The asymmetric gradient discretisation method

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Abstract An asymmetric version of the gradient discretisation method is developed for linear anisotropic elliptic equations. Error estimates and convergence are proved for this method, which is showed to cover all finite volume methods.

Key words: Gradient scheme, gradient discretisation method, error estimates, convergence analysis, finite volume methods.

1 Introduction

The gradient discretisation method (GDM) is a recent framework for the numerical discretisation and analysis of elliptic and parabolic PDEs. The GDM consists in writing the weak formulation of the PDEs with a discrete space of DOFs, and functions and gradients reconstructed from these DOFs. The choice of these space and reconstructions form what is a called a gradient discretisation (GD), and the scheme obtained is a gradient scheme (GS). For many classical schemes, we can find a specific GD such that the corresponding GS is the considered scheme [7]: conforming and non-conforming ("Crouzeix–Raviart") finite element methods, $\mathbb{RT}_k$ mixed finite elements, the multi-point flux approximation-O on rectangles and triangles, mimetic finite differences, and hybrid mimetic mixed methods. The GDM enables a complete and unified convergence analysis of these schemes for linear and nonlinear models of elliptic and parabolic problems, including degenerate equations [3, 4, 6, 8, 9, 11, 12]. The monograph [5] gives a complete presentation of the GDM.

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For all its flexibility, the GDM does not seem to cover some important families of numerical methods, in particular some finite volume schemes such as the two-point flux approximation (TPFA) and the multi-point flux approximation MPFA-L/G schemes. We present here an extension of the GDM that encompasses all finite volume schemes – and possibly others. On the contrary to the usual GDM, this extension uses two different gradient reconstructions. For this reason, we call this method the *asymmetric GDM* (aGDM). This paper is organised as follows. In the following section, we recall the the usual GDM and the corresponding error estimates. Section 3 presents the asymmetric gradient discretisation method and the corresponding error estimate. Section 4 shows that all finite volume methods fit into this framework, and Section 5 gives a conclusion.

2 The (usual) gradient discretisation method

Throughout this paper we consider the standard linear elliptic equation

$$-\mathrm{div}(\Lambda\nabla\overline{u}) = f \text{ in } \Omega, \quad \overline{u} = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a bounded open set of $\mathbb{R}^d$ ($d \geq 1$), $f \in L^2(\Omega)$ and $\Lambda : \Omega \to \mathscr{S}_d(\mathbb{R})$ is a function on Ω with co-domain the set of symmetric $d \times d$ matrices on $\mathbb{R}$, such that

$$\exists \underline{\lambda}, \overline{\lambda} \in (0, +\infty) \text{ s.t., for a.e. } \boldsymbol{x} \in \Omega, \forall \xi \in \mathbb{R}^d,\ \underline{\lambda}|\xi|^2 \leq \Lambda(\boldsymbol{x})\xi \cdot \xi \leq \overline{\lambda}|\xi|^2. \tag{2}$$

The weak formulation of (1) is

$$\text{Find } \overline{u} \in H_0^1(\Omega) \text{ such that, } \forall \overline{v} \in H_0^1(\Omega),\ \int_\Omega \Lambda\nabla\overline{u}\cdot\nabla\overline{v} = \int_\Omega f\overline{v}. \tag{3}$$

A gradient discretisation is the choice of a discrete space (the space of DOFs) and rules to reconstruct functions and gradients from the DOFs.

Definition 1 (Gradient discretisation). A gradient discretisation (GD) for homogeneous Dirichlet BCs is $\mathscr{D} = (X_{\mathscr{D},0}, \Pi_{\mathscr{D}}, \nabla_{\mathscr{D}})$, where

- $X_{\mathscr{D},0}$ is a finite-dimensional space,
- the linear mapping $\Pi_{\mathscr{D}} : X_{\mathscr{D},0} \to L^2(\Omega)$ reconstructs functions,
- the linear mapping $\nabla_{\mathscr{D}} : X_{\mathscr{D},0} \to L^2(\Omega)^d$ reconstruct gradients, and must be chosen such that $\|\nabla_{\mathscr{D}} \cdot\|_{L^2}$ is a norm on $X_{\mathscr{D},0}$.

