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Fractal Conservation Laws: Global Smooth Solutions and Vanishing Regularization

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Abstract. We consider the parabolic regularization of a scalar conservation law in which the Laplacian operator has been replaced by a fractional power of itself. Using a splitting method, we prove the existence of a solution to the problem and, thanks to the Banach fixed point theorem, its uniqueness and regularity. We also show that, as the regularization vanishes, the solution converge to the entropy solution of the scalar conservation law. We only present here the outlines of the proofs; we refer the reader to [4] and [5] for the details.

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1. Introduction

1.1. The equation and its motivations

The scalar conservation law

$$\begin{cases} \partial_t u(t,x) + \operatorname{div}(f(u))(t,x) = 0 & t > 0\,,\ x \in \mathbb{R}^N\,, \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N\,, \end{cases} \tag{1}$$

where $f \in C^\infty(\mathbb{R};\mathbb{R}^N)$ and $u_0 \in L^\infty(\mathbb{R}^N)$, is a well-known equation. S.N. Krushkov introduced in [6] a notion of solution for which existence and uniqueness holds (the entropy solution). A way to prove the existence of such entropy solutions is to consider the parabolic regularization of (1):

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \operatorname{div}(f(u^\varepsilon))(t,x) - \varepsilon\Delta u^\varepsilon(t,x) = 0 & t > 0\,,\ x \in \mathbb{R}^N\,, \\ u^\varepsilon(0,x) = u_0(x) & x \in \mathbb{R}^N \end{cases} \tag{2}$$

(for which existence, uniqueness and regularity of solutions is classical), to establish so-called entropy inequalities (see Subsection 1.2), and to pass to the limit $\varepsilon \to 0$.

We are interested here in the case where we replace $-\Delta$ in the parabolic regularization (2) by a fractional power $(-\Delta)^{\lambda/2}$ of the Laplacian; precisely, we consider

$$\begin{cases} \partial_t u(t,x) + \mathrm{div}(f(u))(t,x) + g[u(t,\cdot)](x) = 0 & t>0\,,\ x\in\mathbb{R}^N\,, \\ u(0,x) = u_0(x) & x\in\mathbb{R}^N\,, \end{cases} \tag{3}$$

where the operator g is defined through Fourier transform by

$$\mathcal{F}(g[v])(\xi) = |\xi|^\lambda \mathcal{F}(v)(\xi) \quad \text{with } \lambda \in]1,2]. \tag{4}$$

The motivation for the study of this problem comes from a question of P. Clavin; he shows in [2] that, in some cases of gas detonation, the wave front satisfies an equation which is close to (3) but with $\lambda = 1$; numerical tests indicate that shocks can occur in this case. The question was: if $\lambda > 1$, do we have for (3) the same regularization effect as for (2)? Curiously enough, the regularity of the solutions to (3) is quite easy to obtain; their global existence, on the other hand, is much harder (see Subsection 1.2). Some other motivations for (3) appear in [9].

1.2. Main difficulty

Some partial existence results for (3) can be found in [1], but they are either limited to the case $N = 1$ and $f(u) = u^2$ (and with quite regular initial data), or to results of local existence in time.

The main problem when considering (3) is the lack of *a priori* estimates (which would allow to pass from local existence to global existence). If we consider this equation as a regularization of (1), a natural space for the solutions is L^∞. Let us briefly recall how L^∞ estimates are obtained on the solutions to (2): if η is a convex function and $\phi' = \eta' f'$, multiplying (2) by $\eta'(u^\varepsilon)$ and taking into account (thanks to the convexity of η)

$$\Delta(\eta(u^\varepsilon)) = \eta''(u^\varepsilon)|\nabla u^\varepsilon|^2 + \eta'(u^\varepsilon)\Delta u^\varepsilon \geq \eta'(u^\varepsilon)\Delta u^\varepsilon$$

leads to

$$\partial_t \eta(u^\varepsilon)(t,x) + \mathrm{div}(\phi(u^\varepsilon))(t,x) - \varepsilon\Delta(\eta(u^\varepsilon))(t,x) \leq 0. \tag{5}$$

Then, taking $\eta \equiv 0$ on $[-||u_0||_\infty, ||u_0||_\infty]$ and $\eta > 0$ outside $[-||u_0||_\infty, ||u_0||_\infty]$, the integration of (5) gives $||u^\varepsilon(t)||_\infty \leq ||u_0||_\infty$ for all $t > 0$.

