Next generation methods for the simulation of geophysical flows (and more...)

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...and many collaborators.

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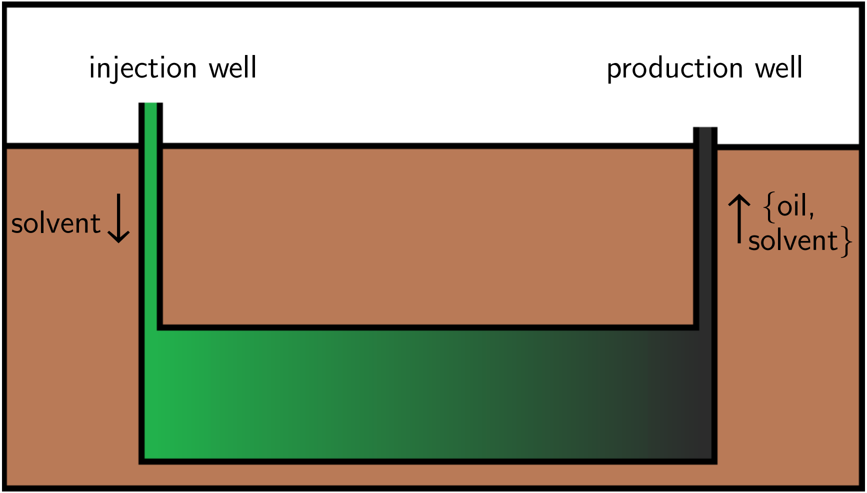
Outline

Enhanced oil recovery



Model for enhanced oil recovery

$$\begin{cases} \operatorname{div} \boldsymbol{u} &= q^+ - q^- := q \\ \boldsymbol{u} &= -\dfrac{\boldsymbol{K}}{\mu(c)} \,\mathbf{grad}\, p \end{cases}$$

$$\phi \frac{\partial c}{\partial t} + \operatorname{div}(\boldsymbol{u} c - \boldsymbol{D}(\boldsymbol{x}, \boldsymbol{u}) \,\mathbf{grad}\, c) + q^- c = q^+$$

Unknowns

- $p(\boldsymbol{x}, t)$ - pressure of the mixture
- $\boldsymbol{u}(\boldsymbol{x}, t)$ - Darcy velocity
- $c(\boldsymbol{x}, t)$ - concentration of the injected solvent

Parameters

- $\boldsymbol{K}(\boldsymbol{x})$ - permeability tensor
- $\phi(\boldsymbol{x})$ - porosity

- **Features**: non-linear, coupled, convection-dominated, anisotropic and heterogeneous.

Incompressible magnetohydrodynamics

$$
\begin{aligned}
-\frac{\partial \boldsymbol{u}}{\partial t} - \nu_k \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \mathbf{grad})\boldsymbol{u} + \mathbf{grad}\,\frac{p}{\rho} - (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} &= \boldsymbol{f}, \\
\frac{\partial \boldsymbol{b}}{\partial t} + \nu_m\,\mathbf{curl}(\mathbf{curl}\,\boldsymbol{b}) - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) &= \mathbf{0}, \\
\operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{b} &= 0,
\end{aligned}
$$

Unknowns

- $\boldsymbol{u}$ - fluid velocity
- p - fluid pressure
- $\boldsymbol{b}$ - magnetic field

Parameters

- $\boldsymbol{f}$ - external body force
- ρ - fluid density
- ν_k, ν_m - kinematic and magnetic diffusivity

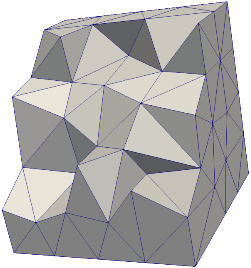
- **Features**: non-linear, coupled, incomplete differential operators and convection forces.

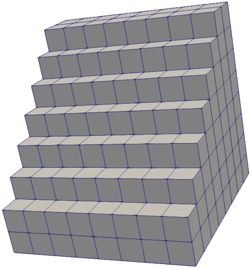
Finite elements

- Approximate using global functions on the domain that are locally polynomials.

Finite elements

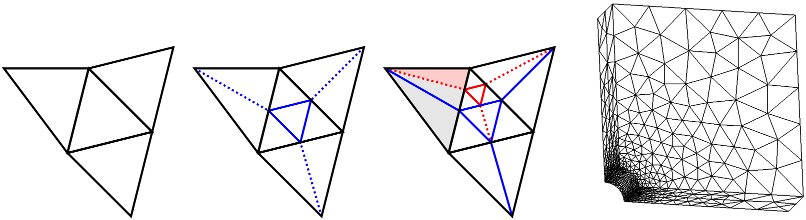
- Approximate using global functions on the domain that are locally polynomials.
- Require specific mesh geometries, mostly tetrahedra or hexahedras, to glue local polynomial functions into global functions.





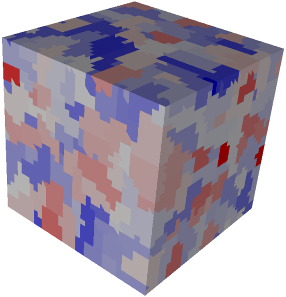
Unless using specific "tricks", e.g. for cut meshes.