The corresponding gradient scheme consists in replacing, in (3), the continuous elements with the discrete ones coming from $\mathscr{D}$:

$$\text{Find } u \in X_{\mathscr{D},0} \text{ such that, } \forall v \in X_{\mathscr{D},0},\ \int_\Omega \Lambda\nabla_{\mathscr{D}}u\cdot\nabla_{\mathscr{D}}v = \int_\Omega f\Pi_{\mathscr{D}}v. \tag{4}$$

The accuracy of a gradient scheme for a linear equation is measured through three indicators: a *coercivity* measure $C_{\mathscr{D}}$; a *GD-consistency* (or consistency, for short) measure $S_{\mathcal{D}}$, similar to an interpolation error in the context of finite element methods; and a *limit-conformity* measure $W_{\mathcal{D}}$, indicating how well $\Pi_{\mathcal{D}}$ and $\nabla_{\mathcal{D}}$ satisfy a discrete divergence theorem.

$$C_{\mathcal{D}} = \max_{v\in X_{\mathcal{D},0}\backslash\{0\}} \frac{\|\Pi_{\mathcal{D}} v\|_{L^2}}{\|\nabla_{\mathcal{D}} v\|_{L^2}}, \tag{5}$$

$$\forall \varphi \in H^1_0(\Omega)\,,\; S_{\mathcal{D}}(\varphi) = \min_{v\in X_{\mathcal{D},0}} \left(\|\Pi_{\mathcal{D}} v - \varphi\|_{L^2} + \|\nabla_{\mathcal{D}} v - \nabla\varphi\|_{L^2}\right), \tag{6}$$

$$\forall \boldsymbol{\xi} \in H_{\rm div}(\Omega)\,,\; W_{\mathcal{D}}(\boldsymbol{\xi}) = \max_{v\in X_{\mathcal{D},0}\backslash\{0\}} \frac{\displaystyle\int_\Omega (\nabla_{\mathcal{D}} v\cdot\boldsymbol{\xi} + \Pi_{\mathcal{D}} v\,{\rm div}(\boldsymbol{\xi}))\,{\rm d}\boldsymbol{x}}{\|\nabla_{\mathcal{D}} v\|_{L^2}}. \tag{7}$$

If $\overline{u}$ solves (1) and u solves (4) then the following error estimate holds [5, 10]:

$$\|\Pi_{\mathcal{D}} u - \overline{u}\|_{L^2} + \|\nabla_{\mathcal{D}} u - \nabla\overline{u}\|_{L^2} \le C(1+C_{\mathcal{D}})\,(S_{\mathcal{D}}(\overline{u}) + W_{\mathcal{D}}(\Lambda\nabla\overline{u})) \tag{8}$$

where C depends only on $\underline{\lambda}$ and $\overline{\lambda}$. As seen in this estimate, a sequence of GDs gives rise to converging GSs if, along the sequence, $C_{\mathcal{D}}$ remains bounded and $S_{\mathcal{D}}, W_{\mathcal{D}} \to 0$.

The notations $S_{\mathcal{D}}$ and $W_{\mathcal{D}}$ have been used since the very first articles on gradient schemes [10, 12], and come from a realisation, at the very onset of the GDM, that two kinds of properties had to be verified by the gradient reconstruction: a *strong* convergence property (the interpolation error goes to 0 in norm), and a *weak* convergence property. This latter property is encoded in the requirement "$W_{\mathcal{D}}(\boldsymbol{\xi}) \to 0$", which imposes in a sense that a formal dual (with respect to $\Pi_{\mathcal{D}}$) of $\nabla_{\mathcal{D}}$ converges to the continuous divergence in a weak sense. Understanding that each of these properties is respectively only required for the gradient on the test function and on the unknown function is at the core of the asymmetric GDM we now present.

3 The asymmetric gradient discretisation method

The asymmetric GDM is built from asymmetric GD (aGD), which define two different gradient reconstructions.

Definition 2 (Asymmetric GD and GS). An asymmetric gradient discretisation for homogeneous Dirichlet BCs is $\mathcal{D}_{as} = (X_{\mathcal{D}_{as},0}, \Pi_{\mathcal{D}_{as}}, \widehat{\nabla}_{\mathcal{D}_{as}}, \overline{\nabla}_{\mathcal{D}_{as}})$, where

- $X_{\mathcal{D}_{as},0}$ is a finite dimensional space,
- the linear mapping $\Pi_{\mathcal{D}_{as}} : X_{\mathcal{D}_{as},0} \to L^2(\Omega)$ reconstructs functions,
- the linear mappings $\widehat{\nabla}_{\mathcal{D}_{as}}, \overline{\nabla}_{\mathcal{D}_{as}} : X_{\mathcal{D}_{as},0} \to L^2(\Omega)^d$ reconstructs gradients and are chosen such that $\|\widehat{\nabla}_{\mathcal{D}_{as}}\cdot\|_{L^2}$ and $\|\overline{\nabla}_{\mathcal{D}_{as}}\cdot\|_{L^2}$ are norms on $X_{\mathcal{D}_{as},0}$.

