Convergence and error estimates of numerical schemes for the porous medium equation

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Joint works with K.-N. Le, C. Cancès, C. Guichard, G. Manzini, M. Bastisdas, and I. S. Pop

Australian Government

Australian Research Council

Discrete Functional Analysis: bridging pure and numerical mathematics

References

- Convergence by compactness [D, Le; 2020]
- Error estimates [Cancès, D, Guichard, Manzini, Bastisdas, Pop; 2021]

Plan

The porous medium equation: strong formulation

$$\begin{aligned} \partial_t \bar{u} - \Delta(|\bar{u}|^{m-1}\bar{u}) &= f && \text{in } (0,T)\times\Omega, \\ \bar{u} &= 0 && \text{on } (0,T)\times\partial\Omega, \\ \bar{u}(0,\cdot) &= u_{\text{ini}} && \text{on } \Omega. \end{aligned}$$

- Ω open bounded connected in $\mathbb{R}^d$ $(d = 2, 3)$, $\Omega_T := (0,T)\times\Omega$.
- $u_{\text{ini}} \in L^{m+1}(\Omega)$, $f \in L^2(\Omega_T)$.
- $m \in (0,\infty)$:

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- $u_{\text{ini}} \in L^{m+1}(\Omega)$, $f \in L^2(\Omega_T)$.
- $m \in (0,\infty)$:
  - $m < 1$: fast diffusion.
  - $m = 1$: heat equation (linear).
  - $m > 1$: slow diffusion.

The porous medium equation: weak formulation

A weak solution to the PME is u such that

1. $\bar{u} \in C([0,T]; L^{m+1}(\Omega)_{\mathrm{w}})$ and $\bar{u}(0,\cdot) = u_0$ in $L^{m+1}(\Omega)$,
2. $|\bar{u}|^{m-1}\bar{u} \in L^2(0,T; H_0^1(\Omega))$,
3. $\partial_t \bar{u} \in L^2(0,T; H^{-1}(\Omega))$, and for any $\phi \in L^2(0,T; H_0^1(\Omega))$

$$\int_0^T \langle \partial_t \bar{u}(t), \phi(t) \rangle_{H^{-1},H_0^1} + \int_{\Omega_T} \nabla(|\bar{u}|^{m-1}\bar{u}) \cdot \nabla\phi = \int_{\Omega_T} f\phi.$$

The porous medium equation: numerical approximation

Components of the spatial discretisation

- $X_{\mathcal{D},0}$ a finite-dimensional space, with canonical basis $(\mathbf{e}_i)_{i\in I}$,
- $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to L^\infty(\Omega)$ a piecewise constant function reconstruction: for some partition $(\Omega_i)_{i\in I}$ of Ω,

$$\forall u = \sum_i u_i \mathbf{e}_i \in X_{\mathcal{D},0}\,, \quad (\Pi_{\mathcal{D}} u)_{|\Omega_i} = u_i\,.$$

- $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to L^\infty(\Omega)^d$ a gradient reconstruction, s.t. $u \mapsto \|\nabla_{\mathcal{D}} u\|_{L^2}$ is a norm on $X_{\mathcal{D},0}$.

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Non-linear transformations in $X_{\mathcal{D},0}$: if $g : \mathbb{R} \to \mathbb{R}$ and $u \in X_{\mathcal{D},0}$, we define $g(u) \in X_{\mathcal{D},0}$ by:

$$g(u) = \sum_{i\in I} g(u_i)\mathbf{e}_i \quad \text{where } u = \sum_{i\in I} u_i \mathbf{e}_i\,.$$

- The piecewise constant function reconstruction property ensures that:

$$\Pi_{\mathcal{D}} g(u) = g(\Pi_{\mathcal{D}} u).$$

The porous medium equation: numerical approximation

Extension as time-space operators: constant-step discretisation of $(0,T)$, with times $t^{(n)} = n\delta t$. If $u = (u^{(n)})_{n=0,\ldots,N} \in X_{\mathcal{D},0}^{N+1}$, set

$$\Pi_{\mathcal{D}} u(t,\cdot) = \Pi_{\mathcal{D}} u^{(n+1)} \text{ and } \nabla_{\mathcal{D}} u(t,\cdot) = \nabla_{\mathcal{D}} u^{(n+1)} \quad \forall t \in (t^{(n)}, t^{(n+1)}].$$

