Convergence of a finite volume - mixed finite element method for an elliptic - hyperbolic system

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Abstract : This paper gives a proof of convergence for the approximate solution of an elliptic-hyperbolic system, describing the conservation of two immiscible incompressible phases flowing in a porous medium. The approximate solution is obtained by a mixed finite element method on a large class of meshes for the elliptic equation and a finite volume method for the hyperbolic equation. Since the considered meshes are not necessarily structured, the proof uses a weak total variation inequality, which cannot yield a BV-estimate. We thus prove, under an L^∞ estimate, the weak convergence of the finite volume approximation. The strong convergence proof is then sketched under regularity assumptions which ensure that the flux is Lipschitz-continuous.

1. Introduction

The purpose of oil reservoir simulation is to account for several phenomena such as chemical reactions, thermodynamical equilibrium and polyphasic flows. Since the full model is too complex, a simplified model, describing the flow of two incompressible immiscible fluids through a porous medium, has been extensively studied. In this simplified model, two fluid phases, oil and water, flow through the pores of some possibly heterogeneous and anisotropic porous medium; water is injected through injection wells in order to displace the oil towards production wells. Here we neglect the gravity effects as well as the capillary pressure. Assuming the total mobility of the two phases to be constant and the mobility of water to be linear, the conservation equations of the two phases in a domain Ω yield the following system of equations:

$$u_t(x,t) - \mathrm{div}(u(x,t)\mathbf{\Lambda}(x)\nabla p(x)) = s(x,t)f^+(x) - u(x,t)f^-(x),$$

$$(1-u)_t(x,t)-\mathrm{div}((1-u(x,t))\mathbf{\Lambda}(x)\nabla p(x)) = (1-s(x,t))f^+(x)-(1-u(x,t))f^-(x),$$

for $(x,t) \in \Omega \times \mathbb{R}^+$. In the above equations, the saturation of the water phase is denoted by u, the common pressure of both phases is denoted by p. The absolute permeability $\mathbf{\Lambda}$ is a symmetric positive definite matrix (in anisotropic media the eigenvalues of the matrix $\mathbf{\Lambda}$ are not all identical) which depends on the space variable in heterogeneous media. The function f represents the internal source terms, corresponding to the presence of wells drilled into the reservoir ($f^+ = \max(f,0)$ and $f^- = \max(-f,0)$ denote the positive and negative parts of f). The positive source term corresponds to an injection well, the negative one corresponds to a production well. The function s represents the fraction of the water phase in the injected source term, and the saturation u of the water in place is the fraction of water in the produced source term. This problem, completed with initial and boundary conditions, is rewritten as follows:

$$\begin{gathered} u_t(x,t) + \mathrm{div}(u\mathbf{q})(x,t) + u(x,t)f^-(x) = s(x,t)f^+(x) \\ \text{for a.e. } (x,t) \in \Omega \times \mathbb{R}^+, \end{gathered} \tag{1.1}$$

$$\mathbf{\Lambda}(x)^{-1}\mathbf{q}(x) + \nabla p(x) = 0 \text{ for a.e. } x \in \Omega, \tag{1.2}$$

$$\mathrm{div}\,\mathbf{q}(x) = f(x) \text{ for a.e. } x \in \Omega, \tag{1.3}$$

$$\mathbf{q}(x) \cdot \mathbf{n}_{\partial\Omega}(x) = g(x) \text{ for a.e. } x \in \partial\Omega, \tag{1.4}$$

$$u(x,t) = \overline{u}(x,t) \text{ for a.e. } (x,t) \in \partial\Omega^- \times \mathbb{R}^+, \tag{1.5}$$

$$u(x,0) = u_0(x) \text{ for a.e. } x \in \Omega, \tag{1.6}$$

Notice that the boundary condition for the saturation is only given on the part $\partial\Omega^-$ of the boundary where the flow enters into the domain, that is, where $\mathbf{q}(x) \cdot \mathbf{n}_{\partial\Omega}(x) = g(x) \leq 0$.

In Eqs (1.1)-(1.6) (referred in the following as Problem (P)) the following hypotheses (referred in the following as Hypotheses (H)) are used.

Hypotheses (H):

1. *Ω is an open bounded subset of $\mathbb{R}^d$ ($d = 2$ or 3 in practical) such that, locally, Ω either has a $C^{1,1}$ regular boundary or is convex.*
2. *$\mathbf{\Lambda}$ is a measurable mapping from Ω to the set of symmetric real $d \times d$ matrices, such that there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ satisfying $\lambda_1|z| \leq |\mathbf{\Lambda}(x)z| \leq \lambda_2|z|$ for almost every $x \in \Omega$ and all $z \in \mathbb{R}^d$.*

3. $f \in L^2(\Omega)$.
4. $g = \mathbf{q}_0 \cdot \mathbf{n}_{\partial\Omega}$ *for some* $\mathbf{q}_0 \in (H^1(\Omega))^d$ *and*

$$\int_\Omega f(x)\,dx - \int_{\partial\Omega} g(x)\,d\gamma(x) = 0.$$

5. $\overline{u} \in L^\infty(\partial\Omega^- \times \mathbb{R}^+)$ *where* $\partial\Omega^- = \{x \in \partial\Omega, g(x) \le 0\}$.
6. $u_0 \in L^\infty(\Omega)$.
7. $s \in L^\infty(\Omega \times \mathbb{R}^+)$.

Here and in the following, when U is an open subset of $\mathbb{R}^d$ with a sufficiently regular boundary (see Definition 2), we denote by $\mathbf{n}_{\partial U}$ the unit outward normal to ∂U and by γ the $(d-1)$-dimensional measure on ∂U. $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$ and $x \cdot y$ denotes the Euclidean scalar product of $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. When X is a subset of $\mathbb{R}^d$, $\delta(X)$ denotes the diameter of X, that is to say $\delta(X) = \sup_{(x,y)\in X^2} |x - y|$. $B(z, r)$ denotes the Euclidean ball of center $z \in \mathbb{R}^d$ and radius $r > 0$.

Remark 1: *Since we allow Ω to have a non-regular boundary, there is no convenient way to characterize the regularity condition on g. Indeed, if Ω has a $C^{1,1}$-regular boundary, it is easy to see that $g = \mathbf{q}_0 \cdot \mathbf{n}_{\partial\Omega}$ if and only if $g \in H^{1/2}(\partial\Omega)$, but on the non-regular parts of $\partial\Omega$, this condition is not necessary and it is not even obvious that it is sufficient. For example, take $\Omega =]0,1[^2$, $g = 1$ on $(\{0\}\times]0,1[) \cup (\{1\}\times]0,1[)$ and $g = 0$ on $(]0,1[\times\{0\}) \cup (]0,1[\times\{1\})$; then g does not belong to $H^{1/2}(\partial\Omega)$, but g can be written as $\mathbf{q}_0 \cdot \mathbf{n}_{\partial\Omega}$ with $\mathbf{q}_0(x, y) = (-1 + 2x, 0) \in (H^1(\Omega))^2$.*

A weak solution of Problem (P) is defined by :

Definition 1: *Under Hypotheses (H), a weak solution of (P) is $(u, p, \mathbf{q}) \in L^\infty(\Omega \times \mathbb{R}^+) \times L^2(\Omega) \times H_g(\mathrm{div}, \Omega)$ such that*

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_\Omega u(x,t)\left(\frac{\partial\phi}{\partial t}(x,t) + \mathbf{q}(x)\cdot\nabla\phi(x,t) - \phi(x,t)f^-(x)\right)dx\,dt = \\
&-\int_\Omega u_0(x)\phi(x,0)\,dx + \int_{\mathbb{R}^+}\int_{\partial\Omega^-} \overline{u}(x,t)\phi(x,t)g(x)\,d\gamma(x)\,dt \\
&-\int_{\mathbb{R}^+}\int_\Omega \phi(x,t)s(x,t)f^+(x)\,dx\,dt
\end{aligned} \tag{1.7}$$

$\forall \phi \in C_c^1(\mathbb{R}^d \times \mathbb{R})$ *such that* $\phi = 0$ *on* $\partial\Omega^+ \times \mathbb{R}^+ = (\partial\Omega \setminus \partial\Omega^-) \times \mathbb{R}^+$,

$$\int_\Omega \mathbf{y}(x)\cdot\mathbf{\Lambda}(x)^{-1}\mathbf{q}(x)\,dx - \int_\Omega p(x)\ \mathrm{div}\,\mathbf{y}(x)\,dx = 0 \quad \forall \mathbf{y} \in H_0(\mathrm{div}, \Omega), \tag{1.8}$$

$$\int_\Omega v(x)\mathrm{div}\,\mathbf{q}(x)\,dx = \int_\Omega f(x)v(x)\,dx \quad \forall v \in L^2(\Omega), \tag{1.9}$$

and

$$\int_{\Omega} p(x)\, dx = 0, \tag{1.10}$$

where the function spaces $H(\mathrm{div}, \Omega)$*,* $H_0(\mathrm{div}, \Omega)$ *and* $H_g(\mathrm{div}, \Omega)$ *are defined by* $H(\mathrm{div}, \Omega) = \{\mathbf{q} \in (L^2(\Omega))^d,\ \mathrm{div}\, \mathbf{q} \in L^2(\Omega)\}$*,* $H_0(\mathrm{div}, \Omega) = \{\mathbf{q} \in H(\mathrm{div}, \Omega),\ \mathbf{q} \cdot \mathbf{n}_{\partial\Omega} = 0 \text{ on } \partial\Omega\}$*, and* $H_g(\mathrm{div}, \Omega) = \{\mathbf{q} \in H(\mathrm{div}, \Omega),\ \mathbf{q} \cdot \mathbf{n}_{\partial\Omega} = g \text{ on } \partial\Omega\}$*.*

The existence and uniqueness of $(p, \mathbf{q}) \in L^2(\Omega) \times H_g(\mathrm{div}, \Omega)$, the solution of (1.8)-(1.10) under Hypotheses (H), is a classical result as long as the equations (1.8)-(1.10) do not depend on u. We could consider the much more complex problem where the function $\mathbf{\Lambda}$ depends on x and u in (1.8); such a problem would be more general than Problem (P), which can only model the case of oil reservoirs in which the viscosity of the oil phase is comparable to that of the water phase (such reservoirs indeed exist). However, it seems that in the case where $\mathbf{\Lambda}$ depends on x and u, it is not yet possible to identify an appropriate weak sense in which the limit of a sequence of numerical approximations can satisfy equation (1.1) (see [Eymard-Gallouët (2002)]). We therefore restrict the present paper to the case where $\mathbf{\Lambda}$ only depends on x. Assuming that the flux $\mathbf{q}$ is given by (1.8)-(1.10) and under Hypotheses (H), the existence of a weak solution u to (1.7) is not standard: indeed, the classical existence and uniqueness theorems for the weak solution of a scalar hyperbolic equation only hold in the case of a Lipschitz continuous flux (the extension of the uniqueness result to more general cases is an open problem). Thus the existence of a solution, in this particular case, appears to be a consequence of the convergence result given in the present paper, and the uniqueness result, sketched in this paper as a necessary step in the direction of a strong convergence property, only holds under additional hypotheses which ensure that $\mathbf{q}$ is Lipschitz continuous. In this last case, we could also handle the case of the problem $u_t(x,t) + \mathrm{div}(F(u)\mathbf{q})(x,t) + u(x,t)f^-(x) = s(x,t)f^+(x)$ with a possibly nonlinear function F (the so-called "fractional flow" function). But this would be somewhat artificial since physical data which lead to a nonlinear fractional flow function also yield dependence of $\mathbf{\Lambda}$ on u.

A number of numerical schemes for this problem in the case of $\mathbf{\Lambda} = Id$ have already been discussed in the literature. Nevertheless, the numerical schemes used to approximate the solution of this simplified model have only recently been studied from a convergence point of view. In particular, the convergence of a numerical scheme, involving a finite volume method for the computation of the saturation u and a standard finite element method for the computation of the pressure p, is proven in [Eymard-Gallouët (1993)], whereas a convergence proof for a finite volume method for the discretization of both equations is given in [Vignal (1996)]. Here we also discretize the conservation law for the saturation by means of a finite volume method but apply the mixed finite element method to discretize the elliptic equation. Error estimates have been derived in [Jaffré-Roberts (1985)] for a semi-discretized problem in the simulation of miscible displacements involving an elliptic equation for the pressure coupled to a parabolic equation for saturation. For the numerical discretization they combine the mixed finite element method with an upstream weighting scheme. More recently, in the case where the finite volume method is applied for the discretization of a parabolic equation instead of the first order conservation law (1.1), error estimates have been proven in [Ohlberger (1997)].

Here we deal with a mixed finite element method with an original basis for the elliptic equation. We use a partition of the domain with very undemanding hypotheses (the elements do not need to be convex, their boundaries do not need to be the union of piecewise planar surfaces), on which we define the generalization of the Raviart-Thomas space. The proof of the “inf-sup” condition and that the interpolation error of regular functions tends to zero with the space step makes use of Lipschitz-continuous homeomorphisms (with Lipschitz-continuous inverse mappings) and of some trace inequalities, for which the constants are given as functions of the size of the domain (the classical proofs, by means of contradiction, of trace inequalities for functions with null averages do not provide the dependence of the trace inequality constants on the domain). An advantage of this framework is that it handles simultaneously the case of domains with piecewise planar or smooth boundary (note that in this paper, some smoothness of the boundary is required in order to ensure the necessary regularity properties of the continuous solution). Note also that the work presented here allows us to handle the case of nonconvex domains with smooth boundary, which is not possible in classical frameworks (because all the meshes on such domains include non convex elements).