The corresponding asymmetric gradient scheme (aGS) is

$$\text{Find } u \in X_{\mathcal{D}_{as},0} \text{ such that, } \forall v \in X_{\mathcal{D}_{as},0},\; \int_\Omega \Lambda\widehat{\nabla}_{\mathcal{D}_{as}} u\cdot\overline{\nabla}_{\mathcal{D}_{as}} v = \int_\Omega f\,\Pi_{\mathcal{D}_{as}} v. \tag{9}$$

An aGS's matrix can be symmetric (see the TPFA scheme below). "Asymmetric" refers to the usage of two different gradients, not to the scheme's properties. The measures $C_{\mathcal{D}}$, $S_{\mathcal{D}}$ and $W_{\mathcal{D}}$ are defined as for GD, but the former two use $\widehat{\nabla}_{\mathcal{D}_{as}}$ while the latter is based on $\overline{\nabla}_{\mathcal{D}_{as}}$.

$$\widehat{C}_{\mathcal{D}_{as}} = \max_{v\in X_{\mathcal{D}_{as},0}\setminus\{0\}} \frac{\|\Pi_{\mathcal{D}_{as}} v\|_{L^2}}{\|\widehat{\nabla}_{\mathcal{D}_{as}} v\|_{L^2}}, \tag{10}$$

$$\forall \varphi \in H^1_0(\Omega)\,,\ \widehat{S}_{\mathcal{D}_{as}}(\varphi) = \min_{v\in X_{\mathcal{D}_{as},0}} \left(\|\Pi_{\mathcal{D}_{as}} v - \varphi\|_{L^2} + \|\widehat{\nabla}_{\mathcal{D}_{as}} v - \nabla\varphi\|_{L^2}\right), \tag{11}$$

$$\forall \boldsymbol{\xi} \in H_{\rm div}(\Omega)\,,\ \overline{W}_{\mathcal{D}_{as}}(\boldsymbol{\xi}) = \max_{v\in X_{\mathcal{D}_{as},0}} \frac{\displaystyle\int_\Omega \left(\overline{\nabla}_{\mathcal{D}_{as}} v\cdot\boldsymbol{\xi} + \Pi_{\mathcal{D}_{as}} v\,{\rm div}(\boldsymbol{\xi})\right){\rm d}\boldsymbol{x}}{\|\overline{\nabla}_{\mathcal{D}_{as}} v\|_{L^2}}. \tag{12}$$

Due to the presence of two different gradient reconstructions, the coercivity of an aGD (and thus well-posedness of the aGS) cannot be solely measured through $\widehat{C}_{\mathcal{D}_{as}}$. An additional compatibility condition, involving Λ, is required.

Definition 3 (Λ-compatibility of aGD). An asymmetric gradient discretisation $\mathcal{D}_{as}$ is *Λ-compatible* if

$$\zeta^\Lambda_{\mathcal{D}_{as}} := \min_{v\in X_{\mathcal{D}_{as},0}\setminus\{0\}} \frac{\displaystyle\int_\Omega \Lambda\widehat{\nabla}_{\mathcal{D}_{as}} v\cdot\overline{\nabla}_{\mathcal{D}_{as}} v\,{\rm d}\boldsymbol{x}}{\|\widehat{\nabla}_{\mathcal{D}} v\|_{L^2(\Omega)}\|\overline{\nabla}_{\mathcal{D}} v\|_{L^2(\Omega)}} > 0. \tag{13}$$

Our main results are the following error estimates.

Theorem 1 (Error estimate for the aGDM). *Let $\mathcal{D}_{as}$ be a Λ-compatible aGD. Then the aGS* (9) *has a unique solution u and, if $\overline{u}$ solves* (3)*,*

$$\|\widehat{\nabla}_{\mathcal{D}_{as}} u - \nabla\overline{u}\|_{L^2} \le \frac{1}{\zeta^\Lambda_{\mathcal{D}_{as}}}\left(\widehat{S}_{\mathcal{D}_{as}}(\overline{u}) + \overline{W}_{\mathcal{D}_{as}}(\Lambda\nabla\overline{u})\right) + \widehat{S}_{\mathcal{D}_{as}}(\overline{u}), \tag{14}$$

$$\|\Pi_{\mathcal{D}_{as}} u - \overline{u}\|_{L^2} \le \frac{\widehat{C}_{\mathcal{D}_{as}}}{\zeta^\Lambda_{\mathcal{D}_{as}}}\left(\widehat{S}_{\mathcal{D}_{as}}(\overline{u}) + \overline{W}_{\mathcal{D}_{as}}(\Lambda\nabla\overline{u})\right) + \widehat{S}_{\mathcal{D}_{as}}(\overline{u}). \tag{15}$$