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Discrete time stepping: for $u \in X_{\mathcal{D},0}^{N+1}$, we define

$$\delta_{\mathcal{D}} u(t,\cdot) = \frac{\Pi_{\mathcal{D}} u^{(n+1)} - \Pi_{\mathcal{D}} u^{(n)}}{\delta t} \quad \forall t \in (t^{(n)}, t^{(n+1)}].$$

The porous medium equation: numerical approximation

Scheme: let $u^{(0)} \in X_{\mathcal{D},0}$ be a suitable interpolate of u_{ini}, and find $u = (u^{(n)})_{n=0,\cdots,N}$ s.t.

$$\int_{\Omega_T} \delta_{\mathcal{D}} u \Pi_{\mathcal{D}} \phi + \int_{\Omega_T} \nabla_{\mathcal{D}} |u|^{m-1} u \cdot \nabla_{\mathcal{D}} \phi = \int_{\Omega_T} f \Pi_{\mathcal{D}} \phi$$

for all 'test function' $\phi = \left(\phi^{(n)}\right)_{n=0,\cdots,N} \in X_{\mathcal{D},0}^{N+1}$.

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for all 'test function' $\phi = \left(\phi^{(n)}\right)_{n=0,\cdots,N} \in X_{\mathcal{D},0}^{N+1}$.

- Large choice of $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$, each one leading to a specific scheme.
- Many classical schemes obtained by specific choices of $\mathcal{D}$: (mass-lumped) conforming and non-conforming finite elements, finite volumes, discontinuous Galerkin, hybrid mimetic mixed, virtual elements, etc.

This approach is called the Gradient Discretisation Method.
$\mathcal{D}$ is a gradient discretisation.

Plan

Key properties of $\mathcal{D}$

A sequence $(\mathcal{D}_\ell)_{\ell\in\mathbb{N}}$ of gradient discretisations is...

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Consistent if, for all $\phi \in H^1_0(\Omega)$, there is $v_\ell \in X_{\mathcal{D}_\ell,0}$ s.t., as $\ell \to \infty$,

$$\Pi_{\mathcal{D}_\ell} v_\ell \to \phi \text{ in } L^2(\Omega) \quad \text{and} \quad \nabla_{\mathcal{D}_\ell} v_\ell \to \nabla\phi \text{ in } L^2(\Omega)^d.$$

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Limit-conforming if, for all $\boldsymbol{\psi} \in L^2(\Omega)^d$ s.t. $\operatorname{div}\boldsymbol{\psi} \in L^2(\Omega)$, as $\ell \to \infty$,

$$\max_{v\in X_{\mathcal{D}_\ell,0}\setminus\{0\}} \frac{\left|\int_\Omega \nabla_{\mathcal{D}_\ell} v \cdot \boldsymbol{\psi} + \Pi_{\mathcal{D}_\ell} v \operatorname{div}\boldsymbol{\psi}\right|}{\left\|\nabla_{\mathcal{D}_\ell} v\right\|_{L^2(\Omega)}} \to 0.$$

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Compact if, for all $v_\ell \in X_{\mathcal{D}_\ell,0}$ such that $(\|\nabla_{\mathcal{D}_\ell} v_\ell\|_{L^2(\Omega)})_{\ell \in \mathbb{N}}$ is bounded, $(\Pi_{\mathcal{D}_\ell} v_\ell)_{\ell \in \mathbb{N}}$ is relatively compact in $L^2(\Omega)$.

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All classical methods satisfy these properties, under standard mesh regularity assumptions.

