Optimal Pointwise Control of Semilinear Parabolic Equations

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Abstract

We study optimal control problems for semilinear parabolic equations with pointwise controls in a bounded domain of R^N. When the nonlinear term in the state equation is of the form $|y|^{\gamma-1}y$, we prove the existence of solutions for such equations when $1 \le \gamma < \frac{N}{N-2}$. We next study a control problem with a terminal observation. We prove existence of optimal controls and a Pontryagin principle for these problems.

Key words. Optimal control, pointwise control, semilinear parabolic equations with measures as data.

AMS subject classification : 49K20, 49J20, 49N25, 35K20.

1 Introduction

Let Ω be a bounded open subset in R^N ($N \ge 2$) with a boundary Γ of class C^2 and A be a second order differential operator defined by $Ay = -\sum_{i,j=1}^N D_i(a_{ij}(x)D_j y) + a_0(x)y$, ($D_i$ denotes the partial derivative with respect to x_i). We consider the following boundary value problem :

$$\frac{\partial y}{\partial t} + Ay + \Phi(x,t,y) = u(t)\delta_{x_0} \text{ in } Q, \qquad \frac{\partial y}{\partial n_A} = 0 \text{ (or } y = 0) \text{ on } \Sigma, \qquad y(0) = y_0 \text{ in } \Omega, \quad (1)$$

where $Q = \Omega\times]0,T[$, $\Sigma = \Gamma\times]0,T[$, δ_{x_0} denotes the Dirac measure at $x_0 \in \Omega$, the control variable u belongs to some subset K_U of $L^q(0,T)$, Φ is a Carathéodory function from $Q \times R$ into R. We are interested in the control problem

$$(P_\Phi) \qquad \inf\{I(y,u) \mid (y,u) \in L^1(0,T;W^{1,1}(\Omega)) \times K_U,\ (y,u) \text{ satisfies (1)}\},$$

where

$$I(y,u) = \beta_1 \int_Q |y - z_d|^r dxdt + \beta_2 \int_\Omega |y(T) - y_d|^s dx + \beta_3 \int_0^T |u|^q(t)dt$$

($\beta_i \ge 0$ for $i = 1,2,3$). For a linear equation (when $\Phi \equiv 0$) and for $q = s = 2$, $\beta_1 = 0$, this problem has been studied by J. -L. Lions in [9], [10]. A characterization of controls u for which $y_u(T)$ (y_u is the solution of (1) corresponding to u) belongs to $L^2(\Omega)$ is given in [10], [15]. Still in the case when $\Phi \equiv 0$, this problem has also been studied by S. Anita [3] for $\beta_2 = \beta_3 = 0$, $q = \infty$, $r = 1$, and K_U is a closed convex bounded subset of $L^\infty(0,T)$ (the existence and the characterization of solutions are established). The case of a nonlinear equation corresponding

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to $\Phi(x,t,y) \equiv y^3$ is addressed in [10], an approximate problem is considered (in which the Dirac mass δ_{x_0} is replaced by the characteristic function of some ball $B(x_0,\varepsilon)$), but optimality conditions for solutions of (P_Φ) by a passage to the limit are not carried out. Moreover, from [5] we known that if $u \equiv 1$ and if $\Phi(x,t,y) \equiv |y|^{\gamma-1}y$, then equation (1) admits a weak solution if and only if $\gamma < \frac{N}{N-2}$. Even if what follows can be extended to nonlinearities more general than $|y|^{\gamma-1}y$, for clarity we consider only this case, with $1 \le \gamma < \frac{N}{N-2}$, and we treat the case of homogeneous Neumann boundary conditions (the results of this paper can be easily adapted to homogeneous Dirichlet boundary conditions). Thus the state equation is

$$\frac{\partial y}{\partial t} + Ay + |y|^{\gamma-1}y = u(t)\delta_{x_0} \text{ in } Q, \qquad \frac{\partial y}{\partial n_A} = 0 \text{ on } \Sigma, \qquad y(0) = y_0 \text{ in } \Omega. \tag{2}$$

To well define the control problem we must indicate in which space y and $y(T)$ may be observed. The answer clearly depends on q [10], [15]. We prove that if $q > \max(1, \frac{2\gamma}{2\gamma - N\gamma + N + 2})$, then $y(T)$ can be observed in $L^s(\Omega)$ for every $1 \le s < \frac{Nq'}{Nq'-2}$ (q' is the conjugate exponent to q). For simplicity, we only consider a terminal observation. Thus we study the control problem

$$(P) \qquad \inf\{J(y,u) \mid (y,u) \in L^1(0,T;W^{1,1}(\Omega)) \times K_U,\ (y,u) \text{ satisfies } (2)\},$$

where

$$J(y,u) = \int_\Omega |y(T) - y_d|^s dx + \beta \int_0^T |u|^q(t)dt$$

($\beta \ge 0$ and y_d is a given function in $L^s(\Omega)$).

Even if, at least in the case $u \in L^\infty(0,T)$, the existence result for (2) can be deduced from [5], the estimates and the regularity results of Theorem 2.1 seems to be new. Moreover our proof is different. By using estimates on analytic semigroups (Lemma 2.1), we first prove estimates in $L^{q'}(0,T;C(\bar{\Omega}))$ for linear equations (Propositions 2.3, 2.5). Next with the so-called transposition method and by taking advantage of the structure of the measure $u(t)\delta_{x_0}$, we obtain estimates for linear equations with right hand side of the form $u(t)\delta_{x_0}$ (Propositions 2.1, 2.2). Existence and regularity results for (2) are stated in Theorem 2.1. Some results concerning the adjoint equation are established in Section 3. The existence of solutions for the control problem and optimality conditions are established in Section 4.

2 State Equation

In all the sequel for $\lambda_1 > 0$ and λ_2 nonnegative, we set $\frac{\lambda_1}{\lambda_2} = \infty$ if $\lambda_2 = 0$. Some constants C_i in estimates in propositions depend on different exponents. Since the constant C_i may intervene in a proof for different exponents, for clarity we sometimes indicate this dependence. The constants K_1 and K_2 are the ones which intervene in semigroup estimates (Lemma 2.1).

(A1) - The coefficient a_0 of A is positive and belongs to $C(\bar{\Omega})$. The coefficients a_{ij} belong to $C^{1,\nu}(\bar{\Omega})$ with $0 < \nu \le 1$, $a_{ij} = a_{ji}$, and they satisfy (for some $m_0 > 0$)

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \ge m_0|\xi|^2 \quad \text{for every } \xi \in R^N \text{ and every } x \in \bar{\Omega}.$$

(A2) - K_U is a closed convex subset of $L^q(0,T)$, $\max(1, \frac{2\gamma}{2\gamma - N\gamma + N + 2}) < q < \infty$ and $1 < s < \frac{Nq'}{Nq'-2}$.

Remark 2.1. It is well known that the condition $a_0 > 0$ is not restrictive. Indeed if y is a solution of (1), then $z = e^{-\lambda t}y$ is the solution of

$$\frac{\partial z}{\partial t} + Az + \lambda z + e^{-\lambda t}\Phi(x,t,e^{\lambda t}z) = e^{-\lambda t}u(t)\delta_{x_0} \text{ in } Q, \quad \frac{\partial z}{\partial n_A} = 0 \text{ (or } z = 0) \text{ on } \Sigma, \quad z(0) = y_0 \text{ in } \Omega.$$

Remark 2.2. When $1 \le \gamma < \frac{N}{N-2}$ and $1 < q$, we can easily verify that the condition $\frac{2\gamma}{2\gamma - N\gamma + N + 2} < q$ is equivalent to $\gamma < \frac{(N+2)q'}{Nq'-2}$ and also to $q' < \frac{2\gamma}{(N\gamma - N - 2)^+}$ (where $(\cdot)^+ = \text{Max}\,(0,\cdot)$). Thus the inequality $\gamma < \min(\frac{N}{N-2}, \frac{(N+2)q'}{Nq'-2})$ is assumed throughout the paper.

In the sequel we consider equations of the form (1) for $\Phi(x,t,y) = |y|^{\gamma-1}y$ or $\Phi(x,t,y) = a(x,t)y$ (where a belongs to some space $L^{\tilde{k}}(0,T;L^k(\Omega))$), and when $u\delta_{x_0}$ is replaced by $f \in L^\ell(0,T;\mathcal{M}_b(\Omega))$, $\ell \ge 1$. For all these equations, we consider solutions in the sense of the following definition.

Definition 2.1 *Let f be in $L^\ell(0,T;\mathcal{M}_b(\Omega))$. We shall say that $y \in L^1(0,T;W^{1,1}(\Omega))$ is a weak solution of the equation*

$$\frac{\partial y}{\partial t} + Ay + \Phi(x,t,y) = f \text{ in } Q, \qquad \frac{\partial y}{\partial n_A} = 0 \text{ on } \Sigma, \qquad y(0) = y_0 \text{ in } \Omega,$$

if $\Phi(\cdot,y(\cdot))$ belongs to $L^1(Q)$ and if

$$\int_Q (-y\frac{\partial \phi}{\partial t} + \sum_{i,j=1}^N a_{ij}(x)D_jyD_i\phi + \Phi(x,t,y)\phi)dxdt = \int_0^T \langle f(t),\phi(t)\rangle_{\mathcal{M}_b(\Omega)\times C_b(\Omega)}dt + \int_\Omega \phi(0)y_0dx$$

for every $\phi \in C^1(\bar{Q})$ such that $\phi(T) = 0$ on $\bar{\Omega}$. ($\langle f(t),\phi(t)\rangle_{\mathcal{M}_b(\Omega)\times C_b(\Omega)}$ denotes the integral over Ω of $\phi(t)$ for the measure $f(t)$.)

To prove the existence of a weak solution for equation (2), we need some estimates for linear equations established below.