Such a manipulation cannot be made if Δ is replaced by g. Thus, to obtain L^∞ bound on the solution to (3), we use a totally different method.

2. Existence of a global solution

The semi-group generated by g is quite easy to understand: passing to Fourier transform, we see that the solution to $\partial_t v + g[v] = 0$ with initial datum $v(0) = v_0$ is given by $v(t,x) = K(t,\cdot) * v_0(x)$, where the kernel K is defined by

$$K(t,x) = \mathcal{F}^{-1}(\xi \to e^{-t|\xi|^\lambda}).$$

A result of [8] states that K is nonnegative, so that $||K(t)||_{L^1(\mathbb{R}^N)} = \mathcal{F}(K)(0) = 1$. As a consequence, we see that

$$||v(t)||_{L^\infty(\mathbb{R}^N)} \le ||v_0||_{L^\infty(\mathbb{R}^N)}\,,\ ||v(t)||_{L^1(\mathbb{R}^N)} \le ||v_0||_{L^1(\mathbb{R}^N)}\,,$$
$$|v(t)|_{BV(\mathbb{R}^N)} \le |v_0|_{BV(\mathbb{R}^N)}.$$

Hence, g "behaves well" (any interesting norm is preserved by g).

It is well known that the same holds for $\partial_t v + \mathrm{div}(f(v)) = 0$: if v evolves according to this scalar conservation law, its L^∞, L^1 and BV norms do not increase.

Hence, since each operator $\partial_t + g$ and $\partial_t + \mathrm{div}(f(\cdot))$ behaves well, we can let them evolve on separate time intervals and, afterwards, try to mix them together in order to get $\partial_t + \mathrm{div}(f(\cdot)) + g$. This idea is well known in numerical analysis, where it is called "splitting", but to our knowledge it has never been used before in order to prove the existence of a solution to a continuous problem.

We take $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$ and, for $\delta > 0$, we define a function $U^\delta : [0,\infty[\times\mathbb{R}^N \to \mathbb{R}$ by (we omit the space variable):

- On $[0,\delta[$, U^δ is the solution to $\partial_t U^\delta + 2g[U^\delta] = 0$ with initial datum $U^\delta(0) = u_0$.
- On $[\delta, 2\delta[$, U^δ is the solution to $\partial_t U^\delta + 2\,\mathrm{div}(f(U^\delta)) = 0$ with initial datum $U^\delta(\delta)$ obtained in the first step.
- On $[2\delta, 3\delta[$, U^δ is the solution to $\partial_t U^\delta + 2g[U^\delta] = 0$ with initial datum $U^\delta(2\delta)$ given by the preceding step.
- etc...

That is to say, on half of the time – but in a set spread throughout $[0,\infty[$ – U^δ evolves according to $\partial_t + 2g = 0$ and, on the other half, it evolves according to $\partial_t + 2\,\mathrm{div}(f(\cdot)) = 0$; the factors "2" come from the fact that each of this operator only appears on half of the time: if we want to recover $\partial_t + \mathrm{div}(f(\cdot)) + g = 0$ on the whole of $[0,\infty[$ at the end, we must give a double weight to the operators on each half of $[0,\infty[$.

Thanks to the preceding considerations on both operators, we see that the L^∞, L^1 and BV norms of $U^\delta(t)$ are bounded by the corresponding norms of u_0. In particular, by Helly's Theorem, $\{U^\delta(t)\,;\ \delta > 0\}$ is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ for each $t \ge 0$. It is possible to prove that $\{U^\delta\,;\ \delta > 0\}$ is equicontinuous $[0,\infty[\to L^1(\mathbb{R}^N)$ and thus, up to a subsequence and as $\delta \to 0$, that U^δ converges in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ to some u. Multiplying by $\varphi \in C_c^\infty([0,\infty[\times\mathbb{R}^N)$ the equations satisfied by U^δ and integrating, we can show that u satisfies (3) in a weak sense:

$$\int_0^\infty \int_{\mathbb{R}^N} u\partial_t\varphi + f(u)\cdot\nabla\varphi - ug[\varphi]\,dtdx + \int_{\mathbb{R}^N} u_0\varphi(0)\,dx = 0.$$

We have thus proved that, if u_0 is regular enough, (3) has a solution in a weak sense; moreover, this solution is bounded by $||u_0||_{L^\infty(\mathbb{R}^N)}$.

