Design and analysis of an extended virtual element method

Jérôme Droniou

joint work with G. Manzini (IMATI CNR) and L. Yemm (Monash University).

IMAG, CNRS & University of Montpellier, France,
School of Mathematics, Monash University, Australia

https://imag.umontpellier.fr/~droniou/

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Reference for this presentation

The eXtended Virtual Element Method for elliptic problems with weakly singular solutions. J. Droniou, G. Manzini, and L. Yemm. Comput. Methods Appl. Mech. Engrg. 429, Paper No. 117129, 16p, 2024. doi: 10.1016/j.cma.2024.117129. https://arxiv.org/abs/2402.02902.

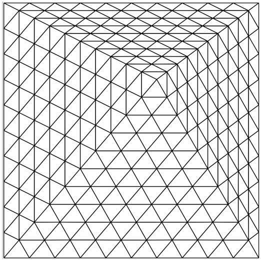
Outline

Principles of mesh-based schemes

Model problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Mesh-based methods: cut domain in small pieces (elements/cells), approximate the solution with piecewise polynomial functions on these pieces.



Error estimate requires smoothness

Typical error estimate: for a method of order $k \geq 1$,

$$\|u - u_h\|_{H^1} \leq Ch^k \|u\|_{H^{k+1}}$$

- Comes from the local approximation properties of polynomial functions.
- Requires smoothness of u.

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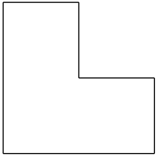
Failure of smoothness: even in non-challenging situation (any non-convex domain!).



$u \notin H^2$, so poor polynomial approximation...

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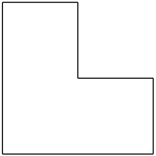

$u = r^{\frac{2}{3}} \sin(\frac{2}{3}(\theta - \frac{\pi}{2})) + u_r$ with $u_r \in H^2$.

Workaround

Extend the local polynomial spaces with the singular part of the solution.

Caveat: far from the singularity, the "singular" part is actually smooth, and too close to polynomials...

Refs.:

- FEM: [Melenk and Babuška, 1996, Babuška and Melenk, 1997, Belytschko and Black, 1999, Laborde et al., 2005, Chin et al., 2017] etc.

Outline

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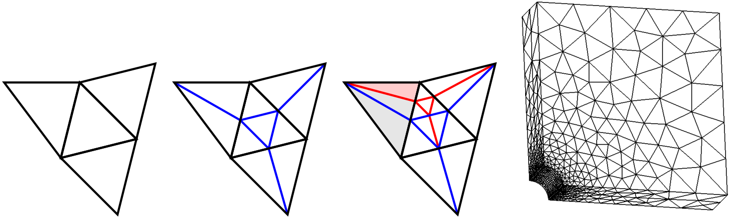
Polytopal methods and virtual elements? I



- Finite Element methods (FEM) have been the golden standard for decades, but their mesh lack flexibility with: local refinement, representation of complex geometries, etc.

- Polytopal methods are inherited from FEM but are applicable on meshes made of generic polygons/polyhedra.

  Refs.: [Beirão da Veiga et al., 2014, Beirão da Veiga et al., 2017, Di Pietro and Droniou, 2020] etc.

Polytopal methods and virtual elements? II

- Virtual Element Method is a polytopal method based on approximation spaces such that:
  - Virtual functions are not fully known,
  - Unisolvent degrees of freedom (DOFs) can be identified for the spaces,
  - Certain projections of virtual functions/gradients can be explicitly computed from the DOFs.

Standard and extended virtual element spaces

Local standard space: Fix $k \geq 1$, $l = \max(0, k-2)$ and define, for each element E,

$$V_{k,h}(E) := \left\{ v_h \in H^1(E) \,:\, \Delta v_h \in \mathbb{P}_l(E), \right.$$
$$\left. v_{h|\partial E} \in C^0(\partial E),\; v_{h|e} \in \mathbb{P}_k(e) \;\forall e \subset \partial E \right\}.$$

Contains polynomials: $\mathbb{P}_k(E) \subset V_{k,h}(E)$.