2.1 Estimates for Linear Equations

First recall some results for analytic semigroups. We denote by $\tilde{A}$ the operator defined by

$$D(\tilde{A}) = \{y \in C^2(\bar{\Omega}) \mid \frac{\partial y}{\partial n_A} = 0 \text{ on } \Gamma\}, \quad \tilde{A}y = Ay.$$

For $1 \le \ell < \infty$, we denote by A_ℓ the closure of $\tilde{A}$ in $L^\ell(\Omega)$. The operator $-A_\ell$ is the generator of a strongly continuous analytic semigroup $S_\ell(t)_{t\ge 0}$ in $L^\ell(\Omega)$ [1]. For $1 < \ell < \infty$ the domain of A_ℓ is $D(A_\ell) = \{y \in W^{2,\ell}(\Omega) \mid \frac{\partial y}{\partial n_A} = 0 \text{ on } \Gamma\}$. For $1 = \ell$, $D(A_1)$ is the set of functions y in $L^1(\Omega)$ such that there exists $z \in L^1(\Omega)$ satisfying $\int_\Omega z(x)v(x)dx = \int_\Omega y(x)Av(x)dx$ for all $v \in D(\tilde{A})$. For any $1 \le \ell < \infty$, 0 belongs to the resolvent of $-A_\ell$ and there exists $\delta > 0$ such $\text{Re}\sigma(A_\ell) \ge \delta$ (it is a consequence of (A1) and of the fact that $\sigma(A_\ell)$ is independent of ℓ). Therefore, for $\alpha > 0$, there exists a constant $K_0 = K_0(\ell,\alpha)$ such that

$$\|A_\ell^\alpha S_\ell(t)\varphi\|_{L^\ell(\Omega)} \le K_0 t^{-\alpha}\|\varphi\|_{L^\ell(\Omega)},$$

for every $t > 0$ and every $\varphi \in L^\ell(\Omega)$ (see [8], [12], A_ℓ^α is the α-power of A_ℓ). Thanks to this result the following lemma can be established. The first part of Lemma 2.1 is stated in [1], the second part is established in [14].

Lemma 2.1 *For every $1 \leq \ell \leq \lambda \leq \infty$ with $\ell < \infty$, there exists a constant $K_1 = K_1(\lambda, \ell)$ such that*

$$\|S_\ell(t)\varphi\|_{L^\lambda(\Omega)} \leq K_1 t^{-\frac{N}{2}(\frac{1}{\ell}-\frac{1}{\lambda})}\|\varphi\|_{L^\ell(\Omega)} \tag{3}$$

for every $\varphi \in L^\ell(\Omega)$ and every $t > 0$. For every $1 \leq \ell \leq \lambda \leq \infty$ with $\ell < \infty$, and every $\alpha > 0$, there exists a constant $K_2 = K_2(\lambda, \ell, \alpha)$ such that

$$\|A_\ell^\alpha S_\ell(t)\varphi\|_{L^\lambda(\Omega)} \leq K_2 t^{-\frac{N}{2}(\frac{1}{\ell}-\frac{1}{\lambda})-\alpha}\|\varphi\|_{L^\ell(\Omega)} \tag{4}$$

for every $\varphi \in L^\ell(\Omega)$ and every $t > 0$.

We consider the linear equation

$$\frac{\partial y}{\partial t} + Ay + ay = f \text{ in } Q, \qquad \frac{\partial y}{\partial n_A} = 0 \text{ on } \Sigma, \qquad y(0) = y_0 \text{ in } \Omega, \tag{5}$$

where f belongs to $L^q(0,T;\mathcal{M}_b(\Omega))$, and a belongs to $L^{\tilde{k}}(0,T;L^k(\Omega))$ for every $(\tilde{k}, k)$ satisfying

$$\frac{q}{\gamma-1} \leq \tilde{k} < \infty, \quad 1 \leq k < \frac{N}{(N-2)(\gamma-1)}, \quad \frac{N}{2} < \frac{1}{q'} + \frac{N}{2(\gamma-1)k} + \frac{1}{(\gamma-1)\tilde{k}} \quad \text{if } \gamma > 1, \tag{6}$$

$$\text{and} \qquad 1 \leq \tilde{k} < \infty, \qquad 1 \leq k < \infty \qquad \text{if } \gamma = 1.$$

We look for estimates for the solution y of (5) in $L^{\tilde{r}}(0,T;L^r(\Omega))$ with

$$q \leq \tilde{r} \leq \infty, \quad 1 < r < \frac{N}{N-2} \quad \text{and} \quad \frac{N}{2} < \frac{1}{q'} + \frac{N}{2r} + \frac{1}{\tilde{r}}. \tag{7}$$

The following lemma will be often used in calculations throughout the paper.

Lemma 2.2 *If a function a belongs to $L^{\tilde{k}}(0,T;L^k(\Omega))$ for every $(\tilde{k}, k)$ satisfying (6), and if y belongs to $L^{\tilde{r}}(0,T;L^r(\Omega))$ for every $(\tilde{r}, r)$ satisfying (7), then ay belongs to $L^\rho(Q)$ for every $1 \leq \rho < inf(\frac{N}{\gamma(N-2)}, \frac{(N+2)q'}{\gamma(Nq'-2)})$. If a sequence $(a_n)_n$ is bounded in $L^{\tilde{k}}(0,T;L^k(\Omega))$ for every $(\tilde{k}, k)$ satisfying (6), and if $(y_n)_n$ is bounded in $L^{\tilde{r}}(0,T;L^r(\Omega))$ for every $(\tilde{r}, r)$ satisfying (7), then $(a_n y_n)_n$ is bounded in $L^\rho(Q)$ for every $1 \leq \rho < inf(\frac{N}{\gamma(N-2)}, \frac{(N+2)q'}{\gamma(Nq'-2)})$.*

Proof. If $q \geq \frac{N}{N-2}$, then for $\tilde{k} = \frac{q}{\gamma-1}$, $\tilde{r} = q$, we can verify that ay belongs to $L^{\tilde{\rho}}(0,T;L^\rho(\Omega))$ for $\tilde{\rho} = \frac{q}{\gamma}$ and for every $1 \leq \rho < \frac{N}{\gamma(N-2)}$. Therefore the first part of the lemma is proved in this case. If $q < \frac{N}{N-2}$, then we can verify that ay belongs to $L^\rho(Q)$ for every $1 \leq \rho < \frac{(N+2)q'}{\gamma(Nq'-2)}$. The second part of the lemma can be proved in the same way.

Proposition 2.1 *If a is a nonnegative function belonging to $L^{\tilde{k}}(0,T;L^k(\Omega))$ for every $(\tilde{k}, k)$ satisfying (6), if f belongs to $L^q(0,T;\mathcal{M}_b(\Omega))$ and if y_0 belongs to $L^{\frac{Nq'}{Nq'-2}}(\Omega)$, then equation (5) admits a unique weak solution in $L^1(0,T;W^{1,1}(\Omega))$, this solution belongs to $L^{\tilde{r}}(0,T;L^r(\Omega))$ for every $(\tilde{r}, r)$ satisfying (7) and there exists a constant $C_1 = C_1(\tilde{r}, r, q)$, not depending on the function a, such that*

$$\|y\|_{L^{\tilde{r}}(0,T;L^r(\Omega))} \leq C_1(\|f\|_{L^q(0,T;\mathcal{M}_b(\Omega))} + \|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)}). \tag{8}$$

In particular for $\tilde{r} = \infty$ and for every $1 < r < \frac{Nq'}{Nq'-2}$, y belongs to $C([0,T];L^r_w(\Omega))$ and we have

$$\|y\|_{L^\infty(0,T;L^r(\Omega))} \leq C_1(\infty, r, q)(\|f\|_{L^q(0,T;\mathcal{M}_b(\Omega))} + \|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)}). \tag{9}$$

($C([0,T];L^r_w(\Omega))$ denotes the space of continuous functions from $[0,T]$ into $L^r(\Omega)$ endowed with its weak topology.)

Proposition 2.2 *Let f be in $L^\ell(0,T;\mathcal{M}_b(\Omega))$. The equation*

$$\frac{\partial y}{\partial t} + Ay = f \ \text{in } Q, \qquad \frac{\partial y}{\partial n_A} = 0 \ \text{on } \Sigma, \qquad y(0) = 0 \ \text{in } \Omega. \tag{10}$$

admits a unique solution in $L^1(0,T;W^{1,1}(\Omega))$, it belongs to $L^{\delta'}(0,T;W^{1,d'}(\Omega))$ for every (δ,d,ℓ) satisfying

$$1 \le \delta \le \ell' \le \infty, \quad N < d < \infty, \qquad \frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2} + \frac{1}{\ell'}, \tag{11}$$

and there exists a constant $C_2 = C_2(\delta,d,\ell)$ such that

$$\|y\|_{L^{\delta'}(0,T;W^{1,d'}(\Omega))} \le C_2 \|f\|_{L^\ell(0,T;\mathcal{M}_b(\Omega))}. \tag{12}$$

For every y_0 in $L^{\frac{Nq'}{Nq'-2}}(\Omega)$, the equation

$$\frac{\partial y}{\partial t} + Ay = 0 \ \text{in } Q, \qquad \frac{\partial y}{\partial n_A} = 0 \ \text{on } \Sigma, \qquad y(0) = y_0 \ \text{in } \Omega, \tag{13}$$

admits a unique solution in $L^1(0,T;W^{1,1}(\Omega))$, it belongs to $L^{\delta'}(0,T;W^{1,d'}(\Omega))$ for every (δ,d) satisfying

$$2 < \delta < \infty, \quad 1 < d \le \frac{Nq'}{2}, \qquad \frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2} + \frac{1}{q'}, \tag{14}$$

and there exists a constant $C_3 = C_3(\delta,d,q)$ such that

$$\|y\|_{L^{\delta'}(0,T;W^{1,d'}(\Omega))} \le C_3 \|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)}. \tag{15}$$

Propositions 2.1 and 2.2 are proved thanks to the so-called transposition method. For this, we prove some estimates in the propositions below.

Proposition 2.3 *Consider the following terminal boundary value problem*

$$-\frac{\partial z}{\partial t} + Az + az = g \ \text{in } Q, \qquad \frac{\partial z}{\partial n_A} = 0 \ \text{on } \Sigma, \qquad z(T) = 0 \ \text{in } \Omega. \tag{16}$$

We suppose that a is a nonnegative function belonging to $L^{\tilde{k}}(0,T;L^k(\Omega))$ for every $(\tilde{k},k)$ satisfying (6), and g belongs to $L^\delta(0,T;L^d(\Omega))$. Then the weak solution of equation (16) belongs to $L^{\ell'}(0,T;C(\bar{\Omega}))$ for every (δ,d,ℓ) satisfying

$$1 \le \delta \le \ell' < \infty, \quad \frac{N}{2} < d < \infty, \quad \frac{N}{2d} + \frac{1}{\delta} < 1 + \frac{1}{\ell'}, \tag{17}$$

and there exists a constant $C_4 = C_4(\ell,\delta,d)$ not depending on the function a such that

$$\|z\|_{L^{\ell'}(0,T;L^\infty(\Omega))} \le C_4 \|g\|_{L^\delta(0,T;L^d(\Omega))}. \tag{18}$$

Proof. 1 - We first consider the case when $a = 0$ and when g is regular. We denote by $\hat{z}$ the solution of (16) corresponding to $a = 0$. As in [13], [14], we use a duality method. Let ϕ be in $\mathcal{D}(\Omega)$ and y be the solution of the Cauchy problem

$$-\frac{\partial y}{\partial t} + Ay = 0 \text{ in } Q, \qquad \frac{\partial y}{\partial n_A} = 0 \text{ on } \Sigma, \qquad y(T) = \phi \text{ in } \Omega.$$