The hyperbolic equation is then discretized by the classical upstream weighting scheme. Under a CFL condition, we prove an L^∞ estimate which allows, up to a subsequence, to pass to the limit in L^∞ weak-$*$; though the hyperbolic equation is linear, such a convergence is not sufficient in order to identify the limit function as a weak solution to (1.1) : we need an additional “weak BV” inequality. Such inequalities have only recently been introduced and used for the proof of convergence of finite volume schemes on unstructured meshes for hyperbolic equations (see e.g. [Eymard-Gallouët-Herbin (2001)])

We note that, in contrast to classical BV estimates on discrete solutions (such as in [Godlewski-Raviart (1991)]) — which cannot be obtained here, since our meshes are not structured and the initial condition does not necessarily have a bounded variation —, the “weak BV” inequality is not a compactness tool; it does not strengthen the L^∞ weak-$*$ convergence: it is only useful for proving that the weak limit is a solution to the continuous hyperbolic equation.

Thus this paper completes a number of previous numerical works in which this scheme has been used on particular meshes (generally triangular meshes).

The organization of this paper is as follows. In Section 2, we present the numerical scheme that we use. In Section 3, we prove a convergence result for the mixed finite element method. In Section 4, we deal with the finite volume scheme, concluding the weak convergence of a subsequence without additional regularity hypotheses on the data, and the strong convergence otherwise.

2. The discretization

2.1. Admissible discretizations

In order to define the scheme, a notion of admissible discretization is given, which is used below in the definition of approximate discrete solutions.

DEFINITION 2: (Admissible discretization of Ω) *Let Ω be an open bounded subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary (see [Droniou (1999)]). An admissible discretization $\mathcal{D}$ of Ω is given by a finite set $\mathcal{M}$ of open subsets $K \subset \Omega$ with weakly Lipschitz-continuous boundaries and a finite set $\mathcal{A}$ of disjoint subsets $a \subset \overline{\Omega}$ such that:*

(i) $\cup_{K\in\mathcal{M}}\overline{K} = \overline{\Omega}$,

(ii) For all $K \in \mathcal{M}$, there exists a Lipschitz-continuous homeomorphism $\mathcal{L}_K$ from $\overline{K}$ to $\overline{B(0,\delta(K))}$ such that the inverse mapping is Lipschitz-continuous as well.

(iii) For all $(K,L) \in \mathcal{M}^2$ with $K \neq L$, one has $K \cap L = \emptyset$.

(iv) For all $a \in \mathcal{A}$, there exists $K \in \mathcal{M}$ such that a is a non-empty open subset of ∂K. By denoting $\mathcal{A}_K = \{a \in \mathcal{A} \mid a \subset \partial K\}$, we assume that $\partial K = \cup_{a\in\mathcal{A}_K}\overline{a}$.

(v) The sets $\mathcal{A}_i \subset \mathcal{A}$ (the interior faces) and $\mathcal{A}_e \subset \mathcal{A}$ (the exterior faces) are defined by $A_i = \{a \in \mathcal{A}, \exists (K,L) \in \mathcal{M}^2,\ K \neq L,\ a \subset \partial K \cap \partial L\}$ and $A_e = \{a \in \mathcal{A}, \exists K \in \mathcal{M}, a \subset \partial K \cap \partial\Omega\}$. One assumes that $(\mathcal{A}_i, \mathcal{A}_e)$ forms a partition of $\mathcal{A}$.

(vi) For all $K \in \mathcal{M}$ and all $a \in \mathcal{A}_K$, one assumes that there exists $x_{K,a} \in a$ and $\zeta_{K,a} > 0$ such that $a \supset \partial K \cap B(x_{K,a}, \zeta_{K,a}\delta(K))$.

We denote by m_K the Lebesgue measure of K and by m_a the $(d-1)$-dimensional measure of a.

Under Properties (iii) and (iv), we can show that, for all $a \in \mathcal{A}_i$, there exists exactly two different control volumes whose boundaries contain a. We select one of these control volumes, that we denote $K(a)$, the other being denoted $L(a)$, and an orientation on the edge a is defined by $\varepsilon_{K(a),a} = 1$, $\varepsilon_{L(a),a} = -1$; and, for $x \in a$, $\mathbf{n}_a(x) = \mathbf{n}_{\partial K(a)}(x) = -\mathbf{n}_{\partial L(a)}(x)$.

We can also prove that, if $a \in \mathcal{A}_e$, there exists exactly one control volume, denoted $K(a)$, whose boundary contains a. We then let $\varepsilon_{K(a),a} = 1$ and, for $x \in a$, $\mathbf{n}_a(x) = \mathbf{n}_{\partial K(a)}(x) = \mathbf{n}_{\partial\Omega}(x)$.

Denoting by $\overline{\mathbf{n}_a}$ the mean value of $\mathbf{n}_a$ on a, the thinness of the discretization $\mathcal{D}$ (controling the size of $\mathcal{D}$ and the behaviour of the faces of $\mathcal{D}$) is defined by

$$\mathrm{thin}(\mathcal{D}) = \max_{K\in\mathcal{M}} \left(\delta(K), \max_{a\in\mathcal{A}_K} \left(\frac{1}{\sqrt{m_a}} ||\mathbf{n}_a - \overline{\mathbf{n}_a}||_{L^2(a)} \right) \right) \tag{2.1}$$

and a geometrical factor, linked to the regularity of the discretization, is defined by

$$\mathrm{regul}(\mathcal{D}) = \max_{K\in\mathcal{M}} \left(\mathrm{lip}(\mathcal{L}_K), \mathrm{lip}(\mathcal{L}_K^{-1}), \max_{a\in\mathcal{A}_K} \left(\frac{1}{\zeta_{K,a}} \right) \right). \tag{2.2}$$

Remark 2: *The definition of an open set with weakly Lipschitz - continuous boundary is given in [Droniou (1999)] (or in [Grisvard (1985)] under the name "d-dimensional Lipschitz-continuous submanifold of $\mathbb{R}^d$"). It is weaker than the definition of Lipschitz-continuous boundary given in [Nečas (1967)].*

Remark 3: *The above definition is easily satisfied for a large variety of meshes. In the case d $= 2$, if we take subsets K such that ∂K is defined in polar coordinates from an origin $M_K \in K$ by a 2π- periodic continuous piecewise C^1 function, then these subsets satisfy condition (ii). This is the case for convex polyhedra, such as triangles or parallelograms for example.*

Remark 4: *According to the above definition,* $\mathrm{thin}(\mathcal{D}) \to 0$ *means that the size of the discretization tends to* 0 *and that the faces become more and more planar. Therefore the faces of the discretization cannot be simply defined by the sets $\partial K \cap \partial L$ or $\partial K \cap \partial\Omega$, which can be highly nonplanar surfaces; in such cases it suffices to cut these surfaces by different faces. Notice that if Ω is polyhedral and the faces are planes, then* $\mathrm{thin}(\mathcal{D}) = \max_{K\in\mathcal{M}} \delta(K)$ *is simply the size of the discretization.*

Remark 5: *Hypothesis (vi) is only used for the study of the convergence of the finite volume scheme to the solution of the hyperbolic equation. It is not used in the proof of convergence of the mixed finite element method. Notice that this hypothesis, along with Hypothesis (ii) and Lemma 11, implies $m_a \geq C\delta(K)^{d-1}$, where C only depends on d and* $\mathrm{regul}(\mathcal{D})$.

2.2. Discrete function spaces

One now defines the set of basis functions for the mixed finite element method, which is a generalization of the Raviart-Thomas space $RT_0^0(\mathcal{M})$ (see [Brezzi-Fortin (1991)], [Raviart-Thomas (1977)] or [Nédélec (1980)]); indeed, one can verify that, if $\mathcal{D}$ is made of triangles (for example), then the following definition gives back the classical Raviart-Thomas space.

Definition 3: (Discrete function spaces) *Let Ω be an open bounded subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary. Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2. For all $K \in \mathcal{M}$ and all $a \in \mathcal{A}_K$, one denotes by $w_{K,a} \in H^1(K)$ the unique variational solution, with $\int_K w_{K,a}(x)\,dx = 0$, of the Neumann problem*

$$\Delta w_{K,a}(x) = \frac{m_a}{m_K} \textit{ for a.e. } x \in K,$$

and

$$\begin{aligned} \nabla w_{K,a}(x)\cdot \mathbf{n}_{\partial K}(x) = 1 \quad & \text{for a.e. } x\in a,\\ \nabla w_{K,a}(x)\cdot \mathbf{n}_{\partial K}(x) = 0 \quad & \text{for a.e. } x\in \partial K\setminus a. \end{aligned}$$

We then define the function $\mathbf{w}_{K,a}$ from Ω to $\mathbb{R}^d$ by $\mathbf{w}_{K,a}(x) = \nabla w_{K,a}(x)$ for a.e. $x \in K$ and $\mathbf{w}_{K,a}(x) = 0$ for all $x \in \Omega \setminus K$.

We also define, for all $a \in \mathcal{A}_i$, $\mathbf{w}_a = \mathbf{w}_{K(a),a} - \mathbf{w}_{L(a),a}$ and, for all $a \in \mathcal{A}_e$, $\mathbf{w}_a = \mathbf{w}_{K(a),a}$. Then one gets $\mathbf{w}_a \in H(\mathrm{div}, \Omega)$. The set $\mathbf{Q}_\mathcal{D} \subset H(\mathrm{div}, \Omega)$ is the space generated by the functions $(\mathbf{w}_a)_{a\in\mathcal{A}}$; the set $\mathbf{Q}_{\mathcal{D},0} \subset H_0(\mathrm{div}, \Omega)$ is the space generated by the functions $(\mathbf{w}_a)_{a\in\mathcal{A}_i}$; for any $b \in L^2(\partial\Omega)$, the set $\mathbf{Q}_{\mathcal{D},b} \subset \mathbf{Q}_\mathcal{D}$ is the space $\left\{\mathbf{q} + \sum_{a\in\mathcal{A}_e} \frac{1}{m_a}\int_a b(x)\, d\gamma(x)\ \mathbf{w}_a\ ,\ \mathbf{q}\in\mathbf{Q}_{\mathcal{D},0}\right\}$.

$V_\mathcal{D} \in L^2(\Omega)$ is the space of functions $f = \sum_{K\in\mathcal{M}} \alpha_K\ \chi_K$ (where, for all $K \in \mathcal{M}$, $\alpha_K \in \mathbb{R}$ and χ_K is the characteristic function of K) such that $\int_\Omega f(x)\, dx = \sum_{K\in\mathcal{M}} m_K\alpha_K = 0$.

2.3. The mixed finite element scheme

The mixed finite element approximation of (1.2)-(1.4) is a pair of functions

$$(p_\mathcal{D}, \mathbf{q}_\mathcal{D}) \in V_\mathcal{D} \times \mathbf{Q}_{\mathcal{D},g},$$

satisfying

$$\int_\Omega v(x)\mathrm{div}\, \mathbf{q}_\mathcal{D}(x)\, dx = \int_\Omega f(x)v(x)\, dx \quad \forall v \in V_\mathcal{D}, \tag{2.3}$$

and

$$\int_\Omega \mathbf{y}(x)\cdot \mathbf{\Lambda}(x)^{-1}\mathbf{q}_\mathcal{D}(x)\, dx - \int_\Omega p_\mathcal{D}(x)\ \mathrm{div}\,\mathbf{y}(x)\, dx = 0 \quad \forall \mathbf{y}\in\mathbf{Q}_{\mathcal{D},0}. \tag{2.4}$$

The unknown functions can be written as

$$\mathbf{q}_\mathcal{D} = \sum_{a\in\mathcal{A}} q_a \mathbf{w}_a$$

and

$$p_\mathcal{D} = \sum_{K\in\mathcal{M}} p_K\, \chi_K.$$

Then equations (2.3) and (2.4) lead to the following system of linear equations, with unknowns $(q_a)_{a\in\mathcal{A}}$ and $(p_K)_{K\in\mathcal{M}}$:

$$\sum_{a'\in\mathcal{A}} q_{a'} \int_\Omega \mathbf{w}_a(x)\cdot \mathbf{\Lambda}(x)^{-1}\mathbf{w}_{a'}(x)\, dx - m_a(p_{K(a)} - p_{L(a)}) = 0 \quad \forall a \in \mathcal{A}_i,$$

$$q_a = g_a \quad \forall a\in\mathcal{A}_e,$$

$$\sum_{a\in\mathcal{A}_K} m_a q_a\,\varepsilon_{K,a} = f_K \quad \forall K\in\mathcal{M}, \tag{2.5}$$

$$\sum_{K\in\mathcal{M}} m_K p_K = 0,$$

where we denote

$$f_K = \int_K f(x)\,dx \quad \forall K\in\mathcal{M}, \tag{2.6}$$

and

$$g_a = \frac{1}{m_a}\int_a g(x)\,d\gamma(x) \quad \forall a\in\mathcal{A}_e. \tag{2.7}$$

The existence and uniqueness of a solution $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}})$ to system (2.3)-(2.4) is stated in the following lemma.

Lemma 1: (Existence and uniqueness of the discrete approximation) *Let us assume hypotheses (H). Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2. Then system (2.3)-(2.4) defines one and only one approximate solution $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}}\times\mathbf{Q}_{\mathcal{D},g}$.*

Proof: Since Lemma 4 (which is proved below) shows that the only solution of a linear system with the same matrix as (2.3)-(2.4) and a zero right-hand side is zero, this matrix is invertible. This proves the lemma.

2.4. The finite volume scheme

We denote, for all $K\in\mathcal{M}$ and $a\in\mathcal{A}_K$, $F_{K,a} = m_a q_a \varepsilon_{K,a}$ (then $F_{K(a),a}+F_{L(a),a} = 0$ holds for all $a\in\mathcal{A}_i$).

