Next generation methods for the simulation of geophysical flows (and more...)

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Outline

1. Some models of interest

2. Polytopal methods

3. Hybrid High-Order methods
   - Design and convergence for Darcy flow
   - Miscible flow
   - Incompressible stationary MHD

4. Cohomology-preserving methods
   - Revisiting MHD through Magnetostatics
   - The de Rham complex

5. Conclusion and perspectives
Enhanced oil recovery

Injection well

Production well

Solvent

{Oil, solvent}
Model for enhanced oil recovery

\[ \begin{cases} \text{div } \mathbf{u} = q^+ - q^- := q \\ \mathbf{u} = -\frac{K}{\mu(c)} \text{ grad } p \end{cases} \]

\[ \phi \frac{\partial c}{\partial t} + \text{div}(\mathbf{u}c - D(x, \mathbf{u}) \text{ grad } c) + q^- c = q^+ \]

**Unknowns**
- \( p(x, t) \) - pressure of the mixture
- \( \mathbf{u}(x, t) \) - Darcy velocity
- \( c(x, t) \) - concentration of the injected solvent

**Parameters**
- \( K(x) \) - permeability tensor
- \( \phi(x) \) - porosity

**Features**: non-linear, coupled, convection-dominated, anisotropic and heterogeneous.
Incompressible magnetohydrodynamics

\[
- \frac{\partial \mathbf{u}}{\partial t} - \nu_k \Delta \mathbf{u} + (\mathbf{u} \cdot \text{grad}) \mathbf{u} + \text{grad} \frac{p}{\rho} - (\text{curl} \mathbf{b}) \times \mathbf{b} = \mathbf{f},
\]

\[
\frac{\partial \mathbf{b}}{\partial t} + \nu_m \text{curl}(\text{curl} \mathbf{b}) - \text{curl}(\mathbf{u} \times \mathbf{b}) = 0,
\]

\[
\text{div} \mathbf{u} = \text{div} \mathbf{b} = 0,
\]

**Unknowns**
- \( \mathbf{u} \) - fluid velocity
- \( p \) - fluid pressure
- \( \mathbf{b} \) - magnetic field

**Parameters**
- \( \mathbf{f} \) - external body force
- \( \rho \) - fluid density
- \( \nu_k, \nu_m \) - kinematic and magnetic diffusivity

**Features**: non-linear, coupled, incomplete differential operators and convection forces.
Finite elements

- Approximate using **global functions** on the domain that are **locally polynomials**.
Finite elements

- Approximate using **global functions** on the domain that are **locally polynomials**.
- Require specific mesh geometries, mostly tetrahedra or hexahedras, to glue local polynomial functions into global functions.

*Unless using specific “tricks”, e.g. for cut meshes.*
Shortcomings of classical Finite Elements

- Limitations of conforming meshes with standard elements
  - local refinement requires to trade mesh size for mesh quality
  - complex geometries may require a large number of elements
  - the element shape cannot be adapted to the solution

- Need for (global) basis functions
  - significant increase of DOFs on hexahedral elements
Meshes for complex problems
What is a polytopal method?

- A discretisation method for PDEs that can be applied to meshes with generic polytopal elements (polygons in 2D, polyhedra in 3D).
- Seamlessly handles non-conformity (“hanging nodes”).

- Sometimes also arbitrary order of accuracy.
Some polytopal methods

- Discontinuous Galerkin (actually started on triangles/tetrahedra), [Arnold, 1982, Brezzi et al., 2000, Di Pietro and Ern, 2010]: 70’s, then 2012+.
- Mixed Finite Volumes, Hybrid Finite Volumes (SUSHI) and Mimetic Finite Differences [D. et al., 2010, Beirão da Veiga et al., 2014]: 2004+.