Proof. The error estimates prove the existence and uniqueness of the solution to (9). Indeed, this equation is a square linear system, and (14) shows that its only solution is $u = 0$ whenever its right-hand side is zero.

We now establish (14) and (15). Let $v \in X_{\mathcal{D}_{as},0}$. By definition of $\overline{W}_{\mathcal{D}_{as}}$ with $\boldsymbol{\xi} = \Lambda\nabla\overline{u}$, since ${\rm div}(\Lambda\overline{u}) = -f$,

$$\left|\int_\Omega \overline{\nabla}_{\mathcal{D}_{as}} v\cdot\Lambda\nabla\overline{u} - \Pi_{\mathcal{D}_{as}} v\,f\,{\rm d}\boldsymbol{x}\right| \le \|\overline{\nabla}_{\mathcal{D}_{as}} v\|_{L^2}\overline{W}_{\mathcal{D}_{as}}(\Lambda\nabla\overline{u}).$$

Since u solves (9), this gives

$$\left|\int_\Omega \Lambda\overline{\nabla}_{\mathscr{D}_{as}} v\cdot(\nabla\overline{u}-\widehat{\nabla}_{\mathscr{D}_{as}} u)\,\mathrm{d}\boldsymbol{x}\right| \le \|\overline{\nabla}_{\mathscr{D}_{as}} v\|_{L^2}\overline{W}_{\mathscr{D}_{as}}(\Lambda\nabla\overline{u}).$$

For $\varphi\in H^1_0(\Omega)$, take $P_{\mathscr{D}_{as}}\varphi\in X_{\mathscr{D}_{as},0}$ such that $S_{\mathscr{D}}(\varphi)=\|\Pi_{\mathscr{D}_{as}}(P_{\mathscr{D}_{as}}\varphi)-\varphi\|_{L^2}+\|\widehat{\nabla}_{\mathscr{D}_{as}}(P_{\mathscr{D}_{as}}\varphi)-\nabla\varphi\|_{L^2}$. The triangle inequality yields

$$\left|\int_\Omega \Lambda\overline{\nabla}_{\mathscr{D}_{as}} v\cdot(\widehat{\nabla}_{\mathscr{D}_{as}} P_{\mathscr{D}}\overline{u}-\widehat{\nabla}_{\mathscr{D}_{as}} u)\,\mathrm{d}\boldsymbol{x}\right| \le \|\overline{\nabla}_{\mathscr{D}_{as}} v\|_{L^2}\left(\overline{W}_{\mathscr{D}_{as}}(\Lambda\nabla\overline{u})+\overline{\lambda}\widehat{S}_{\mathscr{D}_{as}}(\overline{u})\right).$$

Make $v=P_{\mathscr{D}_{as}}\overline{u}-u$ and use the Λ-compatibility to deduce

$$\zeta^\Lambda_{\mathscr{D}_{as}}\|\widehat{\nabla}_{\mathscr{D}_{as}}(P_{\mathscr{D}_{as}}\overline{u}-u)\|_{L^2}\le \overline{W}_{\mathscr{D}_{as}}(\Lambda\nabla\overline{u})+\widehat{S}_{\mathscr{D}_{as}}(\overline{u}). \tag{16}$$

Estimate (14) follows from the triangle inequality and $\|\widehat{\nabla}_{\mathscr{D}_{as}}(P_{\mathscr{D}_{as}}\overline{u})-\nabla\overline{u}\|_{L^2}\le \widehat{S}_{\mathscr{D}_{as}}(\overline{u})$. Equations (10) and (16) yield $\|\Pi_{\mathscr{D}_{as}}(P_{\mathscr{D}_{as}}\overline{u}-u)\|_{L^2}\le \frac{\widehat{C}_{\mathscr{D}_{as}}}{\zeta^\Lambda_{\mathscr{D}_{as}}}(\overline{W}_{\mathscr{D}_{as}}(\Lambda\nabla\overline{u})+\widehat{S}_{\mathscr{D}_{as}}(\overline{u}))$. Estimate (15) follows from the triangle inequality. ■