We now discretize the hyperbolic problem. Let $\Delta t > 0$ be a constant time step. Let us define the discrete source term

$$s_K^n = \frac{1}{\Delta t m_K}\int_{n\Delta t}^{(n+1)\Delta t}\int_K s(x,t)\,dx\,dt \quad \forall K\in\mathcal{M},\ \forall n\in\mathbb{N}. \tag{2.8}$$

Extending by 0 the function $\overline{u}$ on $\partial\Omega^+\times\mathbb{R}_+$, we define

$$\overline{u}_a^n = \frac{1}{\Delta t m_a}\int_{n\Delta t}^{(n+1)\Delta t}\int_a \overline{u}(x)\,d\gamma(x)\,dt \quad \forall a\in\mathcal{A}_e,\ \forall n\in\mathbb{N}. \tag{2.9}$$

The discretization of the initial value (Eq. (1.6)) is given by

$$u_K^0 = \frac{1}{m_K}\int_K u_0(x)dx \quad \forall K\in\mathcal{M}. \tag{2.10}$$

The finite volume scheme discretization of equation (1.1) is written:

$$m_K\frac{u_K^{n+1}-u_K^n}{\Delta t} + \sum_{a\in\mathcal{A}_K} u_a^n F_{K,a} = s_K^n f_K^+ - u_K^n f_K^- \quad \forall K\in\mathcal{M},\ \forall n\in\mathbb{N}, \tag{2.11}$$

where u_a^n is defined by :

$$\begin{array}{llll} u_a^n = u_{K(a)}^n \text{ if } q_a > 0, & \text{else } u_a^n = u_{L(a)}^n & \forall a \in \mathcal{A}_i, \ \forall n \in \mathbb{N} \\ u_a^n = u_{K(a)}^n \text{ if } q_a > 0, & \text{else } u_a^n = \overline{u}_a^n & \forall a \in \mathcal{A}_e, \ \forall n \in \mathbb{N}. \end{array} \tag{2.12}$$

For a given discretization $\mathcal{D}$ and a time step Δt, we can define the approximate solution by:

$$\begin{array}{c} u_{\mathcal{D},\Delta t}(x,t) = u_K^n, \text{ for a.e. } (x,t) \in K \times [n\Delta t, (n+1)\Delta t) \\ \forall K \in \mathcal{M}, \ \forall n \in \mathbb{N}. \end{array} \tag{2.13}$$

3. The convergence of the mixed method

We have the following result.

THEOREM 1: (Convergence of the mixed finite element scheme) *Under Hypotheses (H), let ξ be a fixed positive real value and let $\mathcal{D}$ be a discretization of Ω in the sense of Definition 2 such that* $\text{regul}(\mathcal{D}) \le \xi$. *Let $(p, \mathbf{q}) \in L^2(\Omega) \times H_g(\text{div}, \Omega)$ be the unique weak solution of the problem (1.8) and (1.9) with the condition (1.10) and $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}} \times \mathbf{Q}_{\mathcal{D},g}$ be given by (2.3)-(2.4).*

Then

$$\begin{array}{lll} \lim\limits_{\text{thin}(\mathcal{D})\to 0} \|\mathbf{q} - \mathbf{q}_{\mathcal{D}}\|_{H(\text{div},\Omega)} & = & 0, \\ \lim\limits_{\text{thin}(\mathcal{D})\to 0} \|p - p_{\mathcal{D}}\|_{L^2(\Omega)} & = & 0. \end{array} \tag{3.1}$$

In order to prove Theorem 1, some lemmata must be previously shown. The next lemma deals with an interpolation result for regular functions.

LEMMA 2: (Interpolation of regular functions) *Let Ω be an open bounded subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary, let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2 and let $\xi \ge$* $\text{regul}(\mathcal{D})$. *Let $\mathbf{q} \in (H^1(\Omega))^d$. Let $\mathbf{y} \in H(\text{div}, \Omega)$ be defined by*

$$\mathbf{y} = \sum_{a \in \mathcal{A}} \frac{1}{m_a} \int_a \mathbf{q}(x) \cdot \mathbf{n}_a(x) \, d\gamma(x) \, \mathbf{w}_a.$$

Then we have $\text{div}\, \mathbf{y} = \sum_{K \in \mathcal{M}} \frac{1}{m_K} \int_K \text{div}\, \mathbf{q}(x)\, dx \ \chi_K$ *and there exists $C_1 > 0$ which only depends on d and ξ such that*

$$\|\mathbf{q} - \mathbf{y}\|_{L^2(\Omega)} \le C_1 \, \text{thin}(\mathcal{D}) \|\mathbf{q}\|_{(H^1(\Omega))^d}. \tag{3.2}$$

One can notice then that, when $\text{thin}(\mathcal{D}) \to 0$, the function y so defined tends to q in $H(\text{div}, \Omega)$.

Proof: In the following proof, C_i denotes different positive real values which only depend on ξ and d.

The proof of $\operatorname{div} \mathbf{y} = \sum_{K\in\mathcal{M}} \frac{1}{m_K}\int_K \operatorname{div} \mathbf{q}(x)\,dx\ \chi_K$ is straightforward, since $\operatorname{div} \mathbf{w}_a = 0$ on K if $K \notin \{K(a), L(a)\}$ and $\operatorname{div} \mathbf{w}_a = \varepsilon_{K,a}\frac{m_a}{m_K}$ on K if $K \in \{K(a), L(a)\}$.

Let $K \in \mathcal{M}$. Let us define the function $w \in H^1(K)$ by

$$w = \sum_{a\in\mathcal{A}_K} \left(\frac{1}{m_a}\int_a \mathbf{q}(x)\cdot \mathbf{n}_{\partial K}(x)\,d\gamma(x)\right)\ w_{K,a},$$

which is such that $\nabla w(x) = \mathbf{y}(x)$ for a.e. $x \in K$. Similarly, denoting $\tilde{\mathbf{q}} = \frac{1}{m_K}\int_K \mathbf{q}(x)\,dx$, we define $\tilde{w} \in H^1(K)$ by

$$\tilde{w} = \sum_{a\in\mathcal{A}_K} \left(\frac{1}{m_a}\int_a \tilde{\mathbf{q}}\cdot \mathbf{n}_{\partial K}(x)\,d\gamma(x)\right)\ w_{K,a}.$$

We get

$$\|\mathbf{q}-\mathbf{y}\|^2_{L^2(K)} \le 3\|\mathbf{q}-\tilde{\mathbf{q}}\|^2_{L^2(K)} + 3\|\tilde{\mathbf{q}}-\nabla\tilde{w}\|^2_{L^2(K)} + 3\|\nabla\tilde{w}-\nabla w\|^2_{L^2(K)}.$$

Let us first deal with $A = \|\mathbf{q}-\tilde{\mathbf{q}}\|^2_{L^2(K)}$. Thanks to the Cauchy-Schwarz inequality, one has

$$A \le \frac{1}{m_K}\int_K\int_K |\mathbf{q}(x)-\mathbf{q}(y)|^2\,dx\,dy,$$

which yields, using (A.12) proved in Lemma 13,

$$\|\mathbf{q}-\tilde{\mathbf{q}}\|^2_{L^2(K)} \le C_2\,\delta(K)^2\|\mathbf{q}\|^2_{(H^1(K))^d}. \tag{3.3}$$

We now turn to the study of $B = \|\tilde{\mathbf{q}}-\nabla\tilde{w}\|^2_{L^2(K)}$. We define the function $h \in H^2(K)$ by $h(x) = \tilde{\mathbf{q}}\cdot x - \frac{1}{m_K}\int_K(\tilde{\mathbf{q}}\cdot y)\,dy$. This function thus satisfies $\nabla h = \tilde{\mathbf{q}}$ and $\int_K h(x)\,dx = 0$. Since $h-\tilde{w}$ is the variational solution of a Neumann problem on K with null average and $\Delta(h-\tilde{w})$ is constant, we get

$$B = \sum_{a\in\mathcal{A}_K}\int_a (h(x)-\tilde{w}(x))\left(\tilde{\mathbf{q}}\cdot\mathbf{n}_{\partial K}(x) - \frac{1}{m_a}\int_a \tilde{\mathbf{q}}\cdot\mathbf{n}_{\partial K}(y)\,d\gamma(y)\right)\,d\gamma(x).$$

Thanks to the Cauchy-Schwarz inequality, we deduce that

$$B^2 \le B' \sum_{a\in\mathcal{A}_K}\int_a (\tilde{w}(x)-h(x))^2\,d\gamma(x),$$

where

$$B' = \sum_{a\in\mathcal{A}_K}\int_a \left(\tilde{\mathbf{q}}\cdot\mathbf{n}_{\partial K}(x) - \frac{1}{m_a}\int_a \tilde{\mathbf{q}}\cdot\mathbf{n}_{\partial K}(y)\,d\gamma(y)\right)^2\,d\gamma(x).$$

We use (A.5) proved in Lemma 12. It yields

$$\sum_{a\in\mathcal{A}_K}\int_a (\tilde{w}(x)-h(x))^2\,d\gamma(x)\le C_3\,\delta(K)B,$$

and thus we obtain

$$B\le C_3\,\delta(K)B'. \tag{3.4}$$

We have, by definition of $\mathrm{thin}(\mathcal{D})$,

$$\begin{aligned}
\delta(K)B' &\le \delta(K)|\tilde{\mathbf{q}}|^2\sum_{a\in\mathcal{A}_K}\int_a(\mathbf{n}_a(x)-\overline{\mathbf{n}_a})^2\,d\gamma(x)\\
&\le \frac{\delta(K)}{m_K}\int_K|\mathbf{q}(x)|^2\,dx\sum_{a\in\mathcal{A}_K}\mathrm{thin}(\mathcal{D})^2 m_a\\
&\le C_4\,\mathrm{thin}(\mathcal{D})^2\int_K|\mathbf{q}(x)|^2\,dx\times\frac{\delta(K)m_{\partial K}}{m_K}.
\end{aligned}$$

Using $m_K\ge C_5\,\delta(K)^d$ and $m_{\partial K}\le C_6\,\delta(K)^{d-1}$ (hypothesis (ii) of Definition 2 and Lemma 11), relation (3.4) gives

$$\|\tilde{\mathbf{q}}-\nabla\tilde{w}\|^2_{L^2(K)}\le C_7\,\mathrm{thin}(\mathcal{D})^2\|\mathbf{q}\|^2_{L^2(K)}. \tag{3.5}$$

We finally study the term $C=\|\nabla\tilde{w}-\nabla w\|^2_{L^2(K)}$. We have

$$C=\sum_{a\in\mathcal{A}_K}\int_a(\tilde{w}(x)-w(x))\left(\frac{1}{m_a}\int_a(\tilde{\mathbf{q}}-\mathbf{q}(y))\cdot\mathbf{n}_{\partial K}(y)\,d\gamma(y)\right)d\gamma(x).$$

Thanks to the Cauchy-Schwarz inequality, we get

$$C^2\le C'\sum_{a\in\mathcal{A}_K}\int_a(\tilde{w}(x)-w(x))^2\,d\gamma(x),$$

where

$$C'=\sum_{a\in\mathcal{A}_K}\int_a\left(\frac{1}{m_a}\int_a(\tilde{\mathbf{q}}-\mathbf{q}(y))\cdot\mathbf{n}_{\partial K}(y)\,d\gamma(y)\right)^2 d\gamma(x)$$

Thanks again to (A.5) given by Lemma 12, we get

$$\sum_{a\in\mathcal{A}_K}\int_a(\tilde{w}(x)-w(x))^2\,d\gamma(x)\le C_3\,\delta(K)C,$$

which leads to

$$C\le C_3\,\delta(K)C'. \tag{3.6}$$

Turning to the study of C', and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} C' &\le \sum_{a\in\mathcal{A}_K}\int_a (\tilde{\mathbf{q}}-\mathbf{q}(y))^2\,d\gamma(y) = \int_{\partial K}(\tilde{\mathbf{q}}-\mathbf{q}(y))^2\,d\gamma(y)\\ &\le \int_{\partial K}\frac{1}{m_K}\int_K(\mathbf{q}(z)-\mathbf{q}(y))^2\,dz\,d\gamma(y). \end{aligned} \tag{3.7}$$

Thanks again to Lemma 12, we get

$$C' \le C_2\,\delta(K)\|\mathbf{q}\|^2_{(H^1(K))^d},$$

and therefore, thanks to (3.6) and (3.7), there exists $C_8 > 0$ such that

$$\|\nabla\tilde{w}-\nabla w\|^2_{L^2(K)} \le C_8\,\delta(K)^2\|\mathbf{q}\|^2_{(H^1(K))^d}. \tag{3.8}$$

Summing relations (3.3), (3.5) and (3.8) on $K\in\mathcal{M}$ gives (3.2).

LEMMA 3: *Under Hypotheses (H), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2 and $\xi \ge \mathrm{regul}(\mathcal{D})$. Let $v\in V_{\mathcal{D}}$ and let $h\in H^2(\Omega)$ be the variational solution of $-\Delta h = v$ on Ω, with a homogeneous Neumann boundary condition and $\int_\Omega h(x)\,dx = 0$ (the existence of such a function results from the regularity hypotheses on Ω, see [Grisvard (1985)]). Let us define $\mathbf{y}\in\mathbf{Q}_{\mathcal{D},0}$ by*

$$\mathbf{y} = \sum_{a\in\mathcal{A}}\left(\frac{1}{m_a}\int_a \nabla h(x)\cdot\mathbf{n}_a\,d\gamma(x)dx\right)\mathbf{w}_a. \tag{3.9}$$

Then there exists C_9, only depending on Ω, d and ξ such that $\|\mathbf{y}\|_{(L^2(\Omega))^d} \le C_9\,\|v\|_{L^2(\Omega)}$.

Proof: Using $\|\mathbf{y}\|_{(L^2(\Omega))^d} \le \|\mathbf{y}-\nabla h\|_{(L^2(\Omega))^d}+\|\nabla h\|_{(L^2(\Omega))^d}$, we apply Lemma 2 for $\mathbf{q}=\nabla h$, since $h\in H^2(\Omega)$ implies $\nabla h\in(H^1(\Omega))^d$. We thus obtain $\|\mathbf{y}\|_{(L^2(\Omega))^d} \le (C_1\,\mathrm{thin}(\mathcal{D})+1)\|h\|_{H^2(\Omega)}$. By hypothesis (H), we have $\|h\|_{H^2(\Omega)} \le C_\Omega\|v\|_{L^2(\Omega)}$, which concludes the proof since $\mathrm{thin}(\mathcal{D}) \le \max(\delta(\Omega),2)$.

