High-order methods for linear and non-linear elliptic equations

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Algoritmy 2020

Joint work with D. Anderson, D. Di Pietro, R. Eymard, F. Rapetti...

Australian Government
Australian Research Council

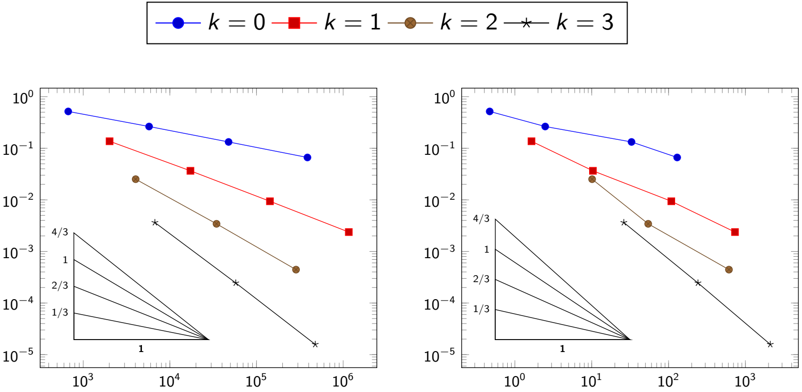
Discrete Functional Analysis: bridging pure and numerical mathematics

Plan

1. **Hybrid High-Order method**
   - An inspiring remark
   - Description of the HHO scheme
   - Miscible flow in porous media
2. **Fully Discrete de Rham sequence**
   - Principles of discrete exact sequence
   - Fully discrete de Rham sequence
   - Application to magnetostatics
3. **High-order schemes for stationnary Stefan/PME models**
   - Towards a stable numerical approximation
   - High-order approximations
   - Numerical tests

What is "high order": the case of $\mathcal{P}^k$ finite elements

Why go for "high order": 3D tests with Hybrid High-Order scheme



(a) Energy error vs. nb degrees of freedom

(b) Energy error vs. total running time

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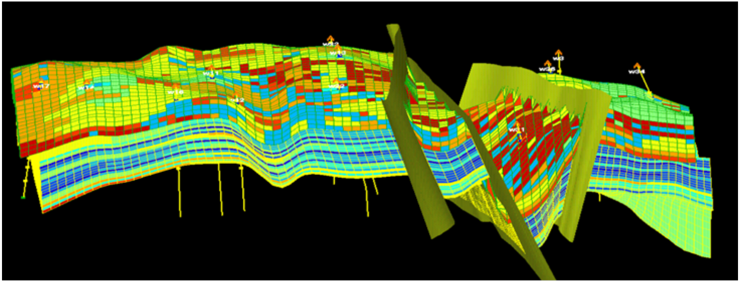
Objectives

Polytopal meshes



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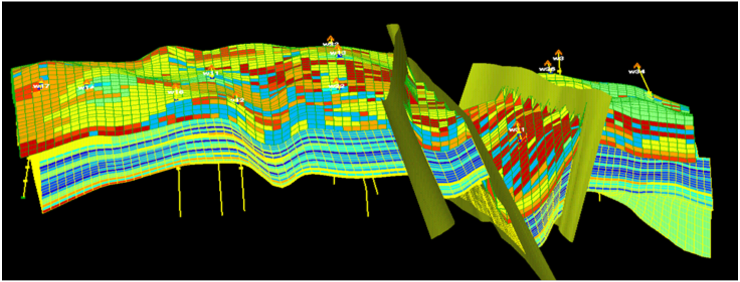

Arbitrary order: choice of an index $k \geq 0$ determining the accuracy of the method

Typically: exactly reproduce solutions that are polynomials of degree $k+1$.

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Where do we put the unknowns?

Model problem: Poisson equation.

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- *Core model in many flows in porous media (including multi-components, multi-phases): oil recovery, CO2 storage, etc.*

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$$\text{Find } u \in H_0^1(\Omega) \text{ s.t. } (\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \text{for all } v \in H_0^1(\Omega).$$

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Computation elliptic projector: for T a mesh element, v smooth and $q \in \mathcal{P}^{k+1}(T)$:

$$(\nabla v, \nabla q)_T = -(v, \Delta q)_T + \sum_{F \in \mathcal{F}_T} (v, \nabla q \cdot \boldsymbol{n}_{TF})_F.$$

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- $\pi^k_{\mathcal{P},Y} : L^2(Y) \to \mathcal{P}^k(Y)$ orthogonal projector.

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The projection $\Pi_{\nabla\mathcal{P}^{k+1}(T)}(\nabla v)$ of ∇v on $\nabla\mathcal{P}^{k+1}(T)$ can be computed from $\pi^k_{\mathcal{P},T} v$ and $(\pi^k_{\mathcal{P},F} v)_{F\in\mathcal{F}_T}$.

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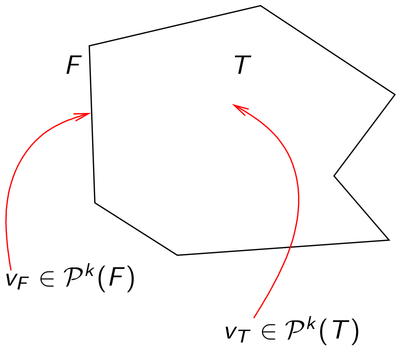
Which unknowns?



Space of local unknowns:

$$\underline{U}_T^k = \{\underline{v}_T = (v_T, (v_F)_{F\in\mathcal{F}_T}) \,:\, v_T \in \mathcal{P}^k(T)\,,\ v_F \in \mathcal{P}^k(F)\}.$$

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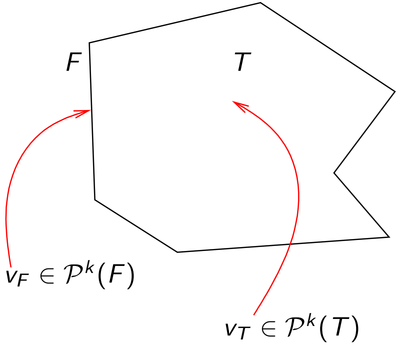

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Interpolator: $\underline{I}_T^k : H^1(T) \to \underline{U}_T^k$ such that

$$\underline{I}_T^k v = (\pi_{\mathcal{P},T}^k v, (\pi_{\mathcal{P},F}^k v)_{F \in \mathcal{F}_T}).$$

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- $\nabla \mathrm{p}_T^{k+1} \underline{I}_T^k v = \Pi_{\nabla \mathcal{P}^{k+1}(T)}(\nabla v)$ for all $v \in H^1(T)$.

Local bilinear form

- $\mathrm{a}_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ such that

$$\mathrm{a}_T(\underline{v}_T, \underline{w}_T) = (\nabla \mathrm{p}_T^{k+1} \underline{v}_T, \nabla \mathrm{p}_T^{k+1} \underline{w}_T)_T + \mathrm{s}_T(\underline{v}_T, \underline{w}_T).$$

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where the local discrete H^1-seminorm is

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- **Polynomial consistency**: $\mathrm{s}_T(\underline{I}_T^k w, \underline{v}_T) = 0$ for all $w \in \mathcal{P}^{k+1}(T)$.

Global space (with boundary conditions): patched local spaces.

$$\underline{U}_{h,0}^k = \big\{\underline{v}_h = ((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_T}) \,:\, v_T \in \mathcal{P}^k(T)\,,\ v_F \in \mathcal{P}^k(F)\,,\\ v_F = 0 \text{ if } F \subset \partial\Omega\big\}.$$

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Scheme:

Find $\underline{u}_h \in \underline{U}_{h,0}^k$ such that, for all $\underline{v}_T \in \underline{U}_{h,0}^k$,

$$\mathrm{a}_h(\underline{u}_h, \underline{v}_h) = \sum_{T\in\mathcal{T}_h} (f, v_T)_T.$$

Error analysis

- *Using a generic framework (3rd Strang lemma) developed for schemes written in fully discrete form (the approximation space is not a space of functions over Ω).*

Energy error estimate: with $\|\cdot\|_{\mathrm{a},h} = \sqrt{\mathrm{a}_h(\cdot,\cdot)}$ norm associated with a_h:

$$\|\underline{I}_h^k u - \underline{u}_h\|_{\mathrm{a},h} \le Ch^{k+1}|u|_{H^{k+2}(\Omega)}.$$

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L^2 error estimate: with $(\mathrm{p}_h^{k+1}\underline{u}_h)_{|T} = \mathrm{p}_T^{k+1}\underline{u}_T$ for all $T \in \mathcal{T}_h$, under elliptic regularity assumption:

$$\|u - \mathrm{p}_h^{k+1}\underline{u}_h\|_\Omega \le C \begin{cases} h^2\|f\|_{H^1(\Omega)} & \text{if } k = 0, \\ h^{k+2}|u|_{H^{k+2}(\Omega)} & \text{if } k \ge 1. \end{cases}$$

More about HHO

- This is a **finite volume method**: we can define fluxes that are conservative and satisfy, up to high order volumic term, the balance relation.