Convergence result

Theorem (D.-Le, 2020)

There is a solution to the gradient scheme and, if $(\mathcal{D}_\ell)_{\ell\in\mathbb{N}}$ is consistent, limit-conforming and compact and u_ℓ is a solution for $\mathcal{D}_\ell$, then, up to a subsequence as $\ell \to \infty$,

$$\Pi_{\mathcal{D}_\ell} u_\ell \to \bar{u} \qquad \text{strongly in } L^\infty(0,T;L^{m+1}(\Omega)),$$
$$\nabla_{\mathcal{D}_\ell}(|u_\ell|^{m-1}u_\ell) \to \nabla(|\bar{u}|^{m-1}\bar{u}) \qquad \text{strongly in } L^2(\Omega_T).$$

where $\bar{u}$ is a solution to the PME.

- First uniform-in-time convergence result in $L^{m+1}(\Omega)$ (semi-group approach only provides $L^1(\Omega)$).
- Valid in whole range: $m > 1$ (slow diffusion) and $m < 1$ (fast diffusion).
- Convergence for more general models: $\Delta(|\bar{u}|^{m-1}\bar{u}) \rightsquigarrow \operatorname{div}(\Lambda(\bar{u})\nabla\beta(\bar{u}))$.

Plan

Special case: nodal gradient discretisation

DOFs attached to nodes: we assume that the coefficients $(u_i)_{i\in I}$ on the basis $(\mathbf{e}_i)_{i\in I}$ of $X_{\mathcal{D},0}$ are attached to points $(\boldsymbol{x}_i)_{i\in I}$ in $\overline{\Omega}$.

A suitable interpolator is then $I_{\mathcal{D}} : C(\overline{\Omega}) \longrightarrow X_{\mathcal{D},0}$ defined by

$$(I_{\mathcal{D}}\phi)_i = \phi(\boldsymbol{x}_i) \qquad \forall i \in I.$$

Two measures of consistency

Strong consistency of gradient: for $\phi \in H_0^1(\Omega)$,

$$S_{\mathcal{D}}^{\nabla}(\phi) = \|\nabla_{\mathcal{D}}(I_{\mathcal{D}}\phi) - \nabla\phi\|_{L^2(\Omega)} .$$

Two measures of consistency

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$$S_{\mathcal{D}}^{\nabla}(\phi) = \|\nabla_{\mathcal{D}}(I_{\mathcal{D}}\phi) - \nabla\phi\|_{L^2(\Omega)}\,.$$

Weak consistency of function reconstruction: for $\omega \in L^2(\Omega)$,

$$S_{\mathcal{D}}^{\Pi,\star}(\omega) = |\Pi_{\mathcal{D}}(I_{\mathcal{D}}\omega) - \omega|_{\mathcal{D},\star},$$

where $|\cdot|_{\mathcal{D},\star}$ is the discrete H^{-1}-seminorm defined for $\xi \in L^2(\Omega)$ by

$$|\xi|_{\mathcal{D},\star} = \max\left\{\int_\Omega \xi\Pi_{\mathcal{D}}v \;:\; v \in X_{\mathcal{D},0}\,,\ \|\nabla_{\mathcal{D}}v\|_{L^2(\Omega)} \le 1\right\}.$$

Error estimates

Set $\zeta(s) = |s|^{m-1}s$.

Measure of error: with $\bar{u}$ solution of the PME,

$$E_{\mathcal{D}}(\bar{u}) = \left[\sum_{n=0}^{N-1} \delta t \, E_{\mathcal{D}}^{n}(\bar{u})^2\right]^{1/2}$$

where

$$E_{\mathcal{D}}^{n}(\bar{u}) = \left|\frac{1}{\delta t}\int_{t^{(n)}}^{t^{(n+1)}} \Delta\zeta(\bar{u}(s))\, ds - \Delta\zeta(\bar{u}(t^{(n+1)}))\right|_{\mathcal{D},\star}$$

$$+ S_{\mathcal{D}}^{\Pi,\star}\left(\frac{\bar{u}(t^{(n+1)}) - \bar{u}(t^{(n)})}{\delta t}\right)$$

$$+ S_{\mathcal{D}}^{\nabla}(\zeta(\bar{u}(t^{(n+1)}))) + W_{\mathcal{D}}(\nabla\zeta(\bar{u}(t^{(n+1)})))$$