Since g is regular, by a straightforward calculation we obtain

$$\int_\Omega \phi(x)\hat{z}(x,t)dx = -\int_t^T \frac{d}{d\tau}(\int_\Omega y(x,T+t-\tau)\hat{z}(x,\tau)dx)d\tau$$

$$= \int_t^T \int_\Omega (Ay(x,T+t-\tau)\hat{z}(x,\tau) - y(x,T+t-\tau)A\hat{z}(x,\tau))dxd\tau + \int_t^T \int_\Omega y(x,T+t-\tau)g(x,\tau)dxd\tau$$

$$= \int_t^T \int_\Omega y(x,T+t-\tau)g(x,\tau)dxd\tau. \tag{19}$$

Thanks to (3) we have

$$\|y(t)\|_{L^{d'}(\Omega)} \le K_1(d',1)(T-t)^{-\frac{N}{2d}}\|\phi\|_{L^1(\Omega)} \tag{20}$$

for all $1 \le d \le \infty$ and all $\phi \in \mathcal{D}(\Omega)$. From (19) and (20), it follows

$$\|\hat{z}(t)\|_{L^\infty(\Omega)} = \mathrm{Sup}\{\int_\Omega \phi(x)\hat{z}(x,t)dx \mid \|\phi\|_{L^1(\Omega)} = 1\} \le K_1 \int_t^T (T+t-\tau)^{-\frac{N}{2d}}\|g(\tau)\|_{L^d(\Omega)}d\tau.$$

Notice that $t \to \|g(t)\|_{L^d(\Omega)}$ belongs to $L^\delta(0,T)$ and $t \to t^{-\frac{N}{2d}}$ belongs to $L^i(0,T)$ for every $1 \le i < 2d/N$. If we set $\hat{g}(t) = \|g(t)\|_{L^d(\Omega)}\chi_{]-\infty,T]}(t)$, $\hat{h}(t) = (T+t)^{-\frac{N}{2d}}\chi_{]-\infty,T]}(T+t)$ (where $\chi_{]-\infty,T]}$ is the characteristic function of $]-\infty,T]$), then $\int_t^T (T+t-\tau)^{-\frac{N}{2d}}\|g(\tau)\|_{L^d(\Omega)}d\tau = \hat{g}*\hat{h}(t)$. Thus $t \to \int_t^T (T+t-\tau)^{-\frac{N}{2d}}\|g(\tau)\|_{L^d(\Omega)}d\tau$ belongs to $L^{\ell'}(0,T)$ if $\frac{1}{\ell'} = \frac{1}{\delta} + \frac{1}{i} - 1$. Therefore, $\hat{z}$ belongs to $L^{\ell'}(0,T;L^\infty(\Omega))$ for every ℓ' satisfying (17) and we have

$$\|\hat{z}\|_{L^{\ell'}(0,T;L^\infty(\Omega))} \le \tilde{C}\|g\|_{L^\delta(0,T;L^d(\Omega))}$$

for some constant $\tilde{C} = \tilde{C}(\ell,\delta,d)$. The same estimate can be obtained if g is not regular. For that it is sufficient to use an approximation process.

2 - Now we suppose that a is regular and nonnegative. We set $g^+ = \max(0,g)$, $g^- = \max(0,-g)$, we denote by $\hat{z}_1$ (resp. $\hat{z}_2$) the solution of (16) corresponding to $a = 0$ and to g^+ (resp. g^-) and by z_1 (resp. z_2) the solution of (16) corresponding to g^+ (resp. g^-). The function $w = z_1 - \hat{z}_1$ is the solution of

$$-\frac{\partial w}{\partial t} + Aw + aw = -a\hat{z}_1 \text{ in } Q, \qquad \frac{\partial w}{\partial n_A} = 0 \text{ on } \Sigma, \qquad w(T) = 0 \text{ in } \Omega.$$

Since $a\hat{z}_1$ belongs to $L^1(Q)$ and is nonnegative, thanks to a classical comparison theorem, we obtain $0 \le z_1 \le \hat{z}_1$ a.e. on Q, and with Step 1 we have

$$\|z_1\|_{L^{\ell'}(0,T;L^\infty(\Omega))} \le \|\hat{z}_1\|_{L^{\ell'}(0,T;L^\infty(\Omega))} \le \tilde{C}\|g^+\|_{L^\delta(0,T;L^d(\Omega))}.$$

We can prove a similar estimate for z_2 and we have

$$\|z\|_{L^{\ell'}(0,T;L^\infty(\Omega))} \le 2\tilde{C}\|g\|_{L^\delta(0,T;L^d(\Omega))}.$$

Notice that $\tilde{C}$ is the constant of Step 1 and it is independent of a. Moreover we can easily verify that z belongs to $L^{\ell'}(0,T;C(\bar{\Omega}))$. Indeed if g is regular the result is obvious, if not, we can consider a sequence $(g_n)_n$ of regular functions converging to g in $L^\delta(0,T;L^d(\Omega))$. If z_n is the solution of (16) corresponding to g_n, then with the previous estimate we see that $(z_n)_n$ is a

Cauchy sequence in $L^{\ell'}(0,T;C(\bar\Omega))$, converging to z (the solution of (16) corresponding to g) in $L^{\ell'}(0,T;C(\bar\Omega))$.

3 - We suppose that g belongs to $L^\infty(\Omega)$ and that a belongs to $L^{\tilde k}(0,T;L^k(\Omega))$ for every $(\tilde k,k)$ satisfying (6). Let $(a_n)_n$ be a sequence of nonnegative regular functions converging to a in $L^{\tilde k}(0,T;L^k(\Omega))$ for every $(\tilde k,k)$ satisfying (6). Denote by z^n the solution of

$$-\frac{\partial z}{\partial t}+Az+a_nz=g \text{ in } Q,\qquad \frac{\partial z}{\partial n_A}=0 \text{ on } \Sigma,\qquad z(T)=0. \tag{21}$$

The sequence $(z_n)_n$ is bounded in $L^{\ell'}(0,T;L^\infty(\Omega))$. The sequence $(z_n)_n$ is bounded in $L^\infty(Q)$ (because $g\in L^\infty(Q)$), the sequence $(a_nz_n)_n$ is bounded in $L^{\tilde k}(0,T;L^k(\Omega))$ for every $(\tilde k,k)$ satisfying (6). The sequence $(z_n)_n$ is also bounded in $L^s(0,T;W^{1,r}(\Omega))$ for $\frac{N}{2}+\frac{1}{2}<\frac{1}{s}+\frac{N}{2r}$ [13], therefore the vector distribution $\frac{dz_n}{dt}$ is bounded in $L^\lambda(0,T;(W^{1,\sigma}(\Omega))')$ for $\sigma>1$ big enough and some $\lambda>1$. Thus from Aubin Theorem ([11] Theorem 1.5.1), we deduce that $(z_n)_n$ is relatively compact in $L^1(Q)$. Thus we can pass to the limit in the variational formulation satisfied by z_n and we see that $(z_n)_n$ converges to the weak solution z of (16) for the weak-star topology of $L^{\ell'}(0,T;L^\infty(\Omega))$, and that z satisfies (18) with $C_4=2\tilde C$. To prove that z belongs to $L^{\ell'}(0,T;C(\bar\Omega))$, we notice that $z\in L^\infty(Q)$ (because $g\in L^\infty(Q)$) and that

$$-\frac{\partial z}{\partial t}+Az=-az+g \text{ in } Q,\qquad \frac{\partial z}{\partial n_A}=0 \text{ on } \Sigma,\qquad z(T)=0. \tag{22}$$

Thanks to Step 1, z belongs to $L^\lambda(0,T;C(\bar\Omega))$ for every $\lambda\ge1$ satisfying $\frac{N}{2k}+\frac{1}{\tilde k}<1+\frac{1}{\lambda}$, where $(\tilde k,k)$ satisfies (6) and $\tilde k\le\lambda$ (the triplet $(\tilde k,k,\lambda)$ exists because $\gamma<\frac{N}{N-2}$). We have proved that z belongs to $L^\lambda(0,T;C(\bar\Omega))\cap L^{\ell'}(0,T;L^\infty(\Omega))$, therefore z belongs to $L^{\ell'}(0,T;C(\bar\Omega))$.

4 - We suppose that g belongs to $L^\delta(0,T;L^d(\Omega))$ and that a belongs to $L^{\tilde k}(0,T;L^k(\Omega))$ for every $(\tilde k,k)$ satisfying (6). Let $(g_n)_n$ be a sequence of regular functions converging to g in $L^\delta(0,T;L^d(\Omega))$ for every (δ,d,ℓ) satisfying (17). Let z^n be the solution of (16) corresponding to g_n. Thanks to estimate (18) proved in Step 3 for z^n, we can prove that $(z^n)_n$ is a Cauchy sequence in $L^{\ell'}(0,T;C(\bar\Omega))$, that it converges in $L^{\ell'}(0,T;C(\bar\Omega))$ to the weak solution z of (16) and that (18) is satisfied.

Proposition 2.4 *Consider the following equation*

$$-\frac{\partial z}{\partial t}+Az=-div\,\xi \text{ in } Q,\qquad \frac{\partial z}{\partial n_A}=0 \text{ on } \Sigma,\qquad z(T)=0 \text{ in } \Omega. \tag{23}$$

Suppose that $\xi\in(\mathcal D(Q))^N$. Then the weak solution of equation (23) belongs to $L^{\ell'}(0,T;C(\bar\Omega))$ for every (δ,d,ℓ) satisfying (11), and there exists a constant $C_6=C_6(q,\delta,d)$ not depending on a such that

$$\|z\|_{L^{\ell'}(0,T;L^\infty(\Omega))}\le C_6\|\xi\|_{L^\delta(0,T;(L^d(\Omega))^N)} \tag{24}$$

for every $\xi\in(\mathcal D(Q))^N$.