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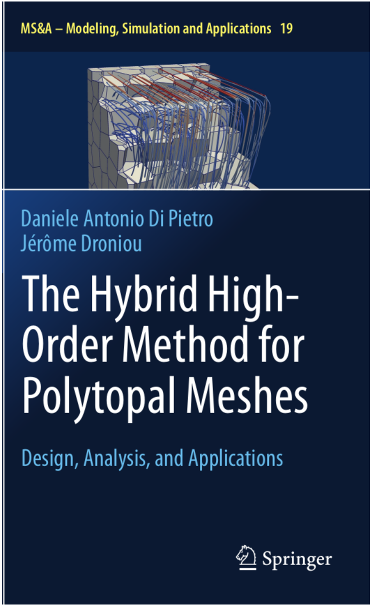
▶ Variant: polynomial degree in element could be $k \pm 1$.
Leads to links with

- non-conforming $\mathcal{P}^1$ FE;
- Virtual Element Methods;
- Hybridizable Discontinuous Galerkin;
- etc.

More about HHO



More about HHO

- Implementation in various libraries, in particular HArD::Core3D library (`https://github.com/jdroniou/HArDCore`).
  - Open source C++ code for numerical schemes on generic polyhedral meshes.
  - Based on Eigen linear algebra library (`http://eigen.tuxfamily.org`).
  - Complete and intuitive description of mesh.
  - Routines for handling polynomial spaces (on edges, faces and cells), for quadrature rules, for Gram-like matrices (mass, stiffness), etc.

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Model for enhanced oil recovery

$$\left\{\begin{array}{rl} \nabla\cdot\mathbf{u} & = q^+ - q^- := q \\ \mathbf{u} & = -\dfrac{\mathbf{K}}{\mu(c)}\nabla p \end{array}\right.$$

$$\phi\frac{\partial c}{\partial t} + \nabla\cdot(\mathbf{u}c - \mathbf{D}(\mathbf{x},\mathbf{u})\nabla c) + q^- c = q^+$$

Unknowns

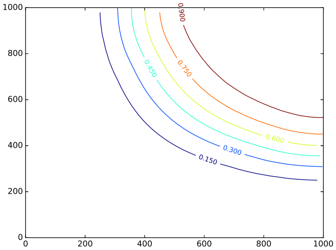
- $p(\mathbf{x}, t)$ - pressure of the mixture
- $\mathbf{u}(\mathbf{x}, t)$ - Darcy velocity
- $c(\mathbf{x}, t)$ - concentration of the injected solvent

Parameters

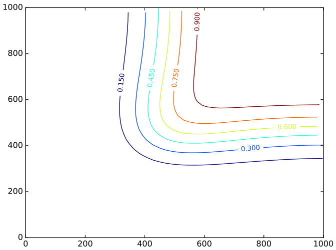
- $\mathbf{K}(\mathbf{x})$ - permeability tensor
- $\phi(\mathbf{x})$ - porosity

- Complemented with no-flow boundary conditions.

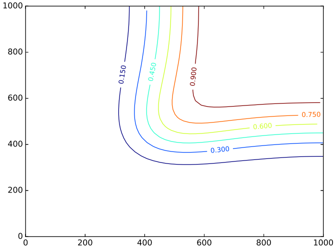
$t = 3$ years, various k, Cartesian mesh



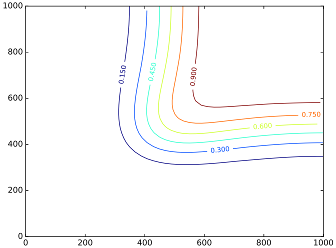
$k = 0$



$k = 1$

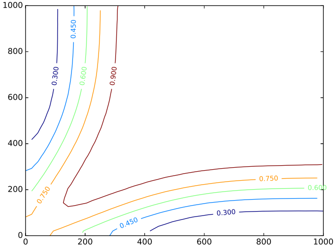


$k = 2$

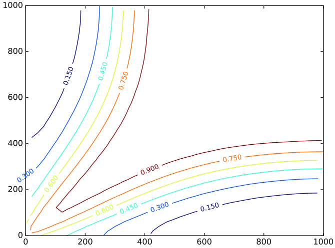


$k = 3$

$t = 10$ years, various k, Cartesian mesh



$k = 0$



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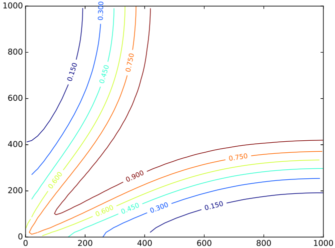

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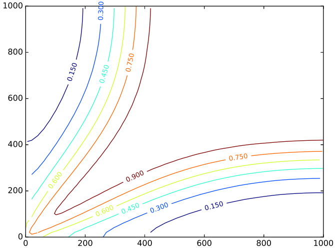

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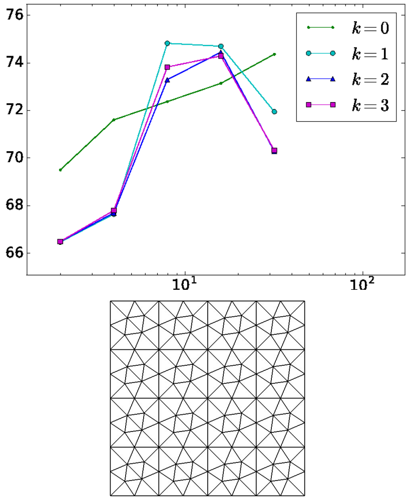
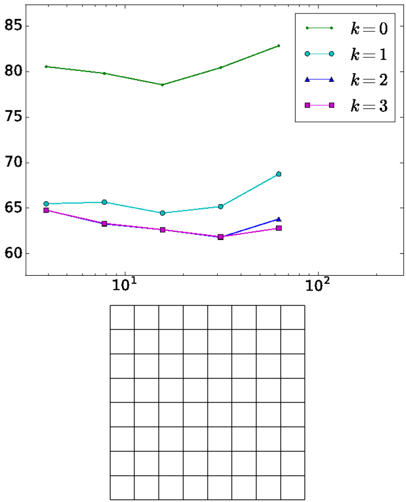
Percentage of oil recovered vs. mesh size, various k





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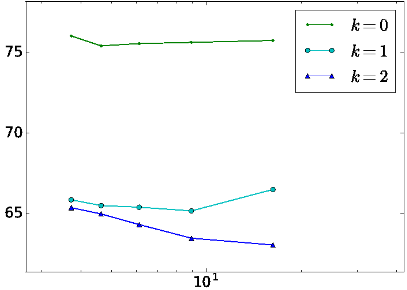
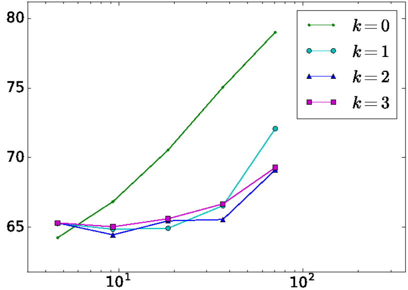
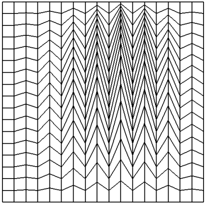
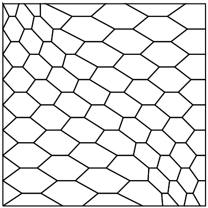
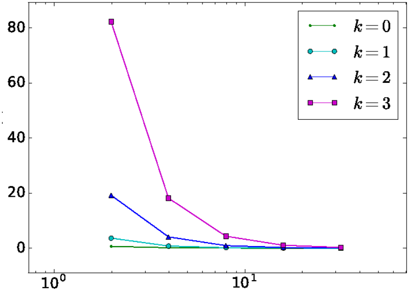


