Noncoercive convection-diffusion elliptic problems with Neumann boundary conditions

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Abstract

We study the existence and uniqueness of solutions of the convective-diffusive elliptic equation

$$-\mathrm{div}(D\nabla u) + \mathrm{div}(\mathbf{V}\, u) = f$$

posed in a bounded domain $\Omega \subset \mathbb{R}^N$, with pure Neumann boundary conditions

$$D\nabla u \cdot \mathbf{n} = (\mathbf{V} \cdot \mathbf{n})\, u \qquad \text{on } \partial\Omega.$$

Under the assumption that $\mathbf{V} \in L^p(\Omega)^N$ with $p = N$ if $N \geq 3$ (resp. $p > 2$ if $N = 2$), we prove that the problem has a solution $u \in H^1(\Omega)$ if $\int_\Omega f \,\mathrm{d}x = 0$, and also that the kernel is generated by a function $\widehat{u} \in H^1(\Omega)$, unique up to a multiplicative constant, which satisfies $\widehat{u} > 0$ a.e. on Ω. We also prove that the equation

$$-\mathrm{div}(D\nabla u) + \mathrm{div}(\mathbf{V}\, u) + \nu\, u = f$$

has a unique solution for all $\nu > 0$ and the map $f \mapsto u$ is an isomorphism of the respective spaces. The study is made in parallel with the dual problem, with equation

$$-\mathrm{div}(D^T\nabla v) - \mathbf{V} \cdot \nabla v = g.$$

The dependence on the data is also examined, and we give applications to solutions of nonlinear elliptic PDE with measure data and to parabolic problems.

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1 Introduction and main result

We study the existence and uniqueness of solutions for the following convective-diffusive elliptic problem with pure Neumann boundary conditions

$$\begin{cases} -\mathrm{div}(D\nabla u) + \mathrm{div}(\mathbf{V}\, u) = f & \text{in } \Omega \\ D\,\nabla u \cdot \mathbf{n} = (\mathbf{V} \cdot \mathbf{n})\, u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

under the following basic assumptions:

(H1) Ω is a bounded connected open set of $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz-continuous boundary,

(H2) The matrix valued function $D : \Omega \to M_N(\mathbb{R})$ is bounded and measurable and there exists $\alpha > 0$ such that

$$D(x)\,\xi \cdot \xi \geq \alpha|\xi|^2 \text{ for all } \xi \in \mathbb{R}^N \text{ and a.e. } x \in \Omega,$$

(H3) $\mathbf{V} \in L^p(\Omega)^N$ with $p = N$ if $N \geq 3$, or $p > 2$ if $N = 2$,

(H4) $f \in (H^1(\Omega))'$.

Here, $\mathbf{n}$ is the unit normal to $\partial\Omega$ and dot indicates scalar product in $\mathbb{R}^N$. We are mainly interested in the effect of the convection term on the existence of solutions and in finding a natural functional framework where the problem is well-posed. Equations of this form where the convection speed $\mathbf{V}$ is the gradient of a potential, $\mathbf{V}(x) = \nabla\Phi(x)$, are known as the stationary version of the Fokker-Planck equation, [20], but we will not be confined here to such gradient speeds.

We consider weak solutions to (1.1) understood in the standard sense, that is

$$\begin{cases} u \in H^1(\Omega)\,, \\ \forall \varphi \in H^1(\Omega)\,,\ \displaystyle\int_\Omega D\nabla u \cdot \nabla\varphi \,\mathrm{d}x - \int_\Omega u\mathbf{V} \cdot \nabla\varphi \,\mathrm{d}x = \langle f, \varphi\rangle_{(H^1)', H^1}. \end{cases} \tag{1.2}$$

Here, $\langle \cdot, \cdot \rangle$ is the duality pairing. Under suitable assumptions on the divergence of $\mathbf{V}$ and the values of $\mathbf{V}$ on $\partial\Omega$, the bilinear form $\mathbf{a}(u, \varphi) = \int_\Omega D\nabla u \cdot \nabla\varphi \,\mathrm{d}x - \int_\Omega u\mathbf{V} \cdot \nabla\varphi \,\mathrm{d}x$ appearing in (1.2) is coercive on $H^1(\Omega)$, and existence of a solution to this problem is then an immediate consequence of the Lax-Milgram theorem.

However, for general vector fields $\mathbf{V}$, this bilinear form fails to be coercive and existence of a solution is less obvious; it is well known that, in any case, such elliptic problems (for Neumann or other boundary conditions) have finite dimensional kernels and that their solvability requires the right-hand side to satisfy a number of equations (as many equations as the dimension of the kernel of the elliptic problem), see e.g. [6], [17]. The question is then to determine how many equations are needed and, more precisely, which ones.

For other boundary conditions (Dirichlet, Fourier or mixed conditions), the answer was given in [12] and is quite simple: no condition on f is required, the convection-diffusion problem always has one and only one solution (it is also the same for some non-linear equations, see [11], and for some singular right-hand sides, see [14] and [13]). But, in these references, the proof of the existence of solutions is made via *a priori* estimates which are not available, at least by direct methods, in the case of pure Neumann boundary conditions. Hence, the question for (1.2) remains open: which necessary and sufficient conditions must be placed on f, and what are the degrees of freedom on the solutions?

The study of non-coercive linear elliptic problems as (1.1) is usually performed simultaneously with the study of the associated dual problem, that is

$$\begin{cases} -\mathrm{div}(D^T\nabla v) - \mathbf{V}\cdot\nabla v = g & \text{in } \Omega, \\ D^T\nabla v\cdot\mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $g \in (H^1(\Omega))'$ and D^T is the transpose of D, cf. for instance [12] for the study with other boundary conditions. The weak formulation of this dual problem is

$$\begin{cases} v \in H^1(\Omega)\,, \\ \forall\varphi \in H^1(\Omega)\,, \ \displaystyle\int_\Omega D^T\nabla v\cdot\nabla\varphi\,\mathrm{d}x - \int_\Omega \varphi\mathbf{V}\cdot\nabla v\,\mathrm{d}x = \langle g,\varphi\rangle_{(H^1)',H^1}. \end{cases} \tag{1.4}$$

Here too, the bilinear form $\widehat{\mathbf{a}}$ in the left-hand side of (1.4) is not coercive without special assumptions on $\mathbf{V}$, so the question is now to understand under which conditions this problem has a solution.

There are however two straightforward facts on these primal and dual problems.

a) In order for (1.2) to have a solution, one must have $\langle f, 1\rangle_{(H^1)',H^1} = 0$.

b) If v is a solution to (1.4), then so is $v + C$.

Assertion a) states at least one condition on the right-hand side of (1.2): we need to determine if it is the only one. Moreover, since there are as many conditions as the dimension of the kernel of this problem, such a kernel is non-trivial: can we describe it? How can we select one and only one solution?

The same holds for (1.4): assertion b) shows that the kernel of the dual problem is non trivial; so we ask ourselves, are there any functions other than the constant functions in this kernel? And since this kernel is non-trivial, there must be some condition, that we want to find, on g for (1.4) to have a solution.

Statement of the main result

We use the notation $H_\star = \{u \in H^1(\Omega) \mid \int_\Omega u\,\mathrm{d}x = 0\}$, and for $\varphi \in H^1(\Omega)$ we define

$$H'_{\sharp\varphi} = \{f \in (H^1(\Omega))' \mid \langle f,\varphi\rangle_{(H^1)',H^1} = 0\}.$$

These spaces are endowed with the topologies of $H^1(\Omega)$ and $(H^1(\Omega))'$ respectively. Here is our main result on Problems (1.1) and (1.3), which answers the preceding questions.

Theorem 1.1 *Assume that* (H1)–(H4) *hold. Then,*

(i) There exists a function $\widehat{u} \in H^1(\Omega)$, unique up to a multiplicative constant, which is a solution of Problem (1.2) *with $f = 0$, satisfies $\widehat{u} > 0$ a.e. on Ω, and moreover*

(ii) Problem (1.2) *has a solution if and only if $f \in H'_{\sharp 1}$; in this case, there exists a unique solution u to* (1.2) *in $H_\star$ and the set of all solutions is $u + \mathbb{R}\,\widehat{u} = \{u + c\widehat{u} : c \in \mathbb{R}\}$. Moreover, the application $\mathcal{T} : f \mapsto u$ is a bounded linear map from $H'_{\sharp 1}$ into $H_\star$.*

(iii) Problem (1.4) *has a solution if and only if $g \in H'_{\sharp\widehat{u}}$; in this case, there exists a unique solution v to* (1.4) *in $H_\star$ and the set of all solutions is $v + \mathbb{R}\,1$. Moreover, the application $\mathcal{T}' : g \mapsto v$ is a bounded linear map from $H'_{\sharp\widehat{u}}$ into $H_\star$.*

A simple case: $\mathbf{V}$ is a gradient

(i) If D is the identity matrix and $\mathbf{V}$ is a gradient, say $\mathbf{V} = \nabla\Phi$ with $\Phi \in C^1(\overline{\Omega})$, then we can take $\widehat{u} = e^{\Phi}$ in Theorem 1.1. Indeed, in this situation,

$$-\Delta e^{\Phi} + \mathrm{div}(\mathbf{V}e^{\Phi}) = -\mathrm{div}(\nabla e^{\Phi} - \mathbf{V}e^{\Phi}) = -\mathrm{div}(e^{\Phi}\nabla\Phi - \nabla\Phi\, e^{\Phi}) = 0 \quad \text{in } \Omega$$

and

$$\nabla e^{\Phi} \cdot \mathbf{n} - \mathbf{V} \cdot \mathbf{n} e^{\Phi} = e^{\Phi}\nabla\Phi \cdot n - \nabla\Phi \cdot \mathbf{n} e^{\Phi} = 0 \quad \text{on } \partial\Omega.$$

In fact, in this situation, solving (1.1) is best done by the change of unknown $w = e^{-\Phi}u$, which leads to

$$\begin{cases} -\mathrm{div}(e^{\Phi}\nabla w) = f & \text{in } \Omega, \\ e^{\Phi}\nabla w \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

that is to say a classical heterogeneous Neumann problem (notice that, for some $\beta > 0$, $\beta \leq e^{\Phi} \leq \beta^{-1}$ on Ω). The usual choice to select one and only one solution to (1.5) is to impose $\int_\Omega w \,\mathrm{d}x = 0$, which means that the chosen solution u to (1.2) satisfies $\int_\Omega e^{-\Phi}u \,\mathrm{d}x = 0$ and not $\int_\Omega u \,\mathrm{d}x = 0$ but, adding a suitable constant to w, we can find back the solution in $H_\star$ of (1.2) given by Theorem 1.1.

(ii) In the case where $D \neq \mathrm{Id}$, the same considerations hold if $\mathbf{V} = D\nabla\Phi$ for some $\Phi \in C^1(\overline{\Omega})$. In this case, we still have $\widehat{u} = e^{\Phi}$ and $w = e^{-\Phi}u$ satisfies $-\mathrm{div}(e^{\Phi}D\nabla w) = f$ in Ω and $e^{\Phi}D\nabla w \cdot \mathbf{n} = 0$ on $\partial\Omega$.