Proof. We denote by y the solution of the Cauchy problem

$$-\frac{\partial y}{\partial t}+Ay=0 \text{ in } Q,\qquad \frac{\partial y}{\partial n_A}=0 \text{ on } \Sigma,\qquad y(0)=\phi \text{ in } \Omega.$$

As in the proof of Proposition 2.3, by a direct calculation we have

$$\int_\Omega \phi(x)z(x,t) = \int_t^T \int_\Omega Dy(x, T+t-\tau)\xi(x,\tau)dxd\tau. \tag{25}$$

From [8], Theorem 1.6.1, we know that the domain of $A^\alpha_{d'}$ is continuously imbedded in $W^{1,d'}(\Omega)$ if $\alpha > \frac{1}{2}$. Thus from (4) we deduce

$$\|y(t)\|_{W^{1,d'}(\Omega)} \le \tilde{C}\|A^\alpha_{d'}y(t)\|_{L^{d'}(\Omega)} \le \tilde{C}K_2(d',1)(T-t)^{-\frac{N}{2d}-\alpha}\|\phi\|_{L^1(\Omega)} \tag{26}$$

for every $\frac{1}{2} < \alpha < 1$. From (25) and (26), it follows

$$\|z(t)\|_{L^\infty(\Omega)} \le \tilde{C}K_2 \int_t^T (T+t-\tau)^{-\frac{N}{2d}-\alpha}\|\xi(\tau)\|_{(L^d(\Omega))^N}d\tau.$$

Notice that $t \to \|\xi(t)\|_{(L^d(\Omega))^N}$ belongs to $L^\delta(0,T)$ and that, for every $1 \le i < \frac{2d}{N+d}$, we can choose $\alpha > \frac{1}{2}$ such that $t \to t^{-\frac{N}{2d}-\alpha}$ belongs to $L^i(0,T)$. Therefore, still by using integrability results for convolution products (as in the proof of Proposition 2.3), we claim that $t \to \int_t^T (T+t-\tau)^{-\frac{N}{2d}-\alpha}\|g(\tau)\|_{(L^d(\Omega))^N}d\tau$ belongs to $L^{\ell'}(0,T)$ for every $\ell' \ge \delta$ satisfying $\frac{1}{2}+\frac{1}{\ell'} > \frac{1}{\delta}+\frac{N}{2d}$ and that

$$\|z\|_{L^{\ell'}(0,T;L^\infty(\Omega))} \le C_6\|\xi\|_{L^\delta(0,T;(L^d(\Omega))^N)}$$

for $C_6 = C_6(\ell,\delta,d)$. Since ξ is regular, it is clear that z belongs to $L^{\ell'}(0,T;C(\bar{\Omega}))$.

Proof of Proposition 2.1. 1 - We first prove the estimate in $L^{\tilde{r}}(0,T;L^r(\Omega))$ in the case when f is regular (for example $f \in C(\bar{Q})$) and $y_0 \equiv 0$. Let z be the solution of (16) corresponding to $g \in L^{\tilde{r}'}(0,T;L^{r'}(\Omega))$. If $(\tilde{r}, r)$ satisfies (7), we see that $\tilde{r}' \le q'$, $r' > \frac{N}{2}$ and $1+\frac{1}{q'} > \frac{N}{2r'}+\frac{1}{\tilde{r}'}$, and Proposition 2.3 yields

$$\|z\|_{L^{q'}(0,T;L^\infty(\Omega))} \le C_4\|g\|_{L^{\tilde{r}'}(0,T;L^{r'}(\Omega))}.$$

Let y be the solution of (5) corresponding to $y_0 \equiv 0$, by a Green formula, we have

$$\int_Q ygdxdt = \int_Q fzdxdt.$$

For every $(\tilde{r}, r)$ verifying (7), we have

$$\|y\|_{L^{\tilde{r}}(0,T;L^r(\Omega))} = \sup\{|\int_Q ygdxdt| \mid \|g\|_{L^{\tilde{r}'}(0,T;L^{r'}(\Omega))} = 1\} \le C_4\|f\|_{L^q(0,T;L^1(\Omega))}$$

for every $f \in C(\bar{Q})$. In particular for $\tilde{r} = \infty$ and every $r < \frac{Nq'}{Nq'-2}$ we obtain

$$\|y\|_{L^\infty(0,T;L^r(\Omega))} \le C_4\|f\|_{L^q(0,T;L^1(\Omega))}.$$

2 - Suppose that f belongs to $L^q(0,T;\mathcal{M}_b(\Omega))$, and let $(f_n)_n$ be a sequence of regular functions converging to f in the following sense

$$\lim_n \int_Q f_n\phi dxdt = \int_0^T \langle f(t), \phi(t)\rangle_{\mathcal{M}_b(\Omega)\times C_b(\Omega)}dt \quad \text{for every } \phi \in C(\bar{Q}),$$

$\lim_n \|f_n\|_{L^q(0,T;L^1(\Omega))} = \|f\|_{L^q(0,T;\mathcal{M}_b(\Omega))}$ and $\|f_n(t)\|_{L^1(\Omega)} \le \|f(t)\|_{\mathcal{M}_b(\Omega)}$ on $[0,T]$.

Let y_n be the solution of

$$\frac{\partial y}{\partial t} + Ay + ay = f_n \text{ in } Q, \qquad \frac{\partial y}{\partial n_A} = 0 \text{ on } \Sigma, \qquad y(0) = 0 \text{ in } \Omega. \tag{27}$$

Thanks to the estimate in Step 1, we obtain

$$\|y_n\|_{L^{\tilde r}(0,T;L^r(\Omega))} \le C_4 \|f_n\|_{L^q(0,T;L^1(\Omega))}. \tag{28}$$

From $(y_n)_n$, we can extract a subsequence, still indexed by n, such that $(y_n)_n$ converges to y for the weak-star topology of $L^{\tilde r}(0,T;L^r(\Omega))$ for every $(\tilde r, r)$ satisfying (7). By passing to the limit in the variational formulation of (27), we can easily verify that y is the solution of (5) corresponding to $y_0 \equiv 0$. By passing to the limit in (28), we obtain (8). Since the solution of (5) is unique, the original sequence $(y_n)_n$ converges to y weakly-star in $L^{\tilde r}(0,T;L^r(\Omega))$, for every $(\tilde r, r)$ satisfying (7).

3 - Now we prove that the solution y of (5) corresponding to $y_0 \equiv 0$ belongs to $C([0,T];L^r_w(\Omega))$. Since y is the weak-star limit of $(y_n)_n$ in $L^\infty(0,T;L^r(\Omega))$ for every $1 < r < \frac{Nq'}{Nq'-2}$, and since

$$\|f_n(t)\|_{L^1(\Omega)} \le \|f(t)\|_{\mathcal{M}_b(\Omega)}$$

for almost every $t \in [0,T]$ and every n, thanks to [4] Theorem 2, we can prove that $y \in C([0,T];L^1(\Omega))$. We already know that y belongs to $L^\infty(0,T;L^r(\Omega))$ for every $1 < r < \frac{Nq'}{Nq'-2}$, therefore y belongs to belongs to $C([0,T];L^r_w(\Omega))$ for every $1 < r < \frac{Nq'}{Nq'-2}$ (see for example [7], Chapter 18, Lemma 5.6).

4 - If $\hat y$ is the solution of

$$\frac{\partial y}{\partial t} + Ay = 0 \text{ in } Q, \qquad \frac{\partial y}{\partial n_A} = 0 \text{ on } \Sigma, \qquad y(0) = y_0 \text{ in } \Omega, \tag{29}$$

then

$$\|\hat y(t)\|_{L^r(\Omega)} \le K_1\Big(r, \frac{Nq'}{Nq'-2}\Big) t^{-\frac{N}{2}\left(\frac{Nq'-2}{Nq'} - \frac{1}{r}\right)} \|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)}$$

for every $\frac{Nq'}{Nq'-2} \le r < \infty$. Therefore $\hat y$ belongs to $L^{\tilde r}(0,T;L^r(\Omega))$ for every $(\tilde r, r)$ satisfying together $\frac{Nq'}{Nq'-2} \le r < \infty$ and (7), and we have

$$\|\hat y\|_{L^{\tilde r}(0,T;L^r(\Omega))} \le C \|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)}$$

for some $C = C(\tilde r, r, q)$. Moreover $\hat y$ belongs to $C([0,T];L^{\frac{Nq'}{Nq'-2}}(\Omega))$. Thus the estimate

$$\|\hat y\|_{L^{\tilde r}(0,T;L^r(\Omega))} \le C \|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)}$$

is true for all $(\tilde r, r)$ satisfying (7).

5 - The estimate of Step 4 can be proved in the case when $a \ne 0$ and a is regular, by using a comparison principle, as in Step 2 of the proof of Proposition 2.3.

6 - Now, we suppose that a belongs to $L^{\tilde{k}}(0,T;L^k(\Omega))$ for every $(\tilde{k},k)$ satisfying (6), and that y_0 belongs to $C(\bar{\Omega})$. Let $(a_n)_n$ be a sequence of nonnegative regular functions, converging to a for every $(\tilde{k},k)$ satisfying (6). Let y_n be the solution of (5) corresponding to a_n and to $f\equiv 0$, and let y be the solution of (5) corresponding to a and to $f\equiv 0$. By arguing as in Step 3 of the proof of Proposition 2.3, we can verify that $(y_n)_n$ converges to y for the weak-star topology of $L^{\tilde{r}}(0,T;L^r(\Omega))$ for every $(\tilde{r},r)$ satisfying (7), and that (15) is satisfied. Therefore y belongs to $L^\infty(0,T;L^{\frac{Nq'}{Nq'-2}}(\Omega))$. We notice that $(y_n)_n$ obeies

$$\frac{\partial y_n}{\partial t}+Ay_n+ay_n=(a-a_n)y_n \text{ in } Q, \qquad \frac{\partial y_n}{\partial n_A}=0 \text{ on } \Sigma, \qquad y_n(0)=y_0 \text{ in } \Omega$$

and that $((a-a_n)y_n)_n$ tends to zero in $L^\rho(Q)$ for some $\rho>1$ (it is a consequence of Lemma 2.2). As in Step 3, we can prove that y belongs to $C([0,T];L^1(\Omega))$. Therefore y belongs to $C([0,T];L_w^{\frac{Nq'}{Nq'-2}}(\Omega))$.

Remark 2.3. With the notation of step 6 in the previous proof, we can write

$$\frac{\partial(y_n-y_m)}{\partial t}+A(y_n-y_m)+a_n(y_n-y_m)=(a_m-a_n)y_m \text{ in } Q,$$

$$\frac{\partial(y_n-y_m)}{\partial n_A}=0 \text{ on } \Sigma,\ (y_n-y_m)(0)=0 \text{ in } \Omega.$$

Thanks to the method developed in the proof of Proposition 2.3, we can prove that

$$\|y_n-y_m\|_{L^{\tilde{\rho}}(0,T;L^\rho(\Omega))}\le C\|(a_n-a_m)y_m\|_{L^{\tilde{\sigma}}(0,T;L^\sigma(\Omega))}$$

with $\sigma\le\rho$, $\tilde{\sigma}\le\tilde{\rho}$ and $\frac{N}{2\sigma}+\frac{1}{\tilde{\sigma}}<\frac{1}{\tilde{\rho}}+\frac{N}{2\rho}+1$. In particular for $\tilde{\rho}=\infty$, $\frac{1}{k}+\frac{1}{r}=\frac{1}{\sigma}$, $\frac{1}{\tilde{k}}+\frac{1}{\tilde{r}}=\frac{1}{\tilde{\sigma}}$, $\sigma<\frac{N}{(N-2)\gamma}$, where $(\tilde{k},k)$ obeies (6) and $(\tilde{r},r)$ obeies (7), we obtain

$$\|y_n-y_m\|_{C([0,T];L^\rho(\Omega))}\le C\|(a_n-a_m)\|_{L^{\tilde{k}}(0,T;L^k(\Omega))}\|y_m\|_{L^{\tilde{r}}(0,T;L^r(\Omega))},$$

with $\frac{N\gamma}{2}-\frac{\gamma}{q'}<\frac{N}{2\sigma}+\frac{1}{\tilde{\sigma}}<\frac{N}{2\rho}+1$. Therefore $(y_n)_n$ is a Cauchy sequence in $C([0,T];L^\rho(\Omega))$, when $\frac{N\gamma}{2}-\frac{\gamma}{q'}<\frac{N}{2\rho}+1$. This condition is in particular satisfied for $\rho=1$ (because $\gamma<\frac{q'(N+2)}{Nq'-2}$) but not in general for $\rho=\frac{Nq'}{Nq'-2}$.