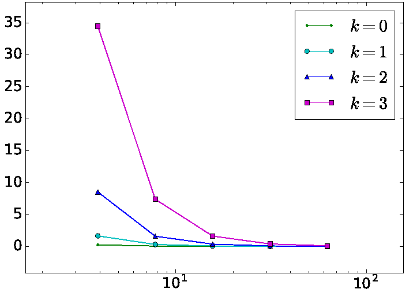


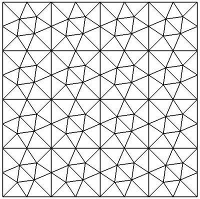


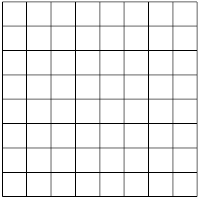


Computational cost for one time step: time (s) vs. mesh sizes









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   - Application to magnetostatics

3. High-order schemes for stationnary Stefan/PME models
   - Towards a stable numerical approximation
   - High-order approximations
   - Numerical tests

- Ω: open simply connected set in $\mathbb{R}^3$ with connected boundary.

Gradient:

$$H^1(\Omega) = \{u \in L^2(\Omega) \,:\, \mathrm{grad}\, u \in L^2(\Omega)^3\},$$
$$\mathrm{grad} : H^1(\Omega) \to L^2(\Omega)^3.$$

Curl:

$$\boldsymbol{H}(\mathrm{curl};\Omega) = \{\boldsymbol{u} \in L^2(\Omega)^3 \,:\, \mathrm{curl}\, \boldsymbol{u} \in L^2(\Omega)^3\},$$
$$\mathrm{curl} : \boldsymbol{H}(\mathrm{curl};\Omega) \to L^2(\Omega)^3.$$

Divergence:

$$\boldsymbol{H}(\mathrm{div};\Omega) = \{\boldsymbol{u} \in L^2(\Omega)^3 \,:\, \mathrm{div}\, \boldsymbol{u} \in L^2(\Omega)\},$$
$$\mathrm{div} : \boldsymbol{H}(\mathrm{div};\Omega) \to L^2(\Omega).$$

Exact sequence

- $i_\Omega : \mathbb{R} \to H^1(\Omega)$ natural embedding.

Theorem (Exactness of de Rham sequence)

The following sequence is exact:

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}(\text{curl};\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\},$$

which means that, if $\mathfrak{D}_i$ and $\mathfrak{D}_{i+1}$ are two consecutive operators in the sequence, then

$$\text{Im}\,\mathfrak{D}_i = \text{Ker}\,\mathfrak{D}_{i+1}.$$

Why is this exactness important?

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}(\text{curl};\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\},$$

Stokes problem

$$\begin{cases} -\Delta \boldsymbol{u} + \text{grad}\, p = f & \text{in } \Omega, \\ \text{div}\, \boldsymbol{u} = 0 & \text{in } \Omega, \\ +\text{BC} & \end{cases}$$

- Inf-sup condition: for all $q \in L^2(\Omega)$,

$$\sup_{\boldsymbol{v} \in \boldsymbol{H}(\text{div};\Omega)} \frac{(\text{div}\, \boldsymbol{v}, q)_{L^2}}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\text{div})}} \geq \beta \|q\|_{L^2}.$$

Proof: Fix $q \in L^2(\Omega)$*, and let* $\boldsymbol{v} \in \boldsymbol{H}(\text{div};\Omega)$ *such that* $\text{div}\, \boldsymbol{v} = q$*...*

Why is this exactness important?

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}(\text{curl};\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\},$$

Magnetostatic problem

$$\begin{cases} \boldsymbol{\sigma} - \text{curl}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \text{curl}\,\boldsymbol{\sigma} = \boldsymbol{f} & \text{in } \Omega, \\ \text{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \partial\Omega. \end{cases}$$

- Inf-sup condition: for all $(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{H}(\text{curl};\Omega) \times \boldsymbol{H}(\text{div};\Omega)$,

$$\sup_{(\boldsymbol{\mu},\boldsymbol{w}) \in \boldsymbol{H}(\text{curl}) \times \boldsymbol{H}(\text{div})} \frac{\mathcal{A}((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\mu}, \boldsymbol{w}))}{\|(\boldsymbol{\mu}, \boldsymbol{w})\|_{\boldsymbol{H}(\text{curl}) \times \boldsymbol{H}(\text{div})}} \geq \beta \|(\boldsymbol{\mu}, \boldsymbol{v})\|_{\boldsymbol{H}(\text{curl}) \times \boldsymbol{H}(\text{div})}, \text{ where}$$

$$\mathcal{A}((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\mu}, \boldsymbol{w})) = (\boldsymbol{\tau}, \boldsymbol{\mu})_{L^2} - (\boldsymbol{v}, \text{curl}\,\boldsymbol{\mu})_{L^2} + (\boldsymbol{w}, \text{curl}\,\boldsymbol{\tau})_{L^2} + (\text{div}\,\boldsymbol{v}, \text{div}\,\boldsymbol{w})_{L^2}.$$

Why is this exactness important?

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}(\text{curl};\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\},$$

Magnetostatic problem

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- Inf-sup condition: for all $(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{H}(\text{curl};\Omega) \times \boldsymbol{H}(\text{div};\Omega)$,

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$$\mathcal{A}((\boldsymbol{\tau},\boldsymbol{v}),(\boldsymbol{\mu},\boldsymbol{w})) = (\boldsymbol{\tau},\boldsymbol{\mu})_{L^2} - (\boldsymbol{v}, \text{curl}\,\boldsymbol{\mu})_{L^2} + (\boldsymbol{w}, \text{curl}\,\boldsymbol{\tau})_{L^2} + (\text{div}\,\boldsymbol{v}, \text{div}\,\boldsymbol{w})_{L^2}.$$

Proof: requires two exactness properties in the sequence, to estimate each component of $\boldsymbol{v}$ on $(\text{Ker}\,\text{div})^\perp$ and $\text{Ker}\,\text{div}$.

Plan

Objective

- Mimic exact sequence with discrete spaces and operators.

$\rightsquigarrow$ *To be used to design stable numerical schemes.*

- Local construction (element by element), as in standard FE.

- Arbitrary order, based on polynomial spaces of degree $k \geq 0$.

Objective

Local discrete spaces and operators: for T mesh element,

$$\mathbb{R} \xrightarrow{\underline{I}_{\mathrm{grad},T}^k} \underline{X}_{\mathrm{grad},T}^k \xrightarrow{\underline{\boldsymbol{G}}_T^k} \underline{\boldsymbol{X}}_{\mathrm{curl},T}^k \xrightarrow{\underline{\boldsymbol{C}}_T^k} \underline{\boldsymbol{X}}_{\mathrm{div},T}^k \xrightarrow{D_T^k} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

Objective

Local discrete spaces and operators: for T mesh element,

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

- Finite Element approach:
  - Finite Element Exterior Calculus (FEEC).
  - Requires elements of certain shapes (tetrahedras, hexahedras...) as in usual FE.
  - Designed in very generic setting, with exterior derivatives etc.

Objective

Local discrete spaces and operators: for T mesh element,

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

- Virtual Element approach:
  - Applicable on generic meshes with polyhedral elements.
  - Degree decreases by one at each application of differential operator.
  - Functions not fully known, only certain moments or values are accessible.
  - Exactness not usable in a scheme due to the variational crime in VEM.

Plan

Features

- Applicable on polyhedral elements.

- Arbitrary order of exactness.

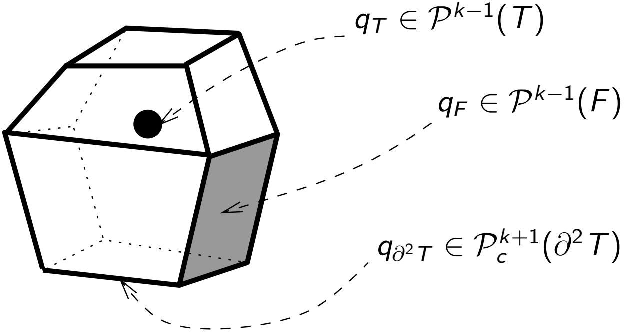
- Same order of accuracy along the entire sequence.