2 Preliminary considerations. Kernels

Let $L : H^1(\Omega) \to (H^1(\Omega))'$ be the bounded linear operator defined by the bilinear form $\mathbf{a}(u,\varphi)$ appearing in (1.2), that is to say

$$\langle Lu, \varphi\rangle_{(H^1)',H^1} = \int_\Omega D\nabla u \cdot \nabla\varphi \,\mathrm{d}x - \int_\Omega u\mathbf{V} \cdot \nabla\varphi \,\mathrm{d}x.$$

Thanks to (H1)–(H3) and the Sobolev embeddings, L is well defined. For $\gamma \in \mathbb{R}$ we denote $L_\gamma = L + \gamma I$ where $I : H^1(\Omega) \to (H^1(\Omega))'$ is the natural embedding through $L^2(\Omega)$, that is to say

$$\langle L_\gamma u, \varphi\rangle_{(H^1)',H^1} = \int_\Omega D\nabla u \cdot \nabla\varphi \,\mathrm{d}x - \int_\Omega u\mathbf{V} \cdot \nabla\varphi \,\mathrm{d}x + \gamma \int_\Omega u\varphi \,\mathrm{d}x.$$

It is well-known that, for $\gamma > 0$ large enough, $L_\gamma : H^1(\Omega) \to (H^1(\Omega))'$ is an isomorphism (see also Lemma 4.1); from now on, we take such a γ fixed. The restriction of $\gamma(L_\gamma)^{-1} : (H^1(\Omega))' \to H^1(\Omega)$ to $L^2(\Omega)$ is a compact operator $K : L^2(\Omega) \to L^2(\Omega)$. Moreover, since K has values in $H^1(\Omega)$,

$$\begin{aligned} \big(u \in L^2(\Omega) \text{ and } Ku = u\big) &\iff \big(u \in H^1(\Omega) \text{ and } Ku = u\big) \\ &\iff \big(u \in H^1(\Omega) \text{ and } \gamma u = L_\gamma u\big) \\ &\iff \big(u \in H^1(\Omega) \text{ and } Lu = 0\big). \end{aligned}$$

In other terms,

$$\ker(\mathrm{Id}_{L^2} - K) = \ker(L). \tag{2.1}$$

The following lemma gives a first description of $\ker(L)$, which will be made precise below. Its proof summons up a technique close to the one used in [3] (to prove the uniqueness of a non-monotone elliptic equation with Dirichlet boundary conditions), but in the framework of Neumann boundary conditions.

Lemma 2.1 *Assume that* (H1)–(H3) *hold. If* $u \in \ker(L)\backslash\{0\}$*, then either* $u > 0$ *a.e. on* Ω *or* $u < 0$ *a.e. on* Ω*. In particular,* $\ker(L)$ *has dimension* 0 *or* 1.

Proof of Lemma 2.1. Since u is not a.e. null, there exists $\eta > 0$ such that $u \le -\eta$ or $u \ge \eta$ on a set of non-null Lebesgue measure; upon changing u into $-u$, we can assume that the case $u \ge \eta$ occurs. Let $\varphi_\varepsilon(r) = 0$ if $r \ge \varepsilon$, $\varphi_\epsilon(r) = r - \varepsilon$ if $0 < r < \varepsilon$ and $\varphi_\varepsilon(r) = -\varepsilon$ if $r \le 0$ (see figure 1).

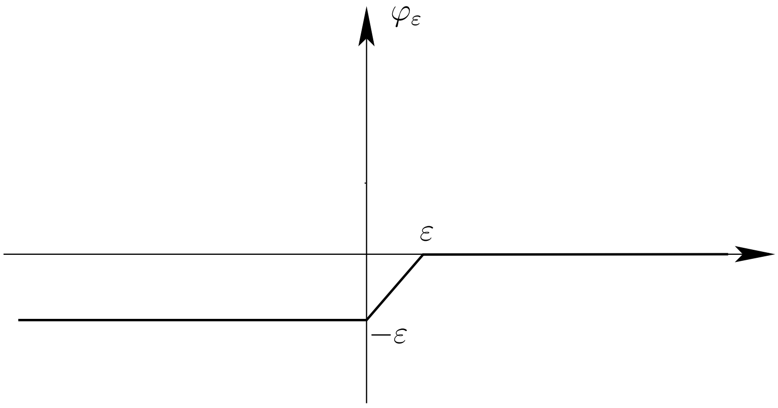

Figure 1: The function φ_ε.

The function u satisfies (1.2) with $f = 0$; using $\varphi_\varepsilon(u) \in H^1(\Omega)$ as a test function in this equation, since $\nabla(\varphi_\varepsilon(u)) = \mathbf{1}_{\{0<u<\varepsilon\}}\nabla u$ (where $\mathbf{1}_B$ stands for the characteristic function of a set B), we obtain

$$\int_\Omega D\nabla(\varphi_\varepsilon(u))\cdot\nabla(\varphi_\varepsilon(u))\,\mathrm{d}x = \int_{\{0<u<\varepsilon\}} u\mathbf{V}\cdot\nabla(\varphi_\varepsilon(u))\,\mathrm{d}x \le \varepsilon||\mathbf{V}||_{L^2(\{0<u<\varepsilon\})^N}||\nabla(\varphi_\varepsilon(u))||_{L^2(\Omega)^N}.$$

Since the measure of $\{0 < u < \varepsilon\}$ tends to 0 as $\varepsilon \to 0$, we deduce from this inequality and (H2) that $||\nabla(\varphi_\varepsilon(u))||_{L^2(\Omega)^N} \le \frac{1}{\alpha}\varepsilon||\mathbf{V}||_{L^2(\{0<u<\varepsilon\})^N} = \varepsilon\omega(\varepsilon)$ where $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let E be a set of non-null Lebesgue measure such that $u \ge \eta$ on E; for all $0 < \varepsilon < \eta$ we have $\varphi_\varepsilon(u) = 0$ on E and thus, by Lemma 8.1 (see Appendix), there exists C_E not depending on $\varepsilon \in]0, \eta[$ such that $||\varphi_\varepsilon(u)||_{L^2(\Omega)} \le C_E||\nabla(\varphi_\varepsilon(u))||_{L^2(\Omega)^N} \le C_E\varepsilon\omega(\varepsilon)$. But $|\varphi_\varepsilon(u)| \ge \varepsilon/2$ on $\{u \le \varepsilon/2\}$, which implies

$$||\varphi_\varepsilon(u)||_{L^2(\Omega)} \ge \left(\int_{\{u\le\varepsilon/2\}} |\varphi_\varepsilon(u)|^2\,\mathrm{d}x\right)^{1/2} \ge \frac{\varepsilon}{2}\left(\mathrm{meas}(\{u \le \varepsilon/2\})\right)^{1/2}$$

and therefore $\left(\mathrm{meas}(\{u \le \varepsilon/2\})\right)^{1/2} \le 2C_E\omega(\varepsilon)$; we deduce that

$$\mathrm{meas}(\{u \le 0\}) = \lim_{\varepsilon\to 0} \mathrm{meas}(\{u \le \varepsilon/2\}) = 0,$$

that is to say $u > 0$ a.e. on Ω.

The proof that the dimension of $\ker(L)$ cannot be more than 1 is rather simple. We take non-null functions $(u,v) \in \ker(L)$ and prove that they are collinear. Since $v \neq 0$, then either $v > 0$ or $v < 0$ a.e. on Ω; in particular, $\lambda = \int_\Omega u\,\mathrm{d}x / \int_\Omega v\,\mathrm{d}x$ is well defined. The function $w = u - \lambda v$ belongs to $\ker(L)$ and, thus, either $w > 0$ or $w < 0$ or $w = 0$ a.e. on Ω; since $\int_\Omega w\,\mathrm{d}x = 0$ by choice of λ, the first two cases cannot occur, and we therefore have $w = 0$, which means that $u = \lambda v$ and concludes the proof. ∎

The operator $L^* : H^1(\Omega) \to (H^1(\Omega))'$ defined by the bilinear form $\widehat{\mathbf{a}}$ in (1.4), that is to say

$$\langle L^* v, \varphi\rangle_{(H^1)',H^1} = \int_\Omega D^T \nabla v \cdot \nabla\varphi\,\mathrm{d}x - \int_\Omega \varphi \mathbf{V}\cdot\nabla v\,\mathrm{d}x\,,$$

is the adjoint of L: it satisfies

$$\langle L\varphi, \psi\rangle_{(H^1)',H^1} = \langle L^*\psi, \varphi\rangle_{(H^1)',H^1} \qquad \text{for all } (\varphi,\psi) \in H^1(\Omega). \tag{2.2}$$

Hence, for $\gamma > 0$ such that L_γ is an isomorphism, $L^*_\gamma = L^* + \gamma I : H^1(\Omega) \to (H^1(\Omega))'$ is also an isomorphism. It is easy to see from (2.2) that the restriction of $\gamma(L^*_\gamma)^{-1} : (H^1(\Omega))' \to H^1(\Omega)$ to $L^2(\Omega)$ is the dual operator $K^* : L^2(\Omega) \to L^2(\Omega)$ to K. Hence, as (2.1) we have

$$\ker(\mathrm{Id}_{L^2} - K^*) = \ker(L^*). \tag{2.3}$$

The properties of compact operators and Lemma 2.1 now allow us to give a precise description of the kernels of L and L^*.

Proposition 2.2 *Assume that* (H1)–(H3) *hold. Then,*

1. $\ker(L)$ *is one-dimensional, and is spanned by a function* $\widehat{u} \in H^1(\Omega)$ *which satisfies* $\widehat{u} > 0$ *a.e. on* Ω*,*
2. $\ker(L^*)$ *is exactly the set of constant functions.*

Proof. Since $K : L^2(\Omega) \to L^2(\Omega)$ is compact, we have $\dim(\ker(\mathrm{Id}_{L^2} - K)) = \dim(\ker(\mathrm{Id}_{L^2} - K^*))$ (see [17]). Properties (2.1) and (2.3) and the second part of Lemma 2.1 then show that $\dim(\ker(L^*)) = \dim(\ker(L)) \leq 1$. But the constant functions obviously are in $\ker(L^*)$, so that $\ker(L^*)$ is exactly made out of these functions and $\dim(\ker(L)) = 1$. The fact that $\ker(L)$ has a generator function which is a.e. positive then follows from the first part of Lemma 2.1. ∎

Remark 2.3 *Note that* $\widehat{u}$ *is constant only if* $\mathrm{div}(\mathbf{V}) = 0$ *in* Ω *and* $\mathbf{V}\cdot\mathbf{n} = 0$ *on* $\partial\Omega$*. In this case, for all* $\varphi \in H^1(\Omega)$ *one has* $\int_\Omega \varphi\mathbf{V}\cdot\nabla\varphi\,\mathrm{d}x = 0$ *and, by the Poincaré-Wirtinger's inequality, the bilinear forms in* (1.2) *and* (1.4) *are coercive on* $H_\star$*; Theorem 1.1 (with* $\widehat{u} = 1$*) is then a trivial consequence of the Lax-Milgram theorem.*

3 Proof of Theorem 1.1

The proof is a direct consequence of the Fredholm alternative and the characterization of the kernels of L and L^*. For the sake of completeness, let us quickly recall the reasoning which leads to these results.