3. Regularity and uniqueness of the solution

3.1. Definition of solution

Another way to handle (3) is to consider that $\operatorname{div}(f(u))$ is a lower order term, and therefore to write $\partial_t u + g[u] = -\operatorname{div}(f(u))$. Since the semi-group generated by g is known, Duhamel's formula then gives

$$u(t,x) = K(t,\cdot) * u_0(x) - \int_0^t K(t-s,\cdot) * \operatorname{div}(f(u(s,\cdot)))(x)\,ds$$

and the properties of the convolution lead to

$$u(t,x) = K(t,\cdot) * u_0(x) - \int_0^t \nabla K(t-s,\cdot) * f(u(s,\cdot))(x)\,ds. \qquad (6)$$

This suggests the following definition.

Definition 3.1. *Let $u_0 \in L^\infty(\mathbb{R}^N)$. A solution to* (3) *is $u \in L^\infty(]0,\infty[\times\mathbb{R}^N)$ which satisfies (6) for a.e. $(t,x) \in]0,\infty[\times\mathbb{R}^N$.*

By the definition of K, it is obvious that $K(t,x) = t^{-N/\lambda}K(1,t^{-1/\lambda}x)$; hence, $||\nabla K(t)||_{L^1(\mathbb{R}^N)} = C_0 t^{-1/\lambda}$ and the integral term in (6) is defined as soon as u is bounded.

It is then easy, by a Banach fixed point theorem, to prove the existence of a solution on a small time interval $[0,T]$ (and its uniqueness on any time interval); but, due to the lack of estimates on this solution, nothing ensures that it can be extended to $[0,\infty[$. However, using its integrability properties, it is possible to prove that the weak solution constructed by a splitting method in Section 2 is also a solution in the sense of Definition 3.1. Hence, we have the existence of a global solution when the initial datum is regular enough, and its uniqueness for any bounded initial condition.

3.2. Regularization effect

The regularity of the solution is not very difficult to obtain. Assume that $u_0 \in L^\infty(\mathbb{R}^N)$ and take u a solution to (6) on $[0,T_0]$ (not necessarily the one constructed before, since we have not assumed that u_0 is integrable and has bounded variation). Since $||\nabla K(t)||_{L^1(\mathbb{R}^N)} = C_0 t^{-1/\lambda}$, the idea is to apply a Banach fixed point theorem on (6) in the space

$$E_T = \{v \in C_b(]0,T[\times\mathbb{R}^N) \mid t^{1/\lambda}\nabla v \in C_b(]0,T[\times\mathbb{R}^N;\mathbb{R}^N)\}.$$

For T small enough and $u_0 \in L^\infty(\mathbb{R}^N)$, we are able to prove the existence of a solution to (6) in E_T; since the solution is unique in $L^\infty(]0,T[\times\mathbb{R}^N)$, this proves that the given solution u is C^1 in space on $]0,T[$; this reasoning can be done from any initial time t_0 (not only $t_0 = 0$), which proves that u is C^1 in space on $]0,T_0[\times\mathbb{R}^N$.

A bootstrap technique, based on integral equations satisfied by the derivatives of u, allows to extend this method and to prove that u is C^∞ in space, and that all its spatial derivatives are bounded on $]t_0,T_0[\times\mathbb{R}^N$, for all $t_0 > 0$, by some constant depending on t_0 and $||u||_{L^\infty(]0,T_0[\times\mathbb{R}^N)}$. It is then possible to give a meaning to $g[u]$ (we prove that, if $2m > N + \lambda$, there exists integrable functions g_1 and g_2 such that $g[u] = g_1 * u + g_2 * \Delta^m u$) and to show that (3) is satisfied in the classical sense; this proves that u is also regular in time.