Error estimates

Theorem (Cancès-D.-Guichard-Mazini-Olivares-Pop, 2020)

- *In case of slow diffusion $m \geq 1$:*

$$\left[\sum_{n=0}^{N-1} \delta t \left\| \Pi_{\mathcal{D}} u^{(n+1)} - \Pi_{\mathcal{D}} I_{\mathcal{D}}^{(n+1)} \bar{u} \right\|_{L^{m+1}(\Omega)}^{m+1}\right]^{\frac{1}{m+1}} \leq C E_{\mathcal{D}}(\bar{u})^{\frac{2}{m+1}}.$$

- *In case of fast diffusion $m < 1$:*

$$\left[\sum_{n=0}^{N-1} \delta t \left\| \zeta(\Pi_{\mathcal{D}} u^{(n+1)}) - \zeta(\Pi_{\mathcal{D}} I_{\mathcal{D}}^{(n+1)} \bar{u}) \right\|_{L^{\frac{m+1}{m}}(\Omega)}^{\frac{m+1}{m}}\right]^{\frac{m}{m+1}} \leq C E_{\mathcal{D}}(\bar{u})^{\frac{2m}{m+1}}.$$

Plan

Numerical tests: schemes

- LEPNC (Locally Enriched Polytopal Non-Conforming): non-conforming pseudo-FE method on generic polygonal meshes.
- HMM (Hybrid Mimetic Mixed scheme): unknowns in cells and on edges, contains mimetic finite differences.
- MLP1 (Mass-Lumped $\mathbb{P}^1$ FE): FE with piecewise constant reconstruction $\Pi_{\mathcal{D}}$.
- VAG (Vertex Averaged Gradient discretisation): unknowns at the vertices, based on $\mathbb{P}^1$ on a triangular subdivision.
- CVFEM (Conforming Virtual Element Method): extension of conforming FE applicable on generic polygonal meshes.
- HDG (Hybridizable Discontinuous Galerking): lowest order discontinuous Galerkin with edge unknowns.

Numerical tests: Errors (at final time)

Discrete H^1-error on $|u|^{m-1}u$:

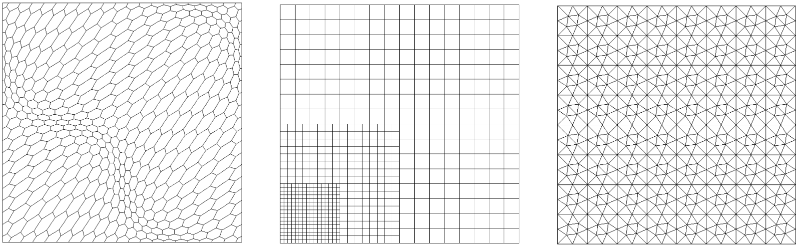
$$E_{H^1,\zeta} = \frac{\left\|\nabla_{\mathcal{D}}(\zeta(u^{(N)}) - I_{\mathcal{D}}\zeta(\bar{u})(T,\cdot))\right\|_{L^2(\Omega)}}{\|\nabla_{\mathcal{D}} I_{\mathcal{D}}\zeta(\bar{u})(T,\cdot)\|_{L^2(\Omega)}}.$$

L^{m+1} error on u:

$$E_{L^{m+1}} = \frac{\left\|\Pi_{\mathcal{D}}(u^{(N)} - I_{\mathcal{D}}\bar{u}(T,\cdot))\right\|_{L^{m+1}(\Omega)}}{\|\Pi_{\mathcal{D}} I_{\mathcal{D}}\bar{u}(T,\cdot)\|_{L^{m+1}(\Omega)}}$$

Numerical tests: mesh and exact solution

Domain and meshes: $\Omega = (0,1)^2$.