*General literature review in the preface of [Di Pietro and D., 2020].*
Model problem: Darcy flow in pressure formulation

- Given $\kappa$ constant symmetric positive definite tensor and $f \in L^2(\Omega)$, the Darcy problem reads:

  Find the velocity $u : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

  \[
  \kappa^{-1}u - \nabla p = 0 \quad \text{in } \Omega, \quad \text{(Darcy’s law)}
  \]

  \[
  - \nabla \cdot u = f \quad \text{in } \Omega, \quad \text{(mass conservation)}
  \]

  \[
  p = 0 \quad \text{on } \partial \Omega \quad \text{(boundary condition)}
  \]
Model problem: Darcy flow in pressure formulation

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- \text{div } u = f \quad \text{in } \Omega, \\
p = 0 \quad \text{on } \partial \Omega
\]

(Darcy’s law)

(mass conservation)

(boundary condition)

- **Primal formulation**: eliminate velocity.

\[
- \text{div}(\kappa \nabla p) = f \quad \text{in } \Omega, \\
p = 0 \quad \text{on } \partial \Omega.
\]
Model problem: Darcy flow in pressure formulation

- Given $\kappa$ constant symmetric positive definite tensor and $f \in L^2(\Omega)$, the Darcy problem reads:

Find the velocity $u : \Omega \rightarrow \mathbb{R}^3$ and pressure $p : \Omega \rightarrow \mathbb{R}$ s.t.

$$\kappa^{-1}u - \text{grad } p = 0 \quad \text{in } \Omega, \quad \text{(Darcy's law)}$$

$$- \text{div } u = f \quad \text{in } \Omega, \quad \text{(mass conservation)}$$

$$p = 0 \quad \text{on } \partial\Omega \quad \text{(boundary condition)}$$

- **Primal formulation:** eliminate velocity.

$$- \text{div}(\kappa \text{grad } p) = f \quad \text{in } \Omega,$$

$$p = 0 \quad \text{on } \partial\Omega.$$  

- **Weak formulation:** Find $p \in H^1_0(\Omega)$ s.t.

$$\int_{\Omega} \kappa \text{grad } p \cdot \text{grad } q = \int_{\Omega} f q \quad \forall q \in H^1_0(\Omega).$$
An inspiring remark

- $T$: mesh element (cell), $\mathcal{F}_T$ set of faces $F$ of $T$.
- $\mathcal{P}^k(X) =$ polynomials of degree $\leq k$ on $X = T, F$.
- $\pi^{0,k}_X$: $L^2$-projector on $\mathcal{P}^k(X)$, satisfies: for $g \in L^2(X)$,
  \[ \int_X g q_k = \int_X (\pi^{0,k}_X g) q_k \quad \forall q \in \mathcal{P}^k(X). \]
- $\pi^{1,k+1}_{k,T}$: (oblique) elliptic projector, defined by: for $g \in H^1(T)$,
  \[ \int_T \kappa \operatorname{grad}(\pi^{1,k+1}_{k,T} g) \cdot \operatorname{grad} q_{k+1} = \int_T \operatorname{grad} g \cdot \operatorname{grad} q_{k+1} \quad \forall q_{k+1} \in \mathcal{P}^{k+1}(T), \]
  \[ \int_T \pi^{1,k+1}_{k,T} g = \int_T g. \]
An inspiring remark

\[ \int_X g q_k = \int_X (\pi^{0,k}_X g) q_k \]
An inspiring remark

\[ \int_X g q_k = \int_X (\pi^{0,k}_X g) q_k \]

- For \( p \in H^1(T) \) and \( q_{k+1} \in \mathcal{P}^{k+1}(T) \):

\[
\int_T \kappa \text{grad}(\pi^{1,k+1}_{\kappa,T} p) \cdot \text{grad} q_{k+1} = \int_T \kappa \text{grad} p \cdot \text{grad} q_{k+1}
\]

\[ = - \int_T p \text{div}(\kappa \text{grad} q_{k+1}) + \sum_{F \in \mathcal{T}_T} \int_F p(\kappa \text{grad} q_{k+1} \cdot n_{TF}). \]
An inspiring remark

\[
\int_X g q_k = \int_X (\pi_X^0 g) q_k
\]

- For \( p \in H^1(T) \) and \( q_{k+1} \in \mathcal{P}^{k+1}(T) \):

\[
\int_T \kappa \text{grad}(\pi_{k,T}^{1,k+1} p) \cdot \text{grad} q_{k+1} = \int_T \kappa \text{grad} p \cdot \text{grad} q_{k+1}
\]

\[
= - \int_T \pi_T^{0,k} p \text{div}(\kappa \text{grad} q_{k+1}) + \sum_{F \in \mathcal{F}_T} \int_F \pi_F^{0,k} p (\kappa \text{grad} q_{k+1} \cdot \mathbf{n}_{TF}) \quad \epsilon \mathcal{P}^k(F)
\]
An inspiring remark