Thus, even if the initial datum is only bounded, the solution is regular and we have a bound on its derivatives which only depends on a bound on the solution itself. Let $u_0 \in L^\infty(\mathbb{R}^N)$; we can approximate it (a.e. and in L^∞ weak-$*$) by regular data u_0^n, for which we have proven the existence of solutions u^n (Section 2); these solutions are bounded by $\sup_n ||u_0^n||_{L^\infty(\mathbb{R}^N)} < +\infty$, which gives a bound on their derivatives; this proves that, up to a subsequence, u^n converge a.e. to some bounded u; it is then easy to pass to the limit in (6), with (u_0^n, u^n) instead of (u_0, u), to see that u is a solution to (3).

3.3. Main result

To sum up, we have obtained the following theorem.

Theorem 3.1. *If $f \in C^\infty(\mathbb{R};\mathbb{R}^N)$ and $u_0 \in L^\infty(\mathbb{R}^N)$, then* (3) *has a unique solution in the sense of Definition* 3.1. *Moreover, this solution u satisfies*

i) *$u \in C^\infty(]0,\infty[\times\mathbb{R}^N)$ and, for all $t_0 > 0$, all the derivatives of u are bounded on $[t_0,\infty[\times\mathbb{R}^N$,*

ii) *for all $t > 0$, $||u(t)||_{L^\infty(\mathbb{R}^N)} \le ||u_0||_{L^\infty(\mathbb{R}^N)}$,*

iii) *as $t \to 0$, we have $u(t) \to u_0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for all $p < \infty$ and in $L^\infty(\mathbb{R}^N)$ weak-$*$.*

Remark 3.1. *The construction via the splitting method proves that the solution to* (3) *has more properties than the one stated above: any property which is satisfied by both equations $\partial_t + g = 0$ and $\partial_t + \mathrm{div}(f(\cdot)) = 0$ is also satisfied by* (3)*; for example: the solution takes its values between the essential lower and upper bounds of u_0, and there is a L^1-contraction principle for* (3)*.*

Remark 3.2. *Since Theorem* 3.1 *only relies on the nonnegativity of K and the integrability properties of K and ∇K, it is also valid for more general g's, such as sums of operators* (4) *or anisotropic operators of the kind*

$$g = \sum_{j=1}^N (-\partial_j^2)^{\frac{\lambda_j}{2}}, \qquad i.e., \qquad \mathcal{F}(g[v])(\xi) = \left(\sum_{j=1}^N |\xi_j|^{\lambda_j}\right) \mathcal{F}(v)(\xi), \qquad with\ \lambda_j \in]1,2].$$

The same holds for Theorem 4.1 *and, in some cases, Theorem* 4.2.

4. Vanishing regularization

Since (3) has been considered as a possible regularization of (1), it seems natural to wonder if, aside from the regularizing effect which has just been proved, the solutions to this equation stay close to the solution of the scalar conservation law when the weight on g is small. Precisely, if we consider

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \mathrm{div}(f(u^\varepsilon))(t,x) + \varepsilon g[u^\varepsilon(t,\cdot)](x) = 0 & t>0\,,\ x\in\mathbb{R}^N\,, \\ u^\varepsilon(0,x) = u_0(x) & x\in\mathbb{R}^N\,, \end{cases} \tag{7}$$

is it true that, as in the case of the parabolic regularization, u^ε converges as $\varepsilon\to 0$ to the entropy solution of (1)?

The answer is not obvious if we recall that some higher-order regularizations of conservation laws can generate too many oscillations, as the regularization vanishes, to allow the convergence towards the entropy solution; an example of this phenomenon, the KdV equation $\partial_t u^\varepsilon + \partial_x((u^\varepsilon)^2) = \varepsilon\partial_x^3 u^\varepsilon$, is mentioned in [3].

The convergence of the parabolic regularization (2) to the conservation law (1) is strongly based on the entropy inequality (5). If we want to prove the convergence of (7) to (1), we need to prove an entropy inequality for the non-local regularization g, and we are back to the problem mentioned in Subsection 1.2.