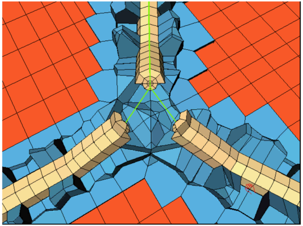
Shortcomings of classical Finite Elements

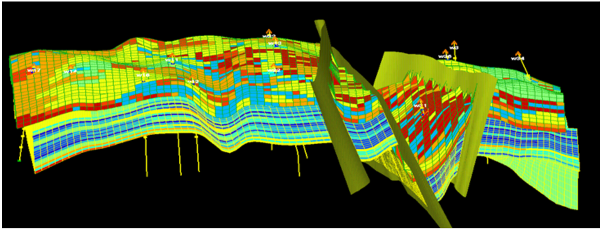


- Limitations of conforming meshes with standard elements
  - ⟹ local refinement requires to trade mesh size for mesh quality
  - ⟹ complex geometries may require a large number of elements
  - ⟹ the element shape cannot be adapted to the solution
- Need for (global) basis functions
  - ⟹ significant increase of DOFs on hexahedral elements

Meshes for complex problems

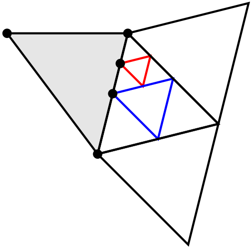






What is a polytopal method?

- A discretisation method for PDEs that can be applied to meshes with generic polytopal elements (polygons in 2D, polyhedra in 3D).
- Seamlessly handles non-conformity ("hanging nodes").



- Sometimes also arbitrary order of accuracy.

Some polytopal methods

- Discontinuous Galerkin *(actually started on triangles/tetrahedra)*, [Arnold, 1982, Brezzi et al., 2000, Di Pietro and Ern, 2010]: 70's, then 2012+.
- Hybridizable Discontinuous Galerkin and Weak Galerkin method [Cockburn et al., 2009, Cockburn, 2018]: 2009+.
- Mixed Finite Volumes, Hybrid Finite Volumes (SUSHI) and Mimetic Finite Differences [D. et al., 2010, Beirão da Veiga et al., 2014]: 2004+.
- Virtual Element Methods [Beirão da Veiga et al., 2013, Ayuso de Dios et al., 2016]: 2013+.
- Hybrid High-Order methods [Di Pietro et al., 2014, Di Pietro and D., 2020]: 2014+.

General literature review in the preface of [Di Pietro and D., 2020].

Model problem: Darcy flow in pressure formulation

- Given κ constant symmetric positive definite tensor and $f \in L^2(\Omega)$, the Darcy problem reads:

  Find the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
\kappa^{-1}\boldsymbol{u} - \mathbf{grad}\, p &= 0 && \text{in } \Omega, && \text{(Darcy's law)}\\
-\operatorname{div} \boldsymbol{u} &= f && \text{in } \Omega, && \text{(mass conservation)}\\
p &= 0 && \text{on } \partial\Omega && \text{(boundary condition)}
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- Primal formulation: eliminate velocity.

$$
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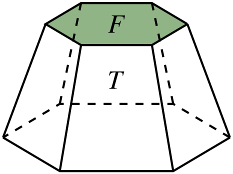
- Weak formulation: Find $p \in H^1_0(\Omega)$ s.t.

$$\int_\Omega \kappa\, \mathbf{grad}\, p \cdot \mathbf{grad}\, q = \int_\Omega f\, q \qquad \forall q \in H^1_0(\Omega).$$

An inspiring remark



- T: mesh element (cell), $\mathcal{F}_T$ set of faces F of T.
- $\mathcal{P}^k(X)$ = polynomials of degree $\le k$ on $X = T, F$.
- $\pi_X^{0,k}$: L^2-projector on $\mathcal{P}^k(X)$, satisfies: for $g \in L^2(X)$,

$$\int_X g q_k = \int_X (\pi_X^{0,k} g) q_k \qquad \forall q \in \mathcal{P}^k(X).$$

- $\pi_{\kappa,T}^{1,k+1}$: (oblique) elliptic projector, defined by: for $g \in H^1(T)$,

$$\int_T \kappa \,\mathbf{grad}(\pi_{\kappa,T}^{1,k+1} g) \cdot \mathbf{grad}\, q_{k+1} = \int_T \mathbf{grad}\, g \cdot \mathbf{grad}\, q_{k+1} \qquad \forall q_{k+1} \in \mathcal{P}^{k+1}(T),$$

$$\int_T \pi_{\kappa,T}^{1,k+1} g = \int_T g.$$

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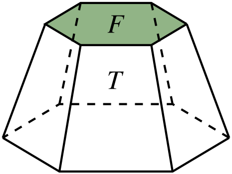

$$\int_X g q_k = \int_X (\pi_X^{0,k} g) q_k$$

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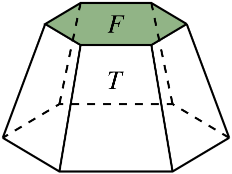
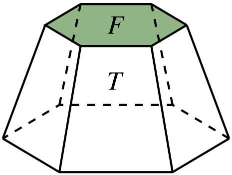


$$\int_X g q_k = \int_X (\pi_X^{0,k} g) q_k$$

- For $p \in H^1(T)$ and $q_{k+1} \in \mathcal{P}^{k+1}(T)$:

$$\begin{aligned}\int_T \kappa \,\mathbf{grad}\,(\pi_{\kappa,T}^{1,k+1} p) \cdot \mathbf{grad}\, q_{k+1} &= \int_T \kappa\,\mathbf{grad}\, p \cdot \mathbf{grad}\, q_{k+1} \\ &= -\int_T p\,\mathrm{div}(\kappa\,\mathbf{grad}\, q_{k+1}) + \sum_{F\in\mathcal{F}_T} \int_F p(\kappa\,\mathbf{grad}\, q_{k+1} \cdot \boldsymbol{n}_{TF}).\end{aligned}$$

An inspiring remark



$$\int_X g q_k = \int_X (\pi_X^{0,k} g) q_k$$

- For $p \in H^1(T)$ and $q_{k+1} \in \mathcal{P}^{k+1}(T)$:

$$\int_T \kappa \,\mathbf{grad}\,(\pi_{\kappa,T}^{1,k+1} p) \cdot \mathbf{grad}\, q_{k+1} = \int_T \kappa \,\mathbf{grad}\, p \cdot \mathbf{grad}\, q_{k+1}$$

$$= -\int_T \pi_T^{0,k} p \underbrace{\mathrm{div}(\kappa \,\mathbf{grad}\, q_{k+1})}_{\in \mathcal{P}^k(T)} + \sum_{F \in \mathcal{F}_T} \int_F \pi_F^{0,k} p \underbrace{(\kappa \,\mathbf{grad}\, q_{k+1} \cdot \boldsymbol{n}_{TF})}_{\in \mathcal{P}^k(F)}.$$