$$\int_X g q_k = \int_X (\pi^{0,k}_X g) q_k$$

- For $p \in H^1(T)$ and $q_{k+1} \in \mathcal{P}^{k+1}(T)$:

$$\int_T \kappa \text{grad} (\pi^{1,k+1}_{k,T} p) \cdot \text{grad} q_{k+1} = \int_T \kappa \text{grad} p \cdot \text{grad} q_{k+1}$$

$$= - \int_T \pi^{0,k}_T p \underbrace{\text{div}(\kappa \text{grad} q_{k+1})}_{\in \mathcal{P}^k(T)} + \sum_{F \in \mathcal{F}_T} \int_F \pi^{0,k}_F p \underbrace{(\kappa \text{grad} q_{k+1} \cdot \mathbf{n}_{TF})}_{\in \mathcal{P}^k(F)}.$$

$$\pi^{1,k+1}_{k,T} p$$ computable from $\pi^{0,k}_T p$ and $(\pi^{0,k}_F p)_{F \in \mathcal{F}_T}$. 
Design: Local space and interpolator

For \( k \geq 0 \) and \( T \in \mathcal{T}_h \), define the local HHO space

\[
U^k_T := \{ v_T = (v_T, (v_F)_{F \in F_T}) : v_T \in P^k(T) \text{ and } v_F \in P^k(F) \text{ for all } F \in F_T \}
\]

The local interpolator \( I^k_T : H^1(T) \to U^k_T \) is s.t., for all \( v \in H^1(T) \),

\[
I^k_T v := (\pi^0_T v, (\pi^0_F v)_F)_{F \in F_T}
\]
Design and convergence for Darcy flow

Design: Potential reconstruction

- Let $T \in \mathcal{T}_h$, the potential reconstruction $r_{T}^{k+1} : \bar{U}_{T}^{k} \rightarrow \mathcal{P}^{k+1}(T)$ is s.t., for all $v_{T} \in \bar{U}_{T}^{k}$ and $q_{k+1} \in \mathcal{P}^{k+1}(T)$,

$$
\int_{T} \kappa \text{grad} r_{T}^{k+1} v_{T} \cdot \text{grad} q_{k+1} = - \int_{T} v_{T} \left( \text{div} \kappa \text{grad} q_{k+1} \right) + \sum_{F \in \mathcal{F}_{T}} \int_{F} v_{F} \left( \kappa \text{grad} q_{k+1} \cdot n_{TF} \right),
$$

$$
\int_{T} r_{T}^{k+1} v_{T} = \int_{T} v_{T}.
$$

- By construction:

$$
r_{T}^{k+1} (I_{T}^{k} v) = \pi_{k,T}^{1,k+1} v \quad \forall v \in H^{1}(T).
$$
Design: Local bilinear form

- Bilinear form in weak formulation:
  \[
  \int_{\Omega} \kappa \, \text{grad} \, p \cdot \text{grad} \, q = \sum_{T \in T_h} \int_{T} \kappa \, \text{grad} \, p \cdot \text{grad} \, q.
  \]
Design: Local bilinear form

- Bilinear form in weak formulation:
  \[ \int_{\Omega} \kappa \text{grad} p \cdot \text{grad} q = \sum_{T \in \mathcal{T}_h} \int_{T} \kappa \text{grad} p \cdot \text{grad} q. \]

- Approximate local term:
  \[ \int_{T} \kappa \text{grad} p \cdot \text{grad} q \]
  \[ \sim a_T (p_T, q_T) := \int_{T} \kappa \text{grad}(r_{T}^{k+1} p_T) \cdot \text{grad}(r_{T}^{k+1} q_T) + s_T (p_T, q_T). \]
Design: Local bilinear form

• Approximate local term:

\[
\int_T \kappa \text{grad} \ p \cdot \text{grad} \ q \\
\sim a_T(p_T, q_T) := \int_T \kappa \text{grad}(r_T^{k+1}p_T) \cdot \text{grad}(r_T^{k+1}q_T) + s_T(p_T, q_T).
\]

• Stabilisation term \( s_T : U_T^k \times U_T^k \rightarrow \mathbb{R} \):

1. Symmetric semi-definite positive,
2. Polynomally consistent:

\[
s_T(I_T^k p_{k+1}, \cdot) = 0 \quad \forall p_{k+1} \in \mathcal{P}^{k+1}(T),
\]
3. Stable: in particular,

\[
a_T(p_T, p_T) = 0 \iff p_T = I_T^k C \quad \text{for some } C \in \mathbb{R}.
\]

