness of Gradient Discretisations for Equation (1.1) are the same as for diffusion equations, taking into account the fact that functions are vector- or tensor-valued in the elasticity model.

The consistency of a sequence of Gradient Discretisations ensure that any function in the energy space can be approximated, along with its gradient, by discrete functions.

Definition 2.4 (Consistency)**.** *Let $\mathcal{D}$ be a Gradient Discretisation in the sense of Definition 2.2, and let $S_{\mathcal{D}} : \mathbf{H}^1_{\Gamma_D}(\Omega) \to [0, +\infty)$ be defined by*

$$\forall \boldsymbol{\varphi} \in \mathbf{H}^1_{\Gamma_D}(\Omega), \quad S_{\mathcal{D}}(\boldsymbol{\varphi}) = \min_{\mathbf{v} \in \mathbf{X}_{\mathcal{D},\Gamma_D}} \left\{ \|\Pi_{\mathcal{D}}\mathbf{v} - \boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)} + \|\nabla_{\mathcal{D}}\mathbf{v} - \nabla\boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)^d} \right\}. \tag{2.6}$$

A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of Gradient Discretisations is said to be ***consistent*** *if, for all $\boldsymbol{\varphi} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$, $S_{\mathcal{D}_m}(\boldsymbol{\varphi}) \to 0$ as $m \to \infty$.*

The limit-conformity of a sequence of Gradient Discretisations ensures that the dual of the discrete gradient behaves as an approximation of the divergence operator. We let

$$\mathbf{H}_{\mathrm{div}}(\Omega, \Gamma_N) = \{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)^d \,:\, \mathrm{div}\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)\,,\ \gamma_{\mathbf{n}}(\boldsymbol{\tau}) \in \mathbf{L}^2(\Gamma_N)\},$$

where $\gamma_{\mathbf{n}}(\boldsymbol{\tau})$ is the normal trace of $\boldsymbol{\tau}$. This normal trace is well defined in $H^{-1/2}(\partial\Omega)^d$ if $\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)^d$ and $\mathrm{div}(\boldsymbol{\tau}) \in \mathbf{L}^2(\Omega)$ ([1]). The meaning of "$\gamma_{\mathbf{n}}(\boldsymbol{\tau}) \in \mathbf{L}^2(\Gamma_N)$" is: there exists $\mathbf{h} \in \mathbf{L}^2(\Gamma_N)$ such that, for any $\boldsymbol{\varphi} \in H^{1/2}(\partial\Omega)^d$ with $\boldsymbol{\varphi} = 0$ on Γ_D,

$$\langle \gamma_{\mathbf{n}}(\boldsymbol{\tau}), \boldsymbol{\varphi}\rangle_{H^{-1/2}(\partial\Omega)^d, H^{1/2}(\partial\Omega)^d} = \int_{\Gamma_N} \mathbf{h}(x)\cdot\boldsymbol{\varphi}(x)\, dS(x). \tag{2.7}$$

Since Γ_N is open in $\partial\Omega$, the restrictions to Γ_N of functions $\boldsymbol{\varphi}$ as above form a dense set in $\mathbf{L}^2(\Gamma_N)$, and Relation (2.7) defines $\mathbf{h}$ uniquely; we can therefore use the notation $\gamma_{\mathbf{n}}(\boldsymbol{\tau})$ for $\mathbf{h}$.

Definition 2.5 (Limit-conformity)**.** *Let $\mathcal{D}$ be a Gradient Discretisation in the sense of Definition 2.2. We define $W_{\mathcal{D}}\colon \mathbf{H}_{\mathrm{div}}(\Omega,\Gamma_N) \to [0,+\infty)$ by*

$$\begin{aligned} &\forall \boldsymbol{\tau} \in \mathbf{H}_{\mathrm{div}}(\Omega,\Gamma_N)\,,\\ &W_{\mathcal{D}}(\boldsymbol{\tau}) = \max_{\substack{\mathbf{v}\in\mathbf{X}_{\mathcal{D},\Gamma_D}\\ \mathbf{v}\neq 0}} \frac{1}{\|\mathbf{v}\|_{\mathcal{D}}}\Bigg| \int_\Omega \big(\nabla_{\mathcal{D}}\mathbf{v}(x) : \boldsymbol{\tau}(x) + \Pi_{\mathcal{D}}\mathbf{v}(x)\cdot \mathrm{div}(\boldsymbol{\tau})(x)\big)\mathrm{d}x \\ &\qquad\qquad - \int_{\Gamma_N} \gamma_{\mathbf{n}}(\boldsymbol{\tau})(x)\cdot \mathcal{T}_{\mathcal{D}}(\mathbf{v})(x)\mathrm{d}S(x)\Bigg|. \end{aligned} \tag{2.8}$$

A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of Gradient Discretisations is said to be ***limit-conforming*** *if, for all $\boldsymbol{\tau} \in \mathbf{H}_{\mathrm{div}}(\Omega,\Gamma_N)$, $W_{\mathcal{D}_m}(\boldsymbol{\tau}) \to 0$ as $m\to\infty$.*

The definition of coercivity of Gradient Discretisations for the elasticity equation starts in the same way as for diffusion equations. However, since the natural energy estimate for elasticity equations is not on $\nabla\overline{\mathbf{u}}$ but on $\boldsymbol{\varepsilon}(\overline{\mathbf{u}})$, as in the continuous case we must add to it some discrete form of Körn's inequality.

Definition 2.6 (Coercivity)**.** *Let $\mathcal{D}$ be a Gradient Discretisation in the sense of Definition 2.2. We define $C_{\mathcal{D}}$ (maximum of the norms of the linear mappings $\Pi_{\mathcal{D}}$ and $\mathcal{T}_{\mathcal{D}}$) by*

$$C_{\mathcal{D}} = \max_{\mathbf{v}\in\mathbf{X}_{\mathcal{D},\Gamma_D}\setminus\{0\}} \left(\frac{\|\Pi_{\mathcal{D}}\mathbf{v}\|_{\mathbf{L}^2(\Omega)}}{\|\mathbf{v}\|_{\mathcal{D}}}, \frac{\|\mathcal{T}_{\mathcal{D}}\mathbf{v}\|_{\mathbf{L}^2(\Gamma_N)}}{\|\mathbf{v}\|_{\mathcal{D}}}\right). \tag{2.9}$$

and $K_{\mathcal{D}}$ (constant of the discrete Körn inequality) by

$$K_{\mathcal{D}} = \max_{\mathbf{v}\in\mathbf{X}_{\mathcal{D},\Gamma_D}\setminus\{0\}} \frac{\|\mathbf{v}\|_{\mathcal{D}}}{\|\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v})\|_{\mathbf{L}^2(\Omega)^d}}. \tag{2.10}$$

A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of Gradient Discretisations is said to be ***coercive*** *if there exists $C_P > 0$ such that $C_{\mathcal{D}_m} + K_{\mathcal{D}_m} \le C_P$ for all $m\in\mathbb{N}$.*

The definition of $C_{\mathcal{D}}$ gives the following discrete Poincaré's inequality:

$$\forall \mathbf{v}\in\mathbf{X}_{\mathcal{D},\Gamma_D}\ :\quad \|\Pi_{\mathcal{D}}\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \le C_{\mathcal{D}}\|\nabla_{\mathcal{D}}\mathbf{v}\|_{\mathbf{L}^2(\Omega)^d}. \tag{2.11}$$

Remark 2.7 (Non-homogeneous Dirichlet boundary conditions)**.** *Non-homogeneous Dirichlet boundary conditions $\overline{\mathbf{u}} = \mathbf{w}$ can also be considered in* (1.1) *and in the framework of Gradient Schemes, upon introducing an interpolation operator and modifying the definition of limit-conformity to take into account this interpolation operator. See [15] for diffusion equations.*

Remark 2.8. *Although it does not seem to relate to any elasticity model we know of, we could also handle a dependency of $\boldsymbol{\sigma}$ on $\overline{\mathbf{u}}$, i.e. $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x,\overline{\mathbf{u}},\boldsymbol{\varepsilon}(\overline{\mathbf{u}}))$, upon adding a compactness property of Gradient Discretisations (see [16] for the handling of such lower order terms in diffusion equations).*

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[1] The divergence of a tensor $\boldsymbol{\tau}$ is taken row by row, i.e. if $\boldsymbol{\tau} = (\boldsymbol{\tau}_{i,j})_{i,j=1,\ldots,d}$ then $\mathrm{div}(\boldsymbol{\tau}) = (\sum_{j=1}^d \partial_j \boldsymbol{\tau}_{i,j})_{i=1,\ldots,d}$. This definition is consistent with our definition of ∇ by column: $-\mathrm{div}$ is the formal dual operator of ∇.

3. Convergence analysis

3.1. **Linear case.** We assume here that the relationship between the strain and stress is linear, and thus given by a stiffness 4th order tensor:

$$
\text{(3.1)}\quad
\begin{array}{ll}
\text{There exists a measurable } \mathbb{C}:\Omega\to\mathbf{R}^{d^4} \text{ such that } \boldsymbol{\sigma}(x,\boldsymbol{\tau})=\mathbb{C}(x)\boldsymbol{\tau} \text{ and} & \\
\exists\sigma^*,\sigma_*>0 \text{ s.t., for a.e. } x\in\Omega\,,\ \forall\boldsymbol{\tau},\boldsymbol{\omega}\in\mathbf{R}^{d\times d}\,, & \\
\mathbb{C}(x)\boldsymbol{\tau}:\boldsymbol{\omega}=\boldsymbol{\tau}:\mathbb{C}(x)\boldsymbol{\omega} \text{ and } (\mathbb{C}(x)\boldsymbol{\tau})^T=\mathbb{C}(x)\boldsymbol{\tau}^T\,, & \text{(symmetry)},\\
|\mathbb{C}(x)|\le\sigma^* & \text{(bound)},\\
\mathbb{C}(x)\boldsymbol{\tau}:\boldsymbol{\tau}\ge\sigma_*|\boldsymbol{\tau}|^2 & \text{(coercivity)}.
\end{array}
$$

Remark 3.1. *These assumptions imply* (2.3) *and cover the classical linear elasticity model* $\boldsymbol{\sigma}(x,\boldsymbol{\varepsilon}(\mathbf{u}))=\lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}+2\mu\boldsymbol{\varepsilon}(\mathbf{u})$ *(i.e. the Hencky–von Mises model with Lamé coefficients not depending on* $\mathbf{u}$*).*

In this linear setting, the Gradient Scheme (2.5) takes the form

$$
\text{(3.2)}\quad
\begin{array}{l}
\text{Find } \mathbf{u}\in\mathbf{X}_{\mathcal{D},\Gamma_D} \text{ such that, } \forall\mathbf{v}\in\mathbf{X}_{\mathcal{D},\Gamma_D},\\
\displaystyle\int_\Omega \mathbb{C}(x)\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{u})(x):\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v})(x)\mathrm{d}x = \int_\Omega \mathbf{F}(x)\cdot\Pi_{\mathcal{D}}\mathbf{v}(x)\mathrm{d}x\\
\displaystyle\qquad\qquad\qquad\qquad + \int_{\Gamma_N}\mathbf{g}(x)\cdot\mathcal{T}_{\mathcal{D}}(\mathbf{v})(x)\mathrm{d}S(x).
\end{array}
$$

The proof of the following error estimate is an adaptation of similar estimates done in [21] for linear diffusion equations with homogeneous Dirichlet boundary conditions.