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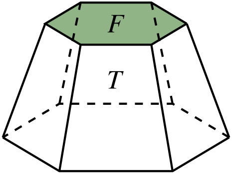

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- For $p \in H^1(T)$ and $q_{k+1} \in \mathcal{P}^{k+1}(T)$:

$$\int_T \kappa\,\mathbf{grad}\,(\pi_{\kappa,T}^{1,k+1} p) \cdot \mathbf{grad}\,q_{k+1} = \int_T \kappa\,\mathbf{grad}\,p \cdot \mathbf{grad}\,q_{k+1}$$
$$= -\int_T \pi_T^{0,k} p\,\underbrace{\mathrm{div}(\kappa\,\mathbf{grad}\,q_{k+1})}_{\in \mathcal{P}^k(T)} + \sum_{F \in \mathcal{F}_T} \int_F \pi_F^{0,k} p\,\underbrace{(\kappa\,\mathbf{grad}\,q_{k+1} \cdot \boldsymbol{n}_{TF})}_{\in \mathcal{P}^k(F)}.$$

$\pi_{\kappa,T}^{1,k+1} p$ computable from $\pi_T^{0,k} p$ and $(\pi_F^{0,k} p)_{F \in \mathcal{F}_T}$.

Design: Local space and interpolator

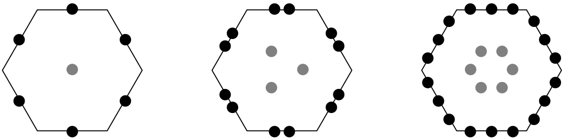

Figure: Degrees of freedom for $k \in \{0, 1, 2\}$ and $d = 2$

- For $k \geq 0$ and $T \in \mathcal{T}_h$, define the local HHO space

$$\underline{U}_T^k \coloneqq \left\{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) \ : \ v_T \in \mathcal{P}^k(T) \text{ and } v_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_T \right\}$$

- The local interpolator $\underline{I}_T^k : H^1(T) \to \underline{U}_T^k$ is s.t., for all $v \in H^1(T)$,

$$\underline{I}_T^k v \coloneqq \left(\pi_T^{0,k} v, (\pi_F^{0,k} v)_{F \in \mathcal{F}_T} \right)$$

Design: Potential reconstruction

- Let $T \in \mathcal{T}_h$, the potential reconstruction $\mathrm{r}_T^{k+1} : \underline{U}_T^k \to \mathcal{P}^{k+1}(T)$ is s.t., for all $\underline{v}_T \in \underline{U}_T^k$ and $q_{k+1} \in \mathcal{P}^{k+1}(T)$,

$$\int_T \kappa\, \mathbf{grad}\, \mathrm{r}_T^{k+1} \underline{v}_T \cdot \mathbf{grad}\, q_{k+1} = -\int_T v_T \,(\mathrm{div}\, \kappa\, \mathbf{grad}\, q_{k+1}) + \sum_{F \in \mathcal{F}_T} \int_F v_F \,(\kappa\, \mathbf{grad}\, q_{k+1} \cdot \boldsymbol{n}_{TF}),$$

$$\int_T \mathrm{r}_T^{k+1} \underline{v}_T = \int_T v_T .$$

- By construction:

$$\mathrm{r}_T^{k+1}(\underline{I}_T^k v) = \pi_{\kappa,T}^{1,k+1} v \qquad \forall v \in H^1(T).$$

Design: Local bilinear form

- Bilinear form in weak formulation:

$$\int_\Omega \kappa\,\mathbf{grad}\,p\cdot\mathbf{grad}\,q=\sum_{T\in\mathcal{T}_h}\int_T \kappa\,\mathbf{grad}\,p\cdot\mathbf{grad}\,q.$$

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- Bilinear form in weak formulation:

$$\int_\Omega \kappa\, \mathbf{grad}\, p \cdot \mathbf{grad}\, q = \sum_{T \in \mathcal{T}_h} \int_T \kappa\, \mathbf{grad}\, p \cdot \mathbf{grad}\, q.$$

- Approximate local term:

$$\int_T \kappa\, \mathbf{grad}\, p \cdot \mathbf{grad}\, q$$

$$\rightsquigarrow \mathrm{a}_T(\underline{p}_T, \underline{q}_T) := \int_T \kappa\, \mathbf{grad}(\mathrm{r}_T^{k+1} \underline{p}_T) \cdot \mathbf{grad}(\mathrm{r}_T^{k+1} \underline{q}_T) + \mathrm{s}_T(\underline{p}_T, \underline{q}_T).$$

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- Stabilisation term $\mathrm{s}_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$:
  1. Symmetric semi-definite positive,
  2. Polynomially consistent:

$$\mathrm{s}_T(\underline{I}_T^k p_{k+1}, \cdot) = 0 \qquad \forall p_{k+1} \in \mathcal{P}^{k+1}(T),$$

  3. Stable: in particular,

$$\mathrm{a}_T(\underline{p}_T, \underline{p}_T) = 0 \Longleftrightarrow \underline{p}_T = \underline{I}_T^k C \quad \text{for some } C \in \mathbb{R}.$$

Many possible choices, not all equally good [D. and Yemm, 2022b].