We now consider the "dual" scheme of (9), obtained by switching the gradients:

$$\text{Find } u\in X_{\mathscr{D}_{as},0} \text{ such that, } \forall v\in X_{\mathscr{D}_{as},0},\ \int_\Omega \Lambda\overline{\nabla}_{\mathscr{D}_{as}} u\cdot\widehat{\nabla}_{\mathscr{D}_{as}} v=\int_\Omega f\Pi_{\mathscr{D}_{as}} v. \tag{17}$$

A weak convergence result can be established for this scheme, by slightly strengthening the definition of $\widehat{C}_{\mathscr{D}_{as}}$ into

$$\widetilde{C}_{\mathscr{D}_{as}}=\max_{v\in X_{\mathscr{D}_{as},0}\setminus\{0\}}\left(\frac{\|\Pi_{\mathscr{D}_{as}} v\|_{L^2}}{\|\widehat{\nabla}_{\mathscr{D}_{as}} v\|_{L^2}}+\frac{\|\Pi_{\mathscr{D}_{as}} v\|_{L^2}}{\|\overline{\nabla}_{\mathscr{D}_{as}} v\|_{L^2}}\right)$$

Theorem 2 (Weak convergence of the dual aGS). *Let $(\mathscr{D}^m_{as})_m$ be a sequence of Λ-compatible aGDs such that $(\widetilde{C}_{\mathscr{D}^m_{as}}+\zeta^\Lambda_{\mathscr{D}^m_{as}})_m$ is bounded, $\widehat{S}_{\mathscr{D}^m_{as}}(\varphi)\to 0$ for all $\varphi\in H^1_0(\Omega)$, and $\overline{W}_{\mathscr{D}^m_{as}}(\boldsymbol{\xi})\to 0$ for all $\boldsymbol{\xi}\in H_{\mathrm{div}}(\Omega)$ (these properties are respectively called the* coercivity, consistency *and* limit-conformity *of $(\mathscr{D}^m_{as})_m$). Then there exists a unique u_m solution to* (17) *with $\mathscr{D}_{as}=\mathscr{D}^m_{as}$ and, as $m\to\infty$, $\Pi_{\mathscr{D}^m_{as}}u_m\to\overline{u}$ and $\nabla_{\mathscr{D}^m_{as}}u_m\to\nabla\overline{u}$ weakly in $L^2(\Omega)$.*

Proof. Make $v=u_m$ in (17) and use the definition of $\widetilde{C}_{\mathscr{D}^m_{as}}$ and $\zeta^\Lambda_{\mathscr{D}^m_{as}}$ to write $\zeta^\Lambda_{\mathscr{D}^m_{as}}\|\widehat{\nabla}_{\mathscr{D}_{as}}u_m\|_{L^2}\|\overline{\nabla}_{\mathscr{D}_{as}}u_m\|_{L^2(\Omega)}\le\widetilde{C}_{\mathscr{D}^m_{as}}\|f\|_{L^2}\|\widehat{\nabla}_{\mathscr{D}_{as}}u_m\|_{L^2}$. Hence, $(\|\overline{\nabla}_{\mathscr{D}_{as}}u_m\|_{L^2})_m$ is bounded and each dual aGS has a unique solution (since these problems boil down to square linear systems). Use then the limit-conformity property as in [5, Lemma 2.12] to infer the existence of $\overline{u}\in H^1_0(\Omega)$ such that, up to a subsequence, $\Pi_{\mathscr{D}^m_{as}}u_m\to\overline{u}$ and $\overline{\nabla}_{\mathscr{D}^m_{as}}u_m\to\nabla\overline{u}$ weakly in $L^2(\Omega)$. Define $P_{\mathscr{D}^m_{as}}$ as in the proof of Theorem 1 and, for a generic $\varphi\in H^1_0(\Omega)$, take $v=P_{\mathscr{D}^m_{as}}\varphi$ in (17). The consistency property and the reasoning in [5, Step 3, proof of Theorem 3.16] show that $\overline{u}$ is the solution to (3). By uniqueness of this solution, the above-mentioned convergences apply to the whole sequence. ■