4.1. Entropy inequality

Therefore, we use again the splitting method. Let η be a convex function, $\phi' = \eta' f'$ and U^δ be the function constructed in Section 2 (with εg instead of g and for u_0 regular enough). On $I_\delta = \cup_{p\ \mathrm{odd}}[p\delta,(p+1)\delta]$, U^δ is the (entropy) solution of a scalar conservation law ([1]), and thus, for a nonnegative $\varphi\in C_c^\infty([0,\infty[\times\mathbb{R}^N)$,

$$\int_{I_\delta}\int_{\mathbb{R}^N}\eta(U^\delta)\partial_t\varphi + 2\phi(U^\delta)\cdot\nabla\varphi\,dtdx = \sum_{p\ \mathrm{odd}} a_{p+1}-a_p = -a_0 + \sum_{p\ \mathrm{even}} a_p - a_{p+1}, \tag{8}$$

where $a_p = \int_{\mathbb{R}^N}\eta(U^\delta(p\delta))\varphi(p\delta)\,dx$.

On $[p\delta,(p+1)\delta]$ for p even, U^δ satisfies $\partial_t U^\delta + 2\varepsilon g[U^\delta] = 0$ and thus $U^\delta(t) = K(2\varepsilon(t-p\delta)) * U^\delta(p\delta)$. Since η is convex and $K(2\varepsilon(t-\delta))$ is nonnegative with total mass 1, Jensen's inequality gives $\eta(U^\delta(t)) \le K(2\varepsilon(t-p\delta)) * \eta(U^\delta(p\delta))$; hence, φ being nonnegative,

$$a_{p+1}-a_p \le \int_{\mathbb{R}^N} K(2\varepsilon\delta) * \eta(U^\delta(p\delta))\varphi((p+1)\delta)\,dx - \int_{\mathbb{R}^N}\eta(U^\delta(p\delta))\varphi(p\delta)\,dx. \tag{9}$$

But $t\to K(2\varepsilon t) * \eta(U^\delta(p\delta))$ is solution to $\partial_t v + 2\varepsilon g[v] = 0$ with initial datum $\eta(U^\delta(p\delta))$, thus

$$\begin{aligned} &\int_{\mathbb{R}^N} K(2\varepsilon\delta) * \eta(U^\delta(p\delta))\varphi((p+1)\delta)\,dx - \int_{\mathbb{R}^N}\eta(U^\delta(p\delta))\varphi(p\delta)\,dx \\ &\qquad = \int_{p\delta}^{(p+1)\delta}\int_{\mathbb{R}^N} K(2\varepsilon(t-p\delta)) * \eta(U^\delta(p\delta))(\partial_t\varphi - 2\varepsilon g[\varphi])\,dtdx. \end{aligned} \tag{10}$$

Since the L^∞, L^1 and BV norms of $U^\delta(s)$ and $\eta(U^\delta(s))$ are bounded independently of δ and s, and since $K(t)_{t>0}$ is an approximate unit as $t\to 0$ ([2]), we have, for

[1] In fact, for δ small enough, U^δ is regular on I_δ.
[2] This comes from the fact that $K(t)$ is nonnegative with mass 1, and that $K(t,x) = t^{-N/\lambda}K(1,t^{-1/\lambda}x)$.

$t \in]p\delta,(p+1)\delta]$,

$$||K(2\varepsilon(t-p\delta)) * \eta(U^\delta(p\delta)) - \eta(U^\delta(p\delta))||_{L^1(\mathbb{R}^N)} \leq \omega_1(\delta)$$
$$||\eta(U^\delta(t)) - \eta(U^\delta(p\delta))||_{L^1(\mathbb{R}^N)} \leq C||U^\delta(t) - U^\delta(p\delta)||_{L^1(\mathbb{R}^N)} \leq \omega_2(\delta),$$

where $\omega_j(\delta) \to 0$ as $\delta \to 0$ (recall that $U^\delta(t) = K(2\varepsilon(t-p\delta)) * U^\delta(p\delta)$); therefore,

$$||K(2\varepsilon(t-p\delta)) * \eta(U^\delta(p\delta)) - \eta(U^\delta(t))||_{L^1(\mathbb{R}^N)} \leq \omega_1(\delta) + \omega_2(\delta) = \omega_3(\delta)$$

and (9) and (10) give

$$\begin{aligned} a_{p+1} - a_p \quad \leq \quad & \int_{p\delta}^{(p+1)\delta} \int_{\mathbb{R}^N} \eta(U^\delta(t))(\partial_t\varphi - 2\varepsilon g[\varphi])\, dtdx \\ & + \omega_3(\delta) \int_{p\delta}^{(p+1)\delta} ||\partial_t\varphi(t)||_{L^\infty(\mathbb{R}^N)} + 2\varepsilon||g[\varphi(t)]||_{L^\infty(\mathbb{R}^N)}\, dt. \end{aligned}$$