Solving (1.2) comes down to finding $u \in H^1(\Omega)$ such that $Lu = f$; this is equivalent to $L_\gamma u = f + \gamma u$ and thus $u = (L_\gamma)^{-1} f + \gamma (L_\gamma)^{-1} u = (L_\gamma^{-1}) f + Ku$ since $u \in L^2(\Omega)$. Let $w = (L_\gamma)^{-1} f \in H^1(\Omega)$; since K has values in $H^1(\Omega)$, finding $u \in H^1(\Omega)$ such that $u = w + Ku$ is equivalent to finding $u \in L^2(\Omega)$ such that $u = w + Ku$; but, K being compact, we have $(\mathrm{Id}_{L^2} - K)(L^2(\Omega)) = (\ker(\mathrm{Id}_{L^2} - K^*))^\perp$ (see [17] ([1])) and therefore, by (2.3), there exists $u \in L^2(\Omega)$ such that $u = w + Ku$ if and only if $w \in (\ker(\mathrm{Id}_{L^2} - K^*))^\perp = (\ker(L^*))^\perp$. But, by definition of $w = (L_\gamma)^{-1} f$, for all $\varphi \in \ker(L^*)$, since both w and φ are in $L^2(\Omega)$,

$$\langle f, \varphi \rangle_{(H^1)', H^1} = \langle Lw + \gamma w, \varphi \rangle_{(H^1)', H^1} = \langle L^* \varphi, w \rangle_{(H^1)', H^1} + \gamma \int_\Omega w \varphi \,\mathrm{d}x = \gamma \int_\Omega w \varphi \,\mathrm{d}x \,,$$

and therefore $w \in (\ker(L^*))^\perp$ if and only if $\langle f, \varphi \rangle_{(H^1)', H^1} = 0$ for all $\varphi \in \ker(L^*)$, i.e. if and only if $\langle f, 1 \rangle_{(H^1)', H^1} = 0$ by Proposition 2.2. Hence, (1.2) has a solution if and only if $f \in H'_{\sharp 1}$.

All the solutions to (1.2) can then be written $u = u_0 + z$ where u_0 is any fixed solution and $z \in \ker(L)$, that is to say, using the $\widehat{u}$ given by Proposition 2.2, $u = u_0 + \lambda \widehat{u}$ for some $\lambda \in \mathbb{R}$. Since $\int_\Omega \widehat{u} \,\mathrm{d}x > 0$ (because $\widehat{u} > 0$ a.e. on Ω), we can select one and only one solution by imposing $\int_\Omega u \,\mathrm{d}x = 0$ (this fixes $\lambda = -\int_\Omega u_0 \,\mathrm{d}x / \int_\Omega \widehat{u} \,\mathrm{d}x$), and all the other solutions lie in $u + \mathbb{R}\widehat{u}$.

We have therefore proved that, for all $f \in H'_{\sharp 1}$, (1.2) has a unique solution $u \in H_\star$. This defines an application $\mathcal{T} : H'_{\sharp 1} \to H_\star$ which is clearly linear, and the proof of the continuity of $\mathcal{T}$ can be made by way of the closed-graph theorem: if $\{f_n\}_{n \geq 1}$ is a sequence in $H'_{\sharp 1}$ that converges to some f in $H'_{\sharp 1}$ and if $\{u_n = \mathcal{T} f_n\}$ converges to some u in $H_\star$ then, writing (1.2) with u_n and f_n and passing to the limit $n \to +\infty$ we see that $u \in H_\star$ is a solution to (1.2) with f as right-hand side, i.e. that $u = \mathcal{T} f$; the graph of $\mathcal{T}$ is therefore closed and $\mathcal{T}$ is continuous.

The study of (1.4) is completely similar, inverting the roles of L and L^*.

4 Estimates and global continuity

Since the application $f \in H'_{\sharp 1} \to u \in H_\star$ solution to (1.2) is linear and continuous, there exists $C > 0$ not depending on f such that $||u||_{H^1(\Omega)} \leq C ||f||_{(H^1(\Omega))'}$; however, the preceding proof does not allow to estimate C and, in particular, to understand how it depends on D or $\mathbf{V}$ (the same holds for the solution to Problem (1.4)). Such estimates can be of importance in the case where one wants to study the behaviour of u as D or $\mathbf{V}$ varies. In this section, we intend to make clearer the way C depends on these data, and to prove that the solution and the kernel of the operator are both continuous with respect to these data.

To study this dependence, we need to vary $\mathbf{V}$; in order to emphasize on the dependence of L on $\mathbf{V}$, we therefore write $L = L^{\mathbf{V}}$ (and similarly $L_\gamma = L_\gamma^{\mathbf{V}}$). Moreover, if E is Banach space and $r > 0$, $B(r; E)$ denotes the closed ball in E of center 0 and radius r.

[1] Here and in the following, the orthogonal sets are always taken with respect to the scalar product in $L^2(\Omega)$.

Let us begin by making precise how γ must be chosen, for a given $\mathbf{V}$, in order for $L_\gamma^{\mathbf{V}}$ to be an isomorphism.

Lemma 4.1 *Assume that* (H1) *and* (H2) *hold. Let $p = N$ if $N \geq 3$ or $p > 2$ if $N = 2$. Let also $s > p$ and $R > 0$. Then, there exist $\varepsilon_0 > 0$, $M > 0$, and $\gamma_0 > 0$ such that for all $\gamma \geq \gamma_0$ and all $\mathbf{V} \in B(\varepsilon_0; L^p(\Omega)^N) + B(R; L^s(\Omega)^N)$, $L_\gamma^{\mathbf{V}} : H^1(\Omega) \to (H^1(\Omega))'$ is an isomorphism and we have*

$$||(L_\gamma^{\mathbf{V}})^{-1}||_{\mathcal{L}((H^1)', H^1)} \leq M. \tag{4.1}$$

More precisely, we can take any $\varepsilon_0 < \alpha/S(\Omega, p)$, where $S(\Omega, p)$ is the Sobolev constant of the embedding $H^1(\Omega) \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)$, and then $\gamma_0 = \gamma_0(\Omega, \alpha, p, s, \varepsilon_0, R)$ and $M = M(\Omega, \alpha, p, \varepsilon_0, R)$.

Proof. For all $u \in H^1(\Omega)$ and all $\gamma > 0$, using Young's inequality, we have

$$\begin{aligned} \langle L_\gamma^{\mathbf{V}} u, u \rangle_{(H^1)', H^1} &= \int_\Omega D\nabla u \cdot \nabla u \, \mathrm{d}x - \int_\Omega u\mathbf{V} \cdot \nabla u \, \mathrm{d}x + \gamma \int_\Omega u^2 \, \mathrm{d}x \\ &\geq \alpha ||\nabla u||^2_{L^2(\Omega)^N} + \gamma ||u||^2_{L^2(\Omega)} - ||u\mathbf{V}||_{L^2(\Omega)^N} ||\nabla u||_{L^2(\Omega)^N}. \end{aligned} \tag{4.2}$$

We then write $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$ with $||\mathbf{V}_1||_{L^p(\Omega)^N} \leq \varepsilon_0$ and $||\mathbf{V}_2||_{L^s(\Omega)^N} \leq R$. By the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)$, we have

$$\begin{aligned} ||u\mathbf{V}||_{L^2(\Omega)^N} &\leq ||u\mathbf{V}_1||_{L^2(\Omega)^N} + ||u\mathbf{V}_2||_{L^2(\Omega)^N} \\ &\leq ||u||_{L^{\frac{2p}{p-2}}(\Omega)} ||\mathbf{V}_1||_{L^p(\Omega)^N} + ||u||_{L^{\frac{2s}{s-2}}(\Omega)} ||\mathbf{V}_2||_{L^s(\Omega)^N} \\ &\leq S(\Omega, p)\varepsilon_0 ||u||_{H^1(\Omega)} + R||u||_{L^{\frac{2s}{s-2}}(\Omega)}. \end{aligned}$$

But, since $\frac{2s}{s-2} < \frac{2N}{N-2}$, $H^1(\Omega)$ is compactly embedded in $L^{\frac{2s}{s-2}}(\Omega)$ and, therefore, for all $\nu > 0$ there exists $C(\Omega, s, \nu) > 0$ such that $||u||_{L^{\frac{2s}{s-2}}(\Omega)} \leq \nu ||\nabla u||_{L^2(\Omega)^N} + C(\Omega, s, \nu)||u||_{L^2(\Omega)}$. We deduce that

$$||u\mathbf{V}||_{L^2(\Omega)^N} \leq (S(\Omega, p)\varepsilon_0 + R\nu)||\nabla u||_{L^2(\Omega)^N} + (S(\Omega, p)\varepsilon_0 + RC(\Omega, s, \nu))||u||_{L^2(\Omega)}$$

and, coming back to (4.2) and using Young's inequality, for all $\eta > 0$,

$$\begin{aligned} \langle L_\gamma^{\mathbf{V}} u, u \rangle_{(H^1)', H^1} \geq\ & (\alpha - S(\Omega, p)\varepsilon_0 - R\nu) ||\nabla u||^2_{L^2(\Omega)^N} + \gamma ||u||^2_{L^2(\Omega)} \\ & - (S(\Omega, p)\varepsilon_0 + RC(\Omega, s, \nu))||u||_{L^2(\Omega)} ||\nabla u||_{L^2(\Omega)^N} \\ \geq\ & (\alpha - S(\Omega, p)\varepsilon_0 - R\nu - \eta) ||\nabla u||^2_{L^2(\Omega)^N} \\ & + \left(\gamma - \frac{1}{4\eta}(S(\Omega, p)\varepsilon_0 + RC(\Omega, s, \nu))^2\right) ||u||^2_{L^2(\Omega)}. \end{aligned}$$

Fixing $\varepsilon_0 < \alpha/S(\Omega, p)$, we can choose $\nu = \nu(\Omega, \alpha, p, \varepsilon_0, R) > 0$ and $\eta = \eta(\Omega, \alpha, p, \varepsilon_0, R) > 0$ such that $\beta(\Omega, \alpha, p, \varepsilon_0, R) = \alpha - S(\Omega, p)\varepsilon_0 - R\nu - \eta > 0$ and, taking $\gamma_0(\Omega, \alpha, p, s, \varepsilon_0, R) = \beta(\Omega, \alpha, p, \varepsilon_0, R) + \frac{1}{4\eta}(S(\Omega, p)\varepsilon_0 + RC(\Omega, s, \nu))^2$ we see that, for all $\gamma \geq \gamma_0(\Omega, \alpha, p, s, \varepsilon_0, R)$,

$$\langle L_\gamma^{\mathbf{V}} u, u \rangle_{(H^1)', H^1} \geq \beta(\Omega, \alpha, p, \varepsilon_0, R)||u||^2_{H^1(\Omega)}. \tag{4.3}$$

Lax-Milgram's theorem then shows that $L_\gamma^{\mathbf{V}} : H^1(\Omega) \to (H^1(\Omega))'$ is an isomorphism and, for all $f \in (H^1(\Omega))'$, applying (4.3) to $u = (L_\gamma^{\mathbf{V}})^{-1} f$, we find $||u||_{H^1(\Omega)} \le \beta(\Omega,\alpha,p,\varepsilon_0,R)^{-1}||f||_{(H^1(\Omega))'}$, which concludes the proof. ■

We can now state and prove some more precise estimates on the bounds of the solution to (1.2) in $H_\star$.