Design: Discrete problem

- Global space patching local ones and enforcing boundary conditions:

$$\underline{U}_{h,0}^k := \Big\{ \underline{v}_h = ((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_h}) \,:\, v_T \in \mathcal{P}^k(T) \quad \forall T \in \mathcal{T}_h\,, \\ v_F \in \mathcal{P}^k(F) \quad \forall F \in \mathcal{F}_h\,, \quad v_F = 0 \quad \forall F \subset \partial\Omega \Big\}.$$

- Global bilinear form assembling local ones:

$$\mathrm{a}_h(\underline{v}_h, \underline{w}_h) := \sum_{T\in\mathcal{T}_h} \mathrm{a}_T(\underline{v}_T, \underline{w}_T).$$

- HHO scheme: find $\underline{p}_h \in \underline{U}_{h,0}^k$ s.t.

$$\mathrm{a}_h(\underline{p}_h, \underline{q}_h) = \sum_{T\in\mathcal{T}_h} \int_T f q_T \qquad \forall \underline{q}_h \in \underline{U}_{h,0}^k.$$

Convergence analysis

- Based on optimal approximation properties of $\pi_{\kappa,T}^{1,k+1}$.
- Errors in energy norm:

$$\|\underline{p}_h - \underline{I}_h^k p\|_{\mathrm{a},h} = \mathcal{O}(h^{k+1})$$

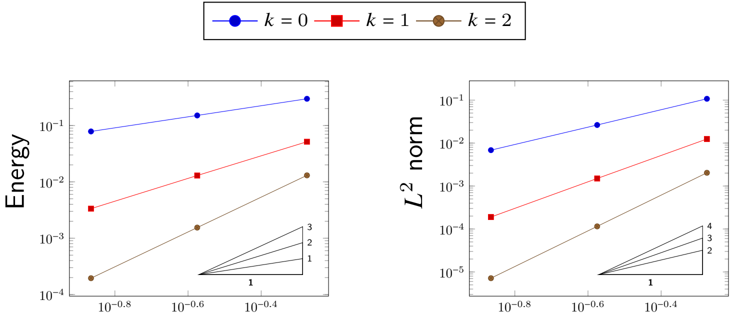
where $\|\underline{v}_h\|_{\mathrm{a},h} = \mathrm{a}_h(\underline{v}_h, \underline{v}_h)^{1/2}$ and $\underline{I}_h^k p = ((\pi_T^{0,k} p)_{T\in\mathcal{T}_h}, \pi_F^{0,k} p)_{F\in\mathcal{F}_h})$ global interpolate of the exact solution p.

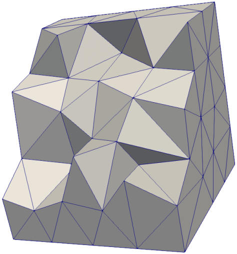
- Errors in L^2-norm (under elliptic regularity of the problem):

$$\|\mathrm{r}_h^{k+1}\underline{p}_h - p\|_{L^2(\Omega)} = \mathcal{O}(h^{k+2})$$

where $(\mathrm{r}_h^{k+1}\underline{p}_h)_{|T} = \mathrm{r}_T^{k+1}\underline{p}_T$.

Numerical results: error vs. h





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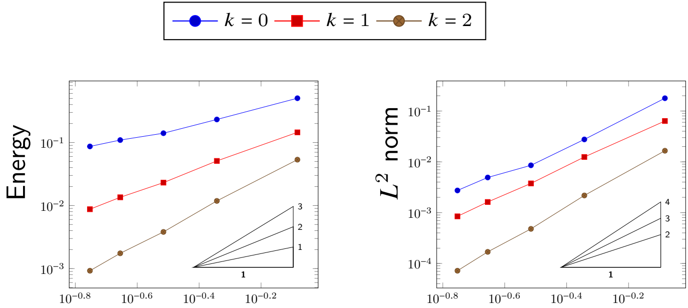

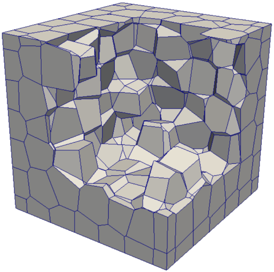

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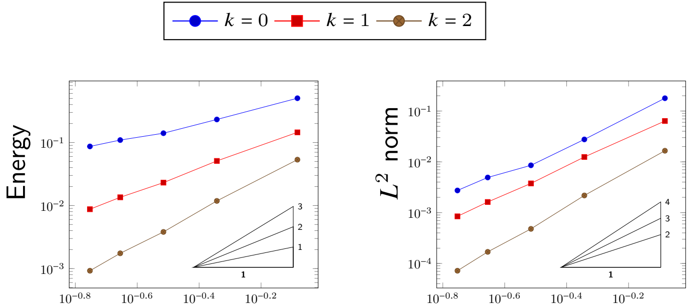

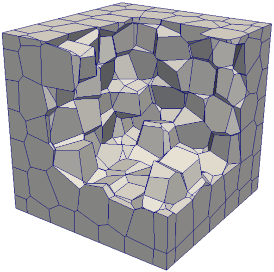

Mesh	Reg. para.
1	107
2	377
3	1.7E+3
4	2.6E+5
5	1.8E+4

Model

From [Anderson and D., 2018].

$$\begin{cases} \operatorname{div} \boldsymbol{u} & = q^+ - q^- := q \\ \boldsymbol{u} & = -\dfrac{\boldsymbol{K}}{\mu(c)} \, \mathbf{grad}\, p \end{cases}$$

$$\phi \frac{\partial c}{\partial t} + \operatorname{div}(\boldsymbol{u} c - \boldsymbol{D}(\boldsymbol{x}, \boldsymbol{u}) \, \mathbf{grad}\, c) + q^- c = q^+$$

- $p(\boldsymbol{x}, t)$ - pressure of the mixture
- $\boldsymbol{u}(\boldsymbol{x}, t)$ - Darcy velocity
- $c(\boldsymbol{x}, t)$ - concentration of the injected solvent