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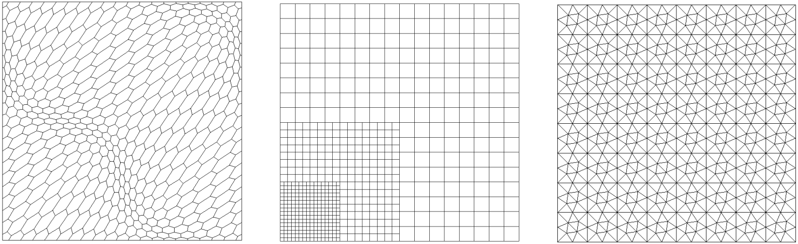


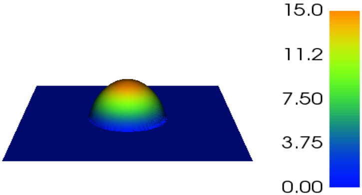
Exact solution: Barenblatt–Pattle solution, with translation in time:
$\bar{u}(t,x) = \mathcal{B}(t_0 + t, x - x_0)$ with

$$\mathcal{B}(t,x) = t^{-\frac{1}{m}} \left\{ \left[C_{\mathcal{B}} - \frac{m-1}{4m^2} \left(\frac{|x|}{t^{\frac{1}{2m}}} \right)^2 \right]_+ \right\}^{\frac{1}{m-1}}$$

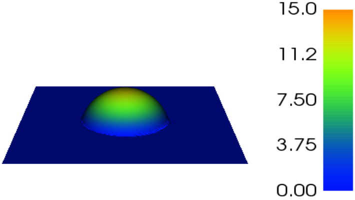
and $x_0 = (0.5, 0.5)$, $t_0 = 0.1$, $C_{\mathcal{B}} = 0.005$.