Theorem 4.2 *Assume that* (H1) *and* (H2) *hold. Let $p = N$ if $N \ge 3$ or $p > 2$ if $N = 2$. Let $s > p$ and $R > 0$, and let $S(\Omega, p)$ and $\varepsilon_0 < \alpha/S(\Omega, p)$ be as in Lemma 4.1. Then, there exists $C(\Omega, D, p, s, \varepsilon_0, R) > 0$ such that, for all $\mathbf{V} \in B(\varepsilon_0; L^p(\Omega)^N) + B(R; L^s(\Omega)^N)$ and all $f \in H'_{\sharp 1}$, if u is the solution to* (1.2) *in $H_\star$ then*

$$||u||_{H^1(\Omega)} \le C(\Omega, D, p, s, \varepsilon_0, R)||f||_{(H^1(\Omega))'}. \tag{4.4}$$

Remark 4.3 *Note that, for all $\varepsilon_0 > 0$ and all $s > N$,*

$$L^p(\Omega)^N = B(\varepsilon_0; L^p(\Omega)^N) + \cup_{R>0} B(R; L^s(\Omega)^N).$$

Hence, this theorem gives an estimate for any $\mathbf{V}$ satisfying (H3)*; this estimate however does not only depend on the norm of $\mathbf{V}$ in $L^p(\Omega)^N$ but also on the way we can split $\mathbf{V}$ as a small function in $L^p(\Omega)^N$ and a function slightly more integrable; it is therefore completely similar to the estimates obtained for non-coercive elliptic problems with other boundary conditions (see* [12]*).*

Proof of Theorem 4.2. By linearity, it is sufficient to make the proof for $||f||_{(H^1(\Omega))'} = 1$. Assume that the result does not hold: then there exists $(\mathbf{V}_n)_{n\ge1} \in B(\varepsilon_0; L^p(\Omega)^N) + B(R; L^s(\Omega)^N)$ such that the solution $u_n \in H_\star$ to (1.2) with $\mathbf{V} = \mathbf{V}_n$ satisfies $||u_n||_{H^1(\Omega)} \to +\infty$ as $n \to +\infty$.

By Lemma 4.1, there exists $\gamma > 0$ and $M > 0$ not depending on n such that, for all $n \ge 1$, $L_\gamma^{\mathbf{V}_n}$ is an isomorphism and

$$||(L_\gamma^{\mathbf{V}_n})^{-1}||_{\mathcal{L}((H^1)',H^1)} \le M. \tag{4.5}$$

As we have seen in the proof of Theorem 1.1, the fact that u_n is a solution to (1.2) with $\mathbf{V} = \mathbf{V}_n$, i.e. that $L^{\mathbf{V}_n} u_n = f$, is equivalent to

$$u_n = w_n + \gamma(L_\gamma^{\mathbf{V}_n})^{-1} u_n, \tag{4.6}$$

where $(w_n)_{n\ge1} = ((L_\gamma^{\mathbf{V}_n})^{-1} f)_{n\ge1}$ is bounded in $H^1(\Omega)$ thanks to (4.5). If $(u_n)_{n\ge1}$ were bounded in $L^2(\Omega)$, and therefore in $(H^1(\Omega))'$, then (4.5) would give a bound on $(\gamma(L_\gamma^{\mathbf{V}_n})^{-1} u_n)_{n\ge1}$ in $H^1(\Omega)$, and, by (4.6), $(u_n)_{n\ge1}$ would be bounded in $H^1(\Omega)$; but we have precisely rejected this from the beginning.

Hence, $(u_n)_{n\ge1}$ is not bounded in $L^2(\Omega)$ and, up to a subsequence, $||u_n||_{L^2(\Omega)} \to +\infty$ as $n \to +\infty$. Let $h_n = u_n/||u_n||_{L^2(\Omega)} \in H_\star$; from (4.6) we deduce

$$h_n = \frac{w_n}{||u_n||_{L^2(\Omega)}} + \gamma(L_\gamma^{\mathbf{V}_n})^{-1} h_n. \tag{4.7}$$

We have $||h_n||_{L^2(\Omega)} = 1$ for all $n \ge 1$ and thus (4.5) implies that $(\gamma(L_\gamma^{\mathbf{V}_n})^{-1} h_n)_{n\ge1}$ is bounded in $H^1(\Omega)$; by (4.7), $(h_n)_{n\ge1}$ is therefore bounded in $H^1(\Omega)$. Up to a subsequence, $(h_n)_{n\ge1}$ converges to some h weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Since $\int_\Omega h_n \,\mathrm{d}x = 0$ and $||h_n||_{L^2(\Omega)} = 1$ for all $n \geq 1$, the same holds for h; in particular, $\int_\Omega h \,\mathrm{d}x = 0$ and $h \neq 0$.

Let $\varphi \in C^\infty(\overline{\Omega})$; by definition of u_n and $h_n = u_n/||u_n||_{L^2(\Omega)}$, we can write

$$\int_\Omega D\nabla h_n \cdot \nabla\varphi \,\mathrm{d}x - \int_\Omega h_n \mathbf{V}_n \cdot \nabla\varphi \,\mathrm{d}x = \langle \frac{f}{||u_n||_{L^2(\Omega)}}, \varphi\rangle_{(H^1)',H^1}. \tag{4.8}$$

Since $(\mathbf{V}_n)_{n\geq 1}$ is bounded in $L^p(\Omega)^N$, up to a subsequence we can assume that $\mathbf{V}_n \to \mathbf{V}$ weakly in $L^p(\Omega)^N$, and in particular weakly in $L^2(\Omega)^N$; the convergence of h_n to h weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ and the fact that $||u_n||_{L^2(\Omega)} \to +\infty$ then allow to pass to the limit $n \to +\infty$ in (4.8) to see that

$$\int_\Omega D\nabla h \cdot \nabla\varphi \,\mathrm{d}x - \int_\Omega h\mathbf{V} \cdot \nabla\varphi \,\mathrm{d}x = 0.$$

This equation has been proved for $\varphi \in C^\infty(\overline{\Omega})$ but, since $h \in H^1(\Omega)$ and $\mathbf{V} \in L^p(\Omega)^N$, by density it also holds for all $\varphi \in H^1(\Omega)$, which proves that $h \in \ker(L^{\mathbf{V}})$. Since $\int_\Omega h \,\mathrm{d}x = 0$ and $h \neq 0$, this is in contradiction with Proposition 2.2 which entails, in particular, that the only function in $\ker(L^{\mathbf{V}})$ which has a null mean value is the null function. Hence, the proof is concluded. ■

Remark 4.4 *This proof shows how C depends on $\mathbf{V}$ (through p, s, ε_0 and R), but it does not give an explicit bound on this constant or on the way it depends on D, contrary to the direct estimates made in the case of other boundary conditions (see [12]). In fact, for other boundary conditions, the existence of a solution is deduced from* a priori *explicit estimates; here, we first proved existence of a solution and then deduced* a posteriori *non-explicit estimates on this solution. This can be a problem when transferring these estimates to a discrete setting, in order to study numerical schemes on* (1.1) *(see e.g.* [15] *for the adaptation of the continuous estimates to the setting of finite volume schemes, in the case of convection-diffusion noncoercive problems with Dirichlet boundary conditions).*

Remark 4.5 *There are however some situations where we can ensure that the constant C in Theorem 4.2 does not explode as D varies: for example, if $(D_n)_{n\geq 1}$ is a bounded sequence of matrix-valued functions which satisfy (H2) (with a uniform $\alpha > 0$) and which converges a.e. to some D, then a straightforward adaptation of the proof of Theorem 4.2 shows that the corresponding $C(\Omega, D_n, p, s, \varepsilon_0, R)$ stays bounded as $n \to \infty$. This particular case will be useful in the following.*

Remark 4.6 *A result similar to Theorem* 4.2 *holds for the solution to* (1.4) *in $H_\star$. It can be deduced from Theorem* 4.2 *using the following simple reasoning: if $g \in H'_{\sharp\widehat{u}}$, and v is the solution to* (1.4) *in $H_\star$ then, for all $F \in (H^1(\Omega))'$, denoting u the solution to* (1.2) *in $H_\star$ with $f = F - \frac{1}{\mathrm{meas}(\Omega)}\langle F, 1\rangle_{(H^1)',H^1} \in H'_{\sharp 1}$, since $\int_\Omega v \,\mathrm{d}x = 0$,*

$$\begin{aligned}
\langle F, v\rangle_{(H^1)',H^1} = \langle f, v\rangle_{(H^1)',H^1} &= \langle L^{\mathbf{V}} u, v\rangle_{(H^1)',H^1} \\
&= \langle (L^{\mathbf{V}})^* v, u\rangle_{(H^1)',H^1} \\
&= \langle g, u\rangle_{(H^1)',H^1} \\
&\leq ||g||_{(H^1(\Omega))'}||u||_{H^1(\Omega)} \\
&\leq C(\Omega, D, p, s, \varepsilon_0, R)||g||_{(H^1(\Omega))'}||f||_{(H^1(\Omega))'}
\end{aligned}$$

and it is clear that $||f||_{(H^1(\Omega))'} \leq 2||F||_{(H^1(\Omega))'}$ so that, taking the supremum on $F \in (H^1(\Omega))'$ having norm smaller than 1, we obtain $||v||_{H^1(\Omega)} \leq 2C(\Omega, D, p, s, \varepsilon_0, R)||g||_{(H^1(\Omega))'}$.

Let us now study the continuity of the kernel of L and the solution to (1.2) with respect to all the data. We normalize the choices of functions $\widehat{u}$ in Theorem 1.1 by imposing $||\widehat{u}||_{H^1} = 1$.

Theorem 4.7 *Let* (H1) *hold and take $p = N$ if $N \geq 3$ or $p > 2$ if $N = 2$. Assume that $(D_n)_{n\geq 1}$ is a bounded sequence of uniformly elliptic matrix-valued functions which converges a.e. to some D, and let $\mathbf{V}_n \to \mathbf{V}$ in $L^p(\Omega)^N$. Then,*

i) The (normalized) function $\widehat{u}_n$ from Theorem 1.1 *corresponding to $(D_n, \mathbf{V}_n)$ converges in $H^1(\Omega)$ to the (normalized) function $\widehat{u}$ corresponding to $(D, \mathbf{V})$,*

ii) If $f_n \to f$ in $H'_{\sharp 1}$, then the solution $u_n \in H_\star$ of (1.2) *for $(D_n, \mathbf{V}_n, f_n)$ converges in $H^1(\Omega)$ to the solution $u \in H_\star$ of* (1.2) *for $(D, \mathbf{V}, f)$.*

Remark 4.8 *We could of course also state a result on the continuity of the solution to (1.4).*

Proof of Theorem 4.7. The proofs of i) and ii) are completely similar, so we only consider ii).