Theorem 3.2 (Error estimate of Gradient Schemes for linear elasticity)**.** *We assume that* (2.1), (2.2) *and* (3.1) *hold and we let* $\overline{\mathbf{u}}$ *be the solution to* (2.4) *(note that* $\boldsymbol{\sigma}(x,\boldsymbol{\varepsilon}(\overline{\mathbf{u}}))=\mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}})\in\mathbf{H}_{\mathrm{div}}(\Omega,\Gamma_N)$ *since* $\mathbf{F}\in\mathbf{L}^2(\Omega)$ *and* $\mathbf{g}\in\mathbf{L}^2(\Gamma_N)$*).*

If $\mathcal{D}$ *is a Gradient Discretization in the sense of Definition 2.2 then the Gradient Scheme* (3.2) *has a unique solution* $\mathbf{u}_{\mathcal{D}}$ *and it satisfies:*

$$
\text{(3.3)}\qquad \|\nabla\overline{\mathbf{u}}-\nabla_{\mathcal{D}}\mathbf{u}_{\mathcal{D}}\|_{\mathbf{L}^2(\Omega)^d}\le\frac{K_{\mathcal{D}}^2}{\sigma_*}W_{\mathcal{D}}(\mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}}))+\left(\frac{K_{\mathcal{D}}^2\sigma^*}{\sigma_*}+1\right)S_{\mathcal{D}}(\overline{\mathbf{u}}),
$$

$$
\text{(3.4)}\qquad \|\overline{\mathbf{u}}-\Pi_{\mathcal{D}}\mathbf{u}_{\mathcal{D}}\|_{\mathbf{L}^2(\Omega)}\le\frac{C_{\mathcal{D}}K_{\mathcal{D}}^2}{\sigma_*}W_{\mathcal{D}}(\mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}}))+\left(\frac{C_{\mathcal{D}}K_{\mathcal{D}}^2\sigma^*}{\sigma_*}+1\right)S_{\mathcal{D}}(\overline{\mathbf{u}}),
$$

where $S_{\mathcal{D}}$*,* $W_{\mathcal{D}}$*,* $C_{\mathcal{D}}$ *and* $K_{\mathcal{D}}$ *are defined by* (2.6)*,* (2.8)*,* (2.9) *and* (2.10)*.*

Proof Let us first notice that if we prove (3.3) for any solution $\mathbf{u}_{\mathcal{D}}$ to the Gradient Scheme (3.2), then the existence and uniqueness of this solution follows. Indeed, (3.2) defines a square linear system and if $\mathbf{F}=0$ and $\mathbf{g}=0$ (meaning that $\overline{\mathbf{u}}=0$) then (3.3) shows that the only solution to this system is 0, since $||\nabla_{\mathcal{D}}\cdot||_{\mathbf{L}^2(\Omega)^d}$ is a norm on $\mathbf{X}_{\mathcal{D},\Gamma_D}$. Hence, this system is invertible and (3.2) has a solution for any right-hand side functions $\mathbf{F}$ and $\mathbf{g}$ satisfying (2.2).

Let us now prove the error estimates. Since $\mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}}) \in \mathbf{H}_{\rm div}(\Omega, \Gamma_N)$, the definition of $W_{\mathcal{D}}$ gives, for any $\mathbf{v} \in \mathbf{X}_{\mathcal{D},\Gamma_D}$,

$$
\begin{aligned}
&||\nabla_{\mathcal{D}}\mathbf{v}||_{\mathbf{L}^2(\Omega)^d} W_{\mathcal{D}}(\mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}}))\\
&\qquad \geq \left| \int_\Omega \big(\nabla_{\mathcal{D}}\mathbf{v}(x) : \mathbb{C}(x)\boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x) + \Pi_{\mathcal{D}}\mathbf{v}(x)\cdot \mathrm{div}(\mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}}))(x)\mathrm{d}x \right.\\
&\qquad\qquad \left. - \int_{\Gamma_N} \gamma_{\mathbf{n}}(\mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}}))(x)\cdot \mathcal{T}_{\mathcal{D}}(\mathbf{v})(x)\mathrm{d}S(x) \right|\\
&\qquad \geq \left| \int_\Omega \big(\nabla_{\mathcal{D}}\mathbf{v}(x) : \mathbb{C}(x)\boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x) - \Pi_{\mathcal{D}}\mathbf{v}(x)\cdot \mathbf{F}(x)\big)\mathrm{d}x \right.\\
&\qquad\qquad \left. - \int_{\Gamma_N} \mathbf{g}(x)\cdot \mathcal{T}_{\mathcal{D}}(\mathbf{v})(x)\mathrm{d}S(x) \right|.
\end{aligned} \tag{3.5}
$$

By symmetry of $\mathbb{C}$ we have $\mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}}) : \nabla_{\mathcal{D}}\mathbf{v} = \mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}}) : \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v})$ and (3.5) therefore gives, since $\mathbf{u}_{\mathcal{D}}$ is a solution to (3.2),

$$
\begin{aligned}
&||\nabla_{\mathcal{D}}\mathbf{v}||_{\mathbf{L}^2(\Omega)^d} W_{\mathcal{D}}(\mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}}))\\
&\qquad \geq \left| \int_\Omega \mathbb{C}(x)\boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x) : \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v})(x) - \mathbb{C}(x)\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{u}_{\mathcal{D}}) : \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v})(x)\mathrm{d}x \right|.
\end{aligned} \tag{3.6}
$$

Defining, for all $\boldsymbol{\varphi} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$,

$$
P_{\mathcal{D}}\boldsymbol{\varphi} = \underset{\mathbf{w}\in\mathbf{X}_{\mathcal{D},\Gamma_D}}{\mathrm{argmin}} \left\{ \|\Pi_{\mathcal{D}}\mathbf{w} - \boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)} + \|\nabla_{\mathcal{D}}\mathbf{w} - \nabla\boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)^d} \right\} \tag{3.7}
$$

and recalling the definition (2.6) of $S_{\mathcal{D}}$, we have

$$
||\boldsymbol{\varepsilon}(\overline{\mathbf{u}}) - \boldsymbol{\varepsilon}_{\mathcal{D}}(P_{\mathcal{D}}\overline{\mathbf{u}})||_{\mathbf{L}^2(\Omega)^d} \leq ||\nabla\overline{\mathbf{u}} - \nabla_{\mathcal{D}}(P_{\mathcal{D}}\overline{\mathbf{u}})||_{\mathbf{L}^2(\Omega)^d} \leq S_{\mathcal{D}}(\overline{\mathbf{u}}). \tag{3.8}
$$

Using the bound of $\mathbb{C}$ in (3.1) and Estimate (3.6), we deduce

$$
\begin{aligned}
&\left| \int_\Omega \mathbb{C}(x)\boldsymbol{\varepsilon}_{\mathcal{D}}(P_{\mathcal{D}}\overline{\mathbf{u}} - \mathbf{u}_{\mathcal{D}})(x) : \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v})(x)\mathrm{d}x \right|\\
&\qquad \leq \left| \int_\Omega \mathbb{C}(x)(\boldsymbol{\varepsilon}_{\mathcal{D}}(P_{\mathcal{D}}\overline{\mathbf{u}}) - \boldsymbol{\varepsilon}(\overline{\mathbf{u}}))(x) : \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v})(x)\mathrm{d}x \right|\\
&\qquad\quad + \left| \int_\Omega \mathbb{C}(x)(\boldsymbol{\varepsilon}(\overline{\mathbf{u}}) - \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{u}_{\mathcal{D}}))(x) : \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v})(x)\mathrm{d}x \right|\\
&\qquad \leq \sigma^* S_{\mathcal{D}}(\overline{\mathbf{u}})||\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v})||_{\mathbf{L}^2(\Omega)^d} + ||\nabla_{\mathcal{D}}\mathbf{v}||_{\mathbf{L}^2(\Omega)^d} W_{\mathcal{D}}(\mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}}))\\
&\qquad \leq ||\nabla_{\mathcal{D}}\mathbf{v}||_{\mathbf{L}^2(\Omega)^d}\big(W_{\mathcal{D}}(\mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}})) + \sigma^* S_{\mathcal{D}}(\overline{\mathbf{u}})\big).
\end{aligned} \tag{3.9}
$$

Plugging $\mathbf{v} = P_{\mathcal{D}}\overline{\mathbf{u}} - \mathbf{u}_{\mathcal{D}} \in \mathbf{X}_{\mathcal{D},\Gamma_D}$ in (3.9) and using the coercivity of $\mathbb{C}$ gives

$$
\begin{aligned}
&\sigma_*||\boldsymbol{\varepsilon}_{\mathcal{D}}(P_{\mathcal{D}}\overline{\mathbf{u}} - \mathbf{u}_{\mathcal{D}})||^2_{\mathbf{L}^2(\Omega)^d}\\
&\qquad \leq ||\nabla_{\mathcal{D}}(P_{\mathcal{D}}\overline{\mathbf{u}} - \mathbf{u}_{\mathcal{D}})||_{\mathbf{L}^2(\Omega)^d}\big(W_{\mathcal{D}}(\mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}})) + \sigma^* S_{\mathcal{D}}(\overline{\mathbf{u}})\big).
\end{aligned} \tag{3.10}
$$

By definition (2.10) of $K_{\mathcal{D}}$, we have

$$
||\nabla_{\mathcal{D}}(P_{\mathcal{D}}\overline{\mathbf{u}} - \mathbf{u}_{\mathcal{D}})||_{\mathbf{L}^2(\Omega)^d} \leq K_{\mathcal{D}}||\boldsymbol{\varepsilon}_{\mathcal{D}}(P_{\mathcal{D}}\overline{\mathbf{u}} - \mathbf{u}_{\mathcal{D}})||_{\mathbf{L}^2(\Omega)^d}
$$

and (3.10) thus leads to

$$
||\nabla_{\mathcal{D}}(P_{\mathcal{D}}\overline{\mathbf{u}}) - \nabla_{\mathcal{D}}\mathbf{u}_{\mathcal{D}}||_{\mathbf{L}^2(\Omega)^d} \leq \frac{K_{\mathcal{D}}^2}{\sigma_*}\big(W_{\mathcal{D}}(\mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}})) + \sigma^* S_{\mathcal{D}}(\overline{\mathbf{u}})\big) \tag{3.11}
$$

and the proof of (3.3) is concluded thanks to (3.8). The Poincaré inequality (2.11) and (3.11) also give

$$
||\Pi_{\mathcal{D}}(P_{\mathcal{D}}\overline{\mathbf{u}}) - \Pi_{\mathcal{D}}\mathbf{u}_{\mathcal{D}}||_{\mathbf{L}^2(\Omega)} \leq \frac{C_{\mathcal{D}}K_{\mathcal{D}}^2}{\sigma_*}\big(W_{\mathcal{D}}(\mathbb{C}\boldsymbol{\varepsilon}(\overline{\mathbf{u}})) + \sigma^* S_{\mathcal{D}}(\overline{\mathbf{u}})\big),
$$

and the estimate $||\Pi_{\mathcal{D}}(P_{\mathcal{D}}\overline{\mathbf{u}}) - \overline{\mathbf{u}}||_{\mathbf{L}^2(\Omega)} \leq S_{\mathcal{D}}(\overline{\mathbf{u}})$ concludes the proof of (3.4). □

The following corollary is a straightforward consequence of Theorem 3.2.

Corollary 3.3 (Convergence of Gradient Schemes for linear elasticity)**.** *We assume that* (2.1), (2.2) *and* (3.1) *hold. We denote by* $\overline{\mathbf{u}}$ *the solution to* (2.4).

If $(\mathcal{D}_m)_{m\in\mathbb{N}}$ *is a sequence of Gradient Discretizations in the sense of Definition 2.2, which is consistent (Definition 2.4), limit-conforming (Definition 2.5) and coercive (Definition 2.6), and if* $\mathbf{u}_m \in \mathbf{X}_{\mathcal{D}_m,\Gamma_D}$ *is the solution to the Gradient Scheme* (3.2) *with* $\mathcal{D} = \mathcal{D}_m$*, then, as* $m \to \infty$*,* $\Pi_{\mathcal{D}_m}\mathbf{u}_m \to \overline{\mathbf{u}}$ *strongly in* $\mathbf{L}^2(\Omega)$ *and* $\nabla_{\mathcal{D}_m}\mathbf{u}_m \to \nabla\overline{\mathbf{u}}$ *strongly in* $\mathbf{L}^2(\Omega)^d$.

Remark 3.4. *This result is valid under no additional regularity assumption on the data or* $\overline{\mathbf{u}}$*. It holds in particular if* $\partial\Omega$ *has singularities or if* $\mathbb{C}$ *is discontinuous with respect to* x*, which corresponds to a body made of several different materials with interfaces (see e.g. [29]).*

However, for most Gradient Schemes (and under reasonable assumptions on the mesh/discretisation), there exists $C > 0$ *not depending on* $\mathcal{D}$ *such that*

$$\forall \boldsymbol{\varphi} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega)\,, \qquad S_{\mathcal{D}}(\boldsymbol{\varphi}) \le C h_{\mathcal{D}} ||\boldsymbol{\varphi}||_{\mathbf{H}^2(\Omega)}\,,$$
$$\forall \boldsymbol{\tau} \in \mathbf{H}^1(\Omega)^d\,, \qquad W_{\mathcal{D}}(\boldsymbol{\tau}) \le C h_{\mathcal{D}} ||\boldsymbol{\tau}||_{\mathbf{H}^1(\Omega)^d}\,,$$

where $h_{\mathcal{D}}$ *measures the scheme's precision (e.g. some mesh size). For such Gradient Schemes, if* $\mathbf{u} \in \mathbf{H}^2(\Omega)$ *and* $\mathbb{C}$ *is Lipschitz continuous then Theorem 3.2 provides* $\mathcal{O}(h_{\mathcal{D}})$ *error estimate for the approximation of* $\overline{\mathbf{u}}$ *and its gradient. We note that the solution is* H^2*-regular when we have a pure Dirichlet problem on a convex polygonal or polyhedral domain [7, 28].*

3.2. **Non-linear case.** In the non-linear case, error estimates cannot be provided in general since Gradient Schemes are not necessarily conforming methods. However, their convergence can still be established without additional regularity assumptions on the data.