Proof of Proposition 2.2. 1 - We first prove the estimate (12) in the case when $f\in C(\bar{Q})$. Let z be the solution of (23). If (δ,d,ℓ) satisfies (11), from Proposition 2.4, we obtain

$$\|z\|_{L^{\ell'}(0,T;L^\infty(\Omega))}\le C_6\|\xi\|_{L^\delta(0,T;(L^d(\Omega))^N)}$$

for every $\xi\in(\mathcal{D}(Q))^N$. Let y be the solution of (10), with a Green formula we have

$$\int_Q Dy\xi dxdt=\int_Q fzdxdt.$$

It follows

$$\|Dy\|_{L^{\delta'}(0,T;(L^{d'}(\Omega))^N)}=\Sigma_i\sup\{|\int_Q D_iy\xi_idxdt|\mid \|\xi_i\|_{L^\delta(0,T;L^d(\Omega))}=1\}\le NC_6\|f\|_{L^\ell(0,T;L^1(\Omega))}$$

for every $f \in C(\bar{Q})$. We finally obtain (12) with this estimate and with Proposition 2.1.

2 - Suppose that f belongs to $L^\ell(0,T;\mathcal{M}_b(\Omega))$. Let $(f_n)_n$ be a sequence of regular functions converging to f in the following sense

$$\lim_n \int_Q f_n \phi dxdt = \int_0^T \langle f(t), \phi(t)\rangle_{\mathcal{M}_b(\Omega)\times C_b(\Omega)} dt \quad \text{for every } \phi \in C(\bar{Q})$$

$$\text{and} \quad \lim_n \|f_n\|_{L^\ell(0,T;L^1(\Omega))} = \|f\|_{L^\ell(0,T;\mathcal{M}_b(\Omega))}.$$

Let y_n be the solution of

$$\frac{\partial y}{\partial t} + Ay = f_n \text{ in } Q, \qquad \frac{\partial y}{\partial n_A} = 0 \text{ on } \Sigma, \qquad y(0) = 0 \text{ in } \Omega. \tag{30}$$

Thanks to estimate (12) proved in Step 1 for regular functions, we obtain

$$\|y_n\|_{L^{\delta'}(0,T;W^{1,d'}(\Omega))} \le C_2 \|f_n\|_{L^\ell(0,T;L^1(\Omega))}. \tag{31}$$

From $(y_n)_n$, we can extract a subsequence, still indexed by n, such that $(y_n)_n$ converges to y for the weak-star topology of $L^{\delta'}(0,T;W^{1,d'}(\Omega))$ for every (δ, d) satisfying (11). By passing to the limit in the variational formulation of (30), we can easily verify that y is the solution of (10). We obtain (12) by passing to the limit in (31).

3 - Let y be the solution of

$$\frac{\partial y}{\partial t} + Ay = 0 \text{ in } Q, \qquad \frac{\partial y}{\partial n_A} = 0 \text{ on } \Sigma, \qquad y(0) = y_0 \text{ in } \Omega.$$

From [8] Theorem 1.6.1, the domain of $A_{d'}^\alpha$ is continuously imbedded in $W^{1,d'}(\Omega)$ if $\alpha > \frac{1}{2}$. With Lemma 2.1, we obtain

$$\|y(t)\|_{W^{1,d'}(\Omega)} \le C\|A_{d'}^\alpha S(t) y_0\|_{L^{d'}(\Omega)} \le CK_2 t^{-\frac{N}{2}(\frac{Nq'-2}{Nq'} - \frac{1}{d'}) - \alpha} \|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)}$$

if $\frac{Nq'}{Nq'-2} \le d' < \infty$. For every (d, δ) satisfying (14), we can choose $\alpha > \frac{1}{2}$ to have

$$\frac{1}{2} + \frac{N}{2} - \frac{1}{q'} < \alpha + \frac{N}{2} - \frac{1}{q'} < \frac{N}{2d'} + \frac{1}{\delta'}.$$

Therefore

$$\|y\|_{L^{\delta'}(0,T;W^{1,d'}(\Omega))} \le C\|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)}.$$

This completes the proof.

2.2 State Equation

Theorem 2.1 *The state equation (2) admits a unique solution, it belongs to $L^{\tilde{r}}(0,T;L^r(\Omega))$ for every $(\tilde{r}, r)$ satisfying (7) and*

$$\|y\|_{L^{\tilde{r}}(0,T;L^r(\Omega))} \le C_1(\|u\|_{L^q(0,T)} + \|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)})$$

(where $C_1 = C_1(\tilde{r}, r, q)$ is the constant in (8)). The solution of (2) also belongs to $C([0,T]; L^r_w(\Omega))$ for every $1 \leq r < \frac{Nq'}{Nq'-2}$ and we have

$$\|y\|_{L^\infty(0,T;L^r(\Omega))} \leq C_1(\infty, r, q)(\|u\|_{L^q(0,T)} + \|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)}).$$

It also belongs to $L^{\delta_1'}(0,T; W^{1,d_1'}(\Omega)) + L^{\delta_2'}(0,T; W^{1,d_2'}(\Omega))$ for every (δ_1, d_1) satisfying

$$1 < \delta_1 \leq \frac{q}{q-\gamma}, \quad N < d_1 < \infty, \quad \frac{N}{2d_1} + \frac{1}{\delta_1} < \frac{3}{2} - \frac{N(\gamma-1)}{2} + \frac{\gamma}{q'}, \tag{32}$$

and every (δ_2, d_2) satisfying (14). There exists a constant $C_7 = C_7(\delta_1, d_1, q)$ such that

$$\|y - \zeta\|_{L^{\delta_1'}(0,T;W^{1,d_1'}(\Omega))} + \|\zeta\|_{L^{\delta_2'}(0,T;W^{1,d_2'}(\Omega))}$$
$$\leq C_7(\|u\|_{L^q(0,T)} + \|u\|^\gamma_{L^q(0,T)} + \|y_0\|^\gamma_{L^{\frac{Nq'}{Nq'-2}}(\Omega)}) + C_3\|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)},$$

where C_3 is the constant in estimate (15) and ζ is the solution of

$$\frac{\partial \zeta}{\partial t} + A\zeta = 0 \text{ in } Q, \qquad \frac{\partial \zeta}{\partial n_A} = 0 \text{ on } \Sigma, \qquad \zeta(0) = y_0 \text{ in } \Omega.$$

Remark 2.4. Since $\gamma < \frac{N}{N-2} < \frac{N+1}{N-2}$, we have $\frac{q-\gamma}{q} < \frac{3}{2} - \frac{N(\gamma-1)}{2} + \frac{\gamma}{q'}$. Therefore the set of pairs (δ_1, d_1) satisfying (32) is nonempty.

Proof. First notice that the uniqueness can be proved as in [6]. Indeed, since the coefficients of the operator A are regular, if y is a solution of (2) in the sense of Definition 2.1, it also satisfies

$$\int_Q (-y\frac{\partial \phi}{\partial t} + yA\phi + |y|^{\gamma-1}y\phi)dxdt = \int_0^T u(t)\phi(x_0,t)dt + \int_\Omega \phi(0)y_0 dx$$

for every $\phi \in C^{2,1}(\bar{Q})$ such that $\phi(T) = 0$ on $\bar{\Omega}$ and $\frac{\partial \phi}{\partial n_A} = 0$ on Σ.

Let $(u_n)_n$ be a sequence of regular functions converging to u in $L^q(0,T)$ such that $\|u_n\|_{L^q(0,T)} \leq \|u\|_{L^q(0,T)}$, let $(v_n)_n$ be a sequence of nonnegative regular functions converging to δ_{x_0} for the narrow topology of $\mathcal{M}_b(\Omega)$ and such that $\|v_n\|_{L^1(\Omega)} = 1$. We denote by y_n the solution of (2) corresponding to $u_n v_n$.

1 - Estimate in $L^{\tilde{r}}(0,T; L^r(\Omega))$. By setting $a_n = |y_n|^{\gamma-1}$, Proposition 2.1 yields

$$\|y_n\|_{L^{\tilde{r}}(0,T;L^r(\Omega))} \leq C_1(\|u\|_{L^q(0,T)} + \|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)})$$

for every n and every $(r, \tilde{r})$ satisfying (7). In particular, the sequence $(y_n)_n$ is bounded in $L^r(Q)$ for all $1 \leq r < \inf(\frac{(N+2)q'}{Nq'-2}, \frac{N}{N-2})$ (see Lemma 2.2). Since $\gamma < \inf(\frac{(N+2)q'}{(Nq'-2)}, \frac{N}{N-2})$, for every $1 \leq \rho < \inf(\frac{(N+2)q'}{\gamma(Nq'-2)}, \frac{N}{\gamma(N-2)})$, $(y_n^\gamma)_n$ is bounded in $L^\rho(Q)$ and

$$\|y_n^\gamma\|_{L^\rho(Q)} \leq \tilde{C} C_1(\gamma\rho, \gamma\rho, q)^\gamma (\|u\|_{L^q(0,T)} + \|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)})^\gamma, \tag{33}$$

where the constant $\tilde{C}$ only depends on the measure of Q. Moreover, still with Proposition 2.1, for every $\ell \geq 1$ satisfying $\gamma\ell \geq q$ and $\frac{1}{q'} + \frac{1}{\gamma\ell} > \frac{N(\gamma-1)}{2\gamma}$, we have

$$\|y_n^\gamma\|_{L^\ell(0,T;L^1(\Omega))} \leq C_1(\gamma\ell, \gamma, q)^\gamma (\|u\|_{L^q(0,T)} + \|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)})^\gamma. \tag{34}$$