Summing on even p's and coming back to (8), we find

$$\begin{aligned} & \int_{I_\delta} \int_{\mathbb{R}^N} \eta(U^\delta)\partial_t\varphi + 2\phi(U^\delta) \cdot \nabla\varphi\, dtdx + \int_{\mathbb{R}^+\backslash I_\delta} \int_{\mathbb{R}^N} \eta(U^\delta)\partial_t\varphi - 2\varepsilon\eta(U^\delta)g[\varphi]\, dtdx \\ & \quad + \int_{\mathbb{R}^N} \eta(u_0)\varphi(0)\, dx \geq -C(\varphi)\omega_3(\delta). \end{aligned}$$

We can then pass to the limit $\delta \to 0$ (recall that $U^\delta \to u^\varepsilon$); since the characteristic functions of I_δ and $\mathbb{R}^+\backslash I_\delta$ weakly converge to $1/2$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^N} \eta(u^\varepsilon)\partial_t\varphi + \phi(u^\varepsilon) \cdot \nabla\varphi - \varepsilon\eta(u^\varepsilon)g[\varphi]\, dtdx + \int_{\mathbb{R}^N} \eta(u_0)\varphi(0)\, dx \geq 0\,, \quad (11)$$

which is the entropy inequality for (7). This relation has been obtained in the case of regular initial data, but it can easily be extended to the case of general bounded initial data by the same idea as in the end of Subsection 3.2.

4.2. Convergence results

Once the entropy inequality for (7) has been obtained, a comparison between u^ε and u can be obtained by means of the doubling variable technique of S.N. Krushkov: we write the entropy inequality (11) with $\eta(u^\varepsilon) = |u^\varepsilon - u(s,y)|$ (s and y fixed) and φ depending on (s,y), we integrate on (s,y), we do the same with the entropy inequality satisfied by u (exchanging the roles of u^ε and u) and we sum the results. Taking φ which forces s to be near t and y to be near x, the term $|u^\varepsilon(t,x) - u(t,x)|$ appears up to an error which can be controlled, and we obtain the following result.

Theorem 4.1. *If $u_0 \in L^\infty(\mathbb{R}^N)$, then the solution to* (7) *converges, as $\varepsilon \to 0$ and in $C([0,T]; L^1_{\rm loc}(\mathbb{R}^N))$ for all $T > 0$, to the entropy solution of* (1).

If we assume more regularity on the initial data, then the error terms which appear in the doubling variable technique can be estimated more precisely and, as in [7] for the parabolic approximation, an optimal rate of convergence can be proved.

Theorem 4.2. *Assume that* $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$*; let* u^ε *be the solution to* (7) *and* u *be the entropy solution to (*1*). Then, for all* $T > 0$, $||u^\varepsilon - u||_{C([0,T];L^1(\mathbb{R}^N))} = \mathcal{O}(\varepsilon^{1/\lambda})$.

Remark 4.1. *We notice that, for* $\lambda < 2$*, the convergence is better than in the case of parabolic approximation. This is due to the fact that, for small times* ([3])*,* g *is less diffusive than* Δ*; this comes from the homogeneity property* $K(t,x) = t^{-N/\lambda}K(1, t^{-1/\lambda}x)$ *of the kernel of* g*, which is to be compared with the homogeneity property* $G(t,x) = t^{-N/2}G(1, t^{-1/2}x)$ *of the heat kernel.*

On the contrary, and because of the same homogeneity properties, g *is more diffusive than* Δ *for large times.*

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[3]The presence of ε entails that εg acts in finite time as g on small time.