Theorem 3.5 (Convergence of Gradient Schemes for non-linear elasticity)**.** *Assume that* (2.1), (2.2) *and* (2.3) *hold and let* $(\mathcal{D}_m)_{m\in\mathbb{N}}$ *be a sequence of Gradient Discretizations in the sense of Definition 2.2, which is consistent (Definition 2.4), limit-conforming (Definition 2.5) and coercive (Definition 2.6).*

Then, for any $m \in \mathbb{N}$ *there exists at least one solution* $\mathbf{u}_m \in \mathbf{X}_{\mathcal{D}_m,\Gamma_D}$ *to the Gradient Scheme* (2.5) *with* $\mathcal{D} = \mathcal{D}_m$ *and, up to a subsequence, as* $m \to \infty$*,* $\Pi_{\mathcal{D}_m}\mathbf{u}_m$ *converges weakly in* $\mathbf{L}^2(\Omega)$ *to some* $\overline{\mathbf{u}}$ *solution of* (2.4) *and* $\nabla_{\mathcal{D}_m}\mathbf{u}_m$ *converges weakly in* $\mathbf{L}^2(\Omega)^d$ *to* $\nabla\overline{\mathbf{u}}$.

Moreover, if we assume that $\boldsymbol{\sigma}$ *is strictly monotone in the following sense:*

(3.12) *For a.e.* $x \in \Omega$*, for all* $\boldsymbol{\tau} \neq \boldsymbol{\omega}$ *in* $\mathcal{S}_{d\times d}$, $(\boldsymbol{\sigma}(x,\boldsymbol{\tau}) - \boldsymbol{\sigma}(x,\boldsymbol{\omega})) : (\boldsymbol{\tau} - \boldsymbol{\omega}) > 0$

then, along the same subsequence, $\Pi_{\mathcal{D}_m}\mathbf{u}_m \to \overline{\mathbf{u}}$ *strongly in* $\mathbf{L}^2(\Omega)$ *and* $\nabla_{\mathcal{D}_m}\mathbf{u}_m \to \nabla\overline{\mathbf{u}}$ *strongly in* $\mathbf{L}^2(\Omega)^d$.

Remark 3.6. *If the sequence of Gradient Discretisations* $(\mathcal{D}_m)_{m\in\mathbb{N}}$ *is* compact *as defined in [16], then the convergence of* $\Pi_{\mathcal{D}_m}\mathbf{u}_m$ *is strong even if the strict monotonicity* (3.12) *is not satisfied.*

Remark 3.7. *Should the solution to* (2.4) *be unique, classical arguments also show that the convergences of* $(\mathbf{u}_m)_{m\in\mathbb{N}}$ *in the senses described in Theorem 3.5 hold for the whole sequence, not only for a subsequence.*

Remark 3.8. *We do not need to assume the existence of a solution to the non-linear elasticity model* (2.4)*. The technique of convergence analysis we use establishes in fact this existence.*

Remark 3.9. *The strict monotonicity assumption* (3.12) *is satisfied by the Hencky–von Mises model (see [2, Lemma 4.1]), and by the damage model when the function f defined in Remark 2.1 is such that $s \in [0,\infty) \to sf(s)$ is (strictly) increasing.*

Note that the strict monotony is weaker than the strong monotony, which is used to prove error estimates on a conforming method in [2].

Proof The proof follows the techniques used in [16] for the non-linear elliptic problem with homogeneous Dirichlet boundary conditions. We adapt those techniques to deal with the non-linear elasticity models with mixed non-homogeneous boundary conditions. In the following steps, we sometimes drop the index m in $\mathcal{D}_m$ to simplify the notations.

Step 1: A priori estimates and existence of a solution to the scheme.

Let us take a scalar product $\langle\cdot,\cdot\rangle$ on $\mathbf{X}_{\mathcal{D},\Gamma_D}$, with associated norm $N(\cdot)$, and let us define $T : \mathbf{X}_{\mathcal{D},\Gamma_D} \to \mathbf{X}_{\mathcal{D},\Gamma_D}$ and $L \in \mathbf{X}_{\mathcal{D},\Gamma_D}$ by: for all $\mathbf{w},\mathbf{v} \in \mathbf{X}_{\mathcal{D},\Gamma_D}$,

$$\langle T(\mathbf{w}),\mathbf{v}\rangle = \int_\Omega \boldsymbol{\sigma}(x,\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{w})(x)) : \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v})(x)\mathrm{d}x$$

and

$$\langle L,\mathbf{v}\rangle = \int_\Omega \mathbf{F}(x)\cdot \Pi_{\mathcal{D}}\mathbf{v}(x)\mathrm{d}x + \int_{\Gamma_N} \mathbf{g}(x)\cdot \mathcal{T}_{\mathcal{D}}(\mathbf{v})(x)\mathrm{d}S(x).$$

Then Assumption (2.3) ensures that T is continuous and that

$$\langle T(\mathbf{w}),\mathbf{w}\rangle \geq \sigma_* ||\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{w})||^2_{\mathbf{L}^2(\Omega)^d} \geq \sigma_* K_{\mathcal{D}}^{-2}||\mathbf{w}||^2_{\mathcal{D}}. \tag{3.13}$$

Since all norms are equivalent on $\mathbf{X}_{\mathcal{D},\Gamma_D}$, we also have $||\mathbf{w}||_{\mathcal{D}} \geq m_{\mathcal{D}}N(\mathbf{w})$ for some $m_{\mathcal{D}} > 0$ and this shows that $\lim_{N(\mathbf{w})\to\infty} \frac{\langle T(\mathbf{w}),\mathbf{w}\rangle}{N(\mathbf{w})} = +\infty$. By [12, Theorem 3.3 (p.19)] or [32, Theorem 1], we see that T is onto and therefore that there exists $\mathbf{u} \in \mathbf{X}_{\mathcal{D},\Gamma_D}$ such that $T(\mathbf{u}) = L$, which precisely states that $\mathbf{u}$ is a solution to (2.5).

From (3.13) and the definition (2.9) of $C_{\mathcal{D}}$, we also deduce that $\mathbf{u}$ satisfies

$$\begin{aligned} ||\mathbf{u}||^2_{\mathcal{D}} &\leq \frac{K^2_{\mathcal{D}}}{\sigma_*}\langle T(\mathbf{u}),\mathbf{u}\rangle = \frac{K^2_{\mathcal{D}}}{\sigma_*}\langle L,\mathbf{u}\rangle \\ &\leq \frac{K^2_{\mathcal{D}}}{\sigma_*}||\mathbf{F}||_{\mathbf{L}^2(\Omega)}||\Pi_{\mathcal{D}}\mathbf{u}||_{\mathbf{L}^2(\Omega)} + \frac{K^2_{\mathcal{D}}}{\sigma_*}||\mathbf{g}||_{\mathbf{L}^2(\Gamma_N)}||\mathcal{T}_{\mathcal{D}}\mathbf{u}||_{\mathbf{L}^2(\Gamma_N)} \\ &\qquad\leq \left(\frac{C_{\mathcal{D}}K^2_{\mathcal{D}}}{\sigma_*}||\mathbf{F}||_{\mathbf{L}^2(\Omega)} + \frac{C_{\mathcal{D}}K^2_{\mathcal{D}}}{\sigma_*}||\mathbf{g}||_{\mathbf{L}^2(\Gamma_N)}\right)||\mathbf{u}||_{\mathcal{D}}, \end{aligned}$$

that is to say

$$||\mathbf{u}||_{\mathcal{D}} \leq \frac{C_{\mathcal{D}}K^2_{\mathcal{D}}}{\sigma_*}||\mathbf{F}||_{\mathbf{L}^2(\Omega)} + \frac{C_{\mathcal{D}}K^2_{\mathcal{D}}}{\sigma_*}||\mathbf{g}||_{\mathbf{L}^2(\Gamma_N)}. \tag{3.14}$$

Step 2: Weak convergences.

By Estimate (3.14), $(||\mathbf{u}_m||_{\mathcal{D}_m})_{m\in\mathbb{N}}$ is bounded and Lemma 3.11 below therefore shows that there exists $\overline{\mathbf{u}} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ such that, up to a subsequence,

$$\begin{aligned} &\Pi_{\mathcal{D}_m}\mathbf{u}_m \to \overline{\mathbf{u}} \text{ weakly in } \mathbf{L}^2(\Omega)\,, \\ &\nabla_{\mathcal{D}_m}\mathbf{u}_m \to \nabla\overline{\mathbf{u}} \text{ weakly in } \mathbf{L}^2(\Omega)^d \text{ and} \\ &\mathcal{T}_{\mathcal{D}_m}\mathbf{u}_m \to \gamma(\overline{\mathbf{u}}) \text{ weakly in } \mathbf{L}^2(\Gamma_N). \end{aligned} \tag{3.15}$$

Let us now prove that $\overline{\mathbf{u}}$ is a solution to (2.4). Assumptions (2.3) and the bound on $\nabla_{\mathcal{D}_m}\mathbf{u}_m$ shows that $(\boldsymbol{\sigma}(\cdot,\boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m))_{m\in\mathbb{N}}$ is symmetric-valued and bounded in $\mathbf{L}^2(\Omega)^d$. There exists therefore a symmetric-valued $\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)^d$ such that, up to a subsequence,

$$\boldsymbol{\sigma}(\cdot,\boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m)) \to \boldsymbol{\tau} \text{ weakly in } \mathbf{L}^2(\Omega)^d. \tag{3.16}$$

Let $\boldsymbol{\varphi} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$. Then $P_{\mathcal{D}_m}\boldsymbol{\varphi}$ defined by (3.7) belongs to $\mathbf{X}_{\mathcal{D}_m,\Gamma_D}$ and, by consistency of $(\mathcal{D}_m)_{m\in\mathbb{N}}$, $\Pi_{\mathcal{D}_m}(P_{\mathcal{D}_m}\boldsymbol{\varphi}) \to \boldsymbol{\varphi}$ strongly in $\mathbf{L}^2(\Omega)$ and $\nabla_{\mathcal{D}_m}(P_{\mathcal{D}_m}\boldsymbol{\varphi}) \to \nabla\boldsymbol{\varphi}$ strongly in $\mathbf{L}^2(\Omega)^d$. By Lemma 3.11, we also deduce that $\mathcal{T}_{\mathcal{D}_m}(P_{\mathcal{D}_m}\boldsymbol{\varphi}) \to \gamma(\boldsymbol{\varphi})$ weakly in $\mathbf{L}^2(\Gamma_N)$. The convergence (3.16) then allows to pass to the limit in (2.5) with $\mathbf{v} = P_{\mathcal{D}_m}\boldsymbol{\varphi}$ as a test function and we obtain

$$\int_\Omega \boldsymbol{\tau}(x) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})(x)\mathrm{d}x = \int_\Omega \mathbf{F}(x)\cdot\boldsymbol{\varphi}(x)\mathrm{d}x + \int_{\Gamma_N} \mathbf{g}(x)\cdot\gamma(\boldsymbol{\varphi})(x)\mathrm{d}S(x). \tag{3.17}$$

We now use the monotonicity assumption on $\boldsymbol{\sigma}$ and Minty's trick [32, 34] to prove that $\boldsymbol{\tau} = \boldsymbol{\sigma}(x, \boldsymbol{\varepsilon}(\overline{\mathbf{u}}))$. We first notice that, plugging $\mathbf{v} = \mathbf{u}_m$ in (2.5) and using (3.15) and (3.17),

$$\begin{aligned}\int_\Omega &\boldsymbol{\sigma}(x, \boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m)(x)) : \boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m)(x)\mathrm{d}x \\ &= \int_\Omega \mathbf{F}(x)\cdot\Pi_{\mathcal{D}_m}\mathbf{u}_m(x)\mathrm{d}x + \int_{\Gamma_N}\mathbf{g}(x)\cdot\mathcal{T}_{\mathcal{D}_m}\mathbf{u}_m(x)\mathrm{d}S(x) \\ &\longrightarrow \int_\Omega \mathbf{F}(x)\cdot\overline{\mathbf{u}}(x)\mathrm{d}x + \int_{\Gamma_N}\mathbf{g}(x)\cdot\gamma(\overline{\mathbf{u}})(x)\mathrm{d}S(x) = \int_\Omega \boldsymbol{\tau}(x) : \boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x)\mathrm{d}x.\end{aligned} \tag{3.18}$$