By noticing that the $\mathbf{y}$ defined by (3.9) satisfies $\mathrm{div}\,\mathbf{y} = -v$, this lemma can also be stated in terms of an "inf-sup" condition.

COROLLARY 1: (Discrete "inf-sup" condition) *Under Hypotheses (H), let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2 and let $\xi \ge \mathrm{regul}(\mathcal{D})$. Then there exists $C_9 > 0$, only depending on Ω, d and ξ such that*

$$\inf_{v\in V_{\mathcal{D}}}\sup_{\mathbf{y}\in\mathbf{Q}_{\mathcal{D},0}}\frac{\displaystyle\int_\Omega v(x)\ \mathrm{div}\,\mathbf{y}(x)\,dx}{\|v\|_{L^2(\Omega)}\ \|\mathbf{y}\|_{(L^2(\Omega))^d}} \ge \frac{1}{C_9}.$$

The following lemmata express the classical proof of the convergence of mixed finite element methods under an "inf-sup" condition and an interpolation result (discussed in [Brezzi-Fortin (1991)] or [Nédélec (1980)] for example). We prove them for the sake of completeness, thus verifying that our hypotheses are sufficient to apply this convergence proof.

LEMMA 4: (Estimate on the discrete approximations) *Under Hypotheses (H), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2 and let $\xi \geq \mathrm{regul}(\mathcal{D})$. Let $h \in L^2(\Omega)$ and $\mathbf{r} \in (L^2(\Omega))^d$ be given.*

Then, there exists one and only one solution $(p_\mathcal{D}, \mathbf{q}_\mathcal{D}) \in V_\mathcal{D} \times \mathbf{Q}_{\mathcal{D},0}$ of

$$\int_\Omega \operatorname{div} \mathbf{q}_\mathcal{D}(x)\ v(x)\,dx = \int_\Omega h(x)\ v(x)\,dx \quad \forall v \in V_\mathcal{D}, \tag{3.10}$$

and

$$\begin{gathered}\int_\Omega \mathbf{y}(x)\cdot \mathbf{\Lambda}(x)^{-1}\mathbf{q}_\mathcal{D}(x)\,dx - \int_\Omega p_\mathcal{D}(x)\ \operatorname{div}\mathbf{y}(x)\,dx = \int_\Omega \mathbf{r}(x)\cdot\mathbf{y}(x)\,dx \\ \forall \mathbf{y} \in \mathbf{Q}_{\mathcal{D},0},\end{gathered} \tag{3.11}$$

and there exists C_{10}, only depending on Ω, d, ξ, λ_1 and λ_2 such that

$$\|\mathbf{q}_\mathcal{D}\|^2_{(L^2(\Omega))^d} + \|p_\mathcal{D}\|^2_{L^2(\Omega)} \leq C_{10}\ (\|\mathbf{r}\|^2_{(L^2(\Omega))^d} + \|h\|^2_{L^2(\Omega)}). \tag{3.12}$$

Proof: We first remark that proving (3.12) for any solution $(p_\mathcal{D}, \mathbf{q}_\mathcal{D}) \in V_\mathcal{D} \times \mathbf{Q}_{\mathcal{D},0}$ to (3.10)-(3.11) is sufficient to prove that for a zero right-hand side, the discrete unknowns are zero, and therefore that the matrix of the linear system is invertible. For the proof of (3.12), we choose, in (3.11), $\mathbf{y} = \mathbf{q}_\mathcal{D}$, and in (3.10), $v = p_\mathcal{D}$. It leads to

$$\frac{1}{\lambda_2}\|\mathbf{q}_\mathcal{D}\|^2_{(L^2(\Omega))^d} \leq \|\mathbf{r}\|_{(L^2(\Omega))^d}\|\mathbf{q}_\mathcal{D}\|_{(L^2(\Omega))^d} + \|h\|_{L^2(\Omega)}\|p_\mathcal{D}\|_{L^2(\Omega)}. \tag{3.13}$$

We then apply Lemma 3, which gives the existence of $\mathbf{y}_0 \in \mathbf{Q}_{\mathcal{D},0}$ such that $\operatorname{div}\mathbf{y}_0 = p_\mathcal{D}$ a.e. in Ω and

$$\|\mathbf{y}_0\|_{(L^2(\Omega))^d} \leq C_9\,\|p_\mathcal{D}\|_{L^2(\Omega)}. \tag{3.14}$$

Introducing $\mathbf{y}_0$ in (3.11), we get

$$\|p_\mathcal{D}\|^2_{L^2(\Omega)} \leq \|\mathbf{r}\|_{(L^2(\Omega))^d}\|\mathbf{y}_0\|_{(L^2(\Omega))^d} + \frac{1}{\lambda_1}\|\mathbf{q}_\mathcal{D}\|_{(L^2(\Omega))^d}\|\mathbf{y}_0\|_{(L^2(\Omega))^d},$$

which gives, thanks to (3.14),

$$\|p_\mathcal{D}\|_{L^2(\Omega)} \leq C_9\left(\|\mathbf{r}\|_{(L^2(\Omega))^d} + \frac{1}{\lambda_1}\|\mathbf{q}_\mathcal{D}\|_{(L^2(\Omega))^d}\right). \tag{3.15}$$

Thanks to (3.13) and (3.15), we get (3.12).

LEMMA 5: (Bound on the approximation error by the interpolation error) *Under Hypotheses (H), let* $\xi > 0$ *and* $\mathcal{D}$ *be a discretization of* Ω *in the sense of Definition 2 such that* $\text{regul}(\mathcal{D}) \le \xi$. *Let* $(p, \mathbf{q}) \in L^2(\Omega) \times H_g(\text{div}, \Omega)$ *be the unique weak solution of the problem (1.8) and (1.9) with the condition (1.10) and* $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}} \times \mathbf{Q}_{\mathcal{D},g}$ *be given by (2.3) and (2.4). Let* $\tilde{\mathbf{q}}_{\mathcal{D}} \in \mathbf{Q}_{\mathcal{D},g}$ *be given and let* $\tilde{p}_{\mathcal{D}} \in V_{\mathcal{D}}$ *be defined by* $\tilde{p}_{\mathcal{D}} = \sum_{K \in \mathcal{M}} \frac{1}{m_K} \int_K p(x)\,dx\ \chi_K$.

Then there exists C_{11}, *only depending on* Ω, d, ξ, λ_1 *and* λ_2 *such that*

$$\|\mathbf{q} - \mathbf{q}_{\mathcal{D}}\|^2_{(L^2(\Omega))^d} + \|p - p_{\mathcal{D}}\|^2_{L^2(\Omega)} \le C_{11}(\|\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{H(\text{div},\Omega)} + \|p - \tilde{p}_{\mathcal{D}}\|^2_{L^2(\Omega)}). \quad (3.16)$$

Proof: We get, using the variational formulations (1.8)-(1.9) and (2.3)-(2.4):

$$\int_\Omega \text{div}(\mathbf{q}_{\mathcal{D}}(x) - \tilde{\mathbf{q}}_{\mathcal{D}}(x))v(x)\,dx = \int_\Omega \text{div}(\mathbf{q}(x) - \tilde{\mathbf{q}}_{\mathcal{D}}(x))v(x)\,dx \quad \forall v \in V_{\mathcal{D}},$$

and

$$\begin{aligned}&\int_\Omega \mathbf{y}(x) \cdot \mathbf{\Lambda}(x)^{-1}(\mathbf{q}_{\mathcal{D}}(x) - \tilde{\mathbf{q}}_{\mathcal{D}}(x))\,dx - \int_\Omega (p_{\mathcal{D}}(x) - \tilde{p}_{\mathcal{D}}(x))\text{div}\,\mathbf{y}(x)\,dx\\ &= \int_\Omega \mathbf{y}(x) \cdot \mathbf{\Lambda}(x)^{-1}(\mathbf{q}(x) - \tilde{\mathbf{q}}_{\mathcal{D}}(x))\,dx - \int_\Omega (p(x) - \tilde{p}_{\mathcal{D}}(x))\text{div}\,\mathbf{y}(x)\,dx\\ &\forall \mathbf{y} \in \mathbf{Q}_{\mathcal{D},0}.\end{aligned}$$

For all $\mathbf{y} \in \mathbf{Q}_{\mathcal{D},0}$, thanks to the definition of $\tilde{p}_{\mathcal{D}}$, we have

$$\int_\Omega (p(x) - \tilde{p}_{\mathcal{D}}(x))\text{div}\,\mathbf{y}(x)\,dx = 0.$$

Thus $(p_{\mathcal{D}} - \tilde{p}_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}} - \tilde{\mathbf{q}}_{\mathcal{D}})$ is the solution of (3.10) and (3.11) with $\mathbf{r} = \mathbf{\Lambda}^{-1}(\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}})$ and $h = \text{div}(\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}})$. Applying Lemma 4 yields

$$\begin{aligned}&\|\mathbf{q}_{\mathcal{D}} - \tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{(L^2(\Omega))^d} + \|p_{\mathcal{D}} - \tilde{p}_{\mathcal{D}}\|^2_{L^2(\Omega)}\\ &\quad \le C_{10}\left(\frac{1}{\lambda_1}\|\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{(L^2(\Omega))^d} + \|\text{div}\,\mathbf{q} - \text{div}\,\tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{L^2(\Omega)}\right).\end{aligned}$$

Using the Cauchy-Schwarz inequality, this leads to

$$\begin{aligned}&\|\mathbf{q} - \mathbf{q}_{\mathcal{D}}\|^2_{(L^2(\Omega))^d} + \|p - p_{\mathcal{D}}\|^2_{L^2(\Omega)}\\ &\quad \le \ 2\left(\frac{C_{10}}{\lambda_1} + 1\right)\|\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{(L^2(\Omega))^d} + 2C_{10}\|\text{div}\,\mathbf{q} - \text{div}\,\tilde{\mathbf{q}}_{\mathcal{D}}\|^2_{L^2(\Omega)}\\ &\qquad +2\|p - \tilde{p}_{\mathcal{D}}\|^2_{L^2(\Omega)},\end{aligned}$$

which gives (3.16).

Proof of Theorem 1. We apply Lemma 5. On the one hand, thanks again to (A.13) proved in Lemma 13, the following inequality holds:

$$\|p - \tilde{p}_{\mathcal{D}}\|^2_{L^2(\Omega)} \leq C_2 \,\mathrm{thin}(\mathcal{D})^2 \|\nabla p\|^2_{L^2(\Omega)},$$

(notice that, when $p \in L^2(\Omega)$ satisfies (1.8), we have, in fact, $p \in H^1(\Omega)$) and therefore $\|p - \tilde{p}_{\mathcal{D}}\|^2_{L^2(\Omega)}$ tends to 0 as $\mathrm{thin}(\mathcal{D})$ tends to 0. On the other hand, it suffices to prove that one can choose $\tilde{\mathbf{q}}_{\mathcal{D}} \in \mathbf{Q}_{\mathcal{D},g}$ such that $\|\mathbf{q} - \tilde{\mathbf{q}}_{\mathcal{D}}\|_{H(\mathrm{div},\Omega)}$ is as small as desired. Notice that, in general, the statement $\mathbf{q} \in (H^1(\Omega))^d \cap H_g(\mathrm{div},\Omega)$ is false. Therefore, we take $\mathbf{q}_0 \in (H^1(\Omega))^d$ such that $\mathbf{q}_0 \cdot \mathbf{n}_{\partial\Omega} = g$; then, $\mathbf{q} - \mathbf{q}_0 \in H_0(\mathrm{div},\Omega)$ and since Hypotheses (H) are sufficient to prove that Ω is locally star-shaped, we can approximate $\mathbf{q} - \mathbf{q}_0$ in $H_0(\mathrm{div},\Omega)$ by regular functions with compact support in Ω (see [Temam (1979)]); thus, $\mathbf{q}$ can be approximated in $H_g(\mathrm{div},\Omega)$ by $\tilde{\mathbf{q}} \in (H^1(\Omega))^d \cap H_g(\mathrm{div},\Omega)$. Then, applying Lemma 2, we can approximate $\tilde{\mathbf{q}}$ by $\tilde{\mathbf{q}}_{\mathcal{D}} \in \mathbf{Q}_{\mathcal{D},g}$ as close as demanded by letting $\mathrm{thin}(\mathcal{D})$ tend to zero.

4. The convergence of the finite volume method

We now show the following theorem.

THEOREM 2: (Convergence of the finite volume scheme) *Under Hypotheses (H), let ξ and $\alpha \in (0,1)$ be fixed positive real values. Let $(p,\mathbf{q}) \in L^2(\Omega) \times H_g(\mathrm{div},\Omega)$ be the unique weak solution of the problem (1.8) and (1.9) with the condition (1.10). Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of discretizations of Ω in the sense of Definition 2 such that for all $m \in \mathbb{N}$, $\mathrm{regul}(\mathcal{D}_m) \leq \xi$ and $\lim_{m\to+\infty} \mathrm{thin}(\mathcal{D}_m) = 0$. For a given $m \in \mathbb{N}$, let us denote by $(p_m, \mathbf{q}_m)$ the solution $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}} \times \mathbf{Q}_{\mathcal{D},g}$ given by (2.3) and (2.4) where $\mathcal{D}$ stands for $\mathcal{D}_m$. Let $\Delta t_m > 0$, denoted Δt, be such that the condition*

$$\Delta t \leq (1-\alpha) \inf_{K\in\mathcal{M}} \frac{m_K}{\displaystyle\sum_{a\in\mathcal{A}_K} m_a (q_a \varepsilon_{K,a})^+ + f_K^-} \tag{4.1}$$

holds. Let $u_m \in L^\infty(\Omega \times \mathbb{R}^+)$ denote the function $u_{\mathcal{D},\Delta t}$ defined by (2.8)-(2.13).

Then, there exists a subsequence of $(u_m)_{m\in\mathbb{N}}$, still denoted $(u_m)_{m\in\mathbb{N}}$, which converges in the weak-$*$ topology of $L^\infty(\Omega \times \mathbb{R}^+)$ to a function $u \in L^\infty(\Omega \times \mathbb{R}^+)$ that is a solution of (1.7).