Many possible choices, not all equally good [D. and Yemm, 2022b].
Design: Discrete problem

- **Global space** patching local ones and enforcing boundary conditions:

  \[ U^k_{h,0} := \left\{ v_h = ((v_T)_T \in \mathcal{T}_h, (v_F)_F \in \mathcal{F}_h) : v_T \in \mathcal{P}^k(T) \quad \forall T \in \mathcal{T}_h, \right. \]

  \[ v_F \in \mathcal{P}^k(F) \quad \forall F \in \mathcal{F}_h, \quad v_F = 0 \quad \forall F \subset \partial \Omega \left\} . \]

- **Global bilinear form** assembling local ones:

  \[ a_h(v_h, w_h) := \sum_{T \in \mathcal{T}_h} a_T(v_T, w_T). \]

- **HHO scheme**: find \( p_h \in U^k_{h,0} \) s.t.

  \[ a_h(p_h, q_h) = \sum_{T \in \mathcal{T}_h} \int_T f q_T \quad \forall q_h \in U^k_{h,0}. \]
Convergence analysis

- Based on optimal approximation properties of $\pi_{k,T}^{1,k+1}$.

- Errors in energy norm:
  \[
  \| p_{\text{h}} - I_h^k p \|_{a,h} = O(h^{k+1})
  \]
  where $\| v_{\text{h}} \|_{a,h} = a_h (v_{\text{h}}, v_{\text{h}})^{1/2}$ and $I_h^k p = ((\pi_T^{0,k} p)_T \in T_h, \pi_F^{0,k} p)_F \in F_h)$ global interpolate of the exact solution $p$.

- Errors in $L^2$-norm (under elliptic regularity of the problem):
  \[
  \| r_{\text{h}}^{k+1} p_{\text{h}} - p \|_{L^2(\Omega)} = O(h^{k+2})
  \]
  where $(r_{\text{h}}^{k+1} p_{\text{h}})_T = r_T^{k+1} p_T$.
Numerical results: error vs. $h$
Numerical results: error vs. $h$

Energy

$L^2$ norm

Mesh Reg. para.

1 $10^7$

2 $377$

3 $1.7E+3$

4 $2.6E+5$

5 $1.8E+4$
Some models of interest

Polytopal methods

Hybrid High-Order methods

Cohomology-preserving methods

Conclusion and perspectives

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Design and convergence for Darcy flow

**Numerical results: error vs. $h$**

![Graphs showing error vs. $h$ for different mesh regularity parameters](image)

<table>
<thead>
<tr>
<th>Mesh Reg. para.</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>107</td>
</tr>
<tr>
<td>2</td>
<td>377</td>
</tr>
<tr>
<td>3</td>
<td>1.7E+3</td>
</tr>
<tr>
<td>4</td>
<td>2.6E+5</td>
</tr>
<tr>
<td>5</td>
<td>1.8E+4</td>
</tr>
</tbody>
</table>
Model

From [Anderson and D., 2018].

\[
\begin{align*}
\text{div } u &= q^+ - q^- := q \\
u &= -\frac{K}{\mu(c)} \text{ grad } p
\end{align*}
\]

\[
\phi \frac{\partial c}{\partial t} + \text{div}(uc - D(x, u) \text{ grad } c) + q^- c = q^+
\]

- \(p(x, t)\) - pressure of the mixture
- \(u(x, t)\) - Darcy velocity
- \(c(x, t)\) - concentration of the injected solvent
Model

From [Anderson and D., 2018].

\[
\begin{aligned}
- \text{div} \left( \frac{K}{\mu(c)} \text{grad} p \right) &= q^+ - q^- := q \\
\mu(c) \frac{\partial c}{\partial t} + \text{div}(uc) - \text{div}(D(x,u) \text{grad} c) + q^- c &= q^+ \\
\end{aligned}
\]

- \( p(x, t) \) - pressure of the mixture
- \( u(x, t) \) - Darcy velocity
- \( c(x, t) \) - concentration of the injected solvent
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Miscible flow

Numerical results

(a) Surface plot at $t = 3$ years

(b) Contour plot at $t = 3$ years

(c) Surface plot at $t = 10$ years

(d) Contour plot at $t = 10$ years

Figure: Concentration of invading solvent, $k = 1$ and $\Delta t = 18j$, discontinuous permeability.
Numerical results: recovery vs. $h$

Recovery: $\int_{\Omega} \phi c_h(T)$ for $T = 10y$. 