Let us now take any symmetric-valued $\boldsymbol{\omega} \in \mathbf{L}^2(\Omega)^d$. The monotonicity of $\boldsymbol{\sigma}$ shows that

$$A_m := \int_\Omega \left[\boldsymbol{\sigma}(x, \boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m)(x)) - \boldsymbol{\sigma}(x, \boldsymbol{\omega}(x))\right] : \left[\boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m)(x) - \boldsymbol{\omega}(x)\right]\mathrm{d}x \ge 0.$$

After developing A_m, we can use (3.15), (3.16) and (3.18) to pass to the limit and we find

$$\lim_{m\to\infty} A_m = \int_\Omega \left[\boldsymbol{\tau}(x) - \boldsymbol{\sigma}(x, \boldsymbol{\omega}(x))\right] : \left[\boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x) - \boldsymbol{\omega}(x)\right]\mathrm{d}x \ge 0. \tag{3.19}$$

The Minty trick then concludes the proof. Applying this inequality to $\boldsymbol{\omega} = \boldsymbol{\varepsilon}(\overline{\mathbf{u}}) + \alpha\boldsymbol{\Delta}$ for some symmetric-valued $\boldsymbol{\Delta} \in \mathbf{L}^2(\Omega)^d$, dividing by α and letting $\alpha \to 0^\pm$ (thanks to Assumption (2.3)), we obtain

$$\int_\Omega \left[\boldsymbol{\tau}(x) - \boldsymbol{\sigma}(x, \boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x))\right] : \boldsymbol{\Delta}(x)\mathrm{d}x = 0,$$

which proves, with $\boldsymbol{\Delta} = \boldsymbol{\tau} - \boldsymbol{\sigma}(\cdot, \boldsymbol{\varepsilon}(\overline{\mathbf{u}}))$, that

$$\boldsymbol{\tau} = \boldsymbol{\sigma}(\cdot, \boldsymbol{\varepsilon}(\overline{\mathbf{u}})). \tag{3.20}$$

Together with (3.17) this shows that $\overline{\mathbf{u}}$ satisfies (2.4).

Step 3: Strong convergences under strict monotonicity.

We now assume that (3.12) holds and we first prove the strong convergence of the strain tensors. We define

$$f_m = \left[\boldsymbol{\sigma}(\cdot, \boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m)) - \boldsymbol{\sigma}(\cdot, \boldsymbol{\varepsilon}(\overline{\mathbf{u}}))\right] : \left[\boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m) - \boldsymbol{\varepsilon}(\overline{\mathbf{u}})\right].$$

The function f_m is non-negative and, by (3.19) with $\boldsymbol{\omega} = \boldsymbol{\varepsilon}(\overline{\mathbf{u}})$ and the identity (3.20), we see that $\lim_{m\to\infty}\int_\Omega f_m(x)\mathrm{d}x = 0$. $(f_m)_{m\in\mathbb{N}}$ thus converges to 0 in $L^1(\Omega)$, and therefore also a.e. on Ω up to a subsequence.

Let us take $x \in \Omega$ such that the above mentioned convergence hold at x. From the coercivity and growth of $\boldsymbol{\sigma}$, developing the products in $f_m(x)$ gives

$$\begin{aligned}f_m(x) \ge \sigma_*|\boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m)(x)|^2 - 2\sigma^*|\boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m)(x)|\,|\boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x)| \\ - |\sigma(x, \boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x)|\,|\boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x)|.\end{aligned}$$

Since the right-hand side is quadratic in $|\boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m)(x)|$ and $(f_m(x))_{m\in\mathbb{N}}$ is bounded, we deduce that the sequence $(\boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m)(x))_{m\in\mathbb{N}}$ is bounded. If $\mathbf{L}_x$ is one of its adherence values then, by passing to the limit in the definition of $f_m(x)$, we see that

$$0 = \big[\boldsymbol{\sigma}(x, \mathbf{L}_x) - \boldsymbol{\sigma}(x, \boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x))\big] : \big[\mathbf{L}_x - \boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x)\big].$$

By (3.12), this forces $\mathbf{L}_x = \boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x)$. The bounded sequence $(\boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m)(x))_{m\in\mathbb{N}}$ only has $\boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x)$ as adherence value and therefore converges in whole to this value. We have therefore established that $\boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m) \to \boldsymbol{\varepsilon}(\mathbf{u})$ a.e. on Ω.

Using then (3.18) and (3.20) and defining

$$F_m = \boldsymbol{\sigma}(\cdot, \boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m)) : \boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m) \geq 0,$$

we see that

$$\lim_{m\to\infty} \int_\Omega F_m(x)\mathrm{d}x = \int_\Omega \boldsymbol{\sigma}(x, \boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x)) : \boldsymbol{\varepsilon}(\overline{\mathbf{u}})(x)\mathrm{d}x.$$

But since $F_m \to \boldsymbol{\sigma}(\cdot, \boldsymbol{\varepsilon}(\overline{\mathbf{u}})) : \boldsymbol{\varepsilon}(\overline{\mathbf{u}})$ a.e. on Ω and is non-negative, we can apply Lemma 3.12 below to deduce that $(F_m)_{m\in\mathbb{N}}$ converges in $L^1(\Omega)$. This sequence is therefore equi-integrable in $L^1(\Omega)$ and, by the coercivity property of $\boldsymbol{\sigma}$, this proves that $(\boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m))_{m\in\mathbb{N}}$ is equi-integrable in $\mathbf{L}^2(\Omega)^d$. As this sequence converges a.e. on Ω to $\boldsymbol{\varepsilon}(\overline{\mathbf{u}})$, Vitali's theorem shows that

$$\boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m) \to \boldsymbol{\varepsilon}(\overline{\mathbf{u}}) \text{ strongly in } \mathbf{L}^2(\Omega)^d. \tag{3.21}$$

We then consider $P_{\mathcal{D}_m}\overline{\mathbf{u}} \in \mathbf{X}_{\mathcal{D}_m,\Gamma_D}$ and write, by definition (2.10) of $K_{\mathcal{D}}$,

$$||\nabla_{\mathcal{D}_m}\mathbf{u}_m - \nabla_{\mathcal{D}_m}(P_{\mathcal{D}_m}\overline{\mathbf{u}})||_{\mathbf{L}^2(\Omega)^d} \leq K_{\mathcal{D}_m}||\boldsymbol{\varepsilon}_{\mathcal{D}_m}(\mathbf{u}_m) - \boldsymbol{\varepsilon}_{\mathcal{D}_m}(P_{\mathcal{D}_m}\overline{\mathbf{u}})||_{\mathbf{L}^2(\Omega)^d}.$$

Since $\nabla_{\mathcal{D}_m}(P_{\mathcal{D}_m}\overline{\mathbf{u}})$ and $\boldsymbol{\varepsilon}_{\mathcal{D}_m}(P_{\mathcal{D}_m}\overline{\mathbf{u}})$ strongly converge in $\mathbf{L}^2(\Omega)^d$ to $\nabla\overline{\mathbf{u}}$ and $\boldsymbol{\varepsilon}(\overline{\mathbf{u}})$, we can pass to the limit in this estimate by using the coercivity of $(\mathcal{D}_m)_{m\in\mathbb{N}}$ and (3.21) and we deduce that $\nabla_{\mathcal{D}_m}\mathbf{u}_m \to \nabla\overline{\mathbf{u}}$ strongly in $\mathbf{L}^2(\Omega)^d$. The definition (2.9) of $C_{\mathcal{D}_m}$ then gives

$$||\Pi_{\mathcal{D}_m}\mathbf{u}_m - \Pi_{\mathcal{D}_m}(P_{\mathcal{D}_m}\overline{\mathbf{u}})||_{\mathbf{L}^2(\Omega)} \leq C_{\mathcal{D}_m}||\nabla_{\mathcal{D}_m}\mathbf{u}_m - \nabla_{\mathcal{D}_m}(P_{\mathcal{D}_m}\overline{\mathbf{u}})||_{\mathbf{L}^2(\Omega)^d}$$

and, since $\Pi_{\mathcal{D}_m}(P_{\mathcal{D}_m}\overline{\mathbf{u}}) \to \overline{\mathbf{u}}$ strongly in $\mathbf{L}^2(\Omega)$, passing to the limit in this estimate proves the strong convergence in $\mathbf{L}^2(\Omega)^d$ of $\Pi_{\mathcal{D}_m}\mathbf{u}_m$ to $\overline{\mathbf{u}}$. □

Remark 3.10. *We saw in the proof that $\mathcal{T}_{\mathcal{D}_m}\mathbf{u}_m \to \gamma(\overline{\mathbf{u}})$ weakly in $\mathbf{L}^2(\Gamma_N)$. If the interpolation $P_{\mathcal{D}}$ defined by (3.7) satisfies, for any $\boldsymbol{\varphi} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$, $\mathcal{T}_{\mathcal{D}_m}(P_{\mathcal{D}_m}\boldsymbol{\varphi}) \to \gamma(\boldsymbol{\varphi})$ strongly in $\mathbf{L}^2(\Gamma_N)$ as $m \to \infty$, the same reasoning as the one used at the end of the proof shows that, in case of strict monotonicity of $\boldsymbol{\sigma}$, $\mathcal{T}_{\mathcal{D}_m}\mathbf{u}_m \to \gamma(\overline{\mathbf{u}})$ strongly in $\mathbf{L}^2(\Gamma_N)$.*

Lemma 3.11. *Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of Gradient Discretizations in the sense of Definition 2.2, which is limit-conforming (Definition 2.5) and coercive (Definition 2.6). For any $m \in \mathbb{N}$ we take $\mathbf{v}_m \in \mathbf{X}_{\mathcal{D}_m,\Gamma_D}$.*

If $(||\mathbf{v}_m||_{\mathcal{D}_m})_{m\in\mathbb{N}}$ is bounded then there exists $\mathbf{v} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ such that, up to a subsequence, $\Pi_{\mathcal{D}_m}\mathbf{v}_m \to \mathbf{v}$ weakly in $\mathbf{L}^2(\Omega)$, $\nabla_{\mathcal{D}_m}\mathbf{v}_m \to \nabla\mathbf{v}$ weakly in $\mathbf{L}^2(\Omega)^d$ and $\mathcal{T}_{\mathcal{D}_m}\mathbf{v}_m \to \gamma(\mathbf{v})$ weakly in $\mathbf{L}^2(\Gamma_N)$.