4 Application to finite volume schemes

Consider a polytopal mesh $\mathfrak{T}=(\mathscr{M},\mathscr{F},\mathscr{P})$ in the sense of [5, Definition 7.2]: $\mathscr{M}$ is the set of cells (generic notation: K), $\mathscr{F}$ is the set of faces (generic notation: σ) and $\mathscr{P}$ is a set made of one point per cell (generic notation: $\boldsymbol{x}_K$). We further let $\mathscr{F}_{\rm int}=\{\sigma\in\mathscr{F}\,:\,\sigma\subset\Omega\}$ be the set of interior faces and $\mathscr{F}_{\rm ext}=\mathscr{F}\setminus\mathscr{F}_{\rm int}$ be the set of boundary faces. For $K\in\mathscr{M}$, $\mathscr{F}_K$ is the set of faces of K. If $\sigma\in\mathscr{F}_K$, $\overline{\boldsymbol{x}}_\sigma$ is the center of mass of σ, $\boldsymbol{n}_{K,\sigma}$ is the outer normal to K on σ, and $D_{K,\sigma}$ is the convex hull of σ and $\boldsymbol{x}_K$. Denoting by $|E|$ the d- or $(d-1)$-dimensional measure of E (depending on the Haussdorff dimension of E), we have $|D_{K,\sigma}|=\frac{|\sigma|d_{K,\sigma}}{d}$, where $d_{K,\sigma}$ is the orthogonal distance between $\boldsymbol{x}_K$ and σ. As a generic notation, if $\sigma\in\mathscr{F}_{\rm int}$, K and L are the two cells on each side of σ. We assume in the following that Λ is constant, equal to Λ_K, in each cell K.

Generic FV scheme, and assumptions. We consider here generic finite volume schemes [2]. The space of cell and face DOFs is $X_{\mathscr{D}_{as},0}=\{v=((v_K)_{K\in\mathscr{M}},(v_\sigma)_{\sigma\in\mathscr{F}})\,:\,v_\sigma=0\text{ if }\sigma\in\mathscr{F}_{\rm ext}\}$ ([1]), and an FV volume scheme is defined from numerical fluxes $F_{K,\sigma}:X_{\mathscr{D}_{as},0}\to\mathbb{R}$ in the following way:

$$\forall K\in\mathscr{M}\,,\ \sum_{\sigma\in\mathscr{F}_K}F_{K,\sigma}(u)=\int_K f\ \text{ and }\ \forall\sigma\in\mathscr{F}_{\rm int}\,,\ F_{K,\sigma}(u)+F_{L,\sigma}(u)=0. \tag{18}$$

Multiplying the cell equations by a generic v_K, the edge equations by a generic v_σ, and summing the resulting equations, this scheme can be recast as

$$\forall v\in X_{\mathscr{D}_{as},0}\,,\ \sum_{K\in\mathscr{M}}\sum_{\sigma\in\mathscr{F}_K}F_{K,\sigma}(u)(v_K-v_\sigma)=\sum_{K\in\mathscr{M}}\int_K fv_K. \tag{19}$$

Define the discrete H^1_0 norm on $X_{\mathscr{D}_{as},0}$ by $\|v\|_{1,\mathfrak{T}}^2=\sum_{K\in\mathscr{M}}\sum_{\sigma\in\mathscr{F}_K}|D_{K,\sigma}|\left|\frac{v_K-v_\sigma}{d_{K,\sigma}}\right|^2$, and the interpolant $I_{\mathscr{D}_{as}}:\varphi\in C(\overline{\Omega})\cap H^1_0(\Omega)\mapsto((\varphi(\boldsymbol{x}_K))_{K\in\mathscr{M}},(\varphi(\overline{\boldsymbol{x}}_\sigma))_{\sigma\in\mathscr{F}})\in X_{\mathscr{D}_{as},0}$. We assume that the fluxes satisfy the following properties.

1. $\mathbb{P}_1$-exactness: for all $K\in\mathscr{M}$, there is $I_K\subset\mathscr{P}\cup\{\overline{\boldsymbol{x}}_\sigma\,:\,\sigma\in\mathscr{F}\}$ such that, if φ is affine on a neighbourhood U of I_K, for all $\sigma\in\mathscr{F}_K$ we have $F_{K,\sigma}(I_{\mathscr{D}_{as}}\varphi)=-|\sigma|\Lambda_K(\nabla\varphi)_{|U}\cdot\boldsymbol{n}_{K,\sigma}$.
2. Stability: there is $C_{\rm stab}>0$ such that

$$\forall v\in X_{\mathscr{D}_{as},0}\,,\ \sum_{K\in\mathscr{M}}\sum_{\sigma\in\mathscr{F}_K}|D_{K,\sigma}|\left(\frac{F_{K,\sigma}(v)}{|\sigma|}\right)^2\le C_{\rm stab}\|v\|_{1,\mathfrak{T}}^2. \tag{20}$$