Since $(\mathbf{V}_n)_{n\geq 1}$ converges in $L^p(\Omega)^N$, it is easy to see that, for all $\varepsilon_0 > 0$, there exists $R > 0$ such that, for all $n \geq 1$, $\mathbf{V}_n \in B(\varepsilon_0; L^p(\Omega)^N) + B(R; L^\infty(\Omega)^N)$ (write $\mathbf{V}_n = (\mathbf{V}_n - T_M(\mathbf{V}_n)) + T_M(\mathbf{V}_n)$ for M large enough, where T_M is the truncation of each component at level M). Hence, by Theorem 4.2 and Remark 4.5, $(u_n)_{n\geq 1}$ is bounded in $H^1(\Omega)$, and converges up to a subsequence weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to some $u \in H_\star$. The strong convergence of the data and the weak convergence of the solutions allow to pass to the limit in the equations (1.2) with $(D_n, \mathbf{V}_n, f_n)$ statisfied by u_n to see that u is the (unique in $H_\star$) solution to this problem with $(D, \mathbf{V}, f)$ (and thus that the whole sequence u_n converges to u, not only a subsequence). It remains to prove that the convergence of u_n to u is strong in $H^1(\Omega)$.

In order to do so, we first notice that

$$\begin{aligned}
\int_\Omega D_n \nabla u_n \cdot \nabla u_n &= \int_\Omega u_n \mathbf{V}_n \cdot \nabla u_n + \langle f_n, u_n \rangle_{(H^1)', H^1} \\
&= \int_\Omega \mathbf{V}_n \cdot \nabla \left(\frac{u_n^2}{2}\right) + \langle f_n, u_n \rangle_{(H^1)', H^1}. \qquad (4.9)
\end{aligned}$$

Since u_n is bounded in $H^1(\Omega)$, $\nabla(u_n^2/2) = u_n \nabla u_n$ is bounded in $L^{p'}(\Omega)^N$ by our choice of p' and the Sobolev embeddings, and thus it weakly converges in this space to some $\mathbf{U}$; but $u_n^2 \to u^2$ strongly in $L^1(\Omega)$ and, therefore, $\nabla(u_n^2/2) \to \nabla(u^2/2)$ in the sense of distributions; this shows that $\mathbf{U} = \nabla(u^2/2)$. By weak convergence of u_n and $\nabla(u_n^2/2)$ and strong convergence of f_n and $\mathbf{V}_n$, we can then pass to the limit in the right-hand side of (4.9) and we deduce, since u is a solution to (1.2),

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega D_n \nabla u_n \cdot \nabla u_n &= \int_\Omega \mathbf{V} \cdot \nabla \left(\frac{u^2}{2}\right) + \langle f, u \rangle_{(H^1)', H^1} \\
&= \int_\Omega u \mathbf{V} \cdot \nabla u + \langle f, u \rangle_{(H^1)', H^1} = \int_\Omega D \nabla u \cdot \nabla u. \qquad (4.10)
\end{aligned}$$

The rest is quite classical: we have

$$\begin{aligned}\alpha||\nabla u_n - \nabla u||^2_{L^2(\Omega)^N} &\le \int_\Omega D_n \nabla(u_n - u)\cdot\nabla(u_n-u)\\ &= \int_\Omega D_n\nabla u_n\cdot\nabla u_n - \int_\Omega D_n\nabla u_n\cdot\nabla u - \int_\Omega D_n\nabla u\cdot\nabla u_n\\ &\quad+\int_\Omega D_n\nabla u\cdot\nabla u\end{aligned}$$

and, by strong convergence of D_n, weak convergence of ∇u_n and (4.10), we can pass to the limit to see that $||\nabla u_n - \nabla u||_{L^2(\Omega)^N} \to 0$. Combined with the strong convergence of u_n in $L^2(\Omega)$, this concludes the proof. ■

As a consequence we can state the continuity of the normalized map $\mathcal{S} : \mathbf{V} \mapsto \widehat{u}$, where $\widehat{u}$ is the unique normalized function of Theorem 1.1 and $\mathbf{V}$ satisfies (H3). It is a nonlinear map into $H_\star$, and we know that for $\mathbf{V} = 0$ we get a positive constant: $\mathcal{S}(0) = |\Omega|^{-1/2}$.

Corollary 4.9 *$\mathcal{S} : \mathbf{V} \mapsto \widehat{u}$ is continuous from L^N if $N \ge 3$ (resp. L^p with $p > 2$ if $N = 2$) to $H_\star$.*

5 Spectral analysis. Optimal solvability

There is an immediate generalization of (2.1) to other possible eigenvalues of K (the restriction to $L^2(\Omega)$ of $\gamma(L_\gamma)^{-1}$):

$$\forall\alpha \neq 0\,,\ \ker(\alpha \mathrm{Id}_{L^2} - K) = \ker\left(L_{\gamma\left(1-\frac{1}{\alpha}\right)}\right). \tag{5.1}$$

This allows in particular to prove that there is no necessity to take γ large in order for L_γ to be invertible.

Proposition 5.1 *Assume that* (H1)–(H3) *hold. Then, $L_\nu : H^1(\Omega) \to (H^1(\Omega))'$ is an isomorphism for every $\nu > 0$.*

Proof. Let us first prove that L_ν is one-to-one. Let $u \in \ker(L_\nu)$ and define $T_\varepsilon(r) = \min(\varepsilon, \max(r, -\varepsilon))$ the truncature function at level ε. Then $\nabla(T_\varepsilon(u)) = \mathbf{1}_{\{-\varepsilon<u<\varepsilon\}}\nabla u$ and thus

$$\begin{aligned}0 = \langle L_\nu u, T_\varepsilon(u)\rangle_{(H^1)',H^1} &= \int_\Omega D\nabla u\cdot\nabla(T_\varepsilon(u))\,\mathrm{d}x - \int_\Omega u\mathbf{V}\cdot\nabla(T_\varepsilon(u))\,\mathrm{d}x + \nu\int_\Omega uT_\varepsilon(u)\,\mathrm{d}x\\ &\ge \alpha||\nabla(T_\varepsilon(u))||^2_{L^2(\Omega)} - \varepsilon||\mathbf{V}||_{L^2(\Omega)^N}||\nabla(T_\varepsilon(u))||_{L^2(\Omega)} + \nu\int_\Omega uT_\varepsilon(u)\,\mathrm{d}x.\end{aligned}$$

Hence, using Young's inequality,

$$\nu\int_\Omega uT_\varepsilon(u)\,\mathrm{d}x \le \frac{\varepsilon^2||\mathbf{V}||^2_{L^2(\Omega)^N}}{4\alpha},$$

so that $\int_\Omega u\, T_\varepsilon(u)/\varepsilon \,\mathrm{d}x \le C\varepsilon$ with C not depending on ε. But, as $\varepsilon \to 0$, $T_\varepsilon(u)/\varepsilon \to \mathrm{sgn}(u)$ while staying bounded by 1 and, passing to the limit thanks to the dominated convergence theorem, we deduce $\int_\Omega |u| \,\mathrm{d}x \le 0$, that is to say $u = 0$. This proves that $\ker(L_\nu) = \{0\}$.

To prove that L_ν is onto, let $\mu \gg 1$ and denote by K the restriction to $L^2(\Omega)$ of $(\nu+\mu)(L_{\nu+\mu})^{-1}$. As at the beginning of the the proof of Theorem 1.1, we notice that, for all $f \in (H^1(\Omega))'$, finding $u \in H^1(\Omega)$ such that $L_\nu u = f$ is equivalent to finding $u \in L^2(\Omega)$ such that $u = w + \frac{\mu}{\nu+\mu} K u$ (where $w = (L_{\nu+\mu})^{-1} f \in H^1(\Omega)$), that is to say $\frac{\nu+\mu}{\mu} u - K u = \frac{\nu+\mu}{\mu} w$. But (5.1) applied to $\gamma = \nu + \mu$ and $\alpha = \frac{\nu+\mu}{\mu}$ shows that

$$\ker\left(\frac{\nu+\mu}{\mu}\mathrm{Id}_{L^2} - K\right) = \ker(L_\nu) = \{0\}.$$

Since K is compact, this implies that $\frac{\nu+\mu}{\mu}\mathrm{Id}_{L^2} - K$ is onto (see [17]) and therefore that there exists $u \in L^2(\Omega)$ such that $\frac{\nu+\mu}{\mu} u - K u = \frac{\nu+\mu}{\mu} w$, which concludes the proof. ■

Remark 5.2 *It is in fact possible to prove this proposition (which comes down to solving (1.1) with an additional term $+\nu u$ in the left-hand side of the PDE) using the techniques in [13]. The proof we give above is however much shorter, although it does not give estimates on the norm of $(L_\nu)^{-1}$.*

From this proposition and (5.1), we deduce that all the eigenvalues of K are in $]0,1]$ (it is quite clear that 0 is not an eigenvalue of K). The spectral analysis of compact operators then tells that the eigenvalues of K form a decreasing sequence which tends to 0. In terms of L, this means that the eigenvalues of L form a sequence of increasing nonnegative numbers which tends to $+\infty$. Moreover, using Proposition 2.2, we see that the first eigenvalue of L is 0, is of multiplicity 1 and is associated with an a.e. positive eigenfunction ([2]).

This property of positivity of the first eigenfunction (and therefore its uniqueness up to multiplication by a scalar) is exactly what is stated in the strong forms of the Krein–Rutman theorem. However, since the data D and $\mathbf{V}$ we consider are irregular, we had to conduct the study of (1.1) in Sobolev spaces in which the interior of the cone of nonnegative functions is empty; hence, on the contrary to what can be done for smooth solutions (see [16]), it does not seem possible to easily deduce the positivity of the first eigenfunction from a strong Krein-Rutman theorem (see [10] in the case of Dirichlet boundary conditions and regular data).

6 An application: uniqueness for nonlinear problems with measure data

One of the main applications of noncoercive convective-diffusive problems (i.e. without any assumption on the divergence of the convection) is the obtention of uniqueness results for nonlinear elliptic equations with measure data, as shown in [14] in the case of Dirichlet, Fourier or mixed boundary conditions. Now that we have shown how to solve convective-diffusive problems with pure Neumann boundary conditions, we can extend the uniqueness result of [14] to this setting. This is however not completely trivial, because of the lack of explicit estimates

[2] The same holds for L^*, by the properties of the spectrums of compact operators and their adjoint.

on the solutions to (1.2) and (1.4) as D varies (see Remark 4.4); therefore, the adaptation of [14] to Neumann boundary conditions demands a few more arguments than those used for other boundary conditions.

Consider the following nonlinear elliptic equation

$$\left\{\begin{array}{ll} -\mathrm{div}(a(x,w,\nabla w)) = \mu & \text{in } \Omega, \\ a(x,w,\nabla w)\cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{array}\right. \tag{6.1}$$

where

$$\bullet\ \mu \text{ is a bounded signed measure on } \Omega \text{ such that } \mu(\Omega)=0, \tag{6.2}$$

$$\begin{array}{c} \bullet\ a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N \text{ is a Caratheodory function satisfying:} \\ \exists\gamma>0\,,\ \exists\Theta\in L^1(\Omega) \text{ such that } a(x,s,\xi)\cdot\xi\ge\gamma|\xi|^2-\Theta(x)\,, \\ \exists\beta>0\,,\ \exists h\in L^2(\Omega) \text{ such that } |a(x,s,\xi)|\le h(x)+\beta|s|+\beta|\xi|\,, \end{array} \tag{6.3}$$

$$\bullet\ \exists\alpha>0 \text{ such that } (a(x,s,\xi)-a(x,s,\eta))\cdot(\xi-\eta)\ge\alpha|\xi-\eta|^2\,, \tag{6.4}$$

$$\bullet\ \exists\Lambda>0 \text{ such that } |a(x,s,\xi)-a(x,s,\eta)|\le\Lambda|\xi-\eta|\,, \tag{6.5}$$

$$\begin{array}{c} \bullet\ \exists\delta>0\,,\ \exists\omega\in[0,\frac{1}{N-2}[\,,\ \exists\chi\in[0,\frac{1}{N-1}[\text{ such that} \\ |a(x,s,\xi)-a(x,t,\xi)|\le\delta|s-t|(1+|s|^\omega+|t|^\omega+|\xi|^\chi). \end{array} \tag{6.6}$$

We refer to [14] for a discussion on these assumptions — notice that Assumption (6.6) is slightly more general here.