Proof The coercivity of $(\mathcal{D}_m)_{m\in\mathbb{N}}$ and the bound on $||\mathbf{v}_m||_{\mathcal{D}_m}$ show that the sequences $||\Pi_{\mathcal{D}_m}\mathbf{v}_m||_{\mathbf{L}^2(\Omega)}$, $||\nabla_{\mathcal{D}_m}\mathbf{v}_m||_{\mathbf{L}^2(\Omega)^d}$ and $||\mathcal{T}_{\mathcal{D}_m}\mathbf{v}_m||_{\mathbf{L}^2(\Gamma_N)}$ remain bounded. There exists therefore $\mathbf{v} \in \mathbf{L}^2(\Omega)$, $\boldsymbol{\omega} \in \mathbf{L}^2(\Omega)^d$ and $\mathbf{w} \in \mathbf{L}^2(\Gamma_N)$ such that, up to a subsequence,

$$\begin{aligned}&\Pi_{\mathcal{D}_m}\mathbf{v}_m \to \mathbf{v} \text{ weakly in } \mathbf{L}^2(\Omega), \quad \nabla_{\mathcal{D}_m}\mathbf{v}_m \to \boldsymbol{\omega} \text{ weakly in } \mathbf{L}^2(\Omega)^d \text{ and}\\ &\mathcal{T}_{\mathcal{D}_m}\mathbf{v}_m \to \mathbf{w} \text{ weakly in } \mathbf{L}^2(\Gamma_N).\end{aligned} \tag{3.22}$$

These convergences and the limit-conformity of $(\mathcal{D}_m)_{m\in\mathbb{N}}$ show that, for any $\boldsymbol{\tau} \in \mathbf{H}_{\rm div}(\Omega,\Gamma_N)$,

$$\left|\int_\Omega \boldsymbol{\omega}(x) : \boldsymbol{\tau}(x) + \mathbf{v}(x)\cdot \mathrm{div}(\boldsymbol{\tau})(x)\mathrm{d}x - \int_{\Gamma_N} \gamma_{\mathbf{n}}(\boldsymbol{\tau})(x)\cdot \mathbf{w}(x)\mathrm{d}S(x)\right|$$
$$= \lim_{m\to\infty}\left|\int_\Omega \nabla_{\mathcal{D}_m}\mathbf{v}_m(x) : \boldsymbol{\tau}(x) + \Pi_{\mathcal{D}_m}\mathbf{v}_m(x)\cdot \mathrm{div}(\boldsymbol{\tau})(x)\mathrm{d}x - \int_{\Gamma_N} \gamma_{\mathbf{n}}(\boldsymbol{\tau})(x)\cdot \mathcal{T}_{\mathcal{D}_m}(\mathbf{v}_m)(x)\mathrm{d}S(x)\right|$$
$$\le \lim_{m\to\infty}\left[||\mathbf{v}_m||_{\mathcal{D}_m} W_{\mathcal{D}_m}(\boldsymbol{\tau})\right] = 0.$$

Hence, for any $\boldsymbol{\tau} \in \mathbf{H}_{\rm div}(\Omega,\Gamma_N)$,

$$\int_\Omega \boldsymbol{\omega}(x) : \boldsymbol{\tau}(x) + \mathbf{v}(x)\cdot \mathrm{div}(\boldsymbol{\tau})(x)\mathrm{d}x - \int_{\Gamma_N} \gamma_{\mathbf{n}}(\boldsymbol{\tau})(x)\cdot \mathbf{w}(x)\mathrm{d}S(x) = 0. \tag{3.23}$$

Applied with $\boldsymbol{\tau} \in C_c^\infty(\Omega)^{d\times d}$, this relation shows that

$$\nabla \mathbf{v} = \boldsymbol{\omega} \text{ in the sense of distributions on } \Omega, \tag{3.24}$$

and thus that $\mathbf{v} \in \mathbf{H}^1(\Omega)$. By using (3.23) with $\boldsymbol{\tau} \in \mathbf{H}^1(\Omega)^d \subset \mathbf{H}_{\rm div}(\Omega,\Gamma_N)$ and by integrating by parts, we obtain

$$\int_{\partial\Omega} \gamma_{\mathbf{n}}(\boldsymbol{\tau})(x)\cdot\gamma(\mathbf{v})(x)\mathrm{d}S(x) - \int_{\Gamma_N} \gamma_{\mathbf{n}}(\boldsymbol{\tau})(x)\cdot \mathbf{w}(x)\mathrm{d}S(x) = 0.$$

As the set $\{\gamma_{\mathbf{n}}(\boldsymbol{\tau}) \,:\, \boldsymbol{\tau} \in \mathbf{H}^1(\Omega)^d\}$ is dense in $\mathbf{L}^2(\partial\Omega)$, we deduce from this that $\gamma(\mathbf{v}) = 0$ on Γ_D and that

$$\gamma(\mathbf{v}) = \mathbf{w} \text{ on } \Gamma_N. \tag{3.25}$$

Thus, $\mathbf{v} \in \mathbf{H}^1_{\Gamma_D}$ and (3.22), (3.24) and (3.25) conclude the proof. □

The proof of the following lemma is classical [14, 18].

Lemma 3.12. *Let $(F_m)_{m\in\mathbb{N}}$ be a sequence of non-negative measurable functions on Ω which converges a.e. on Ω to F and such that $\int_\Omega F_m(x)\mathrm{d}x \to \int_\Omega F(x)\mathrm{d}x$. Then $F_m \to F$ in $L^1(\Omega)$.*

4. Examples of Gradient Schemes

In all the following examples, we assume that Γ_D has non-zero measure and is such that a Körn's inequality holds on $\mathbf{H}^1_{\Gamma_D}(\Omega)$ [6, 11]. This is actually a necessary condition for *coercive* and *consistent* sequences of Gradient Discretisations to exist.

4.1. **Standard displacement-based formulation.** All (conforming) Galerkin methods are Gradient Schemes. If $(\mathbf{V}_n)_{n\ge1}$ is a sequence of finite dimensional subspaces of $\mathbf{H}^1_{\Gamma_D}(\Omega)$ such that $\cup_{n\ge1}\mathbf{V}_n$ is dense in $\mathbf{H}^1_{\Gamma_D}(\Omega)$, then by letting $\mathbf{X}_{\mathcal{D}_n,\Gamma_D} = \mathbf{V}_n$, $\Pi_{\mathcal{D}_n} = \mathrm{Id}$, $\mathcal{T}_{\mathcal{D}_n} = \gamma$ and $\nabla_{\mathcal{D}_n} = \nabla$, we obtain a sequence of Gradient Discretisations whose corresponding Gradient Schemes are Galerkin approximations of (1.1). This sequence of Gradient Discretisations is obviously *consistent* (this is $\overline{\cup_{n\in\mathbb{N}}\mathbf{V}_n} = \mathbf{H}^1_{\Gamma_D}(\Omega)$), *limit-conforming* (as it is a conforming approximation, $W_{\mathcal{D}_n} = 0$ for any n) and *coercive* (since Poincaré's and Körn's inequalities hold in $\mathbf{H}^1_{\Gamma_D}(\Omega)$).

This is in particular the case for conforming Finite Element approximations based on spaces $\mathbf{V}_h$ built on quasi-uniform partitions $\mathcal{T}_h$ of Ω (made of quadrilaterals, hexahedra or simplices [3, 39]).

But non-conforming methods are also included in the framework of Gradient Schemes. For example, the classical nonconforming finite element scheme (often denoted quite improperly as the Crouzeix-Raviart scheme) falls in this framework, with the discrete gradient defined as the classical "broken gradient". Consistency, limit-conformity and the Poincaré's inequality for this scheme are established in [15], and it is known that if $\Gamma_D = \partial\Omega$ then a uniform Körn's inequality holds. This inequality fails for general Γ_D [23] but it is satisfied for higher order non-conforming methods (whose continuity conditions through the edges involve both the zero-th and first order moments) [27]. The consistency, limit-conformity and Poincaré's inequality for such methods can be easily established as for the classical nonconforming method.

4.2. **Stabilised nodal strain formulation.** We consider a nodal strain formulation as presented in [24,30,38] and built on a conforming Finite Element space $\mathbf{V}_h$. Associated with the primal mesh $\mathcal{T}_h$ we let $\mathcal{T}_h^*$ be the dual mesh consisting of dual volumes, where a dual volume is associated with a vertex of $\mathcal{T}_h$ and is constructed as follows. Let $\{T_j^{\boldsymbol{x}_i}\}_{j=1}^{M_i} \subset \mathcal{T}_h$ be the set of all elements touching the vertex $\boldsymbol{x}_i$, and $\{E_j^{\boldsymbol{x}_i}\}_{j=1}^{N_i}$ the set of edges or faces touching $\boldsymbol{x}_i$. Then the dual volume associated with the vertex $\boldsymbol{x}_i$ is the polygonal or polyhedral region joining all the bary-centres of $\{T_j^{\boldsymbol{x}_i}\}_{j=1}^{M_i}$ and $\{E_j^{\boldsymbol{x}_i}\}_{j=1}^{N_i}$. Let S_h^* be the space of vector-valued piecewise constant functions with respect to the dual mesh $\mathcal{T}_h^*$.

Defining the linear form

$$\ell(\mathbf{v}_h) = \int_\Omega \mathbf{F}(x)\cdot\mathbf{v}_h(x)\mathrm{d}x + \int_{\Gamma_N} \mathbf{g}(x)\cdot\gamma(\mathbf{v}_h)(x)\mathrm{d}S(x),$$

the stabilised nodal strain formulation, for a constant stiffness tensor $\mathbb{C}$, is to find $\mathbf{u}_h \in \mathbf{V}_h$ such that, for any $\mathbf{v}_h \in \mathbf{V}_h$,

$$\int_\Omega \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{u}_h)(x) : \mathbb{C}\boldsymbol{\varepsilon}(\mathbf{v}_h)(x)\mathrm{d}x + \int_\Omega \mathbb{D}(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{u}_h))(x) : \boldsymbol{\varepsilon}(\mathbf{v}_h)(x)\mathrm{d}x = \ell(\mathbf{v}_h)$$

where Π_h^* is the orthogonal projection onto S_h^* and $\mathbb{D}$ is a constant stabilisation (symmetric positive definite) tensor. By the properties of the orthogonal projection and since $\mathbb{C}$ and $\mathbb{D}$ are constant, this can be recast as

$$\text{(4.1)}\quad \begin{aligned} &\text{Find } \mathbf{u}_h \in \mathbf{V}_h \text{ such that, } \forall \mathbf{v}_h \in \mathbf{V}_h,\\ &\int_\Omega \mathbb{C}\Pi_h^*\boldsymbol{\varepsilon}(\mathbf{u}_h)(x) : \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{v}_h)(x)\mathrm{d}x\\ &\qquad + \int_\Omega \mathbb{D}(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{u}_h))(x) : (\boldsymbol{\varepsilon}(\mathbf{v}_h) - \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{v}_h))(x)\mathrm{d}x = \ell(\mathbf{v}_h). \end{aligned}$$

We will take this formulation as definition of the stabilised nodal strain formulation in the case where $\mathbb{C}$ and $\mathbb{D}$ are not constant (in which case we assume that $\mathbb{D}$ satisfies Assumption (3.1)).

Let us now construct a Gradient Discretisation $\mathcal{D} = (\mathbf{X}_{\mathcal{D},\Gamma_D}, \Pi_\mathcal{D}, \mathcal{T}_\mathcal{D}, \nabla_\mathcal{D})$ such that this formulation is identical to the corresponding Gradient Scheme (3.2). We start by defining $\mathbf{X}_{\mathcal{D},\Gamma_D}$ and the operators $\Pi_\mathcal{D} : \mathbf{X}_{\mathcal{D},\Gamma_D} \to \mathbf{L}^2(\Omega)$ and $\mathcal{T}_\mathcal{D} : \mathbf{X}_{\mathcal{D},\Gamma_D} \to \mathbf{L}^2(\Gamma_N)$ by

$$\text{(4.2)}\qquad \mathbf{X}_{\mathcal{D},\Gamma_D} = \mathbf{V}_h\,,\ \Pi_\mathcal{D}\mathbf{v}_h = \mathbf{v}_h \text{ and } \mathcal{T}_\mathcal{D}\mathbf{v}_h = \gamma(\mathbf{v}_h)_{|\Gamma_N} \text{ for all } \mathbf{v}_h \in \mathbf{X}_{\mathcal{D},\Gamma_D}.$$

With these choices, $\ell(\mathbf{v}_h)$ is the right-hand side of (3.2) and we therefore just need to find a discrete gradient $\nabla_\mathcal{D}$ such that the left-hand side of (3.2) is equal to the left-hand side of (4.1).