- Based on explicit spaces and reconstructed differential operators, exactness holding for these objects.

Principles

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

Gradient unknowns: $\underline{q}_T = (q_T, (q_F)_{F\in\mathcal{F}_T}, q_{\partial^2 T})$.



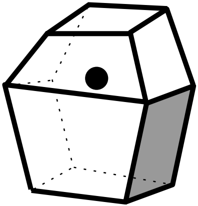
Principles

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

Gradient operator:

$$\underline{\boldsymbol{G}}^k_T \underline{q}_T = (\underbrace{\boldsymbol{G}^k_T \underline{q}_T}_{\in\mathcal{P}^k(T)^3}, (\underbrace{\boldsymbol{G}^k_F(q_F, q_{\partial^2 T})}_{\in\mathcal{P}^k(F)^2})_{F\in\mathcal{F}_T}, (\underbrace{G_E q_E}_{\in\mathcal{P}^k(E)})_{E\in\mathcal{E}_T}).$$

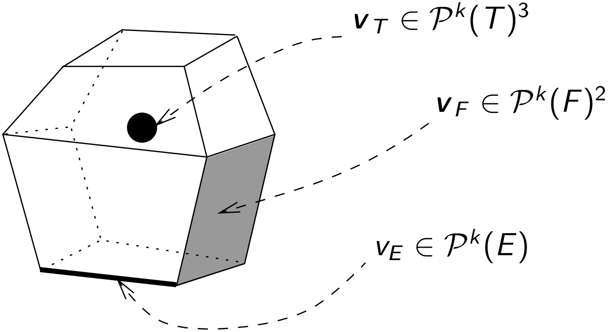
- G_E: derivative along edge.
- $\boldsymbol{G}^k_F$ ($\approx \mathrm{grad}_{|F}$): reconstruction from face and edge, based on formal IBP (divergence formula),
- $\boldsymbol{G}^k_T$ ($\approx$ grad): reconstruction based on formal IBP & face potentials (divergence formula).



Principles

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

Curl unknowns: $\underline{\boldsymbol{v}}_T = (\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F\in\mathcal{F}_T}, (v_E)_{E\in\mathcal{E}_T})$.



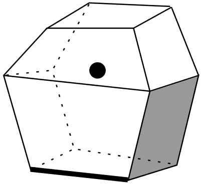
Principles

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

Curl operator:

$$\underline{\boldsymbol{C}}^k_T \underline{\boldsymbol{v}}_T = (\underbrace{\boldsymbol{C}^k_T \underline{\boldsymbol{v}}_T}_{\in \mathcal{P}^k(T)^3}, (\underbrace{C^k_F(\boldsymbol{v}_F, (v_E)_{E\in\mathcal{E}_F})}_{\in \mathcal{P}^k(F)})_{F\in\mathcal{F}_T}).$$

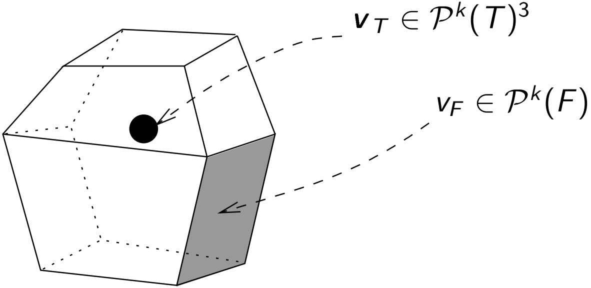
- C^k_F ($\approx$ curl $\cdot \boldsymbol{n}_F$): reconstruction from face and edge, based on formal IBP (rot formula in 2D),
- $\boldsymbol{C}^k_T$ ($\approx$ curl): reconstruction based on formal IBP & face tangential potentials (curl formula).



Principles

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

Divergence unknowns: $\underline{\boldsymbol{v}}_T = (\boldsymbol{v}_T, (v_F)_{F\in\mathcal{F}_T})$.



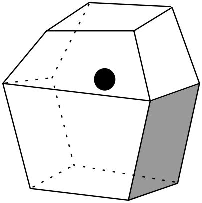
Principles

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

Divergence operator:

$D^k_T\underline{\boldsymbol{v}}_T$ ($\approx$ div) reconstructed in $\mathcal{P}^k(T)$ from divergence formula.

$$\int_T (D^k_T\underline{\boldsymbol{v}}_T)q_T = -\int_T \boldsymbol{v}_T\cdot\mathrm{grad}\,q_T + \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F v_F q_T \quad \forall q_T\in\mathcal{P}^k(T).$$



There's a catch...

$$\mathbb{R} \xrightarrow{\underline{I}_{\mathrm{grad},T}^k} \underline{X}_{\mathrm{grad},T}^k \xrightarrow{\underline{\boldsymbol{G}}_T^k} \underline{\boldsymbol{X}}_{\mathrm{curl},T}^k \xrightarrow{\underline{\boldsymbol{C}}_T^k} \underline{\boldsymbol{X}}_{\mathrm{div},T}^k \xrightarrow{D_T^k} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

- The previous sequence is not exact!

There's a catch...

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

- The previous sequence is not exact!

For $X = F, T$ of dimension $d = 2, 3$ let:

- $\boldsymbol{\mathcal{R}}^k(X) = \mathrm{curl}(\mathcal{P}^{k+1}(X)^d)$, $\boldsymbol{\mathcal{R}}^{\mathrm{c},k}(X)$ complement in $\mathcal{P}^k(X)^d$.
- $\boldsymbol{\mathcal{G}}^k(X) = \mathrm{grad}(\mathcal{P}^{k+1}(X)^d)$, $\boldsymbol{\mathcal{G}}^{\mathrm{c},k}(X)$ complement in $\mathcal{P}^k(X)^d$.

There's a catch...

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

- The previous sequence is not exact!

For $X = F, T$ of dimension $d = 2, 3$ let:

- $\boldsymbol{\mathcal{R}}^k(X) = \mathrm{curl}(\mathcal{P}^{k+1}(X)^d)$, $\boldsymbol{\mathcal{R}}^{\mathrm{c},k}(X)$ complement in $\mathcal{P}^k(X)^d$.
- $\boldsymbol{\mathcal{G}}^k(X) = \mathrm{grad}(\mathcal{P}^{k+1}(X)^d)$, $\boldsymbol{\mathcal{G}}^{\mathrm{c},k}(X)$ complement in $\mathcal{P}^k(X)^d$.

Trimmed spaces: face/cell gradients and curls have to be projected on trimmed spaces.

- Gradients in $\mathcal{P}^k(X)^d$ projected on $\boldsymbol{\mathcal{R}}^{k-1}(X) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(X)$.
- Curls in $\mathcal{P}^k(X)^d$ projected on $\boldsymbol{\mathcal{G}}^{k-1}(X) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(X)$.

Going global

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},\Omega}} \underline{X}^k_{\mathrm{grad},\Omega} \xrightarrow{\underline{\boldsymbol{G}}_\Omega} \underline{\boldsymbol{X}}^k_{\mathrm{curl},\Omega} \xrightarrow{\underline{\boldsymbol{C}}^k_\Omega} \underline{\boldsymbol{X}}^k_{\mathrm{div},\Omega} \xrightarrow{D^k_\Omega} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

Global spaces/operators: by patching local spaces/operators.