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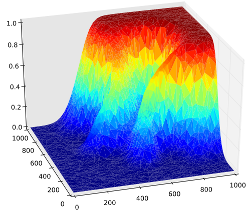
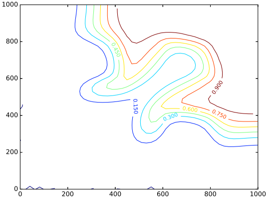
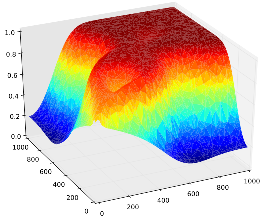
Numerical results



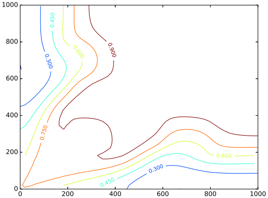
(a) Surface plot at $t = 3$ years



(b) Contour plot at $t = 3$ years



(c) Surface plot at $t = 10$ years

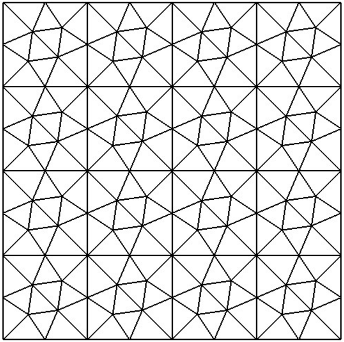


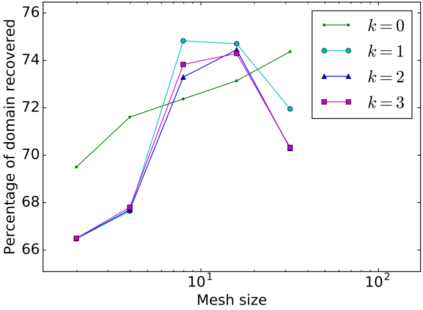
(d) Contour plot at $t = 10$ years

Figure: Concentration of invading solvent, $k = 1$ and $\Delta t = 18j$, discontinuous permeability.

Numerical results: recovery vs. h

Recovery: $\int_\Omega \phi c_h(T)$ for $T = 10y$.





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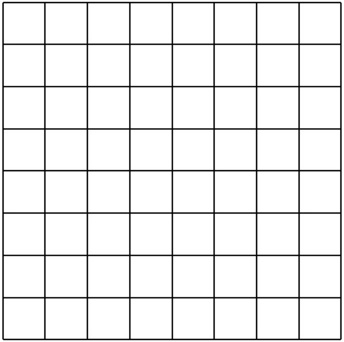
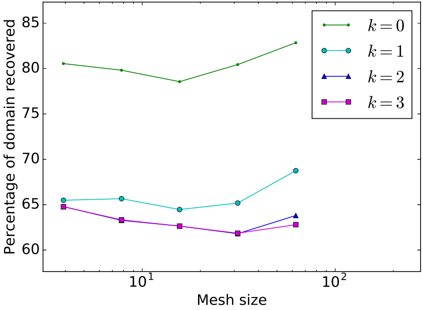
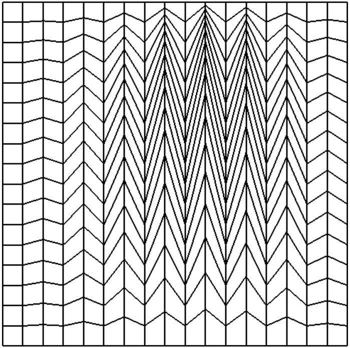
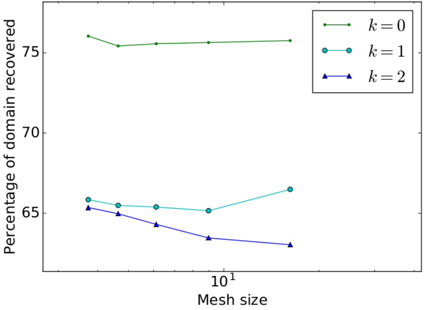




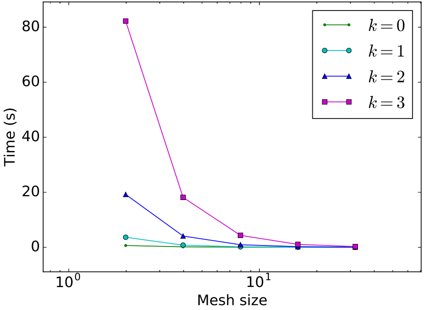
Numerical results: recovery vs. h

Recovery: $\int_\Omega \phi c_h(T)$ for $T = 10y$.

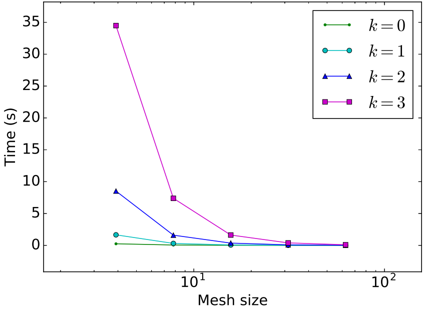




Numerical results: computational cost

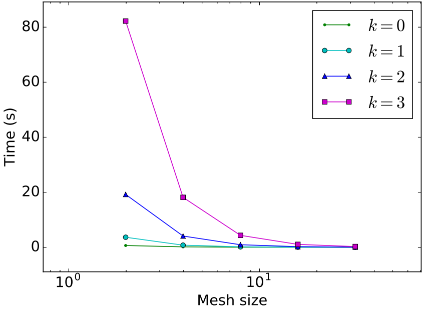


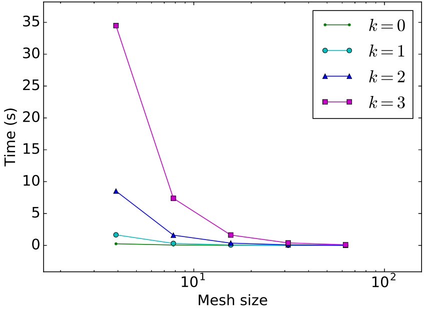
Triangular meshes



Cartesian meshes

Numerical results: computational cost





Triangular meshes

Cartesian meshes

A little bit of higher order approximation is not very expensive, but can make a huge difference.