Numerical tests: mesh and exact solution

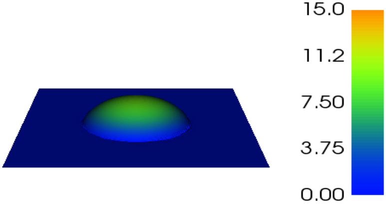
Exact solution: for $m = 2.5$,



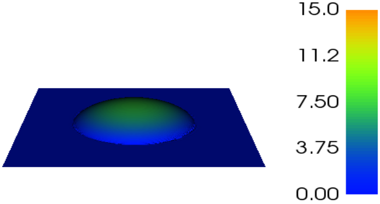
(a) $t = 0.1$



(b) $t = 0.19$

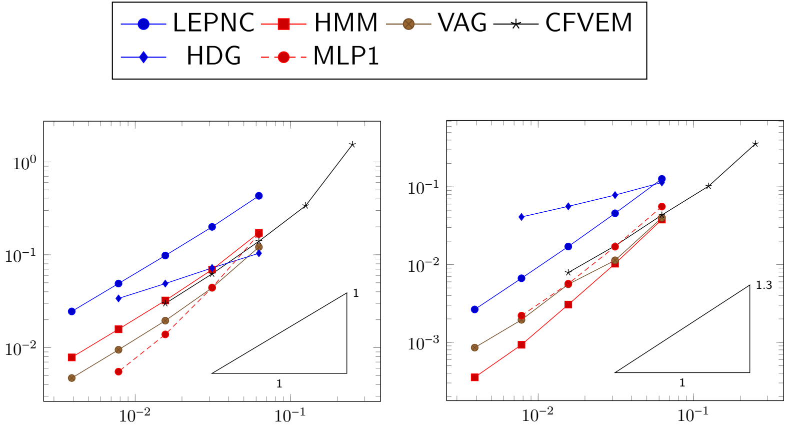


(c) $t = 0.37$



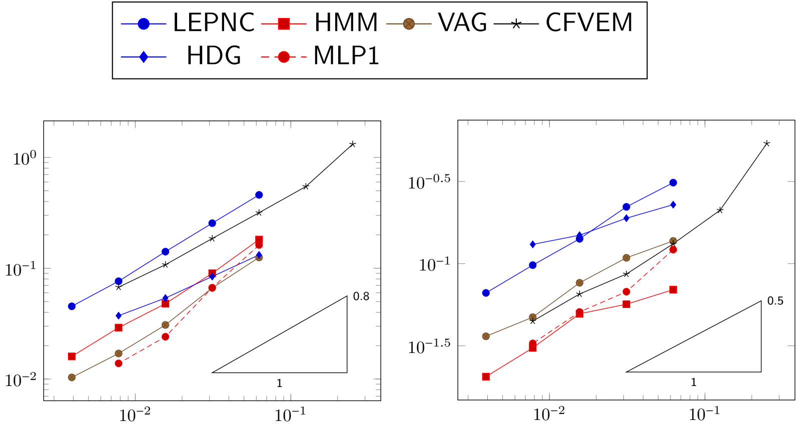
(d) $t = 0.73$

Triangular meshes, $m = 2$



(a) $m = 2$: $E_{H^1,\zeta}$ vs. h.

(b) $m = 2$: $E_{L^{m+1}}$ vs. h (theorem: $h^{0.66}$)



(a) $m = 4$: $E_{H^1,\zeta}$ vs. h.

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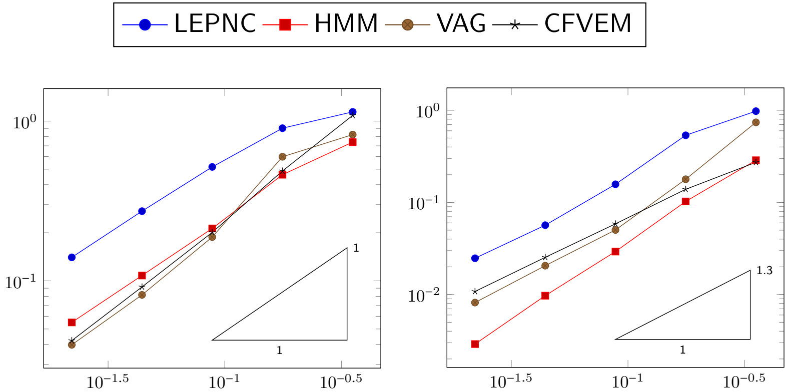
Locally refined cartesian meshes, $m = 2$



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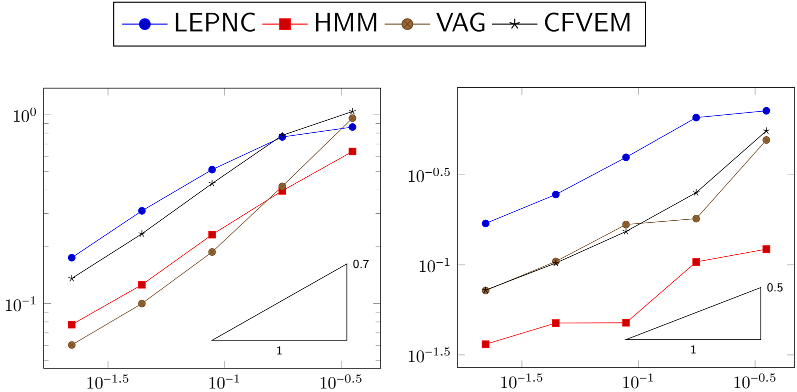

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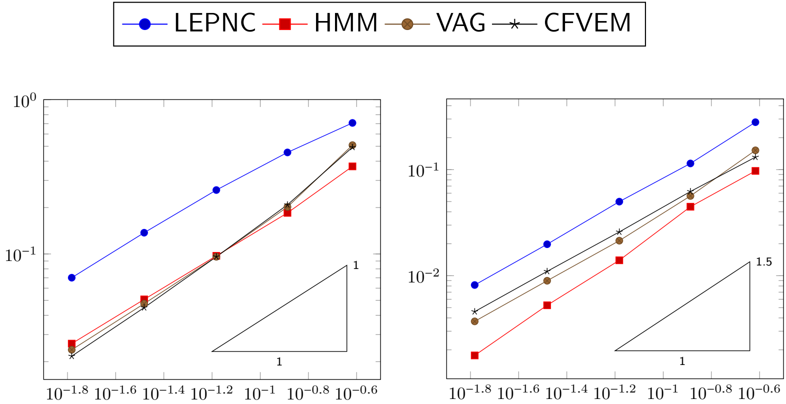
Hexagonal meshes, $m = 2$