We first notice that, by (3.1) on $\mathbb{C}$ and $\mathbb{D}$, for a.e. x the linear mappings $\mathbb{C}(x), \mathbb{D}(x) : \mathbf{R}^{d\times d} \to \mathbf{R}^{d\times d}$ are symmetric positive definite with respect to the inner product ":" and thus $\mathbb{C}(x)^{-1/2}$ and $\mathbb{D}(x)^{1/2}$ make sense. We can therefore define $\nabla_\mathcal{D} : \mathbf{X}_{\mathcal{D},\Gamma_D} \to \mathbf{L}^2(\Omega)^d$ by

$$\text{(4.3)}\qquad \nabla_\mathcal{D}\mathbf{v}_h = \Pi_h^*\nabla\mathbf{v}_h + \mathbb{C}^{-1/2}\mathbb{D}^{1/2}(\nabla\mathbf{v}_h - \Pi_h^*\nabla\mathbf{v}_h).$$

By assumptions on $\mathbb{C}$ and $\mathbb{D}$ and Lemma 4.10, this gives

$$\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v}_h) = \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{v}_h) + \mathbb{C}^{-1/2}\mathbb{D}^{1/2}(\boldsymbol{\varepsilon}(\mathbf{v}_h) - \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{v}_h)).$$

Assuming that $\mathbb{C}$ and $\mathbb{D}$ are piecewise constant on $\mathcal{T}_h^*$, we can then compute

$$\begin{aligned}
&\int_\Omega \mathbb{C}(x)\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{u}_h)(x) : \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v}_h)(x)\mathrm{d}x \\
&\quad = \int_\Omega \mathbb{C}(x)\Pi_h^*\boldsymbol{\varepsilon}(\mathbf{u}_h)(x) : \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{v}_h)(x)\mathrm{d}x \\
&\quad + \int_\Omega \mathbb{C}(x)\Pi_h^*\boldsymbol{\varepsilon}(\mathbf{u}_h)(x) : \mathbb{C}^{-1/2}(x)\mathbb{D}^{1/2}(x)(\boldsymbol{\varepsilon}(\mathbf{v}_h)(x) - \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{v}_h)(x))\mathrm{d}x \qquad (4.4)\\
&\quad + \int_\Omega \mathbb{C}(x)\mathbb{C}^{-1/2}(x)\mathbb{D}^{1/2}(x)(\boldsymbol{\varepsilon}(\mathbf{u}_h)(x) - \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{u}_h)(x)) : \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{v}_h)(x)\mathrm{d}x \qquad (4.5)\\
&\quad + \int_\Omega \mathbb{C}(x)\mathbb{C}^{-1/2}(x)\mathbb{D}^{1/2}(x)(\boldsymbol{\varepsilon}(\mathbf{u}_h)(x) - \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{u}_h)(x)) \\
&\qquad\qquad : \mathbb{C}^{-1/2}(x)\mathbb{D}^{1/2}(x)(\boldsymbol{\varepsilon}(\mathbf{v}_h)(x) - \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{v}_h)(x))\mathrm{d}x.
\end{aligned}$$

But, since $\mathbb{C}$, $\mathbb{D}$ and $\Pi_h^*\boldsymbol{\varepsilon}(\mathbf{u}_h)$ are constant on each cell in $\mathcal{T}_h^*$ and since

$$\Pi_h^*\boldsymbol{\varepsilon}(\mathbf{v}_h) = \frac{1}{\mathrm{meas}(K)}\int_K \boldsymbol{\varepsilon}(\mathbf{v}_h)(x)\mathrm{d}x$$

on $K \in \mathcal{T}_h^*$, we have

$$(4.4) = \sum_{K\in\mathcal{T}_h^*} \mathbb{C}_{|K}\Pi_h^*\boldsymbol{\varepsilon}(\mathbf{u}_h)_{|K} : \mathbb{C}_{|K}^{-1/2}\mathbb{D}_{|K}^{1/2}\int_K (\boldsymbol{\varepsilon}(\mathbf{v}_h)(x) - \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{v}_h)(x))\,\mathrm{d}x = 0.$$

Similarly, (4.5) vanishes and, by using the symmetry of $\mathbb{C}$ and $\mathbb{D}$, we end up with

$$\begin{aligned}
&\int_\Omega \mathbb{C}(x)\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{u}_h)(x) : \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v}_h)(x)\mathrm{d}x \\
&\quad = \int_\Omega \mathbb{C}(x)\Pi_h^*\boldsymbol{\varepsilon}(\mathbf{u}_h)(x) : \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{v}_h)(x) \\
&\quad + \int_\Omega \mathbb{D}(x)(\boldsymbol{\varepsilon}(\mathbf{u}_h)(x) - \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{u}_h)(x)) : (\boldsymbol{\varepsilon}(\mathbf{v}_h)(x) - \Pi_h^*\boldsymbol{\varepsilon}(\mathbf{v}_h)(x))\mathrm{d}x,
\end{aligned}$$

which precisely states that the left-hand sides of (3.2) and (4.1) coincide. Thus, under the assumption that $\mathbb{C}$ and $\mathbb{D}$ are piecewise constant on $\mathcal{T}_h^*$, the stabilised nodal strain formulation (4.1) is the Gradient Scheme, for the linear elasticity equation, corresponding to the Gradient Discretisation defined by (4.2)–(4.3).

Remark 4.1. *If $\mathbb{C}$ or $\mathbb{D}$ are not piecewise constant on $\mathcal{T}_h^*$, then by replacing them with $\Pi_h^*\mathbb{C}$ and $\Pi_h^*\mathbb{D}$ in the stabilised nodal strain formulation* (4.1) *and the definition* (4.3) *of the discrete gradient, the stabilised nodal strain formulation is the Gradient Scheme* (3.2) *in which $\mathbb{C}$ is replaced with $\Pi_h^*\mathbb{C}$.*

4.2.1. *Consistency, limit-conformity and coercivity.* Let us consider $(\mathbf{V}_{h_n})_{n\in\mathbb{N}}$ a sequence of conforming Finite Element spaces on meshes $(\mathcal{T}_{h_n})_{n\in\mathbb{N}}$ with $h_n \to 0$. We prove here that if $\mathcal{D}_n$ is the Gradient Discretisation given by (4.2)–(4.3) for $\mathbf{V}_{h_n}$ then, under the classical quasi-uniform assumptions on $(\mathcal{T}_{h_n})_{n\in\mathbb{N}}$, the sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ is consistent, limit-conforming and coercive. The key point is to notice that the definition (4.3) of the discrete gradient can be recast as

$$\nabla_{\mathcal{D}}\mathbf{v}_h = \nabla\mathbf{v}_h + (\mathbb{C}^{-1/2}\mathbb{D}^{1/2} - \mathrm{Id})(\nabla\mathbf{v}_h - \Pi_h^*\nabla\mathbf{v}_h) = \nabla\mathbf{v}_h + \mathcal{L}_h\nabla\mathbf{v}_h \qquad (4.6)$$

where $\mathcal{L}_h = (\mathbb{C}^{-1/2}\mathbb{D}^{1/2} - \mathrm{Id})(\mathrm{Id} - \Pi_h^*) : \mathbf{L}^2(\Omega)^d \to \mathbf{L}^2(\Omega)^d$ has a norm bounded independently on h and converges pointwise to 0.

Let us first consider the consistency property. For any $\boldsymbol{\varphi} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$, by quasi-uniformity of the sequence of meshes, there exists $\mathbf{v}_n \in \mathbf{V}_{h_n} = \mathbf{X}_{\mathcal{D}_n,\Gamma_D}$ such that $\mathbf{v}_n = \Pi_{\mathcal{D}_n}\mathbf{v}_n \to \boldsymbol{\varphi}$ in $\mathbf{L}^2(\Omega)$ and $\nabla\mathbf{v}_n \to \nabla\boldsymbol{\varphi}$ in $\mathbf{L}^2(\Omega)^d$. We have

$$||\mathcal{L}_{h_n}\nabla\mathbf{v}_n||_{\mathbf{L}^2(\Omega)^d} \le ||\mathcal{L}_{h_n}||_{\mathbf{L}^2(\Omega)^d\to\mathbf{L}^2(\Omega)^d}||\nabla\mathbf{v}_n - \nabla\varphi||_{\mathbf{L}^2(\Omega)^d} + ||\mathcal{L}_{h_n}\nabla\boldsymbol{\varphi}||_{\mathbf{L}^2(\Omega)^d}$$

and, by the properties of $\mathcal{L}_{h_n}$, both terms in the right-hand side tend to 0. Combined with (4.6) this proves that $\nabla_{\mathcal{D}_n}\mathbf{v}_n \to \nabla\boldsymbol{\varphi}$ in $\mathbf{L}^2(\Omega)^d$, which concludes the proof of the consistency of $(\mathcal{D}_n)_{n\in\mathbb{N}}$.

Coercivity follows from the following comparisons between ∇, $\nabla_{\mathcal{D}_n}$ and $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}_{\mathcal{D}_n}$: there exists $C_1, C_2 > 0$ not depending on n such that, for any $\mathbf{v} \in \mathbf{V}_{h_n} = \mathbf{X}_{\mathcal{D}_n,\Gamma_D}$,

$$C_1||\nabla_{\mathcal{D}_n}\mathbf{v}||_{\mathbf{L}^2(\Omega)^d} \le ||\nabla\mathbf{v}||_{\mathbf{L}^2(\Omega)^d} \le C_2||\nabla_{\mathcal{D}_n}\mathbf{v}||_{\mathbf{L}^2(\Omega)^d}\,, \tag{4.7}$$

$$C_1||\boldsymbol{\varepsilon}_{\mathcal{D}_n}(\mathbf{v})||_{\mathbf{L}^2(\Omega)^d} \le ||\boldsymbol{\varepsilon}(\mathbf{v})||_{\mathbf{L}^2(\Omega)^d} \le C_2||\boldsymbol{\varepsilon}_{\mathcal{D}_n}(\mathbf{v})||_{\mathbf{L}^2(\Omega)^d}. \tag{4.8}$$

Indeed, with these two estimates, the coercivity of $(\mathcal{D}_n)_{n\in\mathbb{N}}$ is a straightforward consequence of the Poincaré, trace and Körn's inequalities in $\mathbf{H}^1_{\Gamma_D}(\Omega)$. Since the proofs of (4.7) and (4.8) are similar, we only consider the first one. Using $||\Pi^*_{h_n}\nabla\mathbf{v}||_{\mathbf{L}^2(\Omega)^d} \le ||\nabla\mathbf{v}||_{\mathbf{L}^2(\Omega)^d}$, (4.6) immediately gives the first inequality in (4.7). To establish the second one, we just notice, applying $\Pi^*_{h_n}$ to (4.3) that $\Pi^*_{h_n}\nabla_{\mathcal{D}_n}\mathbf{v} = \Pi^*_{h_n}\nabla\mathbf{v}$, which gives, plugged into (4.3),

$$\nabla\mathbf{v} = \Pi^*_{h_n}\nabla_{\mathcal{D}_n}\mathbf{v} + \mathbb{D}^{-1/2}\mathbb{C}^{1/2}\big(\nabla_{\mathcal{D}_n}\mathbf{v} - \Pi^*_{h_n}\nabla_{\mathcal{D}_n}\mathbf{v}\big).$$

The second estimate of (4.7) follows by taking the $\mathbf{L}^2(\Omega)^d$ norm of this equality and using once more the fact that the orthogonal projection $\Pi^*_{h_n}$ has norm 1.

Limit-conformity is then easy to establish. For any $\boldsymbol{\tau} \in \mathbf{H}_{\rm div}(\Omega,\Gamma_N)$ and any $\mathbf{v} \in \mathbf{V}_{h_n} = \mathbf{X}_{\mathcal{D}_n,\Gamma_D}$, by using (4.6) we have

$$\begin{aligned}
&\left|\int_\Omega \big(\nabla_{\mathcal{D}_n}\mathbf{v}(x) : \boldsymbol{\tau}(x) + \Pi_{\mathcal{D}_n}\mathbf{v}(x)\cdot \mathrm{div}(\boldsymbol{\tau})(x)\big)\mathrm{d}x - \int_{\Gamma_N}\gamma_{\mathbf{n}}(\boldsymbol{\tau})(x)\cdot\mathcal{T}_{\mathcal{D}_n}(\mathbf{v})(x)\mathrm{d}S(x)\right|\\
&\le \left|\int_\Omega \big(\nabla\mathbf{v}(x) : \boldsymbol{\tau}(x) + \mathbf{v}(x)\cdot \mathrm{div}(\boldsymbol{\tau})(x)\big)\mathrm{d}x - \int_{\Gamma_N}\gamma_{\mathbf{n}}(\boldsymbol{\tau})(x)\cdot\gamma(\mathbf{v})(x)\mathrm{d}S(x)\right|\\
&\quad+\left|\int_\Omega \mathcal{L}_{h_n}\nabla\mathbf{v}(x) : \boldsymbol{\tau}(x)\mathrm{d}x\right| = T_1 + T_2.
\end{aligned} \tag{4.9}$$

By conformity of $\mathbf{V}_{h_n}$ we have $T_1 = 0$. Thanks to (4.7) and denoting by $\mathcal{L}^\star_{h_n} = (\mathrm{Id} - \Pi^*_h)(\mathbb{D}^{1/2}\mathbb{C}^{-1/2} - \mathrm{Id})$ the dual operator of $\mathcal{L}_{h_n}$, we can write

$$\begin{aligned}
T_2 &= \left|\int_\Omega \nabla\mathbf{v}(x) : \mathcal{L}^\star_{h_n}\boldsymbol{\tau}(x)\mathrm{d}x\right|\\
&\le ||\nabla\mathbf{v}||_{\mathbf{L}^2(\Omega)^d}||\mathcal{L}^\star_{h_n}\boldsymbol{\tau}||_{\mathbf{L}^2(\Omega)^d} \le C_2||\nabla_{\mathcal{D}_n}\mathbf{v}||_{\mathbf{L}^2(\Omega)^d}||\mathcal{L}^\star_{h_n}\boldsymbol{\tau}||_{\mathbf{L}^2(\Omega)^d}.
\end{aligned}$$

Plugged into (4.9), this estimate on T_2 shows that $W_{\mathcal{D}_n}(\boldsymbol{\tau}) \le C_2||\mathcal{L}^\star_{h_n}\boldsymbol{\tau}||_{\mathbf{L}^2(\Omega)^d}$. As $\mathcal{L}^\star_{h_n} \to 0$ pointwise as $n \to \infty$, this concludes the proof of the limit-conformity of $(\mathcal{D}_n)_{n\in\mathbb{N}}$.