- Additional challenges:
  - Global exactness, especially $\mathrm{Ker}\, D^k_\Omega \subset \mathrm{Im}\, \underline{\boldsymbol{C}}^k_\Omega$.
  - Poincaré inequalities (for stability), e.g.

$$\|\underline{\boldsymbol{v}}_\Omega\|_{\underline{\boldsymbol{X}}^k_{\mathrm{curl},\Omega}} \le M \|\underline{\boldsymbol{C}}^k_\Omega \underline{\boldsymbol{v}}_\Omega\|_{\underline{\boldsymbol{X}}^k_{\mathrm{div},\Omega}} \quad \forall v_\Omega \in (\underline{\boldsymbol{X}}^k_{\mathrm{curl},\Omega})^\perp.$$

Plan

Model and exact solution

$$\begin{cases} \boldsymbol{\sigma} - \operatorname{curl} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \operatorname{curl} \boldsymbol{\sigma} = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \partial\Omega. \end{cases}$$

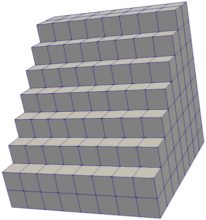
on $\Omega = (0,1)^3$, with exact solution

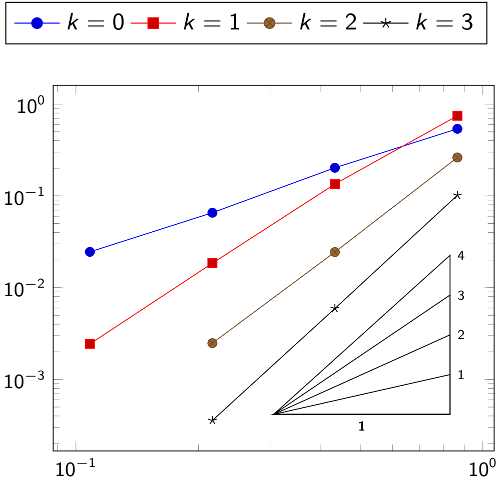
$$\boldsymbol{\sigma}(\boldsymbol{x}) = 3\pi \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \\ 0 \\ -\cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \end{pmatrix},$$

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} \cos(\pi x_1)\sin(\pi x_2)\sin(\pi x_3) \\ -2\sin(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \\ \sin(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \end{pmatrix}.$$

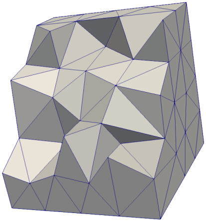
- All spaces and operators implemented in the HArD::Core3D library.

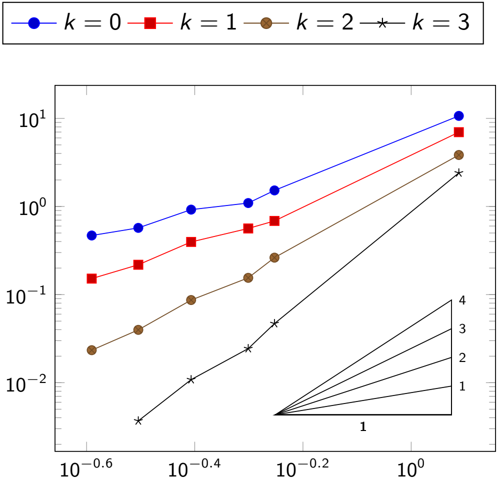
Convergence graphs in energy norm: cubic cells



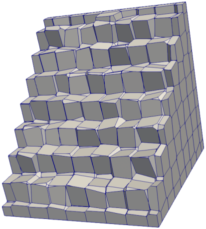


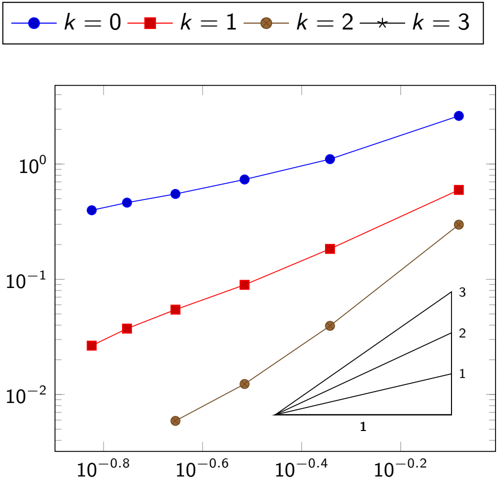
Convergence graphs in energy norm: tetrahedral cells



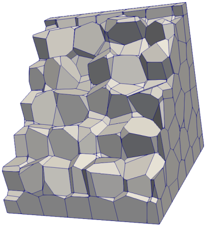


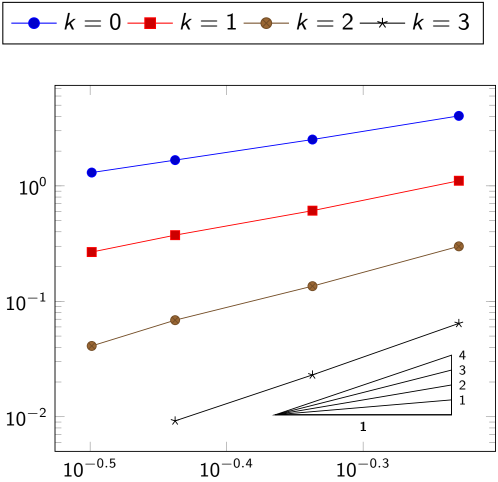
Convergence graphs in energy norm: Voronoi cells 1





Convergence graphs in energy norm: Voronoi cells 2





Plan

1. **Hybrid High-Order method**
   - An inspiring remark
   - Description of the HHO scheme
   - Miscible flow in porous media

2. **Fully Discrete de Rham sequence**
   - Principles of discrete exact sequence
   - Fully discrete de Rham sequence
   - Application to magnetostatics

3. **High-order schemes for stationnary Stefan/PME models**
   - Towards a stable numerical approximation
   - High-order approximations
   - Numerical tests

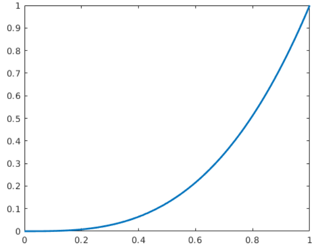
Degenerate equation

Stationary version of Stefan/Porous Medium Equation equations:

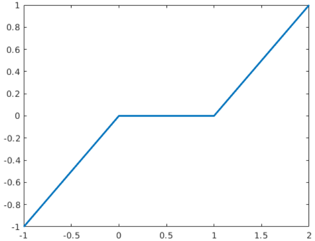
$$u - \mathrm{div}(\Lambda\nabla\zeta(u)) = f - \mathrm{div}(F) \text{ in } \Omega,$$
$$\zeta(u) = 0 \text{ on } \partial\Omega.$$

Non-linearity:

Porous medium: $\zeta(u) = |u|^{m-1}u$



Stefan: ζ with plateau



Plan

First challenge: stability...

$\Lambda = Id$ to simplify.

Continuous model: multiply by $\zeta(u)$, integrate by parts, use $\zeta(s)s \geq 0$:

$$\int_\Omega |\nabla\zeta(u)|^2 \leq \int_\Omega u\zeta(u) + \nabla\zeta(u)\cdot\nabla\zeta(u) = \int_\Omega f\zeta(u) + \int_\Omega F\cdot\nabla\zeta(u).$$

First challenge: stability...

$\Lambda = Id$ to simplify.

Continuous model: multiply by $\zeta(u)$, integrate by parts, use $\zeta(s)s \geq 0$:

$$\int_\Omega |\nabla\zeta(u)|^2 \leq \int_\Omega u\zeta(u) + \nabla\zeta(u)\cdot\nabla\zeta(u) = \int_\Omega f\zeta(u) + \int_\Omega F\cdot\nabla\zeta(u).$$

Use Poincaré inequality on $\zeta(u)$ and Cauchy–Schwarz inequalities:

$$\int_\Omega |\nabla\zeta(u)|^2 \leq C(f,F)$$

$\rightsquigarrow$ Bound on $\zeta(u)$, translates into a bound on u.

First challenge: stability...

Discrete version: think conforming $\mathbb{P}_1$ finite elements: find $u_h \in V_h$ such that, for all $v_h \in V_h$,

$$\int_\Omega u_h v_h + \nabla\zeta(u_h)\cdot\nabla v_h = \int_\Omega f v_h + \int_\Omega F\cdot\nabla v_h.$$

Stability: $v_h = \zeta(u_h)$ not a valid test function, we need to take $v_h = u_h$:

$$\int_\Omega u_h^2 + \zeta'(u_h)|\nabla u_h|^2 \le \int_\Omega f u_h + \int_\Omega F\cdot\nabla u_h.$$

- Last term cannot be estimated by left-hand side...

Towards stability

- For $w_h \in V_h$, define $[\zeta(w)]_h$ by nodal values: unique function in V_h that has the values $\zeta(w_h(\mathrm{s}))$ at the nodes s of V_h *(nodes=degrees of freedom)*.