The existence of a weak solution to (6.1) in the case of homogeneous Dirichlet boundary conditions is now quite classical, see e.g. [2]. One way to prove this existence is to approximate μ in the weak-* topology by a sequence of data $(\mu_n)_{n\ge1}\in L^1(\Omega)\cap H^{-1}(\Omega)$, to consider a weak solution $w_n\in H^1_0(\Omega)$ to $-\mathrm{div}(a(x,w_n,\nabla w_n))=\mu_n$, to prove that $(w_n)_{n\ge1}$ is bounded in $W^{1,q}_0(\Omega)$ for all $q<\frac{N}{N-1}$, and to show the strong convergence in these spaces of this sequence towards a weak solution to $-\mathrm{div}(a(x,w,\nabla w))=\mu$; the solution thus obtained belongs to $\cap_{q<\frac{N}{N-1}}W^{1,q}_0(\Omega)$.

With our assumptions on a, this proof of existence of a solution can be easily adapted to the case of Neumann boundary conditions, using an approximation $\mu_n\in L^1(\Omega)\cap(H^1(\Omega))'$ with null mean value and selecting a solution $w_n\in H^1(\Omega)$ to (6.1) with $\mu=\mu_n$ which also has a null mean value; the proof of the *a priori* estimates on $(w_n)_{n\ge1}$ can be made using the same test-functions and methods as in [5] ([3]) and, to prove the strong convergence of $(w_n)_{n\ge1}$, the method from [1] works fine. This gives a solution to (6.1) in $\cap_{q<\frac{N}{N-1}}W^{1,q}(\Omega)$ with null mean value. A solution constructed this way is called a limit solution, or a SOLA: Solution Obtained as the Limit of Approximations (see [7]).

The next quaestion concerns the uniqueness of the solution, see e.g. [1], [4], [9], [19] for some results on this subject (for Dirichlet boundary conditions). In the framework of SOLAs and with Neumann conditions, being able to solve linear convective-diffusive problems with non-smooth data D and $\mathbf{V}$ (Theorem 1.1) allows to prove the following uniqueness result.

Theorem 6.1 *Under assumptions* (6.2)—(6.6)*, Problem* (6.1) *has a unique SOLA.*

[3] It is not clear that the other techniques used in the literature to prove a priori estimates in the case of Dirichlet boundary conditions can be adapted to Neumann boundary conditions.

Remark 6.2 *We could also state and prove a stability result for the SOLA to* (6.1)*, as it is done in* [14] *for other boundary conditions.*

Proof of Theorem 6.1. Let $(w, \widetilde{w})$ be two SOLA of (6.1). There exists thus $(\mu_n)_{n\geq 1} \in L^1(\Omega) \cap H'_{\sharp 1}$ and $(\widetilde{\mu}_n)_{n\geq 1} \in L^1(\Omega) \cap H'_{\sharp 1}$ which converge weakly-$*$ to μ and such that, for some $w_n \in H_\star$ and $\widetilde{w}_n \in H_\star$ weak solutions to (6.1) with respectively μ_n and $\widetilde{\mu}_n$ instead of μ, we have $w_n \to w$ and $\widetilde{w}_n \to \widetilde{w}$ in $W^{1,q}(\Omega)$ for all $q < \frac{N}{N-1}$.

Thanks to the assumptions on a and the fact that $(w_n)_{n\geq 1}$ and $(\widetilde{w}_n)_{n\geq 1}$ strongly converge in $W^{1,1}(\Omega)$, the functions $a_n(x) = a(x, w_n(x), \nabla w_n(x)) - a(x, w_n(x), \nabla \widetilde{w}_n(x))$ and $b_n(x) = \nabla w_n(x) - \nabla \widetilde{w}_n(x)$ satisfy, up to a subsequence, the assumptions of Lemma 8.2 in the appendix; we still note $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ the subsequences involved and we let $(D_n)_{n\geq 1}$ be the matrix-valued functions given by this lemma. These matrices are thus bounded and uniformly elliptic (independently of n) and there exists $(\varepsilon_n)_{n\geq 1}$ which tends to 0 as $n \to \infty$ such that, on $\{|\nabla w_n(x) - \nabla \widetilde{w}_n(x)| \geq \varepsilon_n\}$,

$$a(x, w_n(x), \nabla w_n(x)) - a(x, w_n(x), \nabla \widetilde{w}_n(x)) = D_n(x)(\nabla w_n(x) - \nabla \widetilde{w}_n(x)). \tag{6.7}$$

Hence, subtracting the equations satisfied by w_n and $\widetilde{w}_n$, and denoting

$$\mathbf{V}_n(x) = -\frac{a(x, w_n(x), \nabla \widetilde{w}_n(x)) - a(x, \widetilde{w}_n(x), \nabla \widetilde{w}_n(x))}{w_n(x) - \widetilde{w}_n(x)},$$

we see that $\Gamma_n = w_n - \widetilde{w}_n \in H_\star$ satisfies, for all $\varphi \in H^1(\Omega)$,

$$\begin{aligned}\int_\Omega D_n \nabla \Gamma_n \cdot \nabla \varphi \,\mathrm{d}x - \int_\Omega \Gamma_n \mathbf{V}_n \cdot \nabla \varphi \,\mathrm{d}x &= \langle \mu_n - \widetilde{\mu}_n, \varphi \rangle_{(H^1)', H^1} \\ &- \int_{\{|\nabla \Gamma_n| < \varepsilon_n\}} (a(x, w_n, \nabla w_n) - a(x, w_n, \nabla \widetilde{w}_n) - D_n \nabla \Gamma_n) \cdot \nabla \varphi \,\mathrm{d}x.\end{aligned} \tag{6.8}$$

Thanks to Assumption (6.6) we see that $|\mathbf{V}_n| \leq \delta(1 + |w_n|^\omega + |\widetilde{w}_n|^\omega + |\nabla \widetilde{w}_n|^\chi)$ and, since $(w_n)_{n\geq 1}$ and $(\widetilde{w}_n)_{n\geq 1}$ are bounded in $W^{1,q}(\Omega)$ for all $q < \frac{N}{N-1}$, we infer that $(\mathbf{V}_n)_{n\geq 1}$ is bounded in $L^s(\Omega)^N$ for some $s > N$.

Let $h \in L^\infty(\Omega)$ and denote by $\widehat{u}_n \in H^1(\Omega)$ a positive function given by Theorem 1.1 for $D = D_n$ and $\mathbf{V} = \mathbf{V}_n$. Defining $t_n = \frac{\int_\Omega h \widehat{u}_n \,\mathrm{d}x}{\int_\Omega \widehat{u}_n \,\mathrm{d}x}$, we see that $h - t_n \in H'_{\sharp \widehat{u}_n}$ and we can therefore consider the solution ψ_n to (1.4) with $D = D_n$, $\mathbf{V} = \mathbf{V}_n$ and $g = h - t_n$, that is to say: $\psi_n \in H_\star$ and

$$\forall \varphi \in H^1(\Omega)\,,\ \int_\Omega D_n^T \nabla \psi_n \cdot \nabla \varphi \,\mathrm{d}x - \int_\Omega \varphi \mathbf{V}_n \cdot \nabla \psi_n \,\mathrm{d}x = \langle h - t_n, \varphi \rangle_{(H^1)', H^1} = \int_\Omega (h - t_n) \varphi \,\mathrm{d}x. \tag{6.9}$$

We notice that $|t_n| \leq ||h||_\infty$ and therefore $(h - t_n)_{n\geq 1}$ is bounded in $L^\infty(\Omega)$ (and thus also in $(H^1(\Omega))'$); since $(\mathbf{V}_n)_{n\geq 1}$ is bounded in $L^s(\Omega)^N$ for some $s > N$ and $(D_n)_{n\geq 1}$ is bounded uniformly elliptic and converges a.e. on Ω, Remark 4.5 (see also Remark 4.6) shows that

$$(\psi_n)_{n\geq 1} \text{ is bounded in } H^1(\Omega). \tag{6.10}$$

Moreover, using again the bound on $(h - t_n)_{n\geq 1}$ in $L^\infty(\Omega)$, Theorem A.1 in [11, Appendix A] shows that there exists $\kappa > 0$ such that

$$(\psi_n)_{n\geq 1} \text{ is bounded in the Hölder space } C^{0,\kappa}(\Omega). \tag{6.11}$$

Pluging $\varphi = \Gamma_n$ in (6.9) and $\varphi = \psi_n$ in (6.8) we deduce, since Γ_n has a null mean value,

$$\begin{aligned}\int_\Omega h\Gamma_n \,\mathrm{d}x &= \int_\Omega (h - t_n)\Gamma_n \,\mathrm{d}x \\ &= \langle \mu_n - \widetilde{\mu}_n, \psi_n\rangle_{(H^1)',H^1} \\ &\quad - \int_{\{|\nabla\Gamma_n|<\varepsilon_n\}} (a(x, w_n, \nabla w_n) - a(x, w_n, \nabla\widetilde{w}_n) - D_n\nabla\Gamma_n)\cdot\nabla\psi_n \,\mathrm{d}x.\end{aligned} \tag{6.12}$$

Assumption (6.5) implies $\left|\mathbf{1}_{\{|\nabla\Gamma_n|<\varepsilon_n\}} (a(x, w_n, \nabla w_n) - a(x, w_n, \nabla\widetilde{w}_n) - D_n\nabla\Gamma_n)\right| \leq C\varepsilon_n$ with C not depending on n and (6.10) therefore shows that the last term of (6.12) tends to 0 as $n \to \infty$. Moreover, by (6.11), $(\psi_n)_{n\geq 1}$ is relatively compact in $C(\overline{\Omega})$ and, since $\mu_n - \widetilde{\mu}_n \to 0$ weakly-$*$ in the space of bounded measures on Ω, we deduce that $\langle \mu_n - \widetilde{\mu}_n, \psi_n\rangle_{(H^1)',H^1} = \int_\Omega \psi_n(\mu_n - \widetilde{\mu}_n)\,dx \to 0$ as $n \to \infty$. As $\Gamma_n \to w - \widetilde{w}$ in $L^1(\Omega)$, we finally obtain, by letting $n \to +\infty$ in (6.12),

$$\int_\Omega h(w - \widetilde{w})\,\mathrm{d}x = 0.$$

This equality being true for all $h \in L^\infty(\Omega)$, we infer that $w = \widetilde{w}$ and the proof is concluded. ■

7 Accretivity and monotonicity. Parabolic equation

We examine here a different perspective for the solution of equation (1.1) with the given boundary conditions. This consists in investigating the properties of the solution map in the L^p-spaces. In view of the application of these properties to the solution of parabolic problems and the generation of continuous semigroups using the Hille-Yosida-Phillips theorem, a fundamental assumption is *accretivity*. This concept can be defined as follows: we consider a Banach space X and a (possibly nonlinear) operator A defined in a subset $D(A) \subset X$ with values in X. The operator is then called *accretive* if for every constant $h > 0$ and every $u_1, u_2 \in D(A)$ we have

$$\|u_1 - u_2\|_X \leq \|(I + hA)u_1 - (I + hA)u_2\|_X$$

In other words, the map $I + hA : D(A) \to X$ is one-to-one and its inverse is a contraction from the range $R(I+hA)$ into $D(A)$, if measured with the X-norm. This inverse is usually denoted as $J_h(A)$ and is called the h-resolvent. When the operator is linear we can replace the differences by simply

$$\|u\|_X \leq \|(I + hA)u\|_X$$

for every $u \in D(A)$ and $h > 0$. Actually, a stronger variant of this concept appears in the practice when X is a space of real-valued functions, and is called T-accretivity. It consists in replacing $\|\cdot\|_X$ by $\|(\cdot)_+\|_X$ in the above formulas. Here, $(\cdot)_+$ denotes positive part (taken pointwise).