If we add some hypotheses to ensure that $\mathbf{q}$ is Lipschitz continuous on $\overline{\Omega}$ (for example, $\partial\Omega$ is of class C^2, $\mathbf{\Lambda}$ is of class C^2, f is of class C^1 and g is of class C^2) then:

- the function u is unique;

- the whole sequence $(u_m)_{m\in\mathbb{N}}$ converges to u in $L^p(\Omega\times]0,T[)$ for all $p \in [1,\infty)$ and all $T > 0$.

The proof of Theorem 2 is classical, and has been developed for various choices of the discretization of the flux $\mathbf{q}$ (see [Champier-Gallouët-Herbin (1993)], [Eymard-Gallouët (1993)] and [Vignal (1996)]). The originality of this proof is the use of the technical Lemma 14, which is nonstandard.

4.1. L^∞ estimate

The purpose of this section is to prove the following result.

LEMMA 6: (L^∞ stability of the finite volume scheme) *Under hypotheses (H), let $\xi > 0$ and let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2 such that $\xi \ge \mathrm{regul}(\mathcal{D})$. Let $(p_\mathcal{D}, \mathbf{q}_\mathcal{D}) \in V_\mathcal{D} \times \mathbf{Q}_{\mathcal{D},g}$ be given by (2.3) and (2.4) and let $\Delta t > 0$ be such that*

$$\Delta t \le \inf_{K\in\mathcal{M}} \frac{m_K}{\displaystyle\sum_{a\in\mathcal{A}_K} m_a (q_a \varepsilon_{K,a})^+ + f_K^-}. \tag{4.2}$$

Then the approximate solution $u_{\mathcal{D},\Delta t}$ given by (2.8)-(2.13) is such that

$$\|u_{\mathcal{D},\Delta t}\|_{L^\infty(\Omega\times\mathbb{R}^+)} \le \max(\|u_0\|_{L^\infty(\Omega)}, \|\overline{u}\|_{L^\infty(\partial\Omega^-\times\mathbb{R}^+)}, \|s\|_{L^\infty(\Omega\times\mathbb{R}^+)}). \tag{4.3}$$

Proof: According to the scheme (2.11), we have

$$u_K^{n+1} = u_K^n - \frac{\Delta t}{m_K}\left(\sum_{a\in\mathcal{A}_K} u_a^n F_{K,a} + u_K^n f_K^- - s_K^n f_K^+\right),$$

which gives

$$\begin{aligned} u_K^{n+1} &= u_K^n\left(1 - \frac{\Delta t}{m_K}\left(\sum_{a\in\mathcal{A}_K} F_{K,a}^+ + f_K^-\right)\right) \\ &\quad + \frac{\Delta t}{m_K}\sum_{a\in\mathcal{A}_K} F_{K,a}^- u_a^n + \frac{\Delta t}{m_K} f_K^+ s_K^n, \end{aligned} \tag{4.4}$$

The discrete elliptic scheme (2.5) is used to get

$$\sum_{a\in\mathcal{A}_K} F_{K,a}^+ + f_K^- = \sum_{a\in\mathcal{A}_K} F_{K,a}^- + f_K^+, \tag{4.5}$$

Thanks to this equation and the stability condition (4.2), (4.4) expresses u_K^{n+1} as a convex combination of the values u_K^n, $\overline{u}_a^n$, s_K^n. An easy proof by induction concludes the proof of the lemma.

REMARK 6: *If the data are regular enough, the term $\sum_{a\in\mathcal{A}_K} m_a|q_a|$ behaves like size$(\mathcal{D})^{d-1}$ as size$(\mathcal{D})$ tends to 0, and the condition (4.2) takes the form $\Delta t \le C$size$(\mathcal{D})$ (where size$(\mathcal{D}) = \max_{K\in\mathcal{M}} \delta(K)$).*

4.2. A weak bound on the spatial variations

LEMMA 7: (Weak bound on spatial variations) *Under hypotheses (H), let $\xi > 0$, $\alpha \in (0,1)$, $T > 0$, and let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2 such that $\xi \geq \mathrm{regul}(\mathcal{D})$. Let $(p_{\mathcal{D}}, \mathbf{q}_{\mathcal{D}}) \in V_{\mathcal{D}} \times \mathbf{Q}_{\mathcal{D},g}$ be given by (2.3) and (2.4) and let $\Delta t > 0$ such that the condition (4.1) holds. Let N_T be such that $N_T \Delta t \leq T < (N_T + 1)\Delta t$ and let $(u_K^n)_{K\in\mathcal{M},\ n\in\mathbb{N}}$, $(u_a^n)_{a\in\mathcal{A},\ n\in\mathbb{N}}$ be defined by (2.8)-(2.12).*

Then there exists C_{12}, which only depends on d, Ω, T, ξ, α, f, s, g, $\overline{u}$ and u_0 (but not on $\mathcal{D}$ or Δt), such that

$$\sum_{n=0}^{N_T} \Delta t \sum_{K\in\mathcal{M}} \left(\sum_{a\in\mathcal{A}_K} m_a (q_a \varepsilon_{K,a})^- (u_a^n - u_K^n)^2 \right) \leq C_{12}. \tag{4.6}$$

REMARK 7: *In [Champier-Gallouët-Herbin (1993)], [Eymard-Gallouët (1993)] and [Vignal (1996)], a weak BV-estimate is obtained from (4.6). We do not do so here, since in the convergence proof, the use of Lemma 14 takes advantage of a local bound of the diameter of each control volume. Otherwise, we should assume the existence of some $\beta > 0$ with*

$$\delta(K) \ \geq\ \beta\ size(\mathcal{D}) \quad \forall K \in \mathcal{M}.$$

Proof: Thanks to (4.5), the scheme (2.11) can be rewritten as

$$\begin{gathered} m_K(u_K^{n+1} - u_K^n) + \Delta t \left(\sum_{a\in\mathcal{A}_K} F_{K,a}^- (u_K^n - u_a^n) + f_K^+ (u_K^n - s_K^n) \right) = 0 \\ \forall K \in \mathcal{M},\ \forall n \in \mathbb{N}. \end{gathered} \tag{4.7}$$

For all $n \in \mathbb{N}$ and $K \in \mathcal{M}$, let us multiply the equation (4.7) by u_K^n and sum the result over $K \in \mathcal{M}$ and $n = 0, \ldots, N_T$. It gives $T_1 + T_2 = 0$ with

$$T_1 = \sum_{n=0}^{N_T} \sum_{K\in\mathcal{M}} m_K (u_K^{n+1} - u_K^n) u_K^n$$

and

$$T_2 = \sum_{n=0}^{N_T} \Delta t \sum_{K\in\mathcal{M}} \left(\sum_{a\in\mathcal{A}_K} F_{K,a}^- (u_K^n - u_a^n) u_K^n + f_K^+ (u_K^n - s_K^n) u_K^n \right).$$

Writing $u_K^{n+1} u_K^n = -\frac{1}{2}(u_K^{n+1} - u_K^n)^2 + \frac{1}{2}(u_K^{n+1})^2 + \frac{1}{2}(u_K^n)^2$, we get

$$T_1 = T_{11} + T_{12},$$

where

$$T_{11} = -\frac{1}{2} \sum_{n=0}^{N_T} \sum_{K\in\mathcal{M}} m_K (u_K^{n+1} - u_K^n)^2$$

and

$$T_{12} = \frac{1}{2}\left(\sum_{K\in\mathcal{M}} m_K((u_K^{N_T+1})^2 - (u_K^0)^2)\right).$$

Using (4.7) and the Cauchy-Schwarz inequality gives, for all $K \in \mathcal{M}$ and all $n \in \mathbb{N}$,

$$\begin{aligned} m_K^2(u_K^{n+1} - u_K^n)^2 \quad &\leq \quad \Delta t\left(\sum_{a\in\mathcal{A}_K} F_{K,a}^- + f_K^+\right) \\ &\times \left(\Delta t\sum_{a\in\mathcal{A}_K} F_{K,a}^-(u_a^n - u_K^n)^2 + f_K^+(s_K^n - u_K^n)^2\right). \end{aligned}$$

Using condition (4.1) and equation (4.5), we get, for all $K \in \mathcal{M}$ and $n \in \mathbb{N}$,

$$\begin{aligned} &m_K(u_K^{n+1} - u_K^n)^2 \\ &\quad \leq (1-\alpha)\left(\Delta t\sum_{a\in\mathcal{A}_K} F_{K,a}^-(u_a^n - u_K^n)^2 + f_K^+(s_K^n - u_K^n)^2\right). \end{aligned} \tag{4.8}$$

Let us consider T_2. We have $T_2 = T_{21} + T_{22}$ with

$$T_{21} = \frac{1}{2}\sum_{n=0}^{N_T}\Delta t\sum_{K\in\mathcal{M}}\left(\sum_{a\in\mathcal{A}_K} F_{K,a}^-(u_a^n - u_K^n)^2 + f_K^+(s_K^n - u_K^n)^2\right)$$

and

$$T_{22} = \frac{1}{2}\sum_{n=0}^{N_T}\Delta t\sum_{K\in\mathcal{M}}\left(\sum_{a\in\mathcal{A}_K} F_{K,a}^-((u_K^n)^2 - (u_a^n)^2) + f_K^+((u_K^n)^2 - (s_K^n)^2)\right).$$

We thus get, thanks to (4.8),

$$T_{11} + T_{21} \geq \alpha T_{21}.$$

According to (4.5), the term T_{22} can be rewritten as

$$T_{22} = \frac{1}{2}\sum_{n=0}^{N_T}\Delta t\sum_{K\in\mathcal{M}}\left(\sum_{a\in\mathcal{A}_K} F_{K,a}(u_a^n)^2 + f_K^-(u_K^n)^2 - f_K^+(s_K^n)^2\right).$$

Thus, gathering by faces, we get

$$T_{22} = \frac{1}{2}\sum_{n=0}^{N_T}\Delta t\left(\sum_{a\in\mathcal{A}_e} m_a g_a(u_a^n)^2 + \sum_{K\in\mathcal{M}}(f_K^-(u_K^n)^2 - f_K^+(s_K^n)^2)\right).$$

Since terms T_{12} and T_{22} can easily be bounded using Lemma 6 (since condition (4.2) is weaker than (4.1)), we thus get (4.6).

4.3. The proof of the convergence theorem, Theorem 2

We first notice that Lemma 6 gives the existence of a subsequence u_m and of a function $u \in L^\infty(\Omega\times\mathbb{R}^+)$ such that u_m converges to u in the weak-$*$ topology of $L^\infty(\Omega\times\mathbb{R}^+)$ as $m\to+\infty$. Recall that we have proved above (Theorem 1) that $\mathbf{q}_m$ tends to $\mathbf{q}$ in $H(\mathrm{div},\Omega)$ as $m\to+\infty$. This section is devoted to the proof that u satisfies (1.7) (the uniqueness part of the proof being studied in the next section).

Let $\phi\in C^1_c(\mathbb{R}^d\times\mathbb{R})$ be such that $\phi=0$ on $\partial\Omega\backslash\partial\Omega^-\times\mathbb{R}^+$. Let $T>0$ be such that

$$\phi = 0 \quad \text{ on } \quad \mathbb{R}^d\times[T,+\infty[. \tag{4.9}$$

In this proof, we denote by C_i various positive real values which only depend on d, Ω, ϕ, T, ξ, α, s, f, g, $\overline{u}$, u_0 and not on $\mathcal{D}$ or Δt.

In the following, we use the notations $\mathcal{D}=\mathcal{D}_m$ and $\Delta t=\Delta t_m$. Let us denote by N_T the integer such that $N_T\Delta t\le T<(N_T+1)\Delta t$. Setting

$$\phi_K^n = \frac{1}{\Delta t\ m_K}\int_K\int_{n\Delta t}^{(n+1)\Delta t}\phi(x,t)\,dx\,dt \quad \forall K\in\mathcal{M},\ \forall n\in\mathbb{N},$$

we multiply the equality (4.7) by ϕ_K^n and sum over $K\in\mathcal{M}$ and $n\in\mathbb{N}$. We obtain $E_1+E_2=0$ with

$$E_1=\sum_{n=0}^{N_T}\sum_{K\in\mathcal{M}} m_K(u_K^{n+1}-u_K^n)\phi_K^n,$$

and

$$E_2=\sum_{n=0}^{N_T}\Delta t\sum_{K\in\mathcal{M}}\left(\sum_{a\in\mathcal{A}_K}F^-_{K,a}(u_K^n-u_a^n)\phi_K^n+f_K^+(u_K^n-s_K^n)\phi_K^n\right).$$

We also define

$$\phi_a^n=\frac{1}{\Delta t\ m_a}\int_a\int_{n\Delta t}^{(n+1)\Delta t}\phi(x,t)\,d\gamma(x)\,dt.$$

Let us study E_1. Thanks to (4.9), for all $K\in\mathcal{M}$, $\phi_K^{N_T+1}=0$ holds and therefore

$$E_1=\sum_{n=1}^{N_T+1}\sum_{K\in\mathcal{M}} m_K u_K^n(\phi_K^{n-1}-\phi_K^n)-\sum_{K\in\mathcal{M}} m_K u_K^0\phi_K^0.$$

Using the weak-$*$ convergence of $(u_m)_{m\in\mathbb{N}}$ to u, we deduce the convergence of E_1 to

$$-\int_{\Omega\times\mathbb{R}^+}u(x,t)\frac{\partial\phi}{\partial t}(x,t)\,dx\,dt-\int_\Omega u_0(x)\phi(x,0)\,dx.$$