Scheme: find $u_h \in V_h$ such that, for all $v_h \in V_h$,

$$\int_\Omega u_h v_h + \nabla[\zeta(u)]_h \cdot \nabla v_h = \int_\Omega f v_h + \int_\Omega F \cdot \nabla v_h.$$

Towards stability

- For $w_h \in V_h$, define $[\zeta(w)]_h$ by nodal values: unique function in V_h that has the values $\zeta(w_h(\mathrm{s}))$ at the nodes s of V_h *(nodes=degrees of freedom)*.

Scheme: find $u_h \in V_h$ such that, for all $v_h \in V_h$,

$$\int_\Omega u_h v_h + \nabla[\zeta(u)]_h \cdot \nabla v_h = \int_\Omega f v_h + \int_\Omega F \cdot \nabla v_h.$$

Stability: $v_h = [\zeta(u)]_h$ is a valid test function!

$$\int_\Omega u_h[\zeta(u)]_h + |\nabla[\zeta(u)]_h|^2 \leq \int_\Omega f[\zeta(u)]_h + \int_\Omega F \cdot \nabla[\zeta(u)]_h.$$

- What to do with the first term? It was ≥ 0 in the continuous case, but now?

Solution to stability: mass-lumping

$$\int_\Omega u_h[\zeta(u)]_h + |\nabla[\zeta(u)]_h|^2 \le \int_\Omega f[\zeta(u)]_h + \int_\Omega F\cdot\nabla[\zeta(u)]_h.$$

At the nodes, $u_h(\mathsf{s})[\zeta(u)]_h(\mathsf{s}) = u_h(\mathsf{s})\zeta(u_h(\mathsf{s})) \ge 0$.

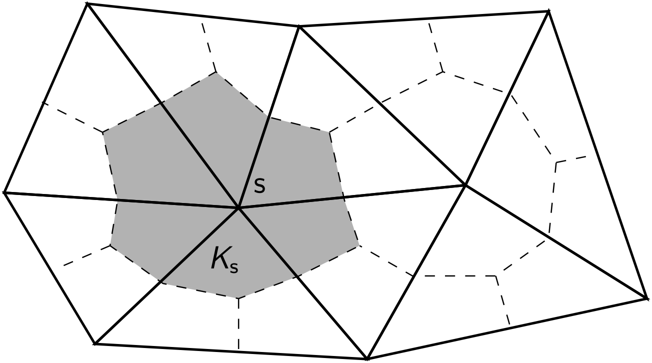
- Replace u_h in the reaction term by a quantity that only uses nodal values.

Solution to stability: mass-lumping

- Let $(K_{\mathrm{s}})_{\mathrm{s\ node}}$ be a partition of Ω, each K_{s} being built "around" s, and set

$$\Pi_h u_h : \Omega \to \mathbb{R} \qquad (\Pi_h u_h)_{|K_{\mathrm{s}}} = u_h(\mathrm{s}) \quad \forall \mathrm{s}.$$

Example for $\mathbb{P}_1$ FE:



Solution to stability: mass-lumping

Mass-lumped scheme: find $u_h \in V_h$ such that, for all $v_h \in V_h$,

$$\int_\Omega \Pi_h u_h \, \Pi_h v_h + \nabla[\zeta(u)]_h \cdot \nabla v_h = \int_\Omega f \Pi_h v_h + \int_\Omega F \cdot \nabla v_h.$$

Stability: make $v_h = [\zeta(u)]_h$ as before, and use (magic of mass-lumping!)

$$\Pi_h[\zeta(u)]_h = \zeta(\Pi_h u_h)$$

to get

$$\int_\Omega \underbrace{\Pi_h u_h \, \zeta(\Pi_h u_h)}_{\geq 0} + |\nabla[\zeta(u)]_h|^2 \leq \int_\Omega f[\zeta(u)]_h + \int_\Omega F \cdot \nabla[\zeta(u)]_h.$$

Plan

Mass-lumping and high-order: a conundrum

- Mass-lumping = approximate reaction terms by piecewise constant functions

  Order 1 consistency error at best...

- How do we recover a high-order scheme?

Setting

Discrete space and nodes: $\mathcal{T}_h$ a mesh of the domain,

$$X_h = \{v = (v_i)_{i\in I} \,:\, v_i \in \mathbb{R}\,,\; v_i = 0 \text{ if } i \in I_{\partial\Omega}\}.$$

There is $(\mathrm{x}_i)_{i\in I}$ and, for each $K \in \mathcal{T}_h$, $I_K \subset I$ such that $\mathrm{x}_i \in \overline{K}$ if $i \in I_K$.

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High-order reconstruction: $\Pi_h^{\rm HO} : X_h \to \mathbb{P}_k(\mathcal{T}_h)$. For all $v \in X_h$, $K \in \mathcal{T}_h$ and $i \in I_K$, $v_i = (\Pi_h^{\rm HO} v)_{|K}(\mathrm{x}_i)$.

Setting

Discrete space and nodes: $\mathcal{T}_h$ a mesh of the domain,

$$X_h = \{v = (v_i)_{i \in I} \;:\; v_i \in \mathbb{R}\,,\; v_i = 0 \text{ if } i \in I_{\partial\Omega}\}.$$

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Mass-lumping: $\mathcal{U}_h = (U_i)_{i \in I}$ partition of Ω and $\Pi_h : X_h \to \mathbb{P}_0(\mathcal{U}_h)$ piecewise constant reconstruction.
For all $i \in I$ and $K \in \mathcal{T}_h$, $U_i \cap K \neq \emptyset$ only if $i \in I_K$.

Setting

Discrete space and nodes: $\mathcal{T}_h$ a mesh of the domain,

$$X_h = \{v = (v_i)_{i\in I} \ :\ v_i \in \mathbb{R}\,,\ v_i = 0 \text{ if } i \in I_{\partial\Omega}\}.$$

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For all $i \in I$ and $K \in \mathcal{T}_h$, $U_i \cap K \neq \emptyset$ only if $i \in I_K$.

High-order gradient reconstruction: $\nabla_h^{\rm HO} : X_h \to L^\infty(\Omega)^d$.

Setting

Discrete space and nodes: $\mathcal{T}_h$ a mesh of the domain,

$$X_h = \{v = (v_i)_{i\in I} \ : \ v_i \in \mathbb{R}\,,\ v_i = 0 \text{ if } i \in I_{\partial\Omega}\}.$$

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For all $i \in I$ and $K \in \mathcal{T}_h$, $U_i \cap K \neq \emptyset$ only if $i \in I_K$.

High-order gradient reconstruction: $\nabla_h^{\mathrm{HO}} : X_h \to L^\infty(\Omega)^d$.

Quadrature: $Q_h : C(\mathcal{T}_h) \to L^\infty(\Omega)$ given by:

$$(Q_h w)_{|K} = \sum_{i\in I_K} w_{|K}(\mathrm{x}_i) 1_{U_i\cap K} \qquad \forall K \in \mathcal{T}_h.$$

Scheme

Non-linear function of vectors: if $v = (v_i)_{i\in I} \in X_h$ and $g : \mathbb{R} \to \mathbb{R}$, we define

$$g(v) \in X_h \text{ such that } (g(v))_i = g(v_i) \quad \forall i \in I.$$

Scheme

Non-linear function of vectors: if $v = (v_i)_{i\in I} \in X_h$ and $g : \mathbb{R} \to \mathbb{R}$, we define

$$g(v) \in X_h \text{ such that } (g(v))_i = g(v_i) \quad \forall i \in I.$$

Scheme:

Find $u_h \in X_h$ such that, for all $v_h \in X_h$,

$$\int_\Omega \Pi_h u_h \, \Pi_h v_h + \Lambda \nabla_h^{\rm HO} \zeta(u_h) \cdot \nabla_h^{\rm HO} v_h = \int_\Omega Q_h f \, \Pi_h v.$$

Assumption on Q_h

Exactness of quadrature: we assume that Q_h is locally exact of degree $k+\ell$, that is

$$\left.\begin{array}{c}\displaystyle\int_K q = \int_K Q_h q \left(= \sum_{i\in I_K} |U_i \cap K| q(\mathrm{x}_i) \right) \\ \forall K \in \mathcal{T}_h\,,\ \forall q \in \mathbb{P}_{k+\ell}.\end{array}\right\} \quad QR_k(\ell).$$