![Graph showing recovery vs. mesh size]
Numerical results: recovery vs. $h$

Recovery: $\int_\Omega \phi c_h(T)$ for $T = 10y$. 

![Graph showing recovery vs. mesh size and percentage of domain recovered for different values of $k$.]
Numerical results: recovery vs. $h$

Recovery: $\int_{\Omega} \phi c_h(T)$ for $T = 10y$. 

Graph showing the percentage of domain recovered versus mesh size with lines for different values of $k$. The graph includes data points for $k = 0$, $k = 1$, and $k = 2$. The percentage of domain recovered decreases as the mesh size increases for $k = 2$.
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Miscible flow

Numerical results: computational cost

![Graphs showing computational cost for Triangular and Cartesian meshes for different polynomial degrees (k = 0, 1, 2, 3).]
Numerical results: computational cost

A little bit of higher order approximation is not very expensive, but can make a huge difference.
Model

From [D. and Yemm, 2022a], based on [Botti et al., 2019] (Navier–Stokes).

\[-\nu_k \Delta u + (u \cdot \text{grad})u + \text{grad } q - (\text{curl } b) \times b = f,\]
\[\nu_m \text{curl(curl } b) - \text{curl(} u \times b) = 0,\]
\[\text{div } u = \text{div } b = 0,\]
Some models of interest

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Incompressible stationary MHD

Model

From [D. and Yemm, 2022a], based on [Botti et al., 2019] (Navier–Stokes).

\[-\nu_k \Delta u + (u \cdot \text{grad})u + \text{grad} q - (\text{curl} b) \times b = f,\]
\[\nu_m \text{curl} (\text{curl} b) - \text{curl} (u \times b) = 0,\]
\[\text{div} u = \text{div} b = 0,\]

... and with a little bit of differential calculus and Lagrange multipliers...

\[-\nu_k \Delta u + (u \cdot \text{grad})u - (b \cdot \text{grad})b + \text{grad} q = f,\]
\[-\nu_m \Delta b + (u \cdot \text{grad})b - (b \cdot \text{grad})u + \text{grad} r = g,\]
\[\text{div} u = \text{div} b = 0.\]
Convergence results

- Small data and smooth solutions:
  
  Optimal convergence rates $O(h^{k+1})$ in energy norm for $u, b$ and in $L^2$-norm for $r, q$.

- Any data and solution:
  
  Convergence of the scheme by compactness techniques.

*Applicable in real-world settings...*
Numerical results: tetrahedral meshes ($\nu_k = \nu_m = 0.1$)

(a) $k = 0$
(b) $k = 1$
(c) $k = 2$
Numerical results: Voronoi meshes \((\nu_k = \nu_m = 0.1)\)
The magnetostatics problem

For $\mu > 0$ and $J \in \text{curl} \; H(\text{curl}; \Omega)$, the magnetostatics problem reads:

Find the magnetic field $H : \Omega \to \mathbb{R}^3$ and vector potential $A : \Omega \to \mathbb{R}^3$ s.t.

\[
\begin{align*}
\mu H - \text{curl} \; A &= 0 \quad \text{in } \Omega, \quad \text{(vector potential)} \\
\text{curl} \; H &= J \quad \text{in } \Omega, \quad \text{(Ampère’s law)} \\
\text{div} \; A &= 0 \quad \text{in } \Omega, \quad \text{(Coulomb’s gauge)} \\
A \times n &= 0 \quad \text{on } \partial \Omega \quad \text{(boundary condition)}
\end{align*}
\]
The magnetostatics problem

- For $\mu > 0$ and $J \in \text{curl} H(\text{curl}; \Omega)$, the magnetostatics problem reads:
  
  Find the magnetic field $H : \Omega \rightarrow \mathbb{R}^3$ and vector potential $A : \Omega \rightarrow \mathbb{R}^3$ s.t.
  
  $\mu H - \text{curl} A = 0 \quad \text{in } \Omega$, \quad (vector potential)
  
  $\text{curl} H = J \quad \text{in } \Omega$, \quad (Ampère’s law)
  
  $\text{div} A = 0 \quad \text{in } \Omega$, \quad (Coulomb’s gauge)
  
  $A \times n = 0 \quad \text{on } \partial \Omega$ \quad (boundary condition)

- Weak formulation: Find $(H, A) \in H(\text{curl}; \Omega) \times H(\text{div}; \Omega)$ s.t.
  