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Contains polynomials: $\mathbb{P}_k(E) \subset V_{k,h}(E)$.

Local extended space: with $\Psi \subset H^1(\Omega)$ the space of singularities of the solution,

$$V^{\Psi}_{k,h}(E) := \big\{ v_h \in H^1(E) \,:\, \Delta v_h \in \mathbb{P}_l(E) + \Delta\Psi_{|E},\\ v_{h|\partial E} \in C^0(\partial E),\ v_{h|e} \in \mathbb{P}_k(e) + \Psi_{|e}\ \forall e \subset \partial E \big\}.$$

Contains polynomials and singular function: $\mathbb{P}_k(E) + \Psi_{|E} \subset V^{\Psi}_{k,h}(E)$.

DOFs for extended virtual element spaces

$$V^{\Psi}_{k,h}(E) := \Big\{ v_h \in H^1(E) \,:\, \Delta v_h \in \mathbb{P}_l(E) + \Delta\Psi_{|E},$$
$$v_{h|\partial E} \in C^0(\partial E),\ v_{h|e} \in \mathbb{P}_k(e) + \Psi_{|e}\ \forall e \subset \partial E \Big\}.$$

Decomposition of boundary space: write $\mathbb{P}_k(e) + \Psi_{|e} = \mathbb{P}_k(e) \oplus \mathfrak{P}_e$ and set

$$\mathbb{P}^{\mathfrak{P}}_{k-2}(e) = \mathbb{P}_{k-2}(e) \oplus \mathfrak{P}_e.$$

Degrees of freedom: for $v_h \in V^{\Psi}_{k,h}(E)$,

- **(D1)** $v_h(x_V)$ for each vertex $V \in \partial E$;
- **(D2)** $\Pi^{\mathfrak{P}}_{k-2,e} v_h$, projection on $\mathbb{P}^{\mathfrak{P}}_{k-2}(e)$, for each edge $e \subset \partial E$;
- **(D3)** $\Pi^{\Delta}_{l,E} v_h$, projection on $\mathbb{P}_l(E) + \Delta\Psi_{|E}$.

$\mathbb{P}_{k-2}(e) + \Psi_{|e}$ *would be natural, but does not ensure unisolvence...*

Extended virtual element scheme I

Extended elliptic projector: $\Pi^{\nabla,\Psi}_{k,E} : V^{\Psi}_{k,h}(E) \to \mathbb{P}_k(E) + \Psi_{|E}$, computable from the DOFs, defined by

$$\int_E \nabla(\Pi^{\nabla,\Psi}_{k,E} v_h) \cdot \nabla q = \int_E \nabla v_h \cdot \nabla q \quad q \in \mathbb{P}_k(E) + \Psi_{|E},$$

$$\int_E \Pi^{\nabla,\Psi}_{k,E} v_h = \int_E v_h.$$

Remark: $\Pi^{\nabla,\Psi}_{k,E} v = v$ for all $v \in \mathbb{P}_k(E) + \Psi_{|E}$.

Global space: $V^{\Psi}_{k,h,0}$ obtained by gluing the local spaces, and

$$V^{\Psi}_{k,h,0} := \{v_h \in H^1_0(\Omega) \,:\, v_{h|E} \in V^{\Psi}_{k,h}(E) \quad \forall E \in \Omega_h\}.$$

Extended virtual element scheme II

Bilinear form: consistent part plus stabilisation.