Model

From [D. and Yemm, 2022a], based on [Botti et al., 2019] (Navier–Stokes).

$$\begin{aligned} -\nu_k \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \mathbf{grad})\boldsymbol{u} + \mathbf{grad}\, q - (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{b} &= \boldsymbol{f}, \\ \nu_m\, \mathbf{curl}(\mathbf{curl}\, \boldsymbol{b}) - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) &= \mathbf{0}, \\ \operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{b} &= 0, \end{aligned}$$

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... and with a little bit of differential calculus and Lagrange multipliers...

$$\begin{aligned} -\nu_k \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \mathbf{grad})\boldsymbol{u} - (\boldsymbol{b} \cdot \mathbf{grad})\boldsymbol{b} + \mathbf{grad}\, q &= \boldsymbol{f}, \\ -\nu_m \Delta \boldsymbol{b} + (\boldsymbol{u} \cdot \mathbf{grad})\boldsymbol{b} - (\boldsymbol{b} \cdot \mathbf{grad})\boldsymbol{u} + \mathbf{grad}\, r &= \boldsymbol{g}, \\ \operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{b} &= 0. \end{aligned}$$

Convergence results

- Small data and smooth solutions:

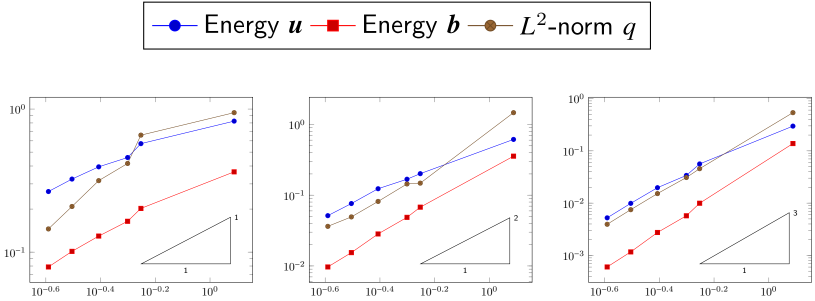
  Optimal convergence rates $\mathcal{O}(h^{k+1})$ in energy norm for $\boldsymbol{u}, \boldsymbol{b}$ and in L^2-norm for r, q.

- Any data and solution:

  Convergence of the scheme by compactness techniques.

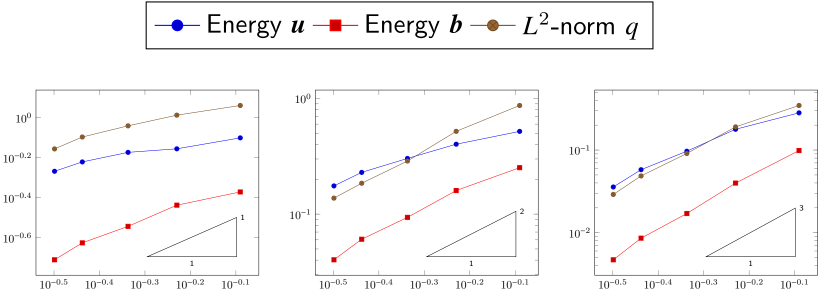
  Applicable in real-worl settings...

Numerical results: tetrahedral meshes ($\nu_k = \nu_m = 0.1$)



(a) $k = 0$ (b) $k = 1$ (c) $k = 2$

Numerical results: Voronoi meshes ($\nu_k = \nu_m = 0.1$)



(a) $k = 0$ (b) $k = 1$ (c) $k = 2$

The magnetostatics problem

- For $\mu > 0$ and $\boldsymbol{J} \in \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl};\Omega)$, the magnetostatics problem reads:
  Find the magnetic field $\boldsymbol{H} : \Omega \to \mathbb{R}^3$ and vector potential $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ s.t.

$$
\begin{aligned}
\mu \boldsymbol{H} - \mathbf{curl}\, \boldsymbol{A} &= \mathbf{0} && \text{in } \Omega, && \text{(vector potential)}\\
\mathbf{curl}\, \boldsymbol{H} &= \boldsymbol{J} && \text{in } \Omega, && \text{(Ampère's law)}\\
\operatorname{div} \boldsymbol{A} &= 0 && \text{in } \Omega, && \text{(Coulomb's gauge)}\\
\boldsymbol{A} \times \boldsymbol{n} &= \mathbf{0} && \text{on } \partial\Omega && \text{(boundary condition)}
\end{aligned}
$$

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$$

- Weak formulation: Find $(\boldsymbol{H}, \boldsymbol{A}) \in \boldsymbol{H}(\mathbf{curl}; \Omega) \times \boldsymbol{H}(\operatorname{div}; \Omega)$ s.t.

$$
\begin{aligned}
\int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl}; \Omega), \\
\int_\Omega \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{v} + \int_\Omega \operatorname{div} \boldsymbol{A} \operatorname{div} \boldsymbol{v} &= \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} && \forall \boldsymbol{v} \in \boldsymbol{H}(\operatorname{div}; \Omega)
\end{aligned}
$$

  with

$$
\boldsymbol{H}(\mathbf{curl}; \Omega) := \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \,:\, \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \right\},
$$
$$
\boldsymbol{H}(\operatorname{div}; \Omega) := \left\{ \boldsymbol{w} \in \boldsymbol{L}^2(\Omega) \,:\, \operatorname{div} \boldsymbol{w} \in L^2(\Omega) \right\}
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  - Bound on $\boldsymbol{A}^*$: requires

$$\mathrm{Im}\, \mathbf{curl} = \mathrm{Ker}\, \mathrm{div}$$

  to write $\boldsymbol{A}^* = -\mathbf{curl}\, \boldsymbol{\tau}$ with $\boldsymbol{\tau} \in (\mathrm{Ker}\, \mathbf{curl})^\perp$, and use $(\boldsymbol{\tau}, \boldsymbol{0})$ as test function.