Remark 4.2. *Reference [38] provides an $\mathcal{O}(h)$ error estimate for* (4.1) *under very strong assumptions on the solution to the continuous equation* (1.1)*, namely $\overline{\mathbf{u}} \in C^2(\overline{\Omega})$. Embedding* (4.1) *into the Gradient Scheme framework allowed us to establish the same error estimate under no regularity assumption on the exact solution (see Theorem 3.2) and that, contrary to what is written in [38, p848], the smoothness of the solution is not required for conducting the error analysis of the method.*

Remark 4.3. *As a consequence of these properties and of Theorem 3.5, we deduce that the Gradient Scheme discretisation* (4.2)–(4.3) *coming from the stabilised nodal strain formulation of the linear elasticity equations can be used to define a "stabilised nodal strain formulation for* non-linear *elasiticity"* (2.5)*, and gives a converging scheme for these equations. In this case, the tensors* $\mathbb{C}$ *and* $\mathbb{D}$ *in* (4.3) *should be chosen accordingly to the considered non-linear equation, e.g. by selecting linear tensors with Lamé's coefficients of the correct order of magnitude with respect to the non-linear model. If we consider for example the Hencky–von Mises stress* $\boldsymbol{\sigma}(x,\boldsymbol{\tau}) = \widetilde{\lambda}(\mathrm{dev}(\boldsymbol{\tau}))\,\mathrm{tr}(\boldsymbol{\tau})\mathbf{I} + 2\widetilde{\mu}(\mathrm{dev}(\boldsymbol{\tau}))\boldsymbol{\tau}$*, then* $\mathbb{C}$ *can be given by* $\mathbb{C}\boldsymbol{\tau} = \lambda\,\mathrm{tr}(\boldsymbol{\tau})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{\tau})$*, where* λ *and* μ *are within the range of* $\widetilde{\lambda}$ *and* $\widetilde{\mu}$ *(see [2] for assumptions on this range), and* $\mathbb{D}$ *can be taken diagonal equal to* 1 *or* 2μ *(see [30]).*

Remark 4.4. *We can also construct the "nodal stabilised" Gradient Discretisation* $\mathcal{D}$ *by* (4.2)–(4.3) *starting from a non-conforming Finite Element discretisation* $\mathbf{V}_h$ *(or, for that matter, any initial Gradient Discretisation built on a polygonal discretisation of* Ω *as defined in [15]). In this case, the preceding reasoning shows that if* $(\mathbf{V}_{h_n})_{n\in\mathbb{N}}$ *is consistent, limit-conforming and coercive then the corresponding nodal stabilised Gradient Discretisation* $(\mathcal{D}_n)_{n\in\mathbb{N}}$ *is also consistent, limit-conforming and coercive.*

4.3. **Hu-Washizu-based formulation on quadrilateral meshes.** We now consider a Finite Element method based on a modified Hu-Washizu formulation [31] for quadrilateral meshes. We start with the statically condensed displacement-based formulation in [31] of the following form: find $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$\int_\Omega P_{S_h}\boldsymbol{\varepsilon}(\mathbf{v}_h)(x) : \mathbb{C}_h P_{S_h}\boldsymbol{\varepsilon}(\mathbf{u}_h)(x)\mathrm{d}x = \ell(\mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{V}_h \ , \tag{4.10}$$

where $\mathbf{V}_h$ is the standard conforming Finite Element space constructed from piecewise bilinear polynomials on a reference element, P_{S_h} is the L^2 orthogonal projection onto the discrete space of stress S_h, and $\mathbb{C}_h$ is some positive-definite symmetric operator approximating the classical linear elasticity tensor $\mathbb{C}$ with constant Lamé coefficients, $\mathbb{C}\boldsymbol{\tau} = \lambda\,\mathrm{tr}(\boldsymbol{\tau})\mathbf{I} + 2\mu\boldsymbol{\tau}$. We note that the space of stress $S_h \subset \mathbf{L}^2(\Omega)^d$ is defined element-wise, and there is no continuity condition for its element across the boundary of cell in $\mathcal{T}_h$. Various Finite Element methods used in alleviating locking effects are derived using this formulation [13, 31]. Among them, the most popular methods are the assumed enhanced strain method of Simo and Rifai [41], the strain gap method of Romano, Marrotti de Sciarra and Diaco [40], and the mixed enhanced strain method of Kasper and Taylor [26]. We now consider the action of the operator $\mathbb{C}_h$ on a tensor $\boldsymbol{d}_h = P_{S_h}\boldsymbol{\varepsilon}(\mathbf{u}_h)$ as derived in [31]. We use an orthogonal decomposition of S_h in the form

$$S_h = S_h^c \oplus S_h^t \ ,$$

where

$$S_h^c := \{\boldsymbol{\tau} \in S_h \mid \ \mathbb{C}\boldsymbol{\tau} \in S_h\}$$

and S_h^t is the orthogonal complement of S_h^c. We consider the case where the operator $\mathbb{C}_h$ is expressed as [31]

$$\mathbb{C}_h\boldsymbol{d}_h = \mathbb{C}P_{S_h^c}\boldsymbol{d}_h + \theta P_{S_h^t}\boldsymbol{d}_h \tag{4.11}$$

where $P_{S_h^c}$ and $P_{S_h^t}$ are the orthogonal projections onto S_h^c and S_h^t and $\theta > 0$ is a constant only depending upon the Lamé coefficients λ, μ of $\mathbb{C}$ and upon the parameter $\alpha > 0$ of the modified three-field Hu-Washizu formulation [31]. When the modified Hu-Washizu formulation is equivalent to the Hellinger-Reissner formulation, θ does not depend on α.

Remark 4.5. *The expression for the action of $\mathbb{C}_h$ is obtained in [31] using Voigt notation for tensors. However, we give here the expression for the discrete space of stress using the full tensor notation so that we have*

$$P_{S_h}\boldsymbol{\varepsilon}(\mathbf{u}_h) = \frac{1}{2}\left(P_{S_h}(\nabla\mathbf{u}_h) + P_{S_h}(\nabla\mathbf{u}_h)^T\right).$$

We restrict ourselves, for simplicity of presentation, to the two-dimensional case, where $\boldsymbol{d}_h$ is a 2 by 2 tensor. We consider three choices for S_h, where this space is generated (through conformal transformations) from bases $S_\Box$ defined on $\hat{K} := (-1,1)^2$. Let these three choices be denoted by S_h^i and $S_\Box^i$, $1 \le i \le 3$.

$$S_\Box^1 := \begin{bmatrix} \text{span}\{1,\hat{y}\} & \text{span}\{1\} \\ \text{span}\{1\} & \text{span}\{1,\hat{x}\} \end{bmatrix}, \quad S_\Box^2 := \begin{bmatrix} \text{span}\{1,\hat{y}\} & \text{span}\{1,\hat{x},\hat{y}\} \\ \text{span}\{1,\hat{x},\hat{y}\} & \text{span}\{1,\hat{x}\} \end{bmatrix},$$

and

$$S_\Box^3 := \begin{bmatrix} \text{span}\{1\} & \text{span}\{1,\hat{x},\hat{y}\} \\ \text{span}\{1,\hat{x},\hat{y}\} & \text{span}\{1\} \end{bmatrix}$$

While the spherical part of the stress might be polluted by checkerboard modes as in the case of the $Q_1 - P_0$ element, it is proved that the error in displacement satisfies a λ-independent *a priori* error estimate [31].

Let us now prove that if $S_h = S_h^i$ for some $1 \le i \le 3$ then (4.10) is a Gradient Scheme. We define

$$\begin{gathered} \mathbf{X}_{\mathcal{D},\Gamma_D} = \mathbf{V}_h\,, \quad \Pi_{\mathcal{D}}\mathbf{v}_h = \mathbf{v}_h\,, \quad \mathcal{T}_{\mathcal{D}}\mathbf{v}_h = \gamma(\mathbf{v}_h)_{|\Gamma_N} \text{ and} \\ \nabla_{\mathcal{D}}\mathbf{v}_h = P_{S_h^c}\nabla\mathbf{v}_h + \sqrt{\theta}\,\mathbb{C}^{-1/2}P_{S_h^t}\nabla\mathbf{v}_h. \end{gathered} \tag{4.12}$$

We note that, by symmetry of $\mathbb{C}$, S_h^c and S_h^t are closed under transposition and therefore the projections onto those spaces commute with the transposition. By Lemma 4.10, the definition of $\nabla_{\mathcal{D}}$ thus shows that

$$\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v}_h) = P_{S_h^c}\boldsymbol{\varepsilon}(\mathbf{v}_h) + \sqrt{\theta}\,\mathbb{C}^{-1/2}P_{S_h^t}\boldsymbol{\varepsilon}(\mathbf{v}_h). \tag{4.13}$$

We now prove that the Gradient Scheme corresponding to the Gradient Discretisation $\mathcal{D} = (\mathbf{X}_{\mathcal{D},\Gamma_D}, \Pi_{\mathcal{D}}, \mathcal{T}_{\mathcal{D}}, \nabla_{\mathcal{D}})$ is precisely the Hu-Washizu scheme (4.10). Let us first start with a lemma.

Lemma 4.6. *For any of the choices S_h^i $(1 \le i \le 3)$ described above and for any linear elasticity tensor $\mathbb{D}$, $(S_h^i)^c$ is closed under $\mathbb{D}$, that is $\mathbb{D}\boldsymbol{\tau} \in (S_h^i)^c$ whenever $\boldsymbol{\tau} \in (S_h^i)^c$. In particular,*

$$\forall \boldsymbol{\tau},\boldsymbol{\omega} \in \mathbf{L}^2(\Omega)^d\,, \quad \int_\Omega \mathbb{D}P_{(S_h^i)^c}\boldsymbol{\tau}(x) : P_{(S_h^i)^t}\boldsymbol{\omega}(x)\mathrm{d}x = 0. \tag{4.14}$$

Proof If $\boldsymbol{\tau} \in (S_h^i)^c$ then $\text{tr}(\boldsymbol{\tau})\mathbf{I} = \lambda^{-1}(\mathbb{C}\boldsymbol{\tau} - 2\mu\boldsymbol{\tau}) \in S_h^i$. The definitions of S_h^i then shows, by examining the coefficients $(1,1)$ and $(2,2)$ of $\text{tr}(\boldsymbol{\tau})\mathbf{I}$, that $\text{tr}(\boldsymbol{\tau}) \in \text{span}\{1,\hat{y}\} \cap \text{span}\{1,\hat{x}\} = \text{span}\{1\}$ and thus that $\text{tr}(\boldsymbol{\tau})$ is constant.