$$a_h(u_h, v_h) := \sum_{E \in \Omega_h} a_E(u_h, v_h)$$

with

$$a_E(u_h, v_h) = \int_E \nabla \Pi^{\nabla,\Psi}_{k,E} u_h \cdot \nabla \Pi^{\nabla,\Psi}_{k,E} v_h + S_E(u_h, v_h)$$

and S_E such that

$$S_E(u_h, u_h) = h_E^{-2} \| \Pi^{\Delta}_{l,E}(u_h - \Pi^{\nabla,\Psi}_{k,E} u_h) \|^2_{L^2(E)} + h_E^{-1} \| u_h - \Pi^{\nabla,\Psi}_{k,E} u_h \|^2_{L^2(\partial E)}.$$

Scheme: Find $u_h \in V^{\Psi}_{k,h,0}$ such that

$$a_h(u_h, v_h) := \sum_{E \in \Omega_h} \int_E f \, \Pi^{\Delta}_{l,E} v_h \quad \forall v_h \in V^{\Psi}_{k,h,0}.$$

Outline

Convergence

Theorem (Discrete Energy Error)

Let $u = u^r + \psi$ solution to PDE with $\psi \in \Psi$ and $u^r \in H^{k+1}(\Omega_h)$. Under standard mesh regularity assumption:

$$\|u_h - \mathcal{I}_{k,h} u\|_{a,h} \lesssim h^k |u^r|_{H^{k+1}(\Omega_h)},$$

where $\|\cdot\|_{a,h}$ is the energy norm associated with a_h and $\mathcal{I}_{k,h} u = \widehat{\mathcal{I}}_{k,h} u^r + \psi$ with $\widehat{\mathcal{I}}_{k,h}$ standard VEM interpolant of u^r.

Consistency analysis – The Virtual Element Approach

$$\begin{aligned}
\sum_{E\in\Omega_h} \int_E -\Delta u\,\Pi^{\Delta}_{l,E} v_h - a_h(\mathcal{I}_{k,h}u, v_h) \\
&= \int_\Omega -\Delta u\, v_h + \mathcal{O}(h^k) - \sum_{E\in\Omega_h}\int_E \nabla(\Pi^{\nabla,\Psi}_{k,E}\mathcal{I}_{k,h}u)\cdot\nabla\cancel{\Pi^{\nabla,\Psi}_{k,E}} v_h + \mathrm{stab} \\
&= \int_\Omega \nabla(u - \Pi^{\nabla,\Psi}_{k,E}\mathcal{I}_{k,h}u)\cdot\nabla v_h + \mathcal{O}(h^k) + \mathrm{stab} \\
&\leq \|\nabla(u - \Pi^{\nabla,\Psi}_{k,E}\mathcal{I}_{k,h}u)\|_{L^2}\|\nabla v_h\|_{L^2} + \mathcal{O}(h^k) + \mathrm{stab}
\end{aligned}$$

To conclude:

- Approximation properties of $\mathcal{I}_{k,h}u$.
- Boudedness $\|\nabla v_h\|_{L^2} \leq C\|v_h\|_{a,h}$ (norm on DOFs of v_h).

Difficult for standard VEM, not known for extended VEM...

Refs: [Benvenuti et al., 2019, Artioli and Mascotto, 2021].

Consistency analysis – The Fully Discrete Approach I

Circumvents these issues by adopting a fully discrete approach: do not introduce virtual functions, express everything in terms of computable quantities.

Refs: Enriched Hybrid High-Order: [Yemm, 2022, Yemm, 2024].

Consistency analysis – The Fully Discrete Approach II

Manipulate source term: introduce elliptic projector, perform IBP, use continuity of fluxes:

$$
\begin{aligned}
-\sum_{E\in\Omega_h}\int_E \Delta u \Pi^{\Delta}_{l,E} v_h
&= -\sum_{E\in\Omega_h}\int_E \Delta u(\Pi^{\Delta}_{l,E} v_h - \Pi^{\nabla,\Psi}_{k,E} v_h) + \sum_{E\in\Omega_h}\int_E \nabla u\cdot\nabla \Pi^{\nabla,\Psi}_{k,E} v_h \\
&\quad + \sum_{E\in\Omega_h}\langle \nabla u\cdot\mathbf{n}, v_h - \Pi^{\nabla,\Psi}_{k,E} v_h\rangle_{\partial E}.
\end{aligned}
$$