  $\int_{\Omega} \mu H \cdot \tau - \int_{\Omega} A \cdot \text{curl} \tau = 0 \quad \forall \tau \in H(\text{curl}; \Omega),$
  
  $\int_{\Omega} \text{curl} H \cdot v + \int_{\Omega} \text{div} A \text{ div } v = \int_{\Omega} J \cdot v \quad \forall v \in H(\text{div}; \Omega)$

  with

  $H(\text{curl}; \Omega) := \{ v \in L^2(\Omega) : \text{curl } v \in L^2(\Omega) \}$,

  $H(\text{div}; \Omega) := \{ w \in L^2(\Omega) : \text{div } w \in L^2(\Omega) \}$
The magnetostatics problem

- **Weak formulation**: Find \((\mathbf{H}, \mathbf{A}) \in H(\text{curl}; \Omega) \times H(\text{div}; \Omega)\) s.t.

\[
\int_{\Omega} \mu \mathbf{H} \cdot \mathbf{\tau} - \int_{\Omega} \mathbf{A} \cdot \text{curl} \mathbf{\tau} = 0 \quad \forall \mathbf{\tau} \in H(\text{curl}; \Omega),
\]

\[
\int_{\Omega} \text{curl} \mathbf{H} \cdot \mathbf{v} + \int_{\Omega} \text{div} \mathbf{A} \text{ div} \mathbf{v} = \int_{\Omega} \mathbf{J} \cdot \mathbf{v} \quad \forall \mathbf{v} \in H(\text{div}; \Omega)
\]

- **Stability (inf–sup) analysis**:
  - Make \((\mathbf{\tau}, \mathbf{v}) = (\mathbf{H}, \mathbf{A}) \leadsto\) bound on \(\mathbf{H}\) and \(\text{div} \mathbf{A}\).
The magnetostatics problem

- **Weak formulation**: Find \((H, A) \in H(\text{curl}; \Omega) \times H(\text{div}; \Omega)\) s.t.

  \[
  \int_{\Omega} \mu H \cdot \tau - \int_{\Omega} A \cdot \text{curl} \tau = 0 \quad \forall \tau \in H(\text{curl}; \Omega),
  \]

  \[
  \int_{\Omega} \text{curl} H \cdot v + \int_{\Omega} \text{div} A \cdot \text{div} v = \int_{\Omega} J \cdot v \quad \forall v \in H(\text{div}; \Omega)
  \]

- **Stability** (inf–sup) analysis:
  - Make \((\tau, v) = (H, A) \mapsto\) bound on \(H\) and \(\text{div} A\).
  - Make \((\tau, v) = (0, \text{curl} H) \mapsto\) bound on \(\text{curl} H\).
The magnetostatics problem

- **Weak formulation:** Find \((H, A) \in H(\text{curl}; \Omega) \times H(\text{div}; \Omega)\) s.t.

\[
\int_{\Omega} \mu H \cdot \tau - \int_{\Omega} A \cdot \text{curl} \tau = 0 \quad \forall \tau \in H(\text{curl}; \Omega),
\]
\[
\int_{\Omega} \text{curl} H \cdot v + \int_{\Omega} \text{div} A \cdot \text{div} v = \int_{\Omega} J \cdot v \quad \forall v \in H(\text{div}; \Omega)
\]

- **Stability (inf–sup) analysis:**
  
  - Make \((\tau, v) = (H, A) \leadsto\) bound on \(H\) and \(\text{div} A\).
  - Make \((\tau, v) = (0, \text{curl} H) \leadsto\) bound on \(\text{curl} H\).
  - Write \(A = A^* + A^\perp \in \text{Ker} \text{div} \oplus (\text{Ker} \text{div})^\perp\).
The magnetostatics problem

- **Weak formulation:** Find \((H, A) \in H(\text{curl}; \Omega) \times H(\text{div}; \Omega)\) s.t.

  \[
  \int_{\Omega} \mu H \cdot \tau - \int_{\Omega} A \cdot \text{curl} \tau = 0 \quad \forall \tau \in H(\text{curl}; \Omega),
  \]

  \[
  \int_{\Omega} \text{curl} H \cdot v + \int_{\Omega} \text{div} A \\text{div} v = \int_{\Omega} J \cdot v \quad \forall v \in H(\text{div}; \Omega)
  \]