3. Coercivity: there is $C_{\rm coer}>0$ such that, for all $v\in X_{\mathscr{D}_{as},0}$,

$$\sum_{K\in\mathscr{M}}\sum_{\sigma\in\mathscr{F}_K}F_{K,\sigma}(v)(v_K-v_\sigma)\ge C_{\rm coer}\|v\|_{1,\mathfrak{T}}^2. \tag{21}$$

[1] Vertex-centered FV methods can easily be considered by changing the DOFs.

The FV scheme (18) **is an aGS of the form** (9). For $v \in X_{\mathscr{D}_{as},0}$, let $\Pi_{\mathscr{D}_{as}} v \in L^2(\Omega)$ be defined by $(\Pi_{\mathscr{D}_{as}} v)_{|K} = v_K$ for all $K \in \mathscr{M}$. Define $\overline{\nabla}_{\mathscr{D}_{as}} v \in L^2(\Omega)^d$ and $\widehat{\nabla}_{\mathscr{D}_{as}} v \in L^2(\Omega)^d$ as the piecewise constant functions such that, for all $K \in \mathscr{M}$ and $\sigma \in \mathscr{F}_K$,

$$(\widehat{\nabla}_{\mathscr{D}_{as}} v)_{|D_{K,\sigma}} = \frac{F_{K,\sigma}(v)}{|\sigma|}\Lambda_K^{-1}\boldsymbol{n}_{K,\sigma} + (\widehat{\nabla}_{\mathscr{D}_{as}} v)_{K,\sigma,\perp} \text{ and } (\overline{\nabla}_{\mathscr{D}_{as}} v)_{|D_{K,\sigma}} = d\frac{v_\sigma - v_K}{d_{K,\sigma}}\boldsymbol{n}_{K,\sigma},$$

with $(\widehat{\nabla}_{\mathscr{D}_{as}} v)_{K,\sigma,\perp} \perp \Lambda_K \boldsymbol{n}_{K,\sigma}$ chosen to be $\mathbb{P}_1$-exact and stable, that is:

$$\begin{cases} \forall \varphi \in C(\overline{\Omega}) \cap H_0^1(\Omega)\,, \forall K \in \mathscr{M}\,, \text{ if } \varphi \text{ is affine on a neighbourhood } U \text{ of } I_K \\ \text{then } (\widehat{\nabla}_{\mathscr{D}_{as}} I_{\mathscr{D}_{as}} \varphi)_{K,\sigma,\perp} \text{ is the component on } (\Lambda_K \boldsymbol{n}_{K,\sigma})^\perp \text{ of } (\nabla\varphi)_{|U}\,, \end{cases}$$

$$\exists C'_{\rm stab} > 0\,,\ \forall v \in X_{\mathscr{D}_{as},0}\,,\ \sum_{K\in\mathscr{M}}\sum_{\sigma\in\mathscr{F}_K} |D_{K,\sigma}|\,|(\widehat{\nabla}_{\mathscr{D}_{as}} v)_{K,\sigma,\perp}|^2 \le C'_{\rm stab}\|v\|_{1,\mathfrak{T}}^2. \quad (22)$$

By definition of $\widehat{\nabla}_{\mathscr{D}_{as}}$ and $\overline{\nabla}_{\mathscr{D}_{as}}$, $\Lambda_K(\widehat{\nabla}_{\mathscr{D}_{as}} u)_{|D_{K,\sigma}} \cdot \boldsymbol{n}_{K,\sigma} = \frac{1}{|\sigma|} F_{K,\sigma}(u)$ and $(\overline{\nabla}_{\mathscr{D}_{as}} v)_{|D_{K,\sigma}} \cdot \Lambda_K(\widehat{\nabla}_{\mathscr{D}_{as}} u)_{K,\sigma,\perp} = 0$. Thus, owing to $|D_{K,\sigma}| = |\sigma| d_{K,\sigma}/d$,

$$\begin{aligned}\int_\Omega \Lambda \widehat{\nabla}_{\mathscr{D}_{as}} u \cdot \overline{\nabla}_{\mathscr{D}_{as}} v \,\mathrm{d}\boldsymbol{x} &= \sum_{K\in\mathscr{M}}\sum_{\sigma\in\mathscr{F}_K} |D_{K,\sigma}| \Lambda_K (\widehat{\nabla}_{\mathscr{D}_{as}} u)_{|D_{K,\sigma}} \cdot \left(d\frac{v_\sigma - v_K}{d_{K,\sigma}}\boldsymbol{n}_{K,\sigma}\right) \\ &= \sum_{K\in\mathscr{M}}\sum_{\sigma\in\mathscr{F}_K} F_{K,\sigma}(u)(v_K - v_\sigma). \end{aligned} \quad (23)$$

This shows that the aGS (9) with $\mathscr{D}_{as}$ constructed above is the FV scheme (19).