A unified tool for well-posedness

$$\mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}$$

- We have key properties depending on the topology of Ω:

$$\begin{gathered}
\Omega \text{ connected } (b_0 = 1) \implies \mathrm{Ker}\,\mathbf{grad} = \mathbb{R},\\
\mathrm{Im}\,\mathbf{grad} \subset \mathrm{Ker}\,\mathbf{curl},\\
\mathrm{Im}\,\mathbf{curl} \subset \mathrm{Ker}\,\mathrm{div},\\
\Omega \subset \mathbb{R}^3 \ (b_3 = 0) \implies \mathrm{Im}\,\mathrm{div} = L^2(\Omega)
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- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\text{Ker}\,\textbf{curl}/\text{Im}\,\textbf{grad} \quad \text{and} \quad \text{Ker}\,\text{div}/\text{Im}\,\textbf{curl}$$

- Key consequences are Hodge decompositions and Poincaré inequalities

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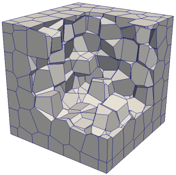
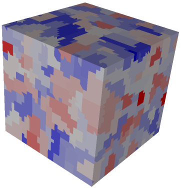
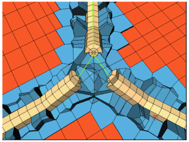
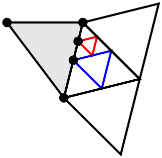
- Key consequences are Hodge decompositions and Poincaré inequalities
- **Emulating these properties is key for stable discretizations**

The discrete de Rham (DDR) approach I

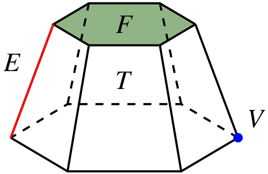





- **Key idea:** replace both spaces and operators by discrete counterparts:

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{G}^k_h} \underline{X}^k_{\mathbf{curl},h} \xrightarrow{\underline{C}^k_h} \underline{X}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

- Support of polyhedral meshes (CW complexes) and high-order
- Key exactness and consistency properties proved at the discrete level
- Several strategies to reduce the number of unknowns on general shapes

The discrete de Rham (DDR) approach II



- DDR spaces are spanned by vectors of polynomials
- Polynomial components enable consistent reconstructions of
  - vector calculus operators
  - the corresponding scalar or vector potentials
- These reconstructions emulate integration by parts (Stokes) formulas

Works on DDR

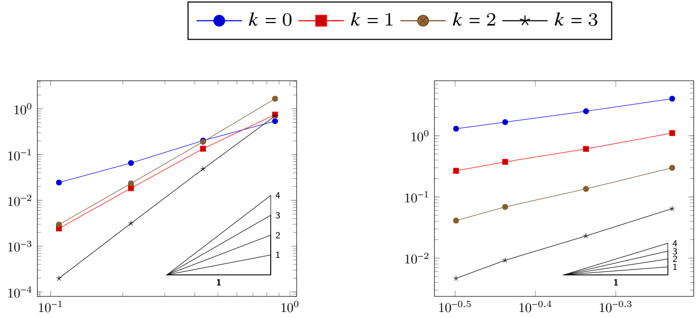
- Introduction of DDR [Di Pietro et al., 2020]
- Analytical properties [Di Pietro and D., 2021a]
- Application to magnetostatics [Di Pietro and D., 2021b]
- Bridges with VEM [Beirão da Veiga et al., 2021]
- Serendipity technique (reduction DOFs) [Di Pietro and D., 2022b]
- Cohomology analysis: ongoing...
- Other recent developments include:
  - Reissner–Mindlin plates [Di Pietro and D., 2021c]
  - The 2D plates complex and Kirchhoff–Love plates [Di Pietro and D., 2022a]

$$\boldsymbol{\mathcal{RT}}^1(F) \hookrightarrow \boldsymbol{H}^1(\Omega;\mathbb{R}^2) \xrightarrow{\mathrm{sym}\,\mathbf{rot}} \boldsymbol{H}(\mathrm{div}\,\mathbf{div},\Omega;\mathbb{S}) \xrightarrow{\mathrm{div}\,\mathbf{div}} L^2(\Omega) \xrightarrow{0} 0$$

  - The 2D Stokes complex [Hanot, 2021]

$$\mathbb{R} \hookrightarrow H^2(\Omega) \xrightarrow{\mathbf{rot}} \boldsymbol{H}^1(\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} 0$$

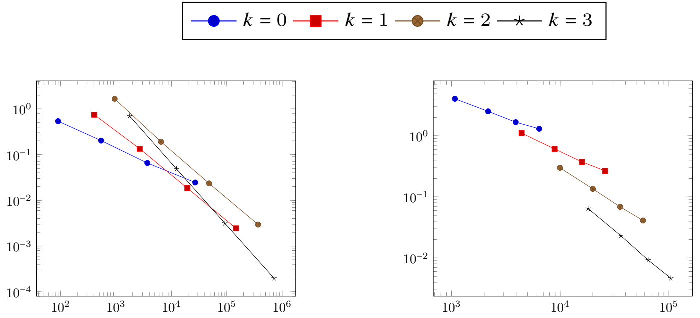
Numerical results for magnetostatics model



(a) Cubic mesh, error vs. h

(b) Voronoi mesh, error vs. h

Numerical results for magnetostatics model



(a) Cubic mesh, error vs. #DOFs

(b) Voronoi mesh, error vs. #DOFs

Stokes in curl-curl form: robustness, serendipity efficiency

- Pressure-robust discretisations: optimal error estimates depend only on the velocity.
- Strong computational gain with serendipity DDR.