Next we consider the term E_2. It can be written, using (2.12) and gathering by faces, as

$$\begin{aligned} E_2 &= \sum_{n=0}^{N_T} \Delta t \sum_{a\in\mathcal{A}_i} m_a q_a \phi^n_{K_d(a)}(u^n_{L(a)} - u^n_{K(a)}) \\ &\quad + \sum_{n=0}^{N_T} \Delta t \sum_{a\in\mathcal{A}_e} m_a q_a \phi^n_{K_d(a)}(u^n_a - u^n_{K(a)}) \\ &\quad + \sum_{n=0}^{N_T} \Delta t \sum_{K\in\mathcal{M}} f^+_K \phi^n_K (u^n_K - s^n_K), \end{aligned} \tag{4.10}$$

where we define, for all $a \in \mathcal{A}_i$, $K_d(a)$ (the "downstream" control volume) by $K_d(a) = K(a)$ if $q_a \le 0$, else $K_d(a) = L(a)$, and for all $a \in \mathcal{A}_e$, $K_d(a) = K(a)$. We set

$$\begin{aligned} f_{\mathcal{D}}(x) &= \frac{1}{m_K} f_K, && \text{for a.e.} \quad x \in K \quad \forall K \in \mathcal{M}, \\ s_{\mathcal{D},\Delta t}(x,t) &= s^n_K, && \text{for a.e.} \quad (x,t) \in K \times [n\Delta t, (n+1)\Delta t) \quad \forall K \in \mathcal{M},\ \forall n \in \mathbb{N}, \\ \overline{u}_{\mathcal{D},\Delta t}(x,t) &= \overline{u}^n_a, && \text{for a.e.} \quad (x,t) \in a \times [n\Delta t, (n+1)\Delta t) \quad \forall a \in \mathcal{A}_e,\ \forall n \in \mathbb{N}, \\ g_{\mathcal{D}}(x) &= g_a, && \text{for a.e.} \quad x \in a \quad \forall a \in \mathcal{A}_e, \end{aligned}$$

where f_K, g_a, s^n_K and $\overline{u}^n_a$ are respectively defined by (2.6), (2.7), (2.8) and (2.9). We define E_3 by

$$\begin{aligned} E_3 &= -\int_{\Omega\times\mathbb{R}^+} u_{\mathcal{D},\Delta t}(x,t)\mathbf{q}_{\mathcal{D}}(x)\cdot\nabla\phi(x,t)\,dx\,dt \\ &\quad + \int_{\partial\Omega\times\mathbb{R}^+} \overline{u}_{\mathcal{D},\Delta t}(x,t) g_{\mathcal{D}}(x)\phi(x,t)\,d\gamma(x)\,dt \\ &\quad + \int_{\Omega\times\mathbb{R}^+} (u_{\mathcal{D},\Delta t}(x,t) f^-_{\mathcal{D}}(x) - s_{\mathcal{D},\Delta t}(x,t) f^+_{\mathcal{D}}(x))\phi(x,t)\,dx\,dt. \end{aligned}$$

Since $u_{\mathcal{D},\Delta t}$ converges to u in the weak-$*$ topology of $L^\infty(\Omega\times\mathbb{R}^+)$ and since $\mathbf{q}_{\mathcal{D}}$ converges strongly to $\mathbf{q}$ in $L^2(\Omega)$ as $m \to +\infty$, in view of the definitions of $\overline{u}_{\mathcal{D},\Delta t}$ and $g_{\mathcal{D}}$, we deduce the convergence, as $m \to \infty$, of E_3 to

$$\begin{aligned} &-\int_{\Omega\times\mathbb{R}^+} u(x,t)\mathbf{q}(x)\cdot\nabla\phi(x,t)\,dx\,dt + \int_{\partial\Omega^-\times\mathbb{R}^+} \overline{u}(x,t)g(x)\phi(x,t)\,d\gamma(x)\,dt \\ &+ \int_{\Omega\times\mathbb{R}^+} (u(x,t)f^-(x) - s(x,t)f^+(x))\phi(x,t)\,dx\,dt. \end{aligned}$$

Using (2.5) and the definition of $\mathbf{w}_{K,a}$, we can rewrite E_3 as

$$\begin{aligned} E_3 \;=\; & \sum_{n=0}^{N_T} \Delta t \sum_{a\in\mathcal{A}_i} m_a q_a \phi_a^n (u_{L(a)}^n - u_{K(a)}^n) \\ & + \sum_{n=0}^{N_T} \Delta t \sum_{a\in\mathcal{A}_e} m_a q_a \phi_a^n (\overline{u}_a^n - u_{K(a)}^n) \\ & + \sum_{n=0}^{N_T} \Delta t \sum_{K\in\mathcal{M}} (u_K^n - s_K^n) f_K^+ \phi_K^n. \end{aligned} \tag{4.11}$$

From (4.10) and (4.11), we deduce that

$$|E_3 - E_2| \leq E_4 + E_5,$$

with

$$E_4 = \sum_{n=0}^{N_T} \Delta t \sum_{a\in\mathcal{A}_e} m_a |q_a| |\phi_a^n| |\overline{u}_a^n - u_a^n|$$

and

$$\begin{aligned} E_5 \;=\; & \sum_{n=0}^{N_T} \Delta t \sum_{a\in\mathcal{A}_i} m_a |q_a| \; |\phi_a^n - \phi_{K_d(a)}^n| \; |u_{K(a)}^n - u_{L(a)}^n| \\ & + \sum_{n=0}^{N_T} \Delta t \sum_{a\in\mathcal{A}_e} m_a |q_a| \; |\phi_a^n - \phi_{K_d(a)}^n| \; |u_a^n - u_{K(a)}^n|. \end{aligned}$$

Let us first study E_4. Since, for all $a \in \mathcal{A}_e$, relation (2.12) implies $u_a^n = \overline{u}_a^n$ when $q_a \leq 0$, we can write

$$E_4 = \sum_{n=0}^{N_T} \Delta t \sum_{a\in\mathcal{A}_e\,,\, q_a>0} m_a |q_a| |\phi_a^n| |\overline{u}_a^n - u_a^n|.$$

For all $a \in \mathcal{A}_e$ such that $q_a = m_a^{-1} \int_a g(x)\, d\gamma(x) > 0$, we have $\partial\Omega^+ \cap a \neq \emptyset$ (recall that $\partial\Omega^+ = \{x \in \partial\Omega \mid g(x) > 0\}$); thus, since $\phi = 0$ on $\partial\Omega^+ \times \mathbb{R}^+$, there exists $x \in a$ such that $\phi(x,t) = 0$ for all $t \geq 0$. Denoting by C_{13} the Lipschitz constant of ϕ, we then have $|\phi(y,t)| \leq C_{13}\, \delta(a)$ for all $y \in a$ and $t \geq 0$, which implies $|\phi_a^n| \leq C_{13}\, \delta(a)$. Using (4.3), we then deduce

$$\begin{aligned} E_4 \;&\leq\; C_{14}\, \mathrm{thin}(\mathcal{D}) \sum_{n=0}^{N_T} \Delta t \sum_{a\in\mathcal{A}_e} m_a |q_a| \\ &\leq\; C_{14}\, \mathrm{thin}(\mathcal{D})(T + \Delta t) \sum_{a\in\mathcal{A}_e} \int_a |g(x)|\, d\gamma(x) \\ &=\; C_{14}\, \mathrm{thin}(\mathcal{D})(T + \Delta t) \int_{\partial\Omega} |g(x)|\, d\gamma(x), \end{aligned}$$

which shows that E_4 tends to 0 as $m \to +\infty$.

We turn now to the study of E_5. Thanks to the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} E_5^2 &\le C_{13}^2 \left(\sum_{n=0}^{N_T} \Delta t \sum_{a \in \mathcal{A}} m_a |q_a| \delta(K_d(a))^2 \right) \\ &\quad \times \left(\sum_{n=0}^{N_T} \Delta t \sum_{K \in \mathcal{M}} \sum_{a \in \mathcal{A}_K} m_a (q_a \varepsilon_{K,a})^- (u_a^n - u_K^n)^2 \right). \end{aligned}$$

This gives, using Lemma 7 and the Cauchy-Schwarz inequality,

$$E_5^2 \le C_{15}\, \mathrm{thin}(\mathcal{D}) \left(\sum_{a \in \mathcal{A}} m_a q_a^2 \delta(K_d(a)) \right)^{1/2} \left(\sum_{a \in \mathcal{A}} m_a \delta(K_d(a)) \right)^{1/2}.$$

We can then apply Lemma 14, which yields

$$\begin{aligned} &\sum_{a \in \mathcal{A}} m_a q_a^2 \delta(K_d(a)) \\ &\qquad \le C_{16} \sum_{a \in \mathcal{A}} \left(\int_{K_d(a)} \mathbf{q}_{\mathcal{D}}^2(x)\, dx + \delta(K_d(a))^2 \int_{K_d(a)} (\mathrm{div} \mathbf{q}_{\mathcal{D}}(x))^2\, dx \right). \end{aligned}$$

Under Hypotheses (H) and the item (vi) of Definition 2, we get that $\mathrm{card} \mathcal{A}_K \le C_{17}$. Therefore, since $\mathbf{q}_{\mathcal{D}}$ converges to $\mathbf{q}$ in $H(\mathrm{div}, \Omega)$, it is bounded and

$$\sum_{a \in \mathcal{A}} m_a q_a^2 \delta(K_d(a)) \le C_{18}.$$

Item (ii) of Definition 2 allows to write $\delta(K) m_{\partial K} \le C_{19}\, \delta(K)^d \le C_{20}\, m_K$ (see also Remark 8) . Thus,

$$\sum_{a \in \mathcal{A}} m_a \delta(K_d(a)) \le \sum_{K \in \mathcal{M}} \delta(K) m_{\partial K} \le C_{20} \sum_{K \in \mathcal{M}} m_K = C_{20} m_\Omega.$$

Therefore, we can conclude that

$$E_5 \le C_{21} \sqrt{\mathrm{thin}(\mathcal{D})},$$

which shows that E_2 tends to

$$\begin{aligned} &- \int_{\Omega \times \mathbb{R}^+} u(x,t) \mathbf{q}(x) \cdot \nabla \phi(x,t)\, dx\, dt + \int_{\partial \Omega^- \times \mathbb{R}^+} \overline{u}(x,t) g(x) \phi(x,t)\, d\gamma(x)\, dt \\ &+ \int_{\Omega \times \mathbb{R}^+} (u(x,t) f^-(x) - s(x,t) f^+(x)) \phi(x,t)\, dx\, dt \end{aligned}$$

as $m \to +\infty$. That concludes the proof of Theorem 2.

5. Uniqueness of the weak solution under regularity on the data

We do not discuss in detail this part, since it does not involve the particular discrete framework we have developed in this paper. Some details can be found in [Eymard-Gallouët-Ghilani-Herbin (1998)], [Chainais (1999)], [Eymard-Gallouët-Herbin (2001)], for example. We first state the following result.

Lemma 8: *Under hypotheses (H), let $\xi > 0$ and let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2 such that $\xi \geq \mathrm{regul}(\mathcal{D})$. Let $(p_\mathcal{D}, \mathbf{q}_\mathcal{D}) \in V_\mathcal{D} \times \mathbf{Q}_{\mathcal{D},g}$ be given by (2.3) and (2.4) and let $\delta t > 0$ such that the CFL condition (4.2) holds.*

Then, the approximate solution $u_{\mathcal{D},\delta t}$ given by (2.8)-(2.13) is such that

$$\begin{aligned}&m_K(\eta(u_K^{n+1}) - \eta(u_K^n)) +\\ &\delta t\left(\sum_{a\in\mathcal{A}_K} F_{K,a}^-(\eta(u_K^n) - \eta(u_a^n)) + f_K^+\eta'(u_K^n)(u_K^n - s_K^n)\right) \leq 0\\ &\forall K \in \mathcal{M},\ \forall n \in \mathbb{N},\ \forall \eta \in C^1(\mathbb{R},\mathbb{R}) \text{ with } \eta'' \geq 0.\end{aligned}$$

The proof of this lemma is easy, starting from the discrete relation (4.7) and multiplying it by $\eta'(u_K^n)$. From this lemma, we get, letting $\mathrm{thin}(\mathcal{D}) \to 0$, the following result, which proves the convergence of the scheme to a solution of the hyperbolic problem in a very weak sense ([Eymard-Gallouët-Herbin (1995)], [DiPerna (1985)]).

Lemma 9: (Convergence of the finite volume scheme to an entropy process solution) *Under Hypotheses (H), let $\xi > 0$ and $\alpha \in (0,1)$ be fixed real values. Let $(p, \mathbf{q}) \in L^2(\Omega) \times H_g(\mathrm{div}, \Omega)$ be the unique weak solution of the problem (1.8) and (1.9) with the condition (1.10). Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of discretizations of Ω in the sense of Definition 2 such that for all $m \in \mathbb{N}$, $\mathrm{regul}(\mathcal{D}_m) \leq \xi$ and $\lim_{m\to+\infty} \mathrm{thin}(\mathcal{D}_m) = 0$. For a given $m \in \mathbb{N}$, let us denote by $(p_m, \mathbf{q}_m)$ the solution $(p_\mathcal{D}, \mathbf{q}_\mathcal{D}) \in V_\mathcal{D} \times \mathbf{Q}_{\mathcal{D},g}$ given by (2.3) and (2.4) where $\mathcal{D}$ stands for $\mathcal{D}_m$. Let $\delta t_m > 0$, denoted δt, such that the CFL condition (4.1) holds. Let $u_m \in L^\infty(\Omega \times \mathbb{R}^+)$ denote the function $u_{\mathcal{D},\delta t}$ defined by (2.8)-(2.13).*

Then there exists a subsequence of $(u_m)_{m\in\mathbb{N}}$, again denoted $(u_m)_{m\in\mathbb{N}}$, which converges in the nonlinear weak-$*$ topology of $L^\infty(\Omega \times \mathbb{R}^+)$ to a function $u \in L^\infty(\Omega \times \mathbb{R}^+ \times (0,1))$, that is a solution of

$$\begin{aligned}&\int_{\mathbb{R}^+}\int_\Omega\int_0^1 \Big(\eta(u(x,t,\alpha))\frac{\partial \phi}{\partial t}(x,t) + \eta(u(x,t,\alpha))\mathrm{div}(\phi(x,t)\mathbf{q}(x))\\ &\qquad +\eta'(u(x,t,\alpha))\phi(x,t)f^+(x)(s(x,t) - u(x,t,\alpha))\Big)\,d\alpha\,dx\,dt\\ &+\int_\Omega \eta(u_0(x))\phi(x,0)\,dx - \int_{\mathbb{R}^+}\int_{\partial\Omega^-}\eta(\overline{u}(x,t))\phi(x,t)g(x)\,d\gamma(x)\,dt \geq 0\\ &\forall \phi \in C_c^1(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}^+) \text{ such that } \phi = 0 \text{ on } \partial\Omega^+ \times \mathbb{R}^+,\\ &\forall \eta \in C^1(\mathbb{R},\mathbb{R}) \text{ with } \eta'' \geq 0.\end{aligned} \tag{5.1}$$

The proof of the above lemma is completely similar to the one which is given in Section 4.3. Use of the classical "variable doubling technique" and Krushkov entropies (Krushkov, 1970) lead to a uniqueness result, under sufficiently strong hypotheses on the data giving that $\mathbf{q}$ is Lipschitz-continuous (see [Otto (1996)] or [Vovelle (2001)] for the particular problem of handling the boundary conditions).