2 - Estimate in $L^{\delta_1'}(0,T;W^{1,d_1'}(\Omega)) + L^{\delta_2'}(0,T;W^{1,d_2'}(\Omega))$. Let ξ_n be the solution of

$$\frac{\partial \xi_n}{\partial t} + A\xi_n = u_n\, v_n - |y_n|^{\gamma-1}y_n \text{ in } Q, \qquad \frac{\partial \xi_n}{\partial n_A} = 0 \text{ on } \Sigma, \qquad \xi_n(0) = 0 \text{ in } \Omega.$$

Then we have $y_n = \xi_n + \zeta$. With Proposition 2.2, for every (δ_1, d_1, ℓ) satisfying (11), we obtain

$$\|\xi_n\|_{L^{\delta_1'}(0,T;W^{1,d_1'}(\Omega))} \le C_2(\delta_1, d_1, \ell)(\|u\|_{L^\ell(0,T)} + \|y_n^\gamma\|_{L^\ell(0,T;L^1(\Omega))}) \tag{35}$$

for every n. If $\gamma < \frac{N}{N-2}$, we have $1 + \frac{\gamma}{q'} - \frac{N(\gamma-1)}{2} \ge \frac{q-\gamma}{q}$. Therefore, if (32) is satisfied, there exists $\ell \ge \frac{q}{\gamma}$ such that $\frac{3}{2} + \frac{\gamma}{q'} - \frac{N(\gamma-1)}{2} \ge \frac{3}{2} - \frac{1}{\ell} = \frac{1}{2} + \frac{1}{\ell'} > \frac{N}{2d_1} + \frac{1}{\delta_1}$ and $1 < \delta_1 \le \ell' \le \frac{q}{q-\gamma}$. Thanks to (34), (35), for every (δ_1, d_1) satisfying (32), there exists a constant $C_7 = C_7(\delta_1, d_1, q)$ such that

$$\|\xi_n\|_{L^{\delta_1'}(0,T;W^{1,d_1'}(\Omega))} \le C_7(\|u\|_{L^q(0,T)} + \|u\|^\gamma_{L^q(0,T)} + \|y_0\|^\gamma_{L^{\frac{Nq'}{Nq'-2}}(\Omega)})$$

for every n. With Proposition 2.2, we also have

$$\|\zeta\|_{L^{\delta_2'}(0,T;W^{1,d_2'}(\Omega))} \le C_3\|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)}$$

for every (δ_2, d_2) satisfying (14).

3 - Estimates of $\frac{dy_n}{dt}$. From the previous estimates we can prove that $\frac{dy_n}{dt}$ is a distribution with values in $(W^{1,\sigma}(\Omega))'$ and that $(\frac{dy_n}{dt})_n$ is bounded in $L^\ell(0,T;(W^{1,\sigma}(\Omega))')$ for some $\ell > 1$ and for σ big enough. Thus, from Aubin Theorem [11] Theorem 1.5.1, the sequence $(y_n)_n$ is relatively compact in $L^1(Q)$.

4 - Passage to the limit. From $(y_n)_n$ we can extract a subsequence, still denoted by $(y_n)_n$, weakly-star convergent to y in $L^{\tilde r}(0,T;L^r(\Omega))$ for every $(r, \tilde r)$ satisfying (7), and such that $(y_n - \zeta)_n$ weakly converges to $y - \zeta$ in $L^{\delta_1'}(0,T;W^{1,d_1'}(\Omega))$ for every (δ_1, d_1) satisfying (32). Since $(y_n)_n$ is relatively compact in $L^1(Q)$, we can also suppose that $(y_n)_n$ converges to y almost everywhere in Q. Because of (33), we can easily prove (with Hölder's inequality and Vitali's Theorem) that $(|y_n|^{\gamma-1}y_n)_n$ converges to $|y|^{\gamma-1}y$ in $L^\rho(Q)$ for all $1 \le \rho < \inf(\frac{(N+2)q'}{\gamma(Nq'-2)}, \frac{N}{\gamma(N-2)})$. Now by passing to the limit in the variational formulation satisfied by y_n, we see that y is a solution of (2). The estimates on y clearly follow from estimates on y_n proved in Step 1 and Step 2.

3 Adjoint Equation

3.1 Adjoint Equation

We consider the following terminal boundary value problem

$$-\frac{\partial p}{\partial t} + Ap + ap = 0 \text{ in } Q, \qquad \frac{\partial p}{\partial n_A} = 0 \text{ on } \Sigma, \qquad p(T) = p_T \text{ in } \Omega. \tag{36}$$

If u is a solution of (P), if y_u is the solution of (2) corresponding to u, if we set $a = \gamma|y_u|^{\gamma-1}$ and $p_T = s|y_u(T) - y_d|^{s-2}(y_u(T) - y_d)$, then equation (36) corresponds to the adjoint equation for (P) associated with (u, y_u). Since $y_u(T)$ belongs to $L^r(\Omega)$ for all $1 \le r < \frac{Nq'}{Nq'-2}$, we have to study (36) for $p_T \in L^\sigma(\Omega)$ with $\sigma < \frac{Nq'}{(Nq'-2)(s-1)}$.

Theorem 3.1 *Suppose that a is nonnegative and belongs to $L^{\tilde{k}}(0,T;L^k(\Omega))$ for every $(\tilde{k},k)$ satisfying (6). If p_T belongs to $L^\sigma(\Omega)$ for all $1 \le \sigma < \frac{Nq'}{(Nq'-2)(s-1)}$, the weak solution of (36) belongs to $L^\ell(0,T;C(\bar{\Omega}))$ for every $\ell < \frac{2q'}{(Nq'-2)(s-1)}$. Moreover, for every $\ell < \frac{2\sigma}{N}$, there exists a constant $C_8 = C_8(\ell,\sigma)$, not depending on the function a such that*

$$\|p\|_{L^\ell(0,T;C(\bar{\Omega}))} \le C_8 \|p_T\|_{L^\sigma(\Omega)}. \tag{37}$$

Proof.
1 - We first prove the result for $a \equiv 0$ by using estimates on analytic semigroups. We denote by $\hat{p}$ the weak solution of (36) corresponding to $a = 0$. From Lemma 2.1, we deduce

$$\|\hat{p}(t)\|_{L^\infty(\Omega)} \le K_1(\infty,\sigma)(T-t)^{-\frac{N}{2\sigma}} \|p_T\|_{L^\sigma(\Omega)}.$$

Therefore $\hat{p}$ belongs to $L^\ell(0,T;L^\infty(\Omega))$ if $\ell < \frac{2\sigma}{N}$ and we have

$$\|\hat{p}\|_{L^\ell(0,T;L^\infty(\Omega))} \le C_8 \|p_T\|_{L^\sigma(\Omega)}$$

for some $C_8 = C_8(\ell,\sigma)$. Thanks to this estimate and by using approximation arguments, we can easily prove that $\hat{p}$ belongs to $L^\ell(0,T;C(\bar{\Omega}))$.

2 - We suppose that $a \neq 0$ and a is regular. If p is the weak solution of (36), then $w = p - \hat{p}$ satisfies

$$-\frac{\partial w}{\partial t} + Aw + aw = -a\hat{p} \text{ in } Q, \qquad \frac{\partial w}{\partial n_A} = 0 \text{ on } \Sigma, \qquad w(T) = 0. \tag{38}$$

Since $\hat{p}$ belongs to $L^\ell(0,T;C(\bar{\Omega}))$ for every $\ell < \frac{2\sigma}{N}$ and a belongs to $L^\infty(Q)$, then $a\hat{p}$ belongs to $L^1(Q)$. As in the proof of Proposition 2.3, we can use a comparison principle to prove that p belongs to $L^\ell(0,T;L^\infty(\Omega))$ and that (37) is satisfied.

3 - Now, we suppose that a belongs to $L^{\tilde{k}}(0,T;L^k(\Omega))$ for every $(\tilde{k},k)$ satisfying (6), and that p_T belongs to $C(\bar{\Omega})$. Let $(a_n)_n$ be a sequence of regular functions, converging to a for every $(\tilde{k},k)$ satisfying (6). Let p_n be the solution of (36) corresponding to a_n and let p be the solution of (36) corresponding to a. By arguing as in Step 3 of the proof of Proposition 2.3, we can verify that, for every $1 < \ell < \frac{2q'}{(Nq'-2)(s-1)}$, $(p_n)_n$ converges to p for the weak-star topology of $L^\ell(0,T;L^\infty(\Omega))$ and that $\|p\|_{L^\ell(0,T;L^\infty(\Omega))} \le C_8\|p_T\|_{L^\sigma(\Omega)}$ is satisfied if $\ell < \frac{2\sigma}{N} < \frac{2q'}{(Nq'-2)(s-1)}$. To prove that p belongs to $L^\ell(0,T;C(\bar{\Omega}))$, we notice that $p \in L^\infty(Q)$ (because p_T belongs to $C(\bar{\Omega})$) and

$$-\frac{\partial p}{\partial t} + Ap = -ap \text{ in } Q, \qquad \frac{\partial p}{\partial n_A} = 0 \text{ on } \Sigma, \qquad p(T) = p_T \text{ in } \Omega. \tag{39}$$

Thanks to Step 1 and to Proposition 2.3, we can prove that p belongs to $L^\lambda(0,T;C(\bar{\Omega}))$ for every $\lambda \ge \tilde{k}$, where $\tilde{k}$ is chosen so that $\frac{N}{2k} + \frac{1}{\tilde{k}} < 1 + \frac{1}{\lambda}$ and so that $(\tilde{k},k)$ obeies (6) (since $\gamma < \frac{N}{N-2}$ such a λ exists). We already know that p belongs to $L^\ell(0,T;L^\infty(\Omega))$, therefore p belongs to $L^\ell(0,T;C(\bar{\Omega}))$ for every $1 \le \ell < \frac{2q'}{(Nq'-2)(s-1)}$.

4 - Now, we suppose that p_T belongs to $L^\sigma(\Omega)$. We denote by $(p_T^n)_n$ a sequence of bounded functions converging to p_T in $L^\sigma(\Omega)$. Let p_n be the solution of (36) corresponding to p_T^n. Thanks to the estimate (37) established for p_n in Step 3, we can prove that $(p_n)_n$ is a Cauchy sequence in $L^\ell(0,T;C(\bar{\Omega}))$, that it converges to p in $L^\ell(0,T;C(\bar{\Omega}))$ and that (37) is satisfied by p.