Another important ingredient in the construction of solutions of evolution problems with accretive operators is the so-called range condition. In its stronger version we require $R(I+hA) = X$ for all $h > 0$ and the operator is called m-accretive. But, in fact, the generation of a semigroup happens under the milder condition $D(A) \subset R(I + hA)$ for all $h > 0$.

The L^1 setting

In order to apply these ideas to our problem we have to select the space X. The choice that better suits our convection-diffusion operator is $X = L^1(\Omega)$. Then, we have to adapt the operator to this framework. To be precise, we define $D(A_1)$ as the set of weak solutions u of Problem (1.1) with right-hand f such that $u \in H^1(\Omega) \subset L^1(\Omega)$ and $f \in (H^1(\Omega))' \cap L^1(\Omega)$. If $u \in D(A_1)$ we define $A_1(u) = f$, i.e., A_1 is a restriction of the differential operator plus boundary conditions used up to now in this paper.

Theorem 7.1 *Under assumptions* (H1), (H2) *and* (H3), A_1 *defined in this way is a* T*-accretive operator in* $L^1(\Omega)$*. Moreover, for every* $h > 0$ *we have* $R(I + hA_1) = (H^1(\Omega))' \cap L^1(\Omega)$ *so that it is dense in* $L^1(\Omega)$*. Finally,* $D(A_1) \subset R(I + hA_1)$ *and* $D(A_1)$ *is dense in* $L^1(\Omega)$*.*

Proof. (i) Write A instead of A_1 for brevity. Checking the T-accretivity in $L^1(\Omega)$ means therefore that when $u \in H^1(\Omega)$ is the weak solution of the problem

$$\begin{cases} -h\,\mathrm{div}(D\nabla u - u\,\mathbf{V}) + u = f & \text{in } \Omega \\ (D\nabla u - u\,\mathbf{V}) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

with $f \in (H^1(\Omega))' \cap L^1(\Omega)$, then $\|u_+\|_1 \le \|f_+\|_1$. There is a standard trick in this theory that consists in using as test functions expressions of the form $p_n(u)$ where p_n are nondecreasing piecewise C^1 (with bounded derivative) functions which approximate the positive sign function sign_+. We do that and get

$$-\int_\Omega \mathrm{div}(D\nabla u - u\,\mathbf{V}) p_n(u)\,\mathrm{d}x = \int_\Omega p_n'(u) D\nabla u \cdot \nabla u\,\mathrm{d}x - \int_\Omega u\,\mathbf{V} \cdot p_n'(u)\nabla u\,\mathrm{d}x.$$

The first term in the right-hand side is nonnegative. Choosing for instance $p_n(s) = 0$ on $]-\infty, 0]$, $p_n(s) = ns$ on $[0, 1/n]$ and $p_n(s) = 1$ on $[1/n, \infty[$, the second term in the right-hand side is bounded from above by $\int_\Omega \mathbf{1}_{\{0<u<1/n\}} |\mathbf{V}|\,|\nabla u|$ which tends to 0 as $n \to \infty$ (by the dominated convergence theorem). Hence, passing to the limit we have

$$-\int_\Omega \mathrm{div}(D\nabla u - u\,\mathbf{V})\,\mathrm{sign}_+(u)\,\mathrm{d}x \ge 0.$$

Using the equation, this means that

$$\|u_+\|_1 = \int_\Omega u\,\mathrm{sign}_+(u)\,\mathrm{d}x \le \int_\Omega f\,\mathrm{sign}_+(u)\,\mathrm{d}x \le \|f_+\|_1.$$

(ii) Let us now prove that $R(I + hA) = (H^1(\Omega))' \cap L^1(\Omega)$. Indeed, by Proposition 5.1 we can find a weak solution of the equation with $h > 0$ for every $f \in (H^1(\Omega))'$. If $f \in L^1(\Omega)$ then we write

$$-\mathrm{div}(D\nabla u) + \mathrm{div}(\mathbf{V}u) = \frac{1}{h}(f - u) \in (H^1(\Omega))' \cap L^1(\Omega),$$

to conclude that $u \in D(A)$ in the sense of our definition of operator A, so that $hA(u) + u = f$ and $(H^1(\Omega))' \cap L^1(\Omega) \subset R(I + hA)$ (the other inclusion is trivial). We also derive from this statement that $R(I + hA)$ is dense in $L^1(\Omega)$.

We now examine the definition of $D(A)$ and see that in particular $u \in H^1(\Omega) \subset (H^1(\Omega))' \cap L^1(\Omega)$. We conclude that $D(A) \subset R(I + hA)$.

(iii) Finally, we prove that $D(A_1)$ is dense in $L^1(\Omega)$. Let $u \in H^1(\Omega)$ and define $f \in (H^1(\Omega))'$ by: for all $\varphi \in H^1(\Omega)$,

$$\langle f, \varphi \rangle_{(H^1)', H^1} = \int_\Omega D\nabla u \cdot \nabla \varphi - \int_\Omega u\mathbf{V} \cdot \nabla \varphi + \int_\Omega u\varphi,$$

i.e. $f = L_1 u$ with the notation of Section 2. We can approximate f in $(H^1(\Omega))'$ by functions $f_n \in (H^1(\Omega))' \cap L^1(\Omega)$ (simply approximate $D\nabla u$, $u\mathbf{V}$ and u in $L^2(\Omega)$ by some functions in $C_c^\infty(\Omega)$); by Proposition 5.1, $u_n = (L_1)^{-1} f_n$ converges to $u = (L_1)^{-1} f$ in $H^1(\Omega)$; moreover, $L_1 u_n = f_n$ implies in particular $Lu_n = f_n - u_n \in (H^1(\Omega))' \cap L^1(\Omega)$, and thus $u_n \in D(A_1)$. Hence, $D(A_1)$ is dense in $H^1(\Omega)$ and, since $H^1(\Omega)$ is dense in $L^1(\Omega)$ and the topology of $L^1(\Omega)$ is weaker than the topology of $H^1(\Omega)$, this concludes the proof. ■

The L^2 setting

In Hilbert spaces the concept of accretivity coincides with the better known concept of monotonicity and the theory has better properties. We take as functional space $X = L^2(\Omega)$ and define the operator A_2 in $L^2(\Omega)$ by further restriction of the domain and range so that A_2 goes from $D(A_2) \subset L^2(\Omega)$ into $L^2(\Omega)$. In that case the corresponding estimate amounts to multiply the equation by u instead of $p(u)$. We easily get

$$h\langle A_2 u, u\rangle + \|u\|_2^2 \le \|f\|_2 \|u\|_2$$

Now,

$$\langle A_2 u, u\rangle = \int_\Omega D(x)\nabla u \cdot \nabla u \,\mathrm{d}x - \int_\Omega u\mathbf{V} \cdot \nabla u \,\mathrm{d}x,$$

and the last term equals

$$-\frac{1}{2}\int_\Omega \mathbf{V} \cdot \nabla(u^2)\,\mathrm{d}x = \frac{1}{2}\int_\Omega (\nabla \cdot \mathbf{V})\, u^2\, dx - \frac{1}{2}\int_{\partial\Omega} (\mathbf{V} \cdot \mathbf{n}) u^2 \,\mathrm{d}S.$$

Therefore, the two conditions : (i) $\nabla \cdot \mathbf{V} \ge 0$ in Ω, and (ii) $\mathbf{V} \cdot \mathbf{n} \le 0$ on $\partial\Omega$, imply that the operator A_2 is accretive (i. e., monotone) in $L^2(\Omega)$. The theory can still be done if the first condition is replaced by $\nabla \cdot \mathbf{V} \ge -2\omega$, with ω constant, then $\langle A_2 u, u\rangle \ge -\omega\|u\|^2$, and the operator is only ω-monotone instead of monotone. All this is classical theory that is only mentioned as a reminder.

It is quite easy in that case to prove that $R(I + hA_2) = L^2(\Omega) \subset (H^1(\Omega))'$, and also that $D(A)$ is contained in $R(I + hA_2)$ and is dense in $L^2(\Omega)$.

7.1 Parabolic equation. Semigroup

We now want to solve the evolution problem consisting of the equation

$$\partial_t u = -\mathrm{div}(D\nabla u) + \mathrm{div}(\mathbf{V}u) \qquad \text{for } (x,t) \in Q = \Omega \times (0, \infty),$$

supplied with boundary conditions

$$D\nabla u \cdot \mathbf{n} - u\,(\mathbf{V} \cdot \mathbf{n}) = 0 \qquad \text{on } \partial\Omega \times [0, \infty),$$

and initial data

$$u(x,0) = u_0(x) \qquad \text{for } x \in \Omega.$$

We proceed as follows: using the Hille-Yosida Theorem, [6] for linear operators, or more generally, the Crandall-Liggett theorem [8] in the nonlinear case, for every accretive operator A (linear or nonlinear) acting in a Banach space X and satisfying the range condition $D(A) \subset R(I + hA)$ for all small $h > 0$, we can construct a continuous semigroup S_t that associates to every initial datum $u_0 \in \overline{D(A)}$ a trajectory $u(t) \in C([0,\infty); X)$ that solves the abstract problem

$$\frac{du}{dt} + A(u) = 0$$

in the *mild sense*. Moreover, the accretivity property implies contractivity of the semigroup: for every two solutions u_1, u_2 we have

$$\|u_1(t) - u_2(t)\|_X \le \|u_1(s) - u_2(s)\|_X \qquad \text{for all } \; t \ge s \ge 0.$$

In the case of T-accretivity we get the stronger property

$$\|(u_1(t) - u_2(t))_+\|_X \le \|(u_1(s) - u_2(s))_+\|_X \qquad \text{for all } \; t \ge s \ge 0.$$

Finally, in the case of ω-monotonicity we get

$$\|u_1(t) - u_2(t)\|_X \le e^{\omega t}\|u_1(s) - u_2(s)\|_X \qquad \text{for all } \; t \ge s \ge 0.$$

8 Appendix

8.1 Technical results

The proof of the following lemma can be made in a very classical manner, by way of contradiction; we leave it as an exercise to the reader.