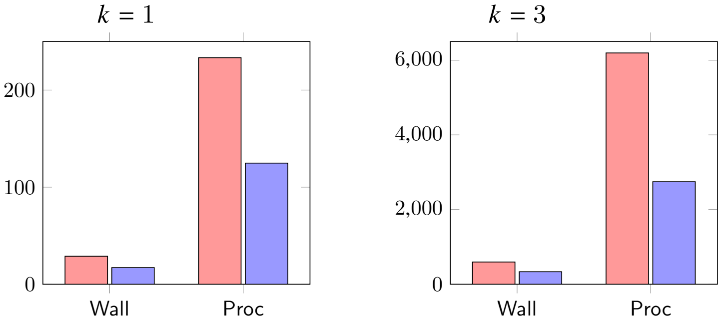


Figure: Voronoi meshes, wall and processor times (s) for the resolution of the linear systems

Benefits

- Increased flexibility for meshing complex domains, or capturing local behaviour of solutions.
- Arbitrary order improves efficiency/cost, especially for steep problems.
- Systematic strategies for reducing the number of DOFs.

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Challenges and perspectives

- Design of efficient polytopal mesh generators.
- Numerical solvers: work currently in infancy.
- Analysis of polytopal methods for incomplete operators (curl, divergence) is very complex.
- Polytopal Exterior Calculus (PEC) to be developed in line of Finite Element Exterior Calculus (FEEC), in the formalism of differential forms.
- Further applications...

References I

Anderson, D. and D., J. (2018).
An arbitrary order scheme on generic meshes for miscible displacements in porous media.
SIAM J. Sci. Comput., 40(4):B1020–B1054.

Arnold, D. N. (1982).
An interior penalty finite element method with discontinuous elements.
SIAM J. Numer. Anal., 19(4):742–760.

Ayuso de Dios, B., Lipnikov, K., and Manzini, G. (2016).
The nonconforming virtual element method.
ESAIM: Math. Model Numer. Anal., 50(3):879–904.

Beirão da Veiga, L., Brezzi, F., Cangiani, A., Manzini, G., Marini, L. D., and Russo, A. (2013).
Basic principles of virtual element methods.
Math. Models Methods Appl. Sci. (M3AS), 199(23):199–214.

Beirão da Veiga, L., Dassi, F., Di Pietro, D. A., and D., J. (2021).
Arbitrary-order pressure-robust DDR and VEM methods for the Stokes problem on polyhedral meshes.
Submitted. URL: `http://arxiv.org/abs/2112.09750`.

Beirão da Veiga, L., Lipnikov, K., and Manzini, G. (2014).
The mimetic finite difference method for elliptic problems, volume 11 of *MS&A. Modeling, Simulation and Applications*.
Springer, Cham.

Botti, L., Di Pietro, D. A., and D., J. (2019).
A Hybrid High-Order method for the incompressible Navier–Stokes equations based on Temam's device.
J. Comput. Phys., 376:786–816.

References II

Brezzi, F., Manzini, G., Marini, L. D., Pietra, P., and Russo, A. (2000).
Discontinuous Galerkin approximations for elliptic problems.
Numer. Methods Partial Differential Equations, 16(4):365–378.

Cockburn, B. (2018).
The Weak Galerkin methods are rewritings of the Hybridizable Discontinuous Galerkin methods.

Cockburn, B., Gopalakrishnan, J., and Lazarov, R. (2009).
Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems.
SIAM J. Numer. Anal., 47(2):1319–1365.

D., J., Eymard, R., Gallouët, T., and Herbin, R. (2010).
A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods.
Math. Models Methods Appl. Sci. (M3AS), 20(2):1–31.

D., J. and Yemm, L. (2022a).
A hybrid high-order scheme for the stationary, incompressible magnetohydrodynamics equations.
page 33p.

D., J. and Yemm, L. (2022b).
Robust hybrid high-order method on polytopal meshes with small faces.
Comput. Methods Appl. Math., 22(1):47–71.

Di Pietro, D. A. and D., J. (2020).
The Hybrid High-Order method for polytopal meshes.
Number 19 in Modeling, Simulation and Application. Springer International Publishing.

References III

Di Pietro, D. A. and D., J. (2021a).
An arbitrary-order discrete de Rham complex on polyhedral meshes: Exactness, Poincaré inequalities, and consistency.
Found. Comput. Math.
Published online. DOI: 10.1007/s10208-021-09542-8.

Di Pietro, D. A. and D., J. (2021b).
An arbitrary-order method for magnetostatics on polyhedral meshes based on a discrete de Rham sequence.
J. Comput. Phys., 429(109991).

Di Pietro, D. A. and D., J. (2021c).
A DDR method for the Reissner–Mindlin plate bending problem on polygonal meshes.
Submitted. URL: http://arxiv.org/abs/2105.11773.

Di Pietro, D. A. and D., J. (2022a).
A fully discrete plates complex on polygonal meshes with application to the Kirchhoff–Love problem.
Submitted. URL: http://arxiv.org/abs/2112.14497.

Di Pietro, D. A. and D., J. (2022b).
Homological- and analytical-preserving serendipity framework for polytopal complexes, with application to the ddr method.
page 31p.

Di Pietro, D. A., D., J., and Rapetti, F. (2020).
Fully discrete polynomial de Rham sequences of arbitrary degree on polygons and polyhedra.
Math. Models Methods Appl. Sci., 30(9):1809–1855.

Di Pietro, D. A. and Ern, A. (2010).
Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier-Stokes equations.
Math. Comp., 79(271):1303–1330.

References IV

Di Pietro, D. A., Ern, A., and Lemaire, S. (2014).
An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators.
Comput. Meth. Appl. Math., 14(4):461–472.

Hanot, M.-L. (2021).
An arbitrary-order fully discrete Stokes complex on general polygonal meshes.
Submitted. URL: https://arxiv.org/abs/2112.03125.