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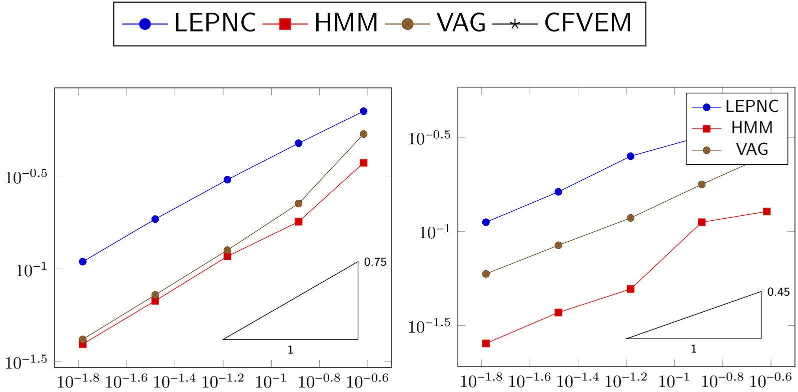

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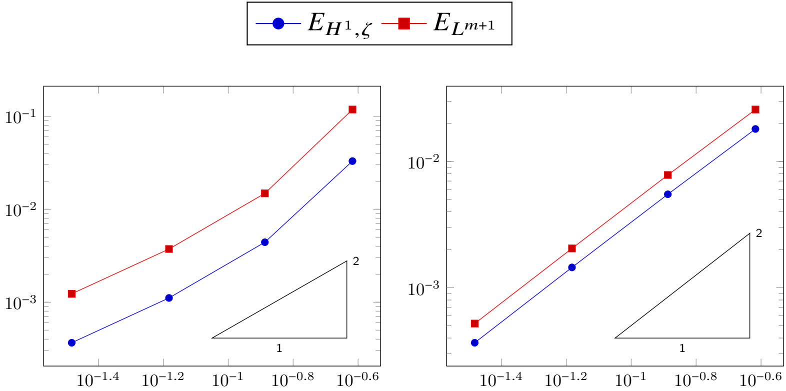
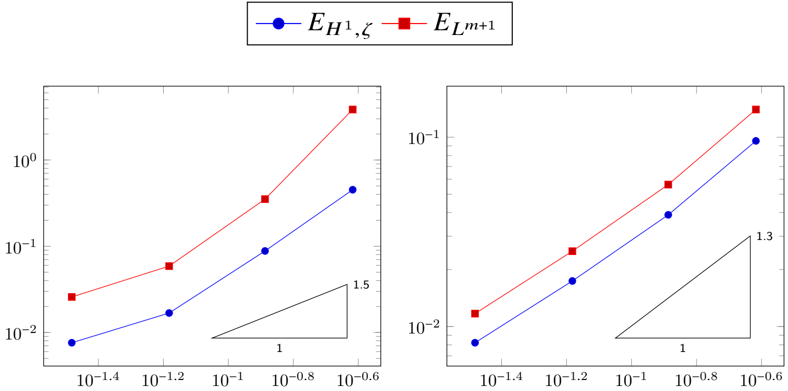
Hexagonal meshes, HMM, $m = 0.3$ and $m = 0.7$



(a) $m = 0.3$ (b) $m = 0.7$

Theorem: $h^{0.46}$ *and* $h^{0.82}$ *in* $L^{\frac{m+1}{m}}(\Omega_T)$*-norm.*

Triangular meshes, MLP1, $m = 0.3$ and $m = 0.7$



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Theorem: $h^{0.46}$ and $h^{0.82}$ in $L^{\frac{m+1}{m}}(\Omega_T)$-norm.

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Conclusion

- Generic discretisation framework for PME (slow and fast diffusion), covers to many numerical methods.
- Uniform-in-time strong L^{m+1} convergence result without regularity assumptions.
- For nodal discretisations: error estimates.
- Numerous numerical tests (and many more results reported in the papers/book chapters: approximation of solution radius, fraction of negartive mass, convergence vs. #DOFs, etc).

Thanks!