Error estimate

- Broken Sobolev space:

$$W^{\ell+2,\infty}(\mathcal{T}_h) = \{w \in L^\infty(\Omega) \,:\, w_{|K} \in W^{\ell+2,\infty}(K) \quad \forall K \in \mathcal{T}_h\}.$$

- Defect of conformity of the (underlying high-order) method: for any $\boldsymbol{\psi} \in L^2(\Omega)^d$ with $\operatorname{div}\boldsymbol{\psi} \in L^2(\Omega)$,

$$W_h^{\rm HO}(\boldsymbol{\psi}) = \max_{w_h \in X_h \setminus \{0\}} \frac{1}{\|\nabla_h^{\rm HO} w_h\|_\Omega} \left| \int_\Omega \Pi_h^{\rm HO} w_h \operatorname{div}\boldsymbol{\psi} q + \nabla_h^{\rm HO} w_h \cdot \psi \right|$$

- Broken Sobolev space:

$$W^{\ell+2,\infty}(\mathcal{T}_h)=\{w\in L^\infty(\Omega)\,:\,w_{|K}\in W^{\ell+2,\infty}(K)\quad\forall K\in\mathcal{T}_h\}.$$

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$$W_h^{\rm HO}(\psi)=\max_{w_h\in X_h\backslash\{0\}}\frac{1}{\|\nabla_h^{\rm HO}w_h\|_\Omega}\left|\int_\Omega \Pi_h^{\rm HO}w_h\,\mathrm{div}\,\psi q+\nabla_h^{\rm HO}w_h\cdot\psi\right|$$

Theorem (D.-Eymard, 2018)

Under $QR_k(\ell)$, if u and f belong to $W^{\ell+2,\infty}(\mathcal{T}_h)$ then

$$\|\nabla_h^{\rm HO}[I_h\zeta(u)-\zeta(u_h)]\|_\Omega \le CW_{h,\rm HO}(\Lambda\nabla\zeta(u))+C\|\nabla_h^{\rm HO}I_h\zeta(u)-\nabla\zeta(u)\|_\Omega+Ch^{\ell+2},$$

with $I_h\zeta(u)=(\zeta(u)(\mathrm{x}_i))_{i\in I}$ interpolate of $\zeta(u)$ on X_h.

Error estimate

Theorem (D.-Eymard, 2018)

Under $QR_k(\ell)$, if u and f belong to $W^{\ell+2,\infty}(\mathcal{T}_h)$ then

$$\|\nabla_h^{\rm HO}[I_h\zeta(u)-\zeta(u_h)]\|_\Omega \le \underbrace{CW_h^{\rm HO}(\Lambda\nabla\zeta(u))}_{\mathcal{O}(h^k)} + \underbrace{C\|\nabla_h^{\rm HO}I_h\zeta(u)-\nabla\zeta(u)\|_\Omega}_{\mathcal{O}(h^k)} + Ch^{\ell+2},$$

with $I_h\zeta(u)=(\zeta(u)(\mathrm{x}_i))_{i\in I}$ interpolate of $\zeta(u)$ on X_h.

- Real limiting factor is $h^{\ell+2}$, dictated by $QR_k(\ell)$ (and regularity of u and f).

Plan

1. Hybrid High-Order method
   - An inspiring remark
   - Description of the HHO scheme
   - Miscible flow in porous media
2. Fully Discrete de Rham sequence
   - Principles of discrete exact sequence
   - Fully discrete de Rham sequence
   - Application to magnetostatics
3. **High-order schemes for stationnary Stefan/PME models**
   - Towards a stable numerical approximation
   - High-order approximations
   - **Numerical tests**

Quadrature rules (dimension 1)

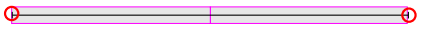
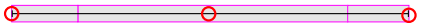
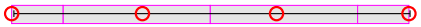
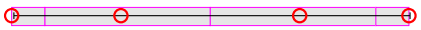
Name	$(\boldsymbol{x}_i)_{i\in I_K}$	$\left(\frac{\lvert U_i\cap K\rvert}{\lvert K\rvert}\right)_{i\in I_K}$	DOE	Illustration
Trapezoidal	(a, b)	$\left(\frac{1}{2}, \frac{1}{2}\right)$	1	
Simpson	$\left(a, \frac{a+b}{2}, b\right)$	$\left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right)$	3	
Equi6	$\left(a, \frac{2a+b}{3}, \frac{a+2b}{3}, b\right)$	$\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right)$	1	
Equi8	$\left(a, \frac{2a+b}{3}, \frac{a+2b}{3}, b\right)$	$\left(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\right)$	3	
Gauss–Lobatto	$\left(a, \frac{5+\sqrt{5}}{10}a + \frac{5-\sqrt{5}}{10}b, \frac{5-\sqrt{5}}{10}a + \frac{5+\sqrt{5}}{10}b, b\right)$	$\left(\frac{1}{12}, \frac{5}{12}, \frac{5}{12}, \frac{1}{12}\right)$	5	

Table: Examples of quadrature rules in dimension $d = 1$ for $K = (a, b)$. DOE stands for degree of exactness (corresponds to $k + \ell$).

Gradient discretisations for $\mathbb{P}_k$ finite elements

Name	Degree k	Quadrature rule	ℓ
$\mathcal{D}_1^{\mathrm{g}}(0)$	1	Trapezoidal	0
$\mathcal{D}_2^{\mathrm{g}}(1)$	2	Simpson	1
$\mathcal{D}_3^{\mathrm{g}}(-)$	3	Equi6	–
$\mathcal{D}_3^{\mathrm{g}}(0)$	3	Equi8	0
$\mathcal{D}_3^{\mathrm{g}}(2)$	3	Gauss–Lobatto	2

Table: Mass-lumped GDs for $\mathbb{P}_k$ Finite Element in dimension $d = 1$. These methods satisfy $QR_k(\ell)$ with the corresponding k, ℓ. $\mathrm{g} = \mathrm{u}$ for uniform meshes, $\mathrm{g} = \mathrm{r}$ for random meshes.

Output

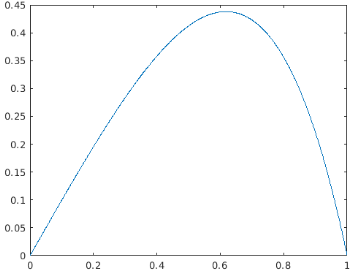
- We provide (C, α) such that

$$\|\nabla_h^{\mathrm{HO}}(I_h\zeta(u) - \zeta(u_h))\|_\Omega \approx C\mathrm{Card}(I)^{-\alpha/d}.$$

$\alpha \sim$ *rate of convergence in meshsize.*

Linear model: $\zeta(u) = u$

Test R: regular exact solution $u(x) = x(1-x)e^x$.

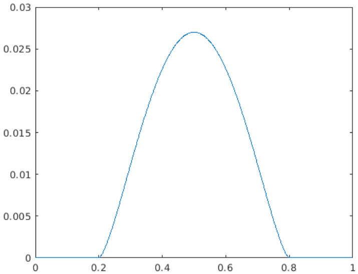


Results

	$\mathcal{D}_1^{\mathrm{u}}(0)$	$\mathcal{D}_1^{\mathrm{r}}(0)$	$\mathcal{D}_2^{\mathrm{u}}(1)$	$\mathcal{D}_2^{\mathrm{r}}(1)$	$\mathcal{D}_3^{\mathrm{u}}(-)$	$\mathcal{D}_3^{\mathrm{r}}(-)$	$\mathcal{D}_3^{\mathrm{u}}(0)$	$\mathcal{D}_3^{\mathrm{r}}(0)$	$\mathcal{D}_3^{\mathrm{u}}(2)$	$\mathcal{D}_3^{\mathrm{u}}(2)$
C	0.44	0.31	0.14	0.13	0.15	0.15	0.2	0.2	0.0002	0.00024
α	2	1.9	3	2.98	1	1	2	1.99	2.95	2.97
$\min(k, \ell+2)$	1	1	2	2	–	–	2	2	3	3

Porous medium equation: $\zeta(u) = \max(u, 0)^2$

Test P1: exact solution with $s^{3/2}$ singularity – piecewise smooth, singularity not aligned with meshes.