3.2 Green Formula

Lemma 3.1 *Let $(a_n)_n$ be a sequence of nonnegative functions converging to a in $L^{\tilde{k}}(0,T;L^k(\Omega))$ for every $(\tilde{k},k)$ satisfying (6). Let z_n be the solution of*

$$\frac{\partial z}{\partial t} + Az + a_n z = v\delta_{x_0} \text{ in } Q, \qquad \frac{\partial z}{\partial n_A} = 0 \text{ on } \Sigma, \qquad z(0) = 0 \text{ in } \Omega, \tag{40}$$

and let z be the solution of

$$\frac{\partial z}{\partial t} + Az + az = v\delta_{x_0} \text{ in } Q, \qquad \frac{\partial z}{\partial n_A} = 0 \text{ on } \Sigma, \qquad z(0) = 0 \text{ in } \Omega. \tag{41}$$

Then $(z_n)_n$ converges to z for the weak-star topology of $L^{\tilde{r}}(0,T;L^r(\Omega))$ for all $(\tilde{r},r)$ satisfying (7), and $(z_n(T))_n$ converges to $z(T)$ for the weak topology of $L^r(\Omega)$ for all $1 < r < \frac{Nq'}{Nq'-2}$.

Proof. Thanks to Proposition 2.1, $(z_n)_n$ is bounded in $L^{\tilde{r}}(0,T;L^r(\Omega))$ for all $(\tilde{r},r)$ satisfying (7), and $(z_n(T))_n$ is bounded in $L^r(\Omega)$ for all $1 < r < \frac{Nq'}{Nq'-2}$. The sequence $(a_n z_n)_n$ is bounded in $L^\rho(Q)$ for some $\rho > 1$ (see Lemma 2.2). Therefore $(z_n)_n$ is also bounded in $L^{\delta'}(0,T;W^{1,d'}(\Omega))$ for every $\delta > 1$, $d > 1$ such that $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$ ([13] Theorem 4.2). From $(z_n)_n$, we can extract a subsequence, still indexed by n, weakly star converging to some z in $L^{\tilde{r}}(0,T;L^r(\Omega))$ for all $(\tilde{r},r)$ satisfying (7), and weakly star converging to z in $L^{\delta'}(0,T;W^{1,d'}(\Omega))$ for every $\delta > 1$, $d > 1$ such that $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$. Moreover, we can suppose that $(z_n(T))_n$ converges to some $\tilde{z}$ for the weak topology of $L^r(\Omega)$ for all $1 < r < \frac{Nq'}{Nq'-2}$. By passing to the limit in the Green formula ([13], Theorem 4.2)

$$\int_Q (-z_n \frac{\partial \phi}{\partial t} + \sum_{i,j=1}^N a_{ij}(x) D_j z_n D_i \phi + a_n z_n \phi) dx dt = \int_0^T v(t)\phi(x_0,t)dt - \int_\Omega \phi(T) z_n(T) dx$$

satisfied for every $\phi \in C^1(\bar{Q})$, we see that z is the solution of (41) and that $\tilde{z} = z(T)$.

Theorem 3.2 *Let p be the solution of (36) (with $p_T \in L^\sigma(\Omega)$ for every $1 \leq \sigma < \frac{Nq'}{(Nq'-2)(s-1)}$) and z be the solution of*

$$\frac{\partial z}{\partial t} + Az + az = v\delta_{x_0} \text{ in } Q, \qquad \frac{\partial z}{\partial n_A} = 0 \text{ on } \Sigma, \qquad z(0) = 0 \text{ in } \Omega. \tag{42}$$

Suppose that a belongs to $L^{\tilde{k}}(0,T;L^k(\Omega))$ for every $(\tilde{k},k)$ satisfying (6) and that v belongs to $L^q(0,T)$, then we have the Green formula

$$\int_0^T p(x_0,t)v(t)dt = \int_\Omega p_T(x) z(x,T) dx. \tag{43}$$

Proof.
1 - If a is regular and $p_T \in C(\bar{\Omega})$, the result can be deduced from Theorem 4.2 in [13].

2 - First suppose that $p_T \in C(\bar{\Omega})$ and $v \in L^\infty(0,T)$. Let $(a_n)_n$ be a sequence of regular functions converging to a for every $(\tilde{k},k)$ satisfying (6). Let p_n (resp. p) be the solution of (36) corresponding to a_n (resp. a). Thanks to Proposition 2.3 we can prove that $(p_n)_n$ is a Cauchy sequence in $L^1(0,T;C(\bar{\Omega}))$ and that $(p_n)_n$ converges to p in $L^1(0,T;C(\bar{\Omega}))$. If z_n (resp. z) is the solution of (42) corresponding to a_n (resp. a), then $(z_n(T))_n$ converges to $z(T)$ for the weak-star

topology of $L^r(\Omega)$ for all $1 < r < \frac{Nq'}{Nq'-2}$ (Lemma 3.1). Therefore we can pass to the limit in the Green formula satisfied by (p_n, z_n) and (43) is satisfied by (p, z).

3 - Let us still suppose that $v \in L^\infty(0,T)$. Let $(p_T^n)_n$ be a sequence of regular functions converging to p_T in $L^\sigma(\Omega)$ for all $1 \le \sigma < \frac{Nq'}{(Nq'-2)(s-1)}$, and let p^n be the solution of (36) corresponding to $p^n(T) = p_T^n$. From Theorem 3.1, we deduce that $(p_n)_n$ converges to the solution p of (36) in $L^\ell(0,T;C(\bar{\Omega}))$ for all $1 \le \ell < \frac{2q'}{(Nq'-2)(s-1)}$, thus $(p_n(x_0))_n$ converges to $p(x_0)$ in $L^{q'}(0,T)$ (indeed $1 \le s < \frac{Nq'}{Nq'-2}$). Thanks to Proposition 2.1, $z(T)$ belongs to $L^r(\Omega)$ for every $1 \le r < \frac{Nq'}{Nq'-2}$. Since $1 \le s < \frac{Nq'}{Nq'-2}$, $(z(T)p_T^n)_n$ converges to $z(T)p_T$ in $L^1(\Omega)$. Thus, to complete the proof in the case when a is not regular but when $v \in L^\infty(0,T)$, we can pass to the limit in the Green formula satisfied by p_n.

4 - Now we consider a sequence $(v_n)_n$ of regular functions converging to v in $L^q(0,T)$. If z_n (resp. z) is the solution of (42) corresponding to v_n (resp. v), thanks to Proposition 2.1, we see that $(z_n(T))_n$ converges to $z(T)$ in $L^r(\Omega)$ for every $1 \le r < \frac{Nq'}{Nq'-2}$. We can pass to the limit in the Green formula satisfied by z_n and the proof is complete.

4 The Control Problem

4.1 Existence of solutions

Lemma 4.1 *If $(y_n)_n$ is a sequence bounded in $L^{\tilde r}(0,T;L^r(\Omega))$ for every $(\tilde r, r)$ satisfying (7), and if $(y_n)_n$ converges to y in $L^1(Q)$, then $(y_n)_n$ converges to y in $L^{\tilde r}(0,T;L^r(\Omega))$ for every $(\tilde r, r)$ satisfying together (7) and $\tilde r < \infty$.*

Proof. Let $\eta > 0$ and A be a measurable subset of Q with $\mathrm{meas}(A) = \eta$. Let $(\tilde r, r)$ satisfying together (7) and $\tilde r < \infty$. Let $(\tilde r_1, r_1)$ satisfying (7) and such that

$$\tilde r < \tilde r_1, \quad r < r_1, \quad \frac{1}{r} - \frac{1}{r_1} = \frac{1}{\tilde r} - \frac{1}{\tilde r_1}.$$

Let us set $A(t) = A \cap (\Omega \times \{t\})$, $|A(t)| = \mathcal{L}^N(A(t))$, where $\mathcal{L}^N$ is the N-dimensional Lebesgue measure, and denote by χ_A the characteristic function of A. Thanks to Hölder's inequality and to the equality satisfied by r, $\tilde r$, r_1, $\tilde r_1$, we have

$$\begin{aligned}
\int_0^T \Big(\int_\Omega |y_n(t) - y(t)|^r \chi_A(t) dx\Big)^{\tilde r/r} dt &= \int_0^T \Big(\int_{A(t)} |y_n(t) - y(t)|^r dx\Big)^{\tilde r/r} dt \\
&\le \int_0^T |A(t)|^{\frac{\tilde r}{r} - \frac{\tilde r}{r_1}} \Big(\int_\Omega |y_n(t) - y(t)|^{r_1} dx\Big)^{\tilde r/r_1} dt \\
&\le \Big(\int_0^T |A(t)| dt\Big)^{1 - \frac{\tilde r}{\tilde r_1}} \Big(\int_0^T \Big(\int_\Omega |y_n(t) - y(t)|^{r_1} dx\Big)^{\tilde r_1/r_1} dt\Big)^{\tilde r/\tilde r_1} \\
&\le \eta^{1 - \frac{\tilde r}{\tilde r_1}} \|y_n - y\|^{\tilde r}_{L^{\tilde r_1}(0,T;L^{r_1}(\Omega))}.
\end{aligned}$$

With this estimate, the lemma can be proved with Egorov's Theorem.

Theorem 4.1 *Let us suppose that either $\beta > 0$ or K_U is bounded in $L^q(0,T)$. Then the control problem (P) admits solutions.*

Proof. Let $(u_n)_n$ be a minimizing sequence in K_U. It is clear that $(u_n)_n$ is bounded in $L^q(0,T)$. We can suppose that $(u_n)_n$ converges to some u weakly-star in $L^q(0,T)$. Since K_U is convex and closed in $L^q(0,T)$, then $u \in K_U$. Thanks to Theorem 2.1, $(y_{u_n})_n$ (the sequence of solutions of (2) corresponding to u_n) is bounded in $L^{\tilde{r}}(0,T;L^r(\Omega))$ for every $(\tilde{r},r)$ satisfying (7) and $(y_{u_n}-\zeta)_n$ (where ζ is the function defined in Theorem 2.1) is bounded in in $L^{\delta_1'}(0,T;W^{1,d_1'}(\Omega))$ for every (δ_1,d_1) satisfying (32). By arguments similar to those in Step 3 in the proof of Theorem 2.1, we can also prove that $(y_{u_n})_n$ is relatively compact in $L^1(Q)$. Thanks to Lemma 4.1 $(y_{u_n})_n$ converges to y_u (the solution of (2) corresponding to u) in $L^{\tilde{r}}(0,T;L^r(\Omega))$ for every $(\tilde{r},r)$ satisfying (7) and $\tilde{r}<\infty$ and $(y_{u_n}-\zeta)_n$ converges to $y_u-\zeta$ weakly-star in $L^{\delta_1'}(0,T;W^{1,d_1'}(\Omega))$ for every (δ_1,d_1) satisfying (32). Since for all $1<s<\frac{Nq'}{Nq'-2}$

$$\|y_{u_n}(T)\|_{L^s(\Omega)} \le C_1(\infty,s,q)(\|u_n\|_{L^q(0,T)} + \|y_0\|_{L^{\frac{Nq'}{Nq'-2}}(\Omega)}),$$

we can also prove that $(y_{u_n}(T))_n$ (or at least a subsequence) converges to some y_T for the weak-star topology of $L^s(\Omega)$. To prove that $y_T = y_u(T)$, we use the Green formula of Theorem 3.2. We introduce the solution p_n of

$$-\frac{\partial p}{\partial t} + Ap + a_n p = 0 \text{ in } Q, \qquad \frac{\partial p}{\partial n_A} = 0 \text{ on } \Sigma, \qquad p(T) = \phi \text{ in } \Omega,$$

where $a_n = |y_{u_n}|^{\gamma-1}$ and $\phi \in \mathcal{D}(\Omega)$, and we introduce the solution p of

$$-\frac{\partial p}{\partial t} + Ap + ap = 0 \text{ in } Q, \qquad \frac{\partial p}{\partial n_A} = 0 \text{ on } \Sigma, \qquad p(T) = \phi \text{ in } \Omega,$$

where $a = |y_u|^{\gamma-1}$. Thanks to previous convergence results, $(a_n)_n$ converges to a in $L^{\tilde{k}}(0,T;L^k(\Omega))$ for all $(\tilde{k},k)$ satisfying (6). We notice that $w = p_n - p$ satisfies

$$-\frac{\partial w}{\partial t} + Aw + a_n w = (a-a_n)p \text{ in } Q, \qquad \frac{\partial w}{\partial n_A} = 0 \text{ on } \Sigma, \qquad w(T) = 0 \text{ in } \Omega.$$