By Lemma 4.9, we see that $\mathbb{C}\mathbb{D}$ is a linear elasticity tensor with some Lamé coefficients (α,β) and therefore $\mathbb{C}\mathbb{D}\boldsymbol{\tau} = \alpha\,\text{tr}(\boldsymbol{\tau})\mathbf{I} + 2\beta\boldsymbol{\tau}$. The second term in this right-hand side clearly belongs to S_h^i and, since $\text{tr}(\boldsymbol{\tau})$ is constant, it is equally obvious that the first term in the right-hand side belongs to S_h^i (which contains $\text{span}\{\mathbf{I}\}$). Hence, $\mathbb{D}\boldsymbol{\tau} \in (S_h^i)^c$ whenever $\boldsymbol{\tau} \in (S_h^i)^c$. Formula (4.14) is a consequence of this and of the orthogonality of S_h^c and S_h^t. □

We now consider the left-hand side of (3.2). Using (4.14) with $\mathbb{D} = \mathbb{C}^{1/2}$ (which is a linear elasticity tensor by Lemma 4.9), the cross-products involving $\mathbb{C}^{1/2}P_{S_h^c}$

and $P_{S_h^t}$ which appear when plugging (4.13) into (3.2) vanish and we obtain

$$\begin{aligned}(4.15)\quad &\int_\Omega \mathbb{C}(x)\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{u}_h)(x) : \boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v}_h)(x)\mathrm{d}x \\ &= \int_\Omega \mathbb{C}(x)P_{S_h^c}\boldsymbol{\varepsilon}(\mathbf{u}_h)(x) : P_{S_h^c}\boldsymbol{\varepsilon}(\mathbf{u}_h)(x)\mathrm{d}x + \int_\Omega \theta P_{S_h^t}\boldsymbol{\varepsilon}(\mathbf{u}_h)(x) : P_{S_h^t}\boldsymbol{\varepsilon}(\mathbf{u}_h)(x)\mathrm{d}x.\end{aligned}$$

Using now the definition (4.11) of $\mathbb{C}_h$ and the orthogonality property (4.14) with $\mathbb{D} = \mathbb{C}$, the left-hand side of (4.10) can be written

$$\begin{aligned}(4.16)\quad &\int_\Omega \mathbb{C}_h P_{S_h}\boldsymbol{\varepsilon}(\mathbf{u}_h)(x) : P_{S_h}\boldsymbol{\varepsilon}(\mathbf{v}_h)(x)\mathrm{d}x \\ &= \int_\Omega \Big[\mathbb{C}(x)P_{S_h^c}\boldsymbol{\varepsilon}(\mathbf{u}_h)(x) + \theta P_{S_h^t}\boldsymbol{\varepsilon}(\mathbf{u}_h)(x)\Big] : \Big[P_{S_h^c}\boldsymbol{\varepsilon}(\mathbf{v}_h)(x) + P_{S_h^t}\boldsymbol{\varepsilon}(\mathbf{v}_h)(x)\Big]\,\mathrm{d}x \\ &= \int_\Omega \mathbb{C}(x)P_{S_h^c}\boldsymbol{\varepsilon}(\mathbf{u}_h)(x) : P_{S_h^c}\boldsymbol{\varepsilon}(\mathbf{v}_h)(x)\mathrm{d}x + \int_\Omega \theta P_{S_h^t}\boldsymbol{\varepsilon}(\mathbf{u}_h)(x) : P_{S_h^t}\boldsymbol{\varepsilon}(\mathbf{v}_h)(x)\mathrm{d}x.\end{aligned}$$

Equations (4.15) and (4.16) show that the left-hand sides of the Gradient Scheme (3.2) and of the Hu-Washizu formulation (4.10) are identical. As the right-hand sides of these equations are trivially identical (by definition of $\Pi_{\mathcal{D}}$ and $\mathcal{T}_{\mathcal{D}}$), this shows that the statically condensed Hu-Washizu formulation [31] is the Gradient Scheme corresponding to the Gradient Discretisation defined by (4.12).

Let us now see that the Gradient Discretisation (4.12) satisfies the properties defined in Section 2. The coercivity is again a consequence of (4.7) and (4.8) that we can prove in the following way. First, since the norms of $P_{S_h^c}$ and $P_{S_h^t}$ are bounded by 1, the definition (4.12) of $\nabla_{\mathcal{D}}$ and the property (4.13) of $\boldsymbol{\varepsilon}_{\mathcal{D}}$ immediately give the first inequalities in (4.7) and (4.8). We then write, from (4.13),

$$(4.17)\qquad \mathbb{C}^{1/2}\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v}_h) = \mathbb{C}^{1/2}P_{S_h^c}\boldsymbol{\varepsilon}(\mathbf{v}_h) + \sqrt{\theta}\,P_{S_h^t}\boldsymbol{\varepsilon}(\mathbf{v}_h).$$

By Lemmas 4.6 and 4.9, we have $\mathbb{C}^{1/2}P_{S_h^c}\nabla\mathbf{v}_h \in S_h^c$ and (4.17) thus shows that $P_{S_h^c}\mathbb{C}^{1/2}\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v}_h) = \mathbb{C}^{1/2}P_{S_h^c}\boldsymbol{\varepsilon}(\mathbf{v}_h)$ and $P_{S_h^t}\mathbb{C}^{1/2}\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v}_h) = \sqrt{\theta}\,P_{S_h^t}\boldsymbol{\varepsilon}(\mathbf{v}_h)$. This allows us to write

$$\begin{aligned}P_{S_h}\boldsymbol{\varepsilon}(\mathbf{v}_h) &= P_{S_h^c}\boldsymbol{\varepsilon}(\mathbf{v}_h) + P_{S_h^t}\boldsymbol{\varepsilon}(\mathbf{v}_h) \\ &= \mathbb{C}^{-1/2}P_{S_h^c}\mathbb{C}^{1/2}\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v}_h) + \sqrt{\theta}^{\,-1}P_{S_h^t}\mathbb{C}^{1/2}\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v}_h).\end{aligned}$$

This relation shows that $||P_{S_h}\boldsymbol{\varepsilon}(\mathbf{v}_h)||_{\mathbf{L}^2(\Omega)^d} \le C_3||\boldsymbol{\varepsilon}_{\mathcal{D}}(\mathbf{v}_h)||_{\mathbf{L}^2(\Omega)^d}$ with C_3 not depending on h or $\mathbf{v}_h$. Since it can be proved (see [31]) that $||\boldsymbol{\varepsilon}(\mathbf{v}_h)||_{\mathbf{L}^2(\Omega)^d} \le C_4||P_{S_h}\boldsymbol{\varepsilon}(\mathbf{v}_h)||_{\mathbf{L}^2(\Omega)^d}$ with C_4 not depending on h or $\mathbf{v}_h$, the second inequality in (4.8) follows immediately. The second inequality in (4.7) can then be established by using the continuous Körn inequality $||\nabla\mathbf{v}_h||_{\mathbf{L}^2(\Omega)^d} \le C_5||\boldsymbol{\varepsilon}(\mathbf{v}_h)||_{\mathbf{L}^2(\Omega)^d}$ and the second inequality of (4.8) that we just established.

To establish the consistency and limit-conformity of the Gradient Discretisation, we notice that

$$(4.18)\qquad \nabla_{\mathcal{D}}\mathbf{v}_h = \nabla\mathbf{v}_h + (P_{S_h^c} - \mathrm{Id})\nabla\mathbf{v}_h + \sqrt{\theta}\,\mathbb{C}^{-1/2}P_{S_h^t}\nabla\mathbf{v}_h = \nabla\mathbf{v}_h + \mathcal{L}_h\nabla\mathbf{v}_h$$

where $\mathcal{L}_h = P_{S_h^c} - \mathrm{Id} + \sqrt{\theta}\,\mathbb{C}^{-1/2}P_{S_h^t} : \mathbf{L}^2(\Omega)^d \to \mathbf{L}^2(\Omega)^d$ is a self-adjoint operator (because $\sqrt{\theta}\,\mathbb{C}^{-1/2}$ is constant) whose norm is bounded independently on h. As S_h^c always contains the set of constant tensors S_h^0 and $P_{S_h^0} \to \mathrm{Id}$ as $h \to 0$, we have $P_{S_h^c} = P_{S_h^c}(\mathrm{Id} - P_{S_h^0}) + P_{S_h^0} \to \mathrm{Id}$ and $P_{S_h^t} = P_{S_h^t}(\mathrm{Id} - P_{S_h^0}) \to 0$ pointwise as $h \to 0$. Hence, $\mathcal{L}_h \to 0$ pointwise as $h \to 0$. Expression (4.18) then allows us to prove the consistency and limit-conformity of the Gradient Discretisation (4.12) by using the same techniques as in Section 4.2.1.

Remark 4.7. *The same construction can be made when $\mathbb{C}$ is only piecewise constant on $\mathcal{T}_h$.*

Remark 4.8. *In contrast to [4, 31], the convergence result of Theorem 3.2 is obtained for the Hu-Washizu scheme without assuming the full H^2-regularity of the solution. Moreover, as in Remark 4.3, this construction also gives a converging Hu-Washizu-based scheme for non-linear elasticity equations; this scheme is obtained by plugging the discrete elements* (4.12)*, with $\mathbb{C}$ chosen for example as in Remark 4.3, in Gradient Scheme* (2.5)*.*

4.4. **Technical lemmas.**

Lemma 4.9. *If $\mathbb{C}_1$ and $\mathbb{C}_2$ are linear elasticity tensors in $\mathbf{R}^d$ with Lamé coefficients (λ_1, μ_1) and (λ_2, μ_2), then, for any $\boldsymbol{\tau} \in \mathbf{R}^{d\times d}$,*

$$\mathbb{C}_1\mathbb{C}_2\boldsymbol{\tau} = (\lambda_1\lambda_2 d + 2\mu_1\lambda_2 + 2\mu_2\lambda_1)\,\mathrm{tr}(\boldsymbol{\tau})\mathbf{I} + 4\mu_1\mu_2\boldsymbol{\tau}. \tag{4.19}$$

If $\mathbb{C}$ is a linear elasticity tensor with Lamé coefficients (λ, μ), then

$$\mathbb{C}^{1/2}\boldsymbol{\tau} = \frac{\sqrt{2\mu + \lambda d} - \sqrt{2\mu}}{d}\,\mathrm{tr}(\boldsymbol{\tau})\mathbf{I} + \sqrt{2\mu}\boldsymbol{\tau}. \tag{4.20}$$

Proof Formula (4.19) is obtained by straightforward computation, and Formula (4.20) by looking for $\mathbb{C}^{1/2}$ as a linear elasticity tensor with coefficients (α, β) such that $\mathbb{C}^{1/2}\mathbb{C}^{1/2} = \mathbb{C}$, which boils down from (4.19) to solving $\alpha^2 d + 4\alpha\beta = \lambda$ and $4\beta^2 = 2\mu$. □

Lemma 4.10. *If $\mathbb{E} : (\mathbf{R}^{d\times d}, :) \to (\mathbf{R}^{d\times d}, :)$ is symmetric positive definite and satisfies, for all $\boldsymbol{\tau} \in \mathbf{R}^{d\times d}$, $(\mathbb{E}\boldsymbol{\tau})^T = \mathbb{E}\boldsymbol{\tau}^T$, then $\mathbb{E}^{1/2}$ also satisfies this property.*

Proof Let $\mathcal{L} : \mathbf{R}^{d\times d} \to \mathbf{R}^{d\times d}$ be the endomorphism $\mathcal{L}\boldsymbol{\tau} = (\mathbb{E}^{1/2}\boldsymbol{\tau}^T)^T$. Using $\boldsymbol{\tau} : \boldsymbol{\omega} = \boldsymbol{\tau}^T : \boldsymbol{\omega}^T$ and the symmetric positive definite character of $\mathbb{E}^{1/2}$, it is easy to check that $\mathcal{L}$ is symmetric positive definite. Moreover, by assumption on $\mathbb{E}$, $\mathcal{L}^2\boldsymbol{\tau} = (\mathbb{E}^{1/2}[(\mathbb{E}^{1/2}\boldsymbol{\tau}^T)^T]^T)^T = (\mathbb{E}^{1/2}\mathbb{E}^{1/2}\boldsymbol{\tau}^T)^T = (\mathbb{E}\boldsymbol{\tau}^T)^T = \mathbb{E}\boldsymbol{\tau}$. Henceforth, $\mathcal{L}$ is the symmetric positive definite square root $\mathbb{E}^{1/2}$ of $\mathbb{E}$ and thus $\mathbb{E}^{1/2}\boldsymbol{\tau}^T = \mathcal{L}(\boldsymbol{\tau}^T) = (\mathbb{E}^{1/2}\boldsymbol{\tau})^T$, which completes the proof. □

5. Conclusion

In this work, we developed the Gradient Scheme framework for linear and non-linear elasticity equations. We proved that this framework makes possible error estimates (for linear equations) and convergence analysis (for non-linear equations) of numerical methods under very few assumptions. In particular, these results hold for conforming as well as non-conforming methods, without assuming the full H^2-regularity of the exact solution (which can be lost in the cases of composite materials or non-linear models).

We showed that many classical and modern numerical schemes developed in the literature for elasticity equations are actually Gradient Schemes. We even established that some three-field schemes, based on a modified Hu-Washizu formulation and designed to be stable in the quasi-incompressible limit, are also Gradient Schemes after being recast in a displacement-only formulation by static condensation.

Since Gradient Schemes are seamlessly applicable to both linear and non-linear equations, the embedding into this framework of numerical methods developed for linear elasticity also allowed us to adapt those methods to non-linear elasticity, while retaining nice stability and convergence properties.

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