LEMMA 10: (Uniqueness of the entropy process solution) *Under Hypotheses (H), and the additional hypotheses that $\partial\Omega$ is of class C^2, $\mathbf{\Lambda}$ is of class C^2, f is of class C^1 and g is of class C^2 (for example), let $(p, \mathbf{q}) \in L^2(\Omega) \times H_g(\mathrm{div}, \Omega)$ be the unique weak solution of the problem (1.8) and (1.9) with the condition (1.10).*

Then, $\mathbf{q}$ is Lipschitz-continuous in $\overline{\Omega}$, there exists one, and only one, function $u \in L^\infty(\Omega \times \mathbb{R}^+ \times (0,1))$ that is a solution of (5.1), and there exists one, and only one, $\tilde{u} \in L^\infty(\Omega \times \mathbb{R}^+)$ solution of (1.7), such that, for a.e. $(x,t,\alpha) \in \Omega \times \mathbb{R}^+ \times (0,1)$, $u(x,t,\alpha) = \tilde{u}(x,t)$.

This result of uniqueness yields the convergence in $L^p(\Omega\times]0,T[)$, for all $p \in [1,\infty)$ and $T > 0$, of $(u_m)_{m\in\mathbb{N}}$ to the unique solution $\tilde{u}$ of the problem.

A. Technical lemmata

LEMMA 11: *Let K be an open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary, such that there exists a Lipschitz-continuous homeomorphism ϕ from $Q_{\delta(K)} =]-\delta(K), \delta(K)[^d$ to K with Lipschitz-continuous inverse mapping; we denote by ξ an upper bound of the Lipschitz constants of ϕ and ϕ^{-1}.*

Then there exists $C_{22} > 0$ only depending on ξ and d such that, for all $f \in L^1(\partial K)$, $f \geq 0$,

$$\frac{1}{C_{22}} \int_{\partial Q_{\delta(K)}} f \circ \phi(x)\, d\gamma(x) \leq \int_{\partial K} f(x)\, d\gamma(x) \leq C_{22} \int_{\partial Q_{\delta(K)}} f \circ \phi(x)\, d\gamma(x). \quad \text{(A.1)}$$

Notice that a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping between two open sets has a unique extension as a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping between the closures of the open sets, and that this extension defines a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping between the boundaries of the open sets.

REMARK 8: *The most useful inequality (and the easiest to obtain) in the following will be the second one of (A.1). We have also stated the first one in order that (A.1) allows to see that, when A is a measurable subset of ∂K, $\gamma(A)$ and $\gamma(\phi^{-1}(A))$ are comparable, with constants only depending on an upper bound on the Lipschitz constants of ϕ and ϕ^{-1} (recall that γ denotes the $(d-1)$-dimensional measure on the boundary of any open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary).*

Proof: We denote $\delta=\delta(K)$.

It is well known (see e.g. [Droniou (1999)]) that the mapping

$$f\in L^1(\partial K)\to f\circ\phi\in L^1(\partial Q_\delta) \tag{A.2}$$

is an isomorphism; here we want to estimate the norm of this mapping (and of its inverse mapping) only in terms of ϕ and ϕ^{-1} (with bounds not depending on δ).

Let us first recall the definition of the integral on ∂K when K is an open set with weakly Lipschitz-continuous boundary: if V is an open set of $\mathbb{R}^d$ and $\tau:]-1,1[^{d-1}\to\partial K\cap V$ is a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping, then for $f\in L^1(\partial K)$, we have

$$\int_{\partial K\cap V} f(x)\,d\gamma(x)=\int_{]-1,1[^{d-1}} f\circ\tau(x)|\partial_1\tau\wedge\cdots\wedge\partial_{d-1}\tau|(x)\,dx,$$

where $\partial_i\tau$ denotes the i-th partial derivative of τ (which is, by the Rademacher Theorem, a function in $(L^\infty(]-1,1[^{d-1}))^d$ and is essentially bounded by $\mathrm{lip}(\tau)$) and $\wedge$ is the vector product of $d-1$ elements of $\mathbb{R}^d$.

With this definition, we can verify that the $(d-1)$-dimensional measure on ∂Q_δ is the $(d-1)$-Lebesgue measure on all the hyperplane pieces the union of which is ∂Q_δ. We can also notice that

$$\partial Q_\delta=A\sqcup\left(\sqcup_{i=1}^d(]-\delta,\delta[^{i-1}\times\{-\delta\}\times]-\delta,\delta[^{d-i}\sqcup]-\delta,\delta[^{i-1}\times\{\delta\}\times]-\delta,\delta[^{d-i})\right)$$

where $\gamma(A)=0$ (A is made of sets of dimension $d-2$).

Since (A.2) is an isomorphism, the sets of zero measure on ∂Q_δ are mapped by ϕ on sets of zero measure on ∂K. Thus, by denoting

$$H_{i,\pm}=]-\delta,\delta[^{i-1}\times\{\pm\delta\}\times]-\delta,\delta[^{d-i},$$

we have, up to a set of zero measure,

$$\partial K=\sqcup_{i=1}^d(\phi(H_{i,+})\sqcup\phi(H_{i,-})).$$

If $f\in L^1(\partial K)$, $f\geq 0$, the integral of f on ∂K can thus be estimated if we estimate the integrals of f on all $\phi(H_{i,\pm})$.

Let us do this for $H_{1,+}$, the other terms being dealt with in the same way.

Define $\tau:]-1,1[^{d-1}\to\partial K\cap\phi(H_{1,+})$ by $\tau(x)=\phi(\delta,\delta x)$. τ is a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping; thus, by definition of the integral on ∂K,

$$\begin{aligned}\int_{\partial K\cap\phi(H_{1,+})} & f(x)\,d\gamma(x)\\ &=\int_{]-1,1[^{d-1}} f\circ\tau(x)|\partial_1\tau\wedge\cdots\wedge\partial_{d-1}\tau|(x)\,dx \qquad (A.3)\\ &=\int_{]-1,1[^{d-1}} f\circ\phi(\delta,\delta x)\delta^{d-1}\left|\frac{\partial\phi}{\partial y_2}(\delta,\delta x)\wedge\cdots\wedge\frac{\partial\phi}{\partial y_d}(\delta,\delta x)\right|(x)\,dx.\end{aligned}$$

Thus, by a change of variable,

$$\int_{\partial K\cap\phi(H_{1,+})} f(x)\,d\gamma(x) = \int_{]-\delta,\delta[^{d-1}} f\circ\phi(\delta,y)\left|\frac{\partial\phi}{\partial y_2}(\delta,y)\wedge\cdots\wedge\frac{\partial\phi}{\partial y_d}(\delta,y)\right|(y)\,dy.$$

Since ϕ is Lipschitz-continuous, we have, for all $i\in[2,d]$,

$$||\frac{\partial\phi}{\partial y_i}||_{L^\infty(H_{1,+})}\le \mathrm{lip}(\phi)$$

and there exists thus C_{23} only depending on ξ and d such that

$$\int_{\partial K\cap\phi(H_{1,+})} f(x)\,d\gamma(x)\le C_{23}\int_{]-\delta,\delta[^{d-1}} f\circ\phi(\delta,y)\,dy.$$

But, as we previously noticed, the $(d-1)$-dimensional measure on $H_{1,+}$ is the $(d-1)$-Lebesgue measure on this piece of hyperplane, and thus

$$\int_{]-\delta,\delta[^{d-1}} f\circ\phi(\delta,y)\,dy = \int_{H_{1,+}} f\circ\phi(x)\,d\gamma(x),$$

which proves the second inequality of (A.1).

The proof of the first inequality of (A.1) relies on a lemma (mainly algebraic) stating that there exists C_{24} only depending on d such that

$$|\partial_1\tau\wedge\cdots\wedge\partial_{d-1}\tau|\ge C_{24}(\mathrm{lip}(\tau^{-1}))^{-(d-1)} \tag{A.4}$$

(see [Droniou (1999)]). Since $\tau^{-1}(z)=\delta^{-1}((\phi^{-1}(z))_2,\dots,(\phi^{-1}(z))_d)$, we have $\mathrm{lip}(\tau^{-1})\le\xi\delta^{-1}$; using this in (A.4) and returning to (A.3) we get, thanks again to a change of variable, the first inequality of (A.1).

Lemma 12: *Let K be an open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary; we denote by m_K the measure of K. One assumes that there exists a Lipschitz-continuous homeomorphism $\mathcal{L}$ from K to $B(0,\delta(K))$ with Lipschitz-continuous inverse mapping. Let ξ be a real value greater than the Lipschitz constants of $\mathcal{L}$ and $\mathcal{L}^{-1}$. Let $g\in H^1(K)$. The trace of g on ∂K is still denoted by g.*

Then there exists $C_3>0$, only depending on ξ and d, such that

$$\frac{1}{m_K}\int_{\partial K}\int_K (g(y)-g(x))^2\,dx\,d\gamma(y)\le C_3\,\delta(K)\int_K(\nabla g(x))^2\,dx.$$

Thus, if $\int_K g(x)\,dx=0$ holds, we have

$$\int_{\partial K} g(x)^2\,d\gamma(x)\le C_3\,\delta(K)\int_K(\nabla g(x))^2\,dx. \tag{A.5}$$

Proof: In the following proof, C_i denotes real values which only depend on d and ξ; δ denotes $\delta(K)$.

The mapping $F : x \to (|x|/\sup_{i\in[1,d]}|x_i|)x$ is a Lipschitz-continuous homeomorphism with Lipschitz continuous inverse mapping between $B(0,\delta)$ and $Q =]-\delta,\delta[^d$; moreover, the Lipschitz constants of F and F^{-1} only depend on d. Thus, there exists a Lipschitz-continuous homeomorphism ϕ from Q to K, with Lipschitz continuous inverse mapping, such that the Lipschitz constants of ϕ and ϕ^{-1} are bounded by C_{25} only depending on d and ξ.

According to Lemma 11, there exists C_{26} only depending on d and ξ such that

$$\begin{aligned}
&\int_{\partial K}\int_K (g(y)-g(x))^2\,dx\,d\gamma(y)\\
&\quad\le\ C_{26}\int_{\partial Q}\int_K (g(\phi(y'))-g(x))^2\,dx\,d\gamma(y')\\
&\quad=\ C_{26}\int_{\partial Q}\int_Q (g(\phi(y'))-g(\phi(x')))^2 J_{\phi,d}(x')\,dx'\,d\gamma(y'),
\end{aligned}$$

where $J_{\phi,d}(x')$ is the absolute value of the jacobian in the change of variable ϕ. Setting $h = g\circ\phi$, we have $h\in H^1(Q)$. Thus we conclude the existence of $C_{27}>0$, only depending on d and ξ, such that

$$\int_{\partial K}\int_K (g(y)-g(x))^2\,dx\,d\gamma(y) \le C_{27}\int_{\partial Q}\int_Q (h(y)-h(x))^2\,dx\,d\gamma(y).$$

The change of variable $x=\phi^{-1}(x')$ proves the existence of $C_{28}>0$, only depending on d and ξ such that

$$\int_Q (\nabla h(x))^2\,dx \le C_{28}\int_K (\nabla g(x'))^2\,dx'. \tag{A.6}$$

Therefore, if we prove the existence of $C_{29}>0$, only depending on d and ξ, such that

$$\int_{\partial Q}\int_Q (h(y)-h(x))^2\,dx\,d\gamma(y) \le C_{29}\,\delta^{d+1}\int_Q (\nabla h(x))^2\,dx, \tag{A.7}$$

we get (12) from (A.6) and (A.7) and the fact that the existence of $\mathcal{L}$ ensures that there exists $C_5>0$ with $m_K \ge C_5\delta^d$.

In order to prove (A.7), we may assume by a classical argument of density that $h\in C^1(\overline{Q})$. Since Q is a cube with $2d$ faces, it suffices to prove the existence of $C_{30}>0$, only depending on d and ξ, such that

$$\int_\sigma\int_Q (h(y)-h(x))^2\,dx\,d\gamma(y) \le C_{30}\,\delta^{d+1}\int_Q (\nabla h(x))^2\,dx, \tag{A.8}$$

where $\sigma = \{-\delta\}\times[-\delta,\delta]^{d-1}$, to get (A.7) with $C_{29} = 2dC_{30}$. Let $H=[-\delta,\delta]^{d-1}$ and $Q^+ = [0,\delta]\times H$.