Let us now check that, under standard regularity assumptions on the mesh, $\mathscr{D}_{as}$ satisfies the coercivity, consistency, limit-conformity and Λ-compatibility properties. From the definition of $\overline{\nabla}_{\mathscr{D}_{as}}$ and the stability properties (20) and (22), $\|\overline{\nabla}_{\mathscr{D}_{as}} v\|_{L^2} + \|\widehat{\nabla}_{\mathscr{D}_{as}} v\|_{L^2} \le C_S\|v\|_{1,\mathfrak{T}}$. Hence, plugging $v = u$ in (23) and using the coercivity (21), we find C_0 and C_1 not depending on u such that

$$C_0\|\widehat{\nabla}_{\mathscr{D}_{as}} u\|_{L^2}\|\overline{\nabla}_{\mathscr{D}_{as}} u\|_{L^2} \ge \int_\Omega \Lambda \widehat{\nabla}_{\mathscr{D}_{as}} u \cdot \overline{\nabla}_{\mathscr{D}_{as}} u \,\mathrm{d}\boldsymbol{x} \ge C_1\|u\|_{1,\mathfrak{T}}^2. \quad (24)$$

Since $\|u\|_{1,\mathfrak{T}}^2 \ge C_S^{-2}\|\widehat{\nabla}_{\mathscr{D}_{as}} u\|_{L^2}\|\overline{\nabla}_{\mathscr{D}_{as}} u\|_{L^2}$, (24) proves the Λ-compatibility, with $\zeta_{\mathscr{D}_{as}}^\Lambda = C_1 C_S^{-2}$. Using now $\|u\|_{1,\mathfrak{T}}^2 \ge C_S^{-1}\|u\|_{1,\mathfrak{T}}\|\overline{\nabla}_{\mathscr{D}_{as}} u\|_{L^2}$, Equation (24) yields $\|\widehat{\nabla}_{\mathscr{D}_{as}} u\|_{L^2} \ge C_0^{-1} C_1 C_S^{-1}\|u\|_{1,\mathfrak{T}}$. Invoking then [5, Lemma B.11] gives the coercivity, i.e. a bound on $C_{\mathscr{D}_{as}}$ ($\widetilde{C}_{\mathscr{D}_{as}}$ can be bounded similarly). The consistency, i.e. the convergence to 0 of $\widehat{S}_{\mathscr{D}_{as}}(\varphi)$ as $h_{\mathscr{M}} \to 0$, follows from [5, Lemma 7.28 and 7.31] by the stability and $\mathbb{P}_1$-exactness of $F_{K,\sigma}(\cdot)$ and of $(\widehat{\nabla}_{\mathscr{D}_{as}}\cdot)_{K,\sigma,\perp}$. The limit-conformity follows from $\frac{1}{|K|}\int_K \overline{\nabla}_{\mathscr{D}_{as}} v \,\mathrm{d}\boldsymbol{x} = \overline{\nabla}_K v$ (see [5, Eq. (7.7e)] for the definition of $\overline{\nabla}_K$) and from [5, Lemma B.8].

The form (18) is the very definition of a FV scheme [2]. For a number of these schemes, such that the TPFA scheme or MPFA schemes, the $\mathbb{P}_1$-exactness and stability of the fluxes are trivial. The coercivity of the fluxes is either easy and wellknown (e.g., for TPFA), or a required assumption to analyse the method (e.g., for MPFA [1]).

5 Conclusion

We developed a generalisation of the gradient discretisation method, which allows for the usage of two different gradients to design numerical schemes for diffusion problems. We showed that this generalisation is adapted to all finite volume methods. Error estimates are obtained in this asymmetric GDM framework. Due to the Λ-compatibility requirement, the aGDM doesn't present all the flexibility of the GDM when it comes to dealing with fully non-linear problems, but it does accommodate some – provided that the non-linearity is isotropic in the diffusion matrix, or in the source/reaction terms.

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