Lemma 8.1 *Assume that* (H1) *holds and let E a subset of Ω with positive Lebesgue measure. Then there exists a constant $C_E > 0$ such that, for all $v \in H^1(\Omega)$ satisfying $v = 0$ a.e. on E, we have $||v||_{L^2(\Omega)} \le C_E ||\nabla v||_{L^2(\Omega)}$.*

The next lemma is used in the proof of Theorem 6.1.

Lemma 8.2 *Let $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ be two sequences of measurable functions $\Omega \to \mathbb{R}^N$ such that there exists $\alpha > 0$ and $\Lambda > 0$ satisfying, for all $n \ge 1$ and a.e. $x \in \Omega$, $a_n(x) \cdot b_n(x) \ge \alpha |b_n(x)|^2$ and $|a_n(x)| \le \Lambda |b_n(x)|$. Assume moreover that $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ converge a.e. on Ω. Then, there exist subsequences $(a_{n_k})_{k\ge1}$ and $(b_{n_k})_{k\ge1}$ and measurable matrix-valued functions $(D_k)_{k\ge1}$ satisfying the following properties:*

$\exists \rho > 0$ *such that, for all $k \ge 1$, a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$,* $D_k(x)\xi \cdot \xi \ge \rho |\xi|^2$,

$\exists C > 0$ *such that, for all $k \ge 1$ and a.e. $x \in \Omega$,* $||D_k(x)|| \le C$,

$\exists \varepsilon_k \to 0$ *such that, for all $k \ge 1$ and a.e. $x \in \Omega$, if $|b_{n_k}(x)| \ge \varepsilon_k$ then* $D_k(x) b_{n_k}(x) = a_{n_k}(x)$,

$(D_k)_{k\ge1}$ *converges a.e. toward a bounded uniformly elliptic matrix-valued function.*

Proof. Let $\gamma > 0$ and $x \in \Omega$ such that $b_n(x) \neq 0$; define the matrix $M_{a_n(x),b_n(x)}$ by: $M_{a_n(x),b_n(x)} b_n(x) = a_n(x)$ and $M_{a_n(x),b_n(x)} = \gamma \mathrm{Id}$ on the orthogonal space of $\mathbb{R} b_n(x)$, i.e.

$$M_{a_n(x),b_n(x)}\xi = \left(\xi \cdot \frac{b_n(x)}{|b_n(x)|}\right)\frac{a_n(x)}{|b_n(x)|} + \gamma\left(\xi - \left(\xi \cdot \frac{b_n(x)}{|b_n(x)|}\right)\frac{b_n(x)}{|b_n(x)|}\right).$$

Let $\xi \in \mathbb{R}^N$; decomposing ξ on $\mathbb{R}b_n(x)$ and $(\mathbb{R}b_n(x))^\perp$ as $\xi = rb_n(x) + y$, we have, using Young's inequality,

$$\begin{aligned} M_{a_n(x),b_n(x)}\xi \cdot \xi &= (ra_n(x) + \gamma y) \cdot (rb_n(x) + y) \\ &= r^2 a_n(x) \cdot b_n(x) + ra_n(x) \cdot y + \gamma|y|^2 \\ &\geq \alpha r^2 |b_n(x)|^2 - r|b_n(x)|\,\Lambda|y| + \gamma|y|^2 \\ &\geq \frac{\alpha}{2} r^2 |b_n(x)|^2 + \left(\gamma - \frac{\Lambda^2}{2\alpha}\right)|y|^2. \end{aligned}$$

Fixing $\gamma = \frac{\Lambda^2}{2\alpha} + \frac{\alpha}{2}$, we obtain

$$M_{a_n(x),b_n(x)}\xi \cdot \xi \geq \frac{\alpha}{2}\left(r^2|b_n(x)|^2 + |y|^2\right) = \frac{\alpha}{2}|\xi|^2. \tag{8.1}$$

Moreover, $|M_{a_n(x),b_n(x)}\xi| \leq |\xi|\left(\frac{|a_n(x)|}{|b_n(x)|} + \gamma\right)$, that is to say

$$||M_{a_n(x),b_n(x)}|| \leq \Lambda + \gamma = \Lambda + \frac{\Lambda^2}{2\alpha} + \frac{\alpha}{2}. \tag{8.2}$$

It is also clear that, if $b_n(x) \to b(x) \neq 0$ and $a_n(x) \to a(x)$, then $M_{a_n(x),b_n(x)} \to M_{a(x),b(x)}$. Hence, if $b_n(x)$ is nowhere null and nowhere converges to 0, the choice $D_n = M_{a_n(x),b_n(x)}$ concludes the proof.

In order to take into account the possibility that $b_n(x) \to 0$, we take $h_l : [0, +\infty[\to [0, 1]$ a continuous function which is null on $[0, 1/(2l)]$ and equal to 1 on $[1/l, +\infty[$ and we define

$$R_{l,n}(x) = h_l(|b_n(x)|)M_{a_n(x),b_n(x)} + (1 - h_l(|b_n(x)|))\frac{\alpha}{2}\mathrm{Id}$$

(with the obvious choice $R_{l,n}(x) = \frac{\alpha}{2}\mathrm{Id}$ if $b_n(x) = 0$). Since, at each point, $R_{l,n}(x)$ is either $\frac{\alpha}{2}\mathrm{Id}$ or a convex combinaison between this function and $M_{a_n(x),b_n(x)}$, we deduce from (8.1) and (8.2) that $R_{l,n}$ is uniformly elliptic and bounded independently of l and n. Moreover, if $|b_n(x)| \geq 1/l$ then $R_{l,n}(x)b_n(x) = M_{a_n(x),b_n(x)}b_n(x) = a_n(x)$. Denoting a and b the respective a.e. limits of a_n and b_n, we see that, for a.e. x,

$$R_{l,n}(x) \to R_l(x) = h_l(|b(x)|)M_{a(x),b(x)} + (1 - h_l(|b(x)|))\frac{\alpha}{2}\mathrm{Id} \quad \text{as } n \to \infty$$

(whatever the definition of $M_{a(x),b(x)}$ if $b(x) = 0$, since $h_l(0) = 0$). It is also clear that, for all x,

$$R_l(x) \to R(x) = \mathbf{1}_{\{|b(x)|\neq 0\}}M_{a(x),b(x)} + \mathbf{1}_{\{|b(x)|=0\}}\frac{\alpha}{2}\mathrm{Id} \quad \text{as } l \to \infty.$$

Since $R_{l,n}$ and R_l are uniformly bounded, these convergences also hold in $L^1(\Omega)^{N\times N}$. We can thus take $\varphi : \mathbb{N} \to \mathbb{N}$ increasing such that, for all $l \geq 1$, $||R_{l,\varphi(l)} - R_l||_{L^1} \leq 1/l$ and we have $||R_{l,\varphi(l)} - R||_{L^1} \leq \frac{1}{l} + ||R_l - R||_{L^1} \to 0$ as $l \to +\infty$. Hence, there is a subsequence $D_k = R_{l_k,\varphi(l_k)}$ which converges a.e. on Ω and the proof is concluded by denoting $n_k = \varphi(l_k)$ and $\varepsilon_k = 1/l_k$. ■

8.2 Maximum principle for other boundary conditions

The maximum principle states that if the right-hand side of the equation is nonnegative, then so is the solution. Of course, it is irrelevant to pure Neumann boundary conditions, because the necessary condition $f \in H'_{\sharp 1}$ prevents any admissible non-trivial right-hand side to be nonnegative. But the question of this principle is relevant for other boundary conditions and, although the existence and uniqueness of solutions to non-coercive problems is known since [12], there does not seem to be any proof of this maximum principle, at least not in the case of irregular data without assumption on $\mathrm{div}(V)$ (for regular data, this principle is known for (1.3) and can therefore be deduced by duality for (1.1) and, for bounded convection V such that $\mathrm{div}(V) \geq 0$, it is proved in [18]). It could be deduced for irregular data from the case of regular data by approximation; however, slightly modifying the test function used in the proof of Lemma 2.1, it is in fact possible to give a direct and very simple proof of this principle for non-coercive elliptic equations with Dirichlet, Fourier or mixed boundary conditions with irregular data, as shown in the following proposition (stated in the case of pure Dirichlet boundary conditions for convenience).

Proposition 8.3 *Assume that* (H1)–(H3) *hold, take* $f \in H^{-1}(\Omega)$ *and let* $u \in H^1_0(\Omega)$ *be the weak solution to*

$$\begin{cases} -\mathrm{div}(D\nabla u) + \mathrm{div}(\mathbf{V}u) = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.3}$$

If $f \geq 0$ *(in the sense that* $\langle f, \varphi\rangle_{H^{-1},H^1_0} \geq 0$ *for any* $\varphi \geq 0$*), then* $u \geq 0$ *a.e. on* Ω*.*

Proof. Let $\varepsilon > 0$ and $\phi_\varepsilon(r) = \varphi_\varepsilon(r + \varepsilon)$ where φ_ε is given in Figure 1, i.e. $\phi_\varepsilon(r) = 0$ if $r \geq 0$, $\phi_\varepsilon(r) = r$ if $-\varepsilon < r < 0$ and $\phi_\varepsilon(r) = -\varepsilon$ if $r \leq -\varepsilon$. Then we can use $\phi_\varepsilon(u) \in H^1_0(\Omega)$ as a test function in the weak formulation satisfied by u and, since $\phi_\varepsilon(u) \leq 0$ on Ω,

$$\int_\Omega D\nabla u \cdot \nabla(\phi_\varepsilon(u))\,\mathrm{d}x - \int_\Omega u\mathbf{V} \cdot \nabla(\phi_\varepsilon(u))\,\mathrm{d}x = \langle f, \phi_\varepsilon(u)\rangle_{H^{-1},H^1_0} \leq 0.$$

But $\nabla(\phi_\varepsilon(u)) = \mathbf{1}_{\{-\varepsilon<u<0\}}\nabla u$ and we deduce

$$||\nabla(\phi_\varepsilon(u))||^2_{L^2(\Omega)^N} \leq \frac{1}{\alpha}\varepsilon||\mathbf{V}||_{L^2(\{-\varepsilon<u<0\})^N}||\nabla(\phi_\varepsilon(u))||_{L^2(\Omega)^N} = \varepsilon\omega(\varepsilon)||\nabla(\phi_\varepsilon(u))||_{L^2(\Omega)^N}$$

with $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. Using Poincaré's inequality in $H^1_0(\Omega)$, and since $|\phi_\varepsilon(u)| \geq \frac{\varepsilon}{2}$ on $\{u \leq -\varepsilon/2\}$, we obtain

$$(\mathrm{meas}(\{u \leq -\varepsilon/2\}))^{1/2} \leq \frac{2}{\varepsilon}||\phi_\varepsilon(u)||_{L^2(\Omega)} \leq \frac{2}{\varepsilon}\mathrm{diam}(\Omega)||\nabla(\phi_\varepsilon(u))||_{L^2(\Omega)^N} \leq 2\mathrm{diam}(\Omega)\omega(\varepsilon).$$

Passing to the limit $\varepsilon \to 0$, this gives $\mathrm{meas}(\{u < 0\}) = 0$ and concludes the proof. ■

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