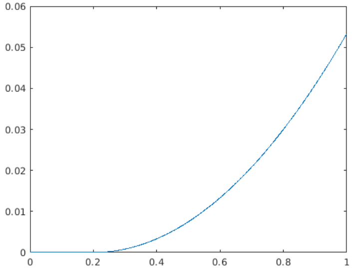


Results

	$\mathcal{D}_1^{\mathrm{u}}(0)$	$\mathcal{D}_1^{\mathrm{r}}(0)$	$\mathcal{D}_2^{\mathrm{u}}(1)$	$\mathcal{D}_2^{\mathrm{r}}(1)$	$\mathcal{D}_3^{\mathrm{u}}(-)$	$\mathcal{D}_3^{\mathrm{r}}(-)$	$\mathcal{D}_3^{\mathrm{u}}(0)$	$\mathcal{D}_3^{\mathrm{r}}(0)$	$\mathcal{D}_3^{\mathrm{u}}(2)$	$\mathcal{D}_3^{\mathrm{u}}(2)$
C	12	12	4.3	16	0.41	0.42	2.7	2.3	1.2	0.22
α	2	1.98	2.45	2.69	1.03	1.03	1.99	1.95	2.42	1.98
$\min(k, \ell+2)$	1	1	2	2	–	–	2	2	3	3

Porous medium equation: $\zeta(u) = \max(u,0)^2$

Test P2: exact solution $u(x) = \max(x - 1.5, 0)^2/12$ corresponding to $f = 0$.

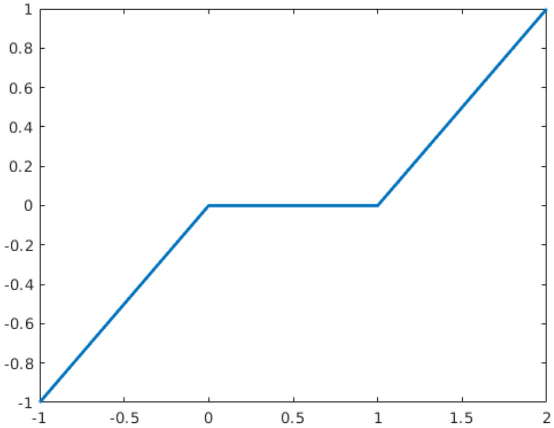


Results

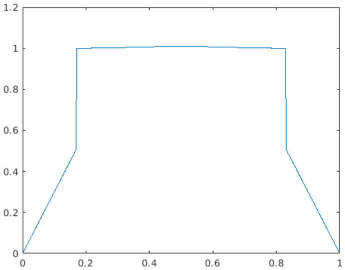
	$\mathcal{D}_1^{\mathrm{u}}(0)$	$\mathcal{D}_1^{\mathrm{r}}(0)$	$\mathcal{D}_2^{\mathrm{u}}(1)$	$\mathcal{D}_2^{\mathrm{r}}(1)$	$\mathcal{D}_3^{\mathrm{u}}(-)$	$\mathcal{D}_3^{\mathrm{r}}(-)$	$\mathcal{D}_3^{\mathrm{u}}(0)$	$\mathcal{D}_3^{\mathrm{r}}(0)$	$\mathcal{D}_3^{\mathrm{u}}(2)$	$\mathcal{D}_3^{\mathrm{u}}(2)$
C	0.19	0.38	0.17	0.18	0.14	0.15	0.24	0.24	0.27	0.0014
α	2	1.97	2.99	2.98	1	1	2	1.99	3.1	3.46
$\min(k, \ell+2)$	1	1	2	2	–	–	2	2	3	3

Stefan model

Nonlinearity: $\zeta(u) = 0$ if $0 \leq u \leq 1$, slope 1 otherwise.



Test S1: $f(x) = 3(0.5 - |0.5 - x|)$, exact solution (computable up to parameters that are numerically evaluated):



Results

	$\mathcal{D}_1^{\mathrm{u}}(0)$	$\mathcal{D}_1^{\mathrm{r}}(0)$	$\mathcal{D}_2^{\mathrm{u}}(1)$	$\mathcal{D}_2^{\mathrm{r}}(1)$	$\mathcal{D}_3^{\mathrm{u}}(-)$	$\mathcal{D}_3^{\mathrm{r}}(-)$	$\mathcal{D}_3^{\mathrm{u}}(0)$	$\mathcal{D}_3^{\mathrm{r}}(0)$	$\mathcal{D}_3^{\mathrm{u}}(2)$	$\mathcal{D}_3^{\mathrm{r}}(2)$
C	12	3.2	0.62	1	0.37	0.38	0.72	0.28	0.36	0.39
α	1.87	1.66	1.54	1.65	1.03	1.05	1.61	1.53	1.58	1.59
$\min(k, \ell+2)$	1	1	2	2	–	–	2	2	3	3

Stefan model

What happens with $\mathcal{D}_3^{\mathrm{g}}(0)$ and $\mathcal{D}_3^{\mathrm{g}}(2)$? No improvement over low order?

- Due to lack of regularity of $\zeta(u)$, only belongs to H^2 as $(\zeta(u))'' = u - f$ is discontinuous.

- Recover $\mathcal{O}(h^2)$ convergence if error calculated far from discontinuity; but not $\mathcal{O}(h^3)$ even for $\mathcal{D}_3^{\mathrm{g}}(2)$.

Degree $k = 3$, various quadrature rules, for PME and Stefan ($f = 0$).

		$\mathcal{D}_3^{\mathrm{u}}(-)$	$\mathcal{D}_3^{\mathrm{r}}(-)$	$\mathcal{D}_3^{\mathrm{u}}(0)$	$\mathcal{D}_3^{\mathrm{r}}(0)$	$\mathcal{D}_3^{\mathrm{u}}(2)$	$\mathcal{D}_3^{\mathrm{u}}(2)$
Test R	C	0.15	0.17	0.22	0.22	0.023	0.011
	α	1.01	1.02	2	1.99	3.25	3.01
Test P1	C	0.42	0.42	2.9	2.9	1.4	1
	α	1.03	1.03	1.98	1.97	2.39	2.32
Test P2	C	0.15	0.15	0.27	0.28	0.039	0.019
	α	1.01	1.01	2	2	3.42	3.08
Test S2	C	0.09	0.085	0.082	0.074	0.054	0.058
	α	1.01	1	1.5	1.57	1.49	1.58
	$\min(k, \ell + 2)$	–	–	2	2	3	3

Conclusion

Hybrid High-Order method

- High-order method for diffusion problems on polytopal meshes.
- Uses element and face unknowns to reconstruct higher-order potential.
- Optimal rates of convergence, for many models of practical interest.
- Finite volume method, provides appropriate fluxes for coupling with transport.

Conclusion

Discrete de Rham sequence

- Preserve exactness property at discrete level: essential in some applications.
- Fully computable (purely polynomial) spaces and operators.
- In its infancy, lot of work remains to be done...

Conclusion

High-order schemes for Stefan/PME

- Stefan/PME requires mass lumping.
- Higher order schemes still possible, provided a key quadrature rule is respected.
- Convergence benefits from high order, unless restricted by regularity of solution (even then, local improvement is possible).

Bibliography

Main books/papers:

- *High-order mass-lumped schemes for nonlinear degenerate elliptic equations*. J. Droniou and R. Eymard. SIAM J. Numer. Anal. 58 (1), pp. 153–188, 2020. doi: 10.1137/19M1244500. https://arxiv.org/abs/1902.04662.
- The Hybrid High-Order Method for Polytopal Meshes: Design, Analysis, and Applications. D. A. Di Pietro and J. Droniou. Springer, MS&A vol. 19, 2020, 551p. doi: 10.1007/978-3-030-37203-3. https://hal.archives-ouvertes.fr/hal-02151813.
- *Fully discrete polynomial de Rham sequences of arbitrary degree on polygons and polyhedra*. D. Di Pietro, J. Droniou, and F. Rapetti. Math. Models Methods Appl. Sci. 44p, 2020. https://arxiv.org/abs/1911.03616.
- *An arbitrary-order method for magnetostatics on polyhedral meshes based on a discrete de Rham sequence*. D. A. Di Pietro and J. Droniou, 31p, submitted, 2020. https://arxiv.org/abs/2005.06890.

Thanks.