We can now write, for all $z \in Q^+$,

$$\begin{aligned}\int_\sigma \int_Q (h(y)-h(x))^2\,dx\,d\gamma(y) \ &\leq\ 2\int_\sigma \int_Q (h(y)-h(z))^2\,dx\,d\gamma(y)\\ &\quad +2\int_\sigma \int_Q (h(z)-h(x))^2\,dx\,d\gamma(y).\end{aligned}$$

An integration with respect to $z \in Q^+$ leads to

$$2^{d-1}\delta^d \int_\sigma \int_Q (h(y)-h(x))^2\,dx\,d\gamma(y) \leq 2(2\delta)^d A + 2(2\delta)^{d-1}B, \tag{A.9}$$

with

$$A = \int_\sigma \int_{Q^+} (h(y)-h(z))^2\,dz\,d\gamma(y),$$

and

$$B = \int_{Q^+} \int_Q (h(z)-h(x))^2\,dx\,dz.$$

Let us first study A. By definition,

$$A = \int_H \int_H \int_0^\delta \left(h((-\delta,y)) - h((a,b))\right)^2\,da\,db\,dy,$$

and therefore, A is equal to

$$\int_H \int_H \int_0^\delta \left(\int_0^1 \nabla h((-\delta+\theta(a+\delta), y+\theta(b-y)))\cdot(a+\delta, b-y)\,d\theta\right)^2\,da\,db\,dy.$$

Using the Cauchy-Schwarz inequality, we get

$$A \leq (2\delta)^2 d \int_H \int_H \int_0^\delta \int_0^1 \left(\nabla h((-\delta+\theta(a+\delta), y+\theta(b-y)))\right)^2\,d\theta\,da\,db\,dy.$$

Using the Fubini Theorem and the two changes of variable $b \to b' = b-y \in H_2 = [-2\delta, 2\delta]^{d-1}$, $y \to y' = y+\theta b' \in H$, we then obtain

$$A \leq (2\delta)^2 d \int_{H_2} \int_0^\delta \int_0^1 \int_H \left(\nabla h((-\delta+\theta(a+\delta), y'))\right)^2\,dy'\,d\theta\,da\,db'.$$

We now change the variable θ into $t = -\delta+\theta(a+\delta)$. This yields:

$$A \leq (2\delta)^2(4\delta)^{d-1} d \int_0^\delta \int_{-\delta}^a \int_H \left(\nabla h((t,y'))\right)^2 \frac{1}{a+\delta}\,dy'\,dt\,da.$$

Since, for all $a \in [0,\delta]$, $\frac{1}{a+\delta} \leq \frac{1}{\delta}$, we get, setting $x = (t,y)$,

$$A \leq 2^{2d}\delta^{d+1} d \int_Q \left(\nabla h(x)\right)^2\,dx. \tag{A.10}$$

Let us now study B. We have

$$B \leq (2\delta)^2 d \int_{Q^+} \int_Q \int_0^1 \left(\nabla h(x+\theta(z-x))\right)^2 \, d\theta \, dx \, dz.$$

Using the Fubini Theorem and the two changes of variable $z \to z' = z - x \in Q_2 = [-2\delta, 2\delta]^d$, $x \to x' = x + \theta z' \in Q$, we get

$$B \leq (2\delta)^2 d \int_{Q_2} \int_Q \left(\nabla h(x')\right)^2 \, dx' \, dz',$$

which gives

$$B \leq 2^{2d+2} \delta^{d+2} d \int_Q \left(\nabla h(x')\right)^2 \, dx'. \tag{A.11}$$

Thus, using (A.9), (A.10) and (A.11), we conclude the proof of (A.8).

Assuming now $\int_K g(x) \, dx = 0$, the proof of (A.5) is then a direct consequence of

$$\begin{aligned} \int_{\partial K} g(x)^2 \, d\gamma(x) &= \int_{\partial K} \left(g(x)^2 - \frac{1}{m_K} \int_K g(y) \, dy \right) d\gamma(x) \\ &\leq \frac{1}{m_K} \int_{\partial K} \int_K \left(g(x) - g(y)\right)^2 \, dx \, d\gamma(y). \end{aligned}$$

Lemma 13: *Let K be an open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary; we denote the measure of K by m_K. We assume that there exists a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping $\mathcal{L}$ from $B(0, \delta(K))$ to K. Let ξ be a real number greater than the Lipschitz constants of $\mathcal{L}$ and $\mathcal{L}^{-1}$. Let $g \in H^1(K)$.*

Then, there exists $C_2 > 0$, only depending on ξ and d, such that

$$\frac{1}{m_K} \int_K \int_K (g(y) - g(x))^2 \, dx \, dy \leq C_2 \, \delta(K)^2 \int_K \left(\nabla g(x)\right)^2 \, dx. \tag{A.12}$$

In particular, if $\int_K g(x) \, dx = 0$ holds, then

$$\int_K g^2(x) \, dx \leq C_2 \, \delta(K)^2 \int_K (\nabla g(x))^2 \, dx. \tag{A.13}$$

Proof: We denote $\delta = \delta(K)$. Using the change of variables $x' = \mathcal{L}(x)$ and $y' = \mathcal{L}(y)$, and writing for simplicity of notation $B = B(0, \delta)$, we get the existence of C_{31}, only depending on d and ξ, such that

$$\int_K \int_K (g(y) - g(x))^2 \, dx \, dy \leq C_{31} \int_B \int_B (g(\mathcal{L}(y')) - g(\mathcal{L}(x')))^2 \, dx' \, dy'.$$

Setting $h = g \circ \mathcal{L}$, we have $h \in H^1(B)$. Then we deduce the existence of $C_{32} > 0$, only depending on d and ξ, such that

$$\int_B (\nabla h(x))^2 \, dx \le C_{32} \int_K (\nabla g(x'))^2 \, dx'. \tag{A.14}$$

Thus, if we prove the existence of $C_{33} > 0$, only depending on d and ξ, such that

$$\int_B \int_B (h(y) - h(x))^2 \, dx \, dy \le C_{33} \, \delta^{d+2} \int_B (\nabla h(x))^2 \, dx, \tag{A.15}$$

we then get (13) from (A.14), (A.15) and the fact that the existence of $\mathcal{L}$ ensures that there exists C_5 with $m_K \ge C_5 \, \delta^d$. In order to prove (A.15), one may assume by a classical argument of density that $h \in C^1(\overline{B})$. We set

$$A = \int_B \int_B (h(z) - h(x))^2 \, dx \, dz.$$

Using the Cauchy-Schwarz inequality, we get

$$A \le (2\delta)^2 d \int_B \int_B \int_0^1 (\nabla h(x + \theta(z - x)))^2 \, d\theta \, dx \, dz.$$

Using the Fubini Theorem and the changes of variable $z \to z' = z - x \in B_2 := B(0, 2\delta)$, $x \to x' = x + \theta z' \in B$, we get

$$A \le (2\delta)^2 d \int_{B_2} \int_B (\nabla h(x))^2 \, dx' \, dz',$$

which gives the existence of some C_{34}, only depending on d, such that

$$A \le C_{34} \, \delta^{d+2} \int_B (\nabla h(x))^2 \, dx.$$

This concludes the proof of (A.15).

Assuming now $\int_K g(x) \, dx = 0$, the proof of (A.13) follows, remarking that in such a case

$$\begin{aligned} \int_K g^2(x) \, dx &= \int_K \left(g(x) - \frac{1}{m_K} \int_K g(y) \, dy \right)^2 dx \\ &\le \frac{1}{m_K} \int_K \int_K (g(x) - g(y))^2 \, dx \, dy. \end{aligned}$$

LEMMA 14: *Let K be an open subset of $\mathbb{R}^d$ with weakly Lipschitz-continuous boundary, such that there exists a Lipschitz-continuous homeomorphism with Lipschitz-continuous inverse mapping $\mathcal{L}$ from K to $B(0, \delta(K))$. We denote by ξ an upper bound on both Lipschitz constants. Let $a \subset \partial K$, such that there exists $x_0 \in a$ and $\zeta > 0$ with*

$$\partial K \cap B(x_0, \zeta\delta(K)) \subset a$$

Let m_a denote the $d-1$ Lebesque measure of a. Let $\mathbf{q} \in H(\mathrm{div}, K)$ such that $\mathbf{q} \cdot \mathbf{n}_{\partial K} \in L^2(\partial K)$ and there exists $q_a \in \mathbb{R}$ with $\mathbf{q}(x) \cdot \mathbf{n}_{\partial K}(x) = q_a$ for a.e. $x \in a$.
Then there exists C_{16}, only depending on d, ξ and ζ, such that

$$m_a q_a^2 \leq C_{16} \left(\frac{1}{\delta(K)} \int_K \mathbf{q}^2(x)\, dx + \delta(K) \int_K (\mathrm{div}\, \mathbf{q}(x))^2\, dx \right) \tag{A.16}$$

Proof: Denoting $\delta = \delta(K)$, let $X \in \partial B(0,\delta)$ and $\eta \in (0,1]$. We have

$$\{Z \in \partial B(0,\delta) \mid Z \cdot X \geq (1-\eta)\delta^2\} = \partial B(0,\delta) \cap B(X, \sqrt{2\eta}\delta).$$

Indeed, take $Z \in \partial B(0,\delta)$ and denote $h = Z - X$. We have, since $|Z|^2 = |X|^2 = \delta^2$, $|h|^2 = 2\delta^2 - 2Z \cdot X$; thus, $|h|^2 \leq 2\eta\delta^2$ if and only if $Z \cdot X \geq (1-\eta)\delta^2$.
Define

$$\mathcal{B}_\eta = \{y \in \partial K \mid \mathcal{L}(y) \cdot \mathcal{L}(x_0) \geq (1-\eta)\delta^2\} = \mathcal{L}^{-1}(\partial B(0,\delta) \cap B(\mathcal{L}(x_0), \sqrt{2\eta}\delta)).$$

Let $F(x) = (|x|/\sup_{i\in[1,d]} |x_i|)x$. $\mathcal{L}^{-1} \circ F^{-1}$ is a Lipschitz continuous homeomorphism with Lipschitz-continuous inverse mapping between K and $Q_\delta =]-\delta,\delta[^d$; moreover, the Lipschitz constants of $\mathcal{L}^{-1} \circ F^{-1}$ and its inverse mapping are bounded by a real number only depending on d and ξ. Thus, by Lemma 11 applied to $f = \chi_{\mathcal{B}_\eta}$,

$$\gamma(\mathcal{B}_\eta) \geq C_{35}\, \gamma(F \circ \mathcal{L}(\mathcal{B}_\eta)) = C_{35}\, \gamma(F(\partial B(0,\delta) \cap B(\mathcal{L}(x_0), \sqrt{2\eta}\delta))),$$

with C_{35} only depending on d and ξ. It is easy to see that

$$\gamma(F(\partial B(0,\delta) \cap B(\mathcal{L}(x_0), \sqrt{2\eta}\delta))) \geq C_{36}\, \delta^{d-1},$$

where C_{36} only depends on d and η (the set $F(\partial B(0,\delta) \cap B(\mathcal{L}(x_0), \sqrt{2\eta}\delta))$ contains a significant part of a $(d-1)$-dimensional ball on ∂Q_δ with radius of order δ). Thus, we have

$$\gamma(B_\eta) \geq C_{37}\, \delta^{d-1}, \tag{A.17}$$

where C_{37} only depends on d, ξ and η.
Now, let $\eta_0 = \inf(1, (\zeta/\xi)^2/2) \in (0,1]$ (η_0 only depends on ζ and ξ); since $\mathcal{L}^{-1}$ is Lipschitz-continuous with constant ξ, we have

$$\mathcal{B}_{\eta_0} \subset \partial K \cap B(x_0, \zeta\delta) \subset a. \tag{A.18}$$

Let us define the function $v \in H^1(K)$ by

$$v(x) = \psi\left(\frac{\mathcal{L}(x) \cdot \mathcal{L}(x_0)}{\delta^2}\right) \quad \forall x \in K,$$

where the function $\psi \in C([-1,1],[0,1])$ is defined by $\psi(s) = 0$ for all $s \in$

$[-1, 1-\eta_0]$, $\psi(s) = \frac{2(s+\eta_0-1)}{\eta_0}$ for all $s \in [1-\eta_0, 1-\eta_0/2]$, $\psi(s) = 1$ for all $s \in [1-\eta_0/2, 1]$. We have therefore $v(x) \in [0,1]$ for all $x \in \overline{K}$, $v = 1$ on $\mathcal{B}_{\eta_0/2}$ and $v = 0$ on $\partial K \setminus \mathcal{B}_{\eta_0} \supset \partial K \setminus a$ and

$$\nabla v(x) = \frac{\psi'\left(\frac{\mathcal{L}(x)\cdot\mathcal{L}(x_0)}{\delta^2}\right)}{\delta^2}(D\mathcal{L}(x))^T\mathcal{L}(x_0).$$

Thus, since $|\mathcal{L}(x_0)| \leq \delta$, we have $\|\nabla v\|_{L^\infty(K)} \leq \frac{C_{38}}{\delta}$ where C_{38} only depends on d, ξ and ζ. For all $x \in \partial K \setminus a$, $v(x) = 0$, and therefore the following relation holds

$$\int_K \nabla v(x)\cdot\mathbf{q}(x)\,dx = -\int_K v(x)\,\operatorname{div}\mathbf{q}(x)\,dx + q_a\int_a v(x)\,d\gamma(x).$$

We have $\int_a v(x)\,d\gamma(x) \geq \gamma(\mathcal{B}_{\eta_0/2})$ (because v is non-negative and has value 1 on $\mathcal{B}_{\eta_0/2}$) and thus, by (A.17), $\int_a v(x)\,d\gamma(x) \geq C_{39}\,\delta^{d-1}$ with C_{39} only depending on d, ξ and ζ. Since $||\nabla v(x)||_{L^\infty(K)} \leq \frac{C_{38}}{\delta}$ and $m_K \leq C_{40}\delta^d$, one therefore gets

$$q_a^2 \leq C_{41}\left(\delta^{d-2-2(d-1)}\int_K \mathbf{q}(x)^2\,dx \;+\; \delta^{d-2(d-1)}\int_K (\operatorname{div}\mathbf{q}(x))^2\,dx\right),$$

which leads to (A.16), since $m_a \leq C_{42}\,\delta^{d-1}$.

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