If $q \le \gamma$, with estimate (18), we have

$$\|p_n - p\|_{L^{q'}(0,T;C(\bar{\Omega}))} \le C_4(q,\delta,d)\|(a-a_n)p\|_{L^\delta(0,T;L^d(\Omega))} \le C_4\|a-a_n\|_{L^\delta(0,T;L^d(\Omega))}\|p\|_{L^\infty(Q)}$$

for $\delta \le q'$ and $\frac{N}{2d}+\frac{1}{\delta} < 1+\frac{1}{q'}$. Since $\gamma < \frac{(N+2)q'}{(Nq'-2)}$, we have $(\frac{N}{2}-\frac{1}{q'})(\gamma-1) < 1+\frac{1}{q'}$ and we can choose $(\tilde{k},k) = (\delta,d) = (q',d)$ satisfying $(\frac{N}{2}-\frac{1}{q'})(\gamma-1) < \frac{N}{2d}+\frac{1}{\delta} < 1+\frac{1}{q'} = 1+\frac{1}{\delta}$. If $q > \gamma$, still with estimate (18), we have

$$\|p_n-p\|_{L^{q'}(0,T;C(\bar{\Omega}))} \le C\|p_n-p\|_{L^{\frac{q}{\gamma-1}}(0,T;C(\bar{\Omega}))} \le CC_4(\frac{q}{q-\gamma+1},\delta,d)\|a-a_n\|_{L^\delta(0,T;L^d(\Omega))}\|p\|_{L^\infty(Q)},$$

if $\delta = \frac{q}{\gamma-1}$ and $\frac{N}{2d} < 1$. Since $\gamma < \frac{N}{(N-2)}$, we choose $(\tilde{k},k) = (\delta,d)$ satisfying $\frac{N}{2d} < 1$ and $(\frac{N}{2}-\frac{1}{q'})(\gamma-1) < \frac{N}{2d}+\frac{1}{\delta}$. Thus $(p_n)_n$ converges to p in $L^{q'}(0,T;C(\bar{\Omega}))$. With the Green formula (43) for p_n and y_{u_n} we have

$$\int_0^T p_n(x_0,t)u_n(t)dt = \int_\Omega \phi(x)y_{u_n}(x,T)dx.$$

By passing to the limit, we obtain

$$\int_0^T p(x_0,t)u(t)dt = \int_\Omega \phi(x)y_T dx.$$

We also have

$$\int_\Omega \phi(x) y_u(x,T)dx = \int_0^T p(x_0,t)u(t)dt$$

for every $\phi \in \mathcal{D}(\Omega)$. Thus we obtain $y_u(T) = y_T$. Since the mapping $u \to \int_0^T |u|^q dt$ is lower semicontinuous for the weak-star topology in $L^q(0,T)$, and the mapping $y \to \int_\Omega |y - y_d|^s dx$ is lower semicontinuous for the weak-star topology in $L^s(\Omega)$, by classical arguments, we prove that u is a solution of (P).

4.2 Optimality conditions

Theorem 4.2 *If u is a solution of (P), then*

$$\int_0^T (p(x_0,t) + \beta q |u(t)|^{q-2} u(t))(v-u)(t)dt \geq 0 \tag{44}$$

for every $v \in K_U$, where p is the solution of

$$-\frac{\partial p}{\partial t} + Ap + \gamma |y_u|^{\gamma-1} p = 0 \text{ in } Q, \qquad \frac{\partial p}{\partial n_A} = 0 \text{ on } \Sigma, \qquad p(T) = s|y_u(T) - y_d|^{s-2}(y_u(T) - y_d) \text{ in } \Omega, \tag{45}$$

and y_u is the solution of (2) corresponding to u.

Proof. Let v be in K_U, $\lambda > 0$, denote by y_λ the solution of (1) corresponding to $u + \lambda(v-u)$. The function $w = y_\lambda - y_u$ satisfies

$$\frac{\partial w}{\partial t} + Aw + a_\lambda w = \lambda(v-u)\delta_{x_0} \text{ in } Q, \qquad \frac{\partial w}{\partial n_A} = 0 \text{ on } \Sigma, \qquad w(0) = 0 \text{ in } \Omega,$$

where $a_\lambda = \gamma \int_0^1 |y_u + \theta(y_\lambda - y_u)|^{\gamma-1} d\theta$. From estimate (8), still true for the above equation, we obtain

$$\|w\|_{L^{\tilde r}(0,T;L^r(\Omega))} \leq C_1(\tilde r, r, q)\lambda \|v-u\|_{L^q(0,T)}$$

for every $(\tilde r, r)$ satisfying (7). Therefore, when λ tends to zero, y_λ tends to y_u in $L^{\tilde r}(0,T;L^r(\Omega))$ (for all $(\tilde r, r)$ satisfying (7)) and $y_\lambda(T)$ tends to $y_u(T)$ in $L^r(\Omega)$ for all $1 \leq r < \frac{Nq'}{Nq'-2}$. In particular $y_\lambda(T)$ tends to $y_u(T)$ in $L^s(\Omega)$. It also follows that a_λ tends to $a = \gamma|y_u|^{\gamma-1}$ in $L^{\tilde k}(0,T;L^k(\Omega))$ for every $(\tilde k, k)$ satisfying (6). Now we set $z_\lambda = (y_\lambda - y_u)/\lambda$ and we denote by z the solution of

$$\frac{\partial z}{\partial t} + Az + \gamma|y_u|^{\gamma-1} z = (v-u)\delta_{x_0} \text{ in } Q, \qquad \frac{\partial z}{\partial n_A} = 0 \text{ on } \Sigma, \qquad z(0) = 0 \text{ in } \Omega.$$

Thanks to Lemma 3.1, $z_\lambda(T)$ tends to $z(T)$ for the weak topology of $L^r(\Omega)$, for all $1 \leq r < \frac{Nq'}{Nq'-2}$. If we set $I(u) = \int_\Omega |y_u(T) - y_d|^s dx$, from the convexity of the mapping $y \to \int_\Omega |y - y_d|^s dx$, it follows

$$s\int_\Omega |y_u(T) - y_d|^{s-2}(y_u(T) - y_d) z_\lambda(T)dx \leq \frac{I(u + \lambda(v-u)) - I(u)}{\lambda}$$
$$\leq s\int_\Omega |y_\lambda(T) - y_d|^{s-2}(y_\lambda(T) - y_d) z_\lambda(T)dx.$$

We set $F(u) = J(y_u, u)$, thanks to the above calculations we obtain

$$F'(u)(v-u) = \int_\Omega s|y_u(T) - y_d|^{s-2}(y_u(T) - y_d)z(T)dx + \int_0^T \beta q|u(t)|^{q-2}u(t)(v-u)(t)dt.$$

Now, if p is the solution of (45), we notice that $\gamma|y_u|^{\gamma-1}$ belongs to $L^{\tilde{k}}(0,T;L^k(\Omega))$ for every $(\tilde{k}, k)$ satisfying (6), and that $s|y(T) - y_d|^{s-2}(y(T) - y_d)$ belongs to $L^\sigma(\Omega)$ for $1 \leq \sigma < \frac{Nq'}{(Nq'-2)(s-1)}$. Therefore we can use the Green formula of Theorem 3.2 to complete the proof. □

References

[1] H. Amann, Dual semigroups and second order linear elliptic boundary value problems, Is. J. Math., Vol. 45 (1983), 225-254.

[2] S. Anita, Optimal impulse-control of population dynamics with diffusion, in "Differential equations and control theory, V. Barbu Ed., Longman, Harlow, 1991, 1-6.

[3] S. Anita, Optimal control of parameter distributed systems with impulses, App. Math. Optim., Vol. 29 (1994), 93-107.

[4] P. Baras, Compacité de l'opérateur $f \rightarrow u$ solution d'une équation non linéaire $f \in (du/dt) + Au$, C. R. Acad. Sc. Paris, Série A, Vol. 286 (1978), 1113-1116.

[5] P. Baras, M. Pierre, Problèmes paraboliques semilinéaires avec données mesures, in Thesis of P. Baras, Paris VI, 1985.

[6] H. Brezis, A. Friedman, Nonlinear parabolic equations involving measures as initial conditions, J. Math. Pures Appl., Vol. 62 (1983), p. 73-97.

[7] R. Dautray, J. -L. Lions, Analyse Mathématique et Calcul Numérique, Tome 8, Masson, Paris, 1987.

[8] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer-Verlag, Berlin/Heidelberg/New York, 1981.

[9] J. -L. Lions, Some Methods in the Mathematical Analysis of Systems and their Control, Science Press, Beijing and Gordon and Breach, New York, 1981.

[10] J. -L. Lions, Pointwise control for distributed systems, in "Control and estimation in distributed parameters sytems", H. T. Banks Ed., SIAM, Philadelphia, 1992, p. 1-39.

[11] J. -L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires, Dunod, Paris, 1969.

[12] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, Berlin/Heidelberg/New York, 1983.

[13] J. P. Raymond, Nonlinear boundary control of semilinear parabolic equation, Disc. and Cont. Dyn. Syst., Vol. 3 (1997), 341-370.

[14] J. P. Raymond, H. Zidani, Hamiltonian Pontryagin's principles for control problems governed by semilinear parabolic equations, to appear in Applied Math. Optim..

[15] J. Simon, Caractérisation d'un espace fonctionnel intervenant en contrôle optimal, Ann. Fac. Sciences Toulouse, Vol. 5 (1983), 149-169.