- **Stability (inf–sup) analysis:**
  - Make \((\tau, v) = (H, A) \mapsto \text{bound on } H\) and \(\text{div} A\).
  - Make \((\tau, v) = (0, \text{curl} H) \mapsto \text{bound on } \text{curl} H\).
  - Write \(A = A^\perp + A^\parallel \in \ker \text{div} \oplus (\ker \text{div})^\perp\).
  - Bound on \(A^\parallel\) through bound on \(\text{div} A = \text{div} A^\parallel\).
The magnetostatics problem

- **Weak formulation**: Find \((H, A) \in H(\text{curl}; \Omega) \times H(\text{div}; \Omega)\) s.t.

\[
\int_{\Omega} \mu H \cdot \tau - \int_{\Omega} A \cdot \text{curl} \tau = 0 \quad \forall \tau \in H(\text{curl}; \Omega),
\]
\[
\int_{\Omega} \text{curl} H \cdot v + \int_{\Omega} \text{div} A \text{ div} v = \int_{\Omega} J \cdot v \quad \forall v \in H(\text{div}; \Omega)
\]

- **Stability** (inf–sup) analysis:
  - Make \((\tau, v) = (H, A) \leadsto\) bound on \(H\) and \(\text{div} A\).
  - Make \((\tau, v) = (0, \text{curl} H) \leadsto\) bound on \(\text{curl} H\).
  - Write \(A = A^* + A^\perp \in \text{Ker div} \oplus (\text{Ker div})^\perp\).
  - Bound on \(A^\perp\) through bound on \(\text{div} A = \text{div} A^\perp\).
  - Bound on \(A^*\): requires
    \[
    \text{Im curl} = \text{Ker div}
    \]
    to write \(A^* = -\text{curl} \tau\) with \(\tau \in (\text{Ker curl})^\perp\), and use \((\tau, 0)\) as test function.
A unified tool for well-posedness

\[ \mathbb{R} \xrightarrow{\text{grad}} H^1(\Omega) \xrightarrow{\text{curl}} H(\text{curl}; \Omega) \xrightarrow{\text{div}} H(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\} \]

We have key properties depending on the topology of \( \Omega \):

- \( \Omega \) connected \( (b_0 = 1) \) \( \Rightarrow \) \( \text{Ker} \text{grad} = \mathbb{R} \),
- \( \text{Im grad} \subset \text{Ker curl} \),
- \( \text{Im curl} \subset \text{Ker div} \),
- \( \Omega \subset \mathbb{R}^3 \) \( (b_3 = 0) \) \( \Rightarrow \) \( \text{Im div} = L^2(\Omega) \)
The de Rham complex

A unified tool for well-posedness

\[ \mathbb{R} \xrightarrow{\text{grad}} H^1(\Omega) \xrightarrow{\text{curl}} H(\text{curl}; \Omega) \xrightarrow{\text{div}} H(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\} \]

- We have key properties depending on the topology of \( \Omega \):
  - \( \Omega \) connected (\( b_0 = 1 \)) \( \implies \) \( \text{Ker grad} = \mathbb{R} \),
  - no “tunnels” crossing \( \Omega \) (\( b_1 = 0 \)) \( \implies \) \( \text{Im grad} = \text{Ker curl} \),
  - no “voids” contained in \( \Omega \) (\( b_2 = 0 \)) \( \implies \) \( \text{Im curl} = \text{Ker div} \),
  - \( \Omega \subseteq \mathbb{R}^3 \) (\( b_3 = 0 \)) \( \implies \) \( \text{Im div} = L^2(\Omega) \)
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  - \( \Omega \subset \mathbb{R}^3 \) (\( b_3 = 0 \)) \( \Rightarrow \) \( \text{Im div} = L^2(\Omega) \)

- When \( b_1 \neq 0 \) or \( b_2 \neq 0 \), de Rham’s cohomology characterizes \( \text{Ker curl}/\text{Im grad} \) and \( \text{Ker div}/\text{Im curl} \)

- Key consequences are Hodge decompositions and Poincaré inequalities
A unified tool for well-posedness

\[ \mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}; \Omega) \xrightarrow{\text{curl}} H(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\} \]