Property of projectors: for all $z\in\mathbb{P}_k(E)+\Psi_{|E}$, we have $\Delta z\in\mathbb{P}_l(E)+\Delta\Psi_{|E}$ so

$$
-\int_E \Delta z(\cancel{\Pi^{\Delta}_{l,E}} v_h - \Pi^{\nabla,\Psi}_{k,E} v_h) + \langle \nabla z\cdot\mathbf{n}, v_h - \Pi^{\nabla,\Psi}_{k,E} v_h\rangle_{\partial E}
$$

$$
\overset{\text{IBP}}{=} \int_E \nabla z\cdot\nabla(v_h - \Pi^{\nabla,\Psi}_{k,E} v_h) \overset{\text{def. } \Pi^{\nabla,\Psi}_{k,E}}{=} 0.
$$

Consistency analysis – The Fully Discrete Approach III

Eliminate singular part of u: recalling that $u = u^r + \psi$, take $z = q_E + \psi \in \mathbb{P}_k(E) + \Psi_{|E}$ and subtract:

$$-\sum_{E\in\Omega_h}\int_E \Delta u \Pi^{\Delta}_{l,E} v_h = \sum_{E\in\Omega_h}\int_E \nabla u\cdot\nabla\Pi^{\nabla,\Psi}_{k,E} v_h$$
$$-\sum_{E\in\Omega_h}\int_E \Delta(u^r - q_E)(\Pi^{\Delta}_{l,E} v_h - \Pi^{\nabla,\Psi}_{k,E} v_h)$$
$$+\sum_{E\in\Omega_h}\int_{\partial E} \nabla(u^r - q_E)\cdot\mathbf{n}(v_h - \Pi^{\nabla,\Psi}_{k,E} v_h).$$

- Last two terms are $\mathcal{O}(h^k)$ by regularity of u^r and polynomial approximation.
- First term combines with consistent term in $a_h(\mathcal{I}_{k,h}u, v_h)$:

$$\int_E \nabla(\Pi^{\nabla,\Psi}_{k,E}\mathcal{I}_{k,h}u)\cdot\nabla\Pi^{\nabla,\Psi}_{k,E} v_h = \int_E \nabla\mathcal{I}_{k,h}u\cdot\nabla\Pi^{\nabla,\Psi}_{k,E} v_h$$

to create $u - \mathcal{I}_{k,h}u = u^r - \widehat{\mathcal{I}}_{k,h}u^r$ (since $\psi = \mathcal{I}_{k,h}\psi$), which is $\mathcal{O}(h^k)$.

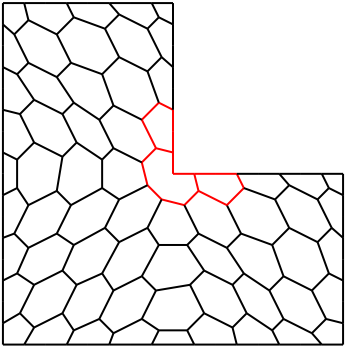
Outline

L-shaped domain $\Omega = (-1,1)^2 \backslash [0,1)^2$

Solution: take $u = \sin(\pi x)\sin(\pi y) + \psi$ with singularity at re-entrant corner

$$\psi(r,\theta) = r^{\frac{2}{3}} \sin(\frac{2}{3}(\theta - \frac{\pi}{2})).$$

Mesh:



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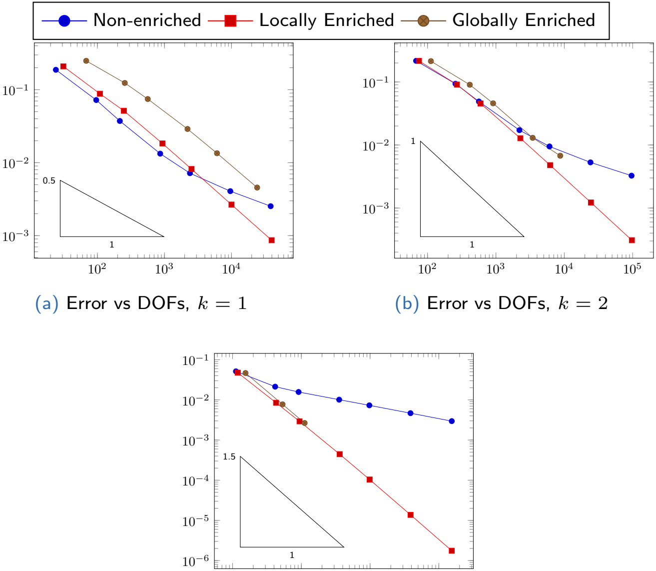


(a) Error vs DOFs, $k = 1$

(b) Error vs DOFs, $k = 2$

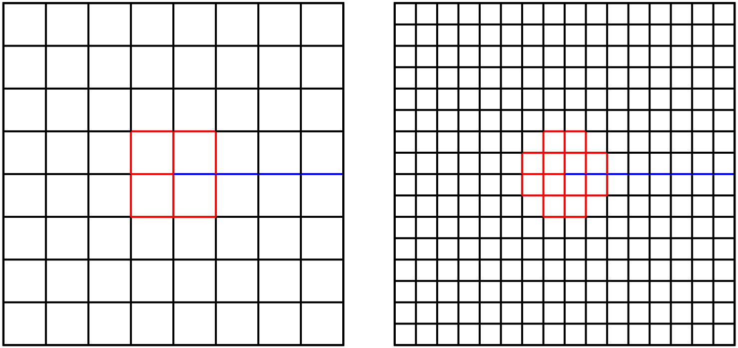
(c) Error vs DOFs, $k = 3$

Fractured domain $\Omega = (0,1)^2 \backslash ([0,1) \times \{0\})$

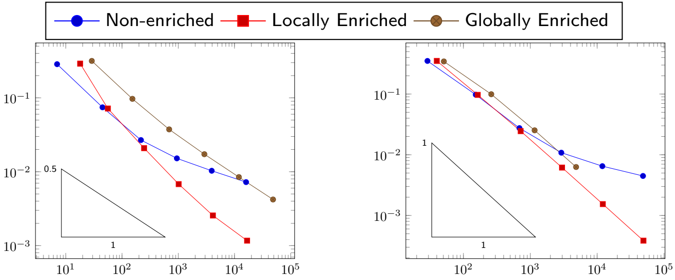
Solution: take $u = \sin(\pi x)\sin(\pi y) + \psi$ with singularity at re-entrant corner

$$\psi(r,\theta) = r^{\frac{1}{2}} \sin(\frac{1}{2}\theta).$$

Mesh:



Fractured domain $\Omega = (0,1)^2 \backslash ([0,1) \times \{0\})$



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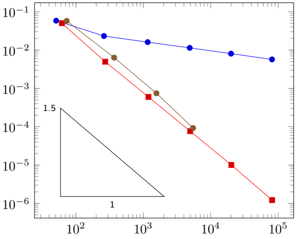

(c) Error vs DOFs, $k = 3$

Conclusion

- Polytopal method, benefitting from the flexibility of general polygonal/polyhedral meshes.
- Extended: includes a singularity space in the design, to better reproduce singular solutions.
- Recovers optimal convergence for problems with re-entrant corners and cracks. But also improves problems with highly oscillatory solutions (PhD L. Yemm).
- First complete analysis, circumvents the issues of a virtual-element based analysis by using a fully discrete approach (only based on DOFs, not virtual functions).

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Thank you for your attention!

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