- We have key properties depending on the topology of $\Omega$:
  
  \begin{align*}
  \Omega \text{ connected (} b_0 = 1 \text{)} & \implies \text{Ker grad} = \mathbb{R}, \\
  \text{no “tunnels” crossing } \Omega \text{ (} b_1 = 0 \text{)} & \implies \text{Im grad} = \text{Ker curl}, \\
  \text{no “voids” contained in } \Omega \text{ (} b_2 = 0 \text{)} & \implies \text{Im curl} = \text{Ker div}, \\
  \Omega \subset \mathbb{R}^3 \text{ (} b_3 = 0 \text{)} & \implies \text{Im div} = L^2(\Omega)
  \end{align*}

- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham’s cohomology characterizes

  \[
  \text{Ker curl}/\text{Im grad} \quad \text{and} \quad \text{Ker div}/\text{Im curl}
  \]

- Key consequences are Hodge decompositions and Poincaré inequalities

- Emulating these properties is key for stable discretizations
The discrete de Rham (DDR) approach I

- **Key idea:** replace both spaces and operators by discrete counterparts:

\[
\mathbb{R} \xrightarrow{I^k_{\text{grad},h}} X^k_{\text{grad},h} \xrightarrow{G^k_h} X^k_{\text{curl},h} \xrightarrow{C^k_h} X^k_{\text{div},h} \xrightarrow{D^k_h} P^k(\mathcal{T}_h) \rightarrow 0 \rightarrow \{0\}
\]

- Support of **polyhedral meshes (CW complexes) and high-order**
- **Key exactness and consistency properties proved at the discrete level**
- Several strategies to **reduce the number of unknowns** on general shapes
The discrete de Rham (DDR) approach II

- DDR spaces are spanned by vectors of polynomials
- Polynomial components enable consistent reconstructions of
  - vector calculus operators
  - the corresponding scalar or vector potentials
- These reconstructions emulate integration by parts (Stokes) formulas
Works on DDR

- Introduction of DDR [Di Pietro et al., 2020]
- Analytical properties [Di Pietro and D., 2021a]
- Application to magnetostatics [Di Pietro and D., 2021b]
- Bridges with VEM [Beirão da Veiga et al., 2021]
- Serendipity technique (reduction DOFs) [Di Pietro and D., 2022b]
- Cohomology analysis: ongoing...
- Other recent developments include:
  - Reissner–Mindlin plates [Di Pietro and D., 2021c]
  - The 2D plates complex and Kirchhoff–Love plates [Di Pietro and D., 2022a]

\[ \mathcal{RT}^1(F) \hookrightarrow H^1(\Omega; \mathbb{R}^2) \xrightarrow{\text{sym rot}} H(\text{div div}, \Omega; \mathbb{S}) \xrightarrow{\text{div div}} L^2(\Omega) \xrightarrow{0} 0 \]

- The 2D Stokes complex [Hanot, 2021]

\[ \mathbb{R} \hookrightarrow H^2(\Omega) \xrightarrow{\text{rot}} H^1(\Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} 0 \]
The de Rham complex

Numerical results for magnetostatics model

(a) Cubic mesh, error vs. $h$

(b) Voronoi mesh, error vs. $h$
Numerical results for magnetostatics model

(a) Cubic mesh, error vs. #DOFs

(b) Voronoi mesh, error vs. #DOFs
Stokes in curl-curl form: robustness, serendipity efficiency

- **Pressure-robust** discretisations: optimal error estimates depend only on the velocity.
- Strong computational gain with serendipity DDR.

**Figure:** Voronoi meshes, wall and processor times (s) for the resolution of the linear systems.
Benefits

- **Increased flexibility** for meshing complex domains, or capturing local behaviour of solutions.
- **Arbitrary order** improves efficiency/cost, especially for steep problems.
- Systematic strategies for **reducing the number of DOFs**.
Benefits

- **Increased flexibility** for meshing complex domains, or capturing local behaviour of solutions.
- **Arbitrary order** improves efficiency/cost, especially for steep problems.
- Systematic strategies for reducing the number of DOFs.

Challenges and perspectives

- Design of efficient **polytopal mesh generators**.
- **Numerical solvers**: work currently in infancy.
- Analysis of polytopal methods for **incomplete operators** (curl, divergence) is very complex.
- **Polytopal Exterior Calculus** (PEC) to be developed in line of Finite Element Exterior Calculus (FEEC), in the formalism of differential forms.
- Further applications...
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