Numerical analysis of two-phase flows models with mechanical deformation in fracture porous media

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joint work with F. Bonaldi, K. Brenner, and R. Masson
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Discrete Functional Analysis: bridging pure and numerical mathematics

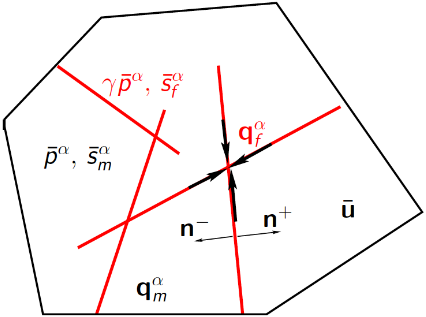
Outline

- Model
  - Modeling concepts
  - Weak formulation
- Discretization
  - Gradient discretization
  - Convergence analysis
- Numerical example
  - Convergence test

Main modeling concepts

- **Hybrid-dimensional model**: fractures reduced to co-dimension 1 surfaces by integration along the width
- **Continuous phase pressures** at matrix fracture interfaces

- Phases: $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$
- Matrix pressures: $\bar{p}_m^\alpha = \bar{p}^\alpha$
- Fracture pressures $\bar{p}_f^\alpha = \gamma \bar{p}^\alpha$
- Matrix saturations: $\bar{s}_m^\alpha$
- Fracture saturations: $\bar{s}_f^\alpha$
- Matrix Darcy velocity: $\mathbf{q}_m^\alpha$
- Fracture tangential velocity: $\mathbf{q}_f^\alpha$
- Displacement field: $\bar{\mathbf{u}}$



Main modeling concepts

- **Poiseuille's law** for the *tangential velocity* in the fractures, extended to two-phase flow using generalized Darcy laws

$$\mathbf{q}_f^\alpha = -\eta_f^\alpha(\bar{s}_f^\alpha)\left(\frac{1}{12}\bar{d}_f^3\right)\nabla_\tau \gamma \bar{p}^\alpha$$

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- **Linear poroelasticity** with small strains assumption

$$\textit{Total stress}: \ \mathbb{\sigma}^T(\bar{\mathbf{u}}) = \mathbb{\sigma}(\bar{\mathbf{u}}) - b\ \bar{p}_m^E \mathbb{I},$$

with $\mathbb{\sigma}(\bar{\mathbf{u}}) = 2\mu\ \mathbb{\epsilon}(\bar{\mathbf{u}}) + \lambda\ \mathrm{div}(\bar{\mathbf{u}})\ \mathbb{I}$ and b the Biot coefficient

- **Open fractures**: no contact

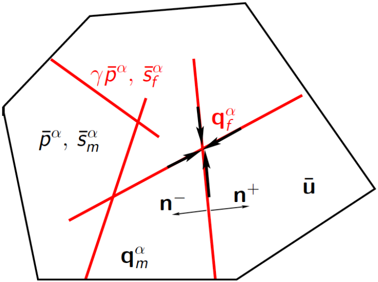
$$\mathbb{\sigma}^T(\bar{\mathbf{u}})\mathbf{n}^\pm = -\bar{p}_f^E \mathbf{n}^\pm,$$

The *equivalent pressures* $\bar{p}_m^E$ and $\bar{p}_f^E$ are defined using the **capillary energy density** concept (Coussy)

Two-phase hybrid-dimensional Darcy flow: $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$

$$\partial_t \left(\bar{\phi}_m \bar{s}_m^\alpha\right) + \mathrm{div}\left(\mathbf{q}_m^\alpha\right) = h_m^\alpha$$

$$\mathbf{q}_m^\alpha = -\eta_m^\alpha(\bar{s}_m^\alpha)\mathbb{K}_m \nabla \bar{p}^\alpha$$



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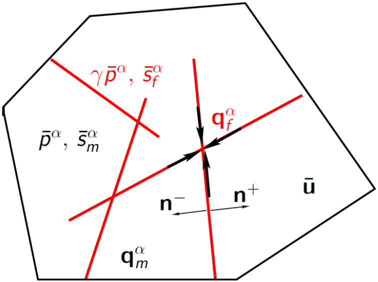

$$\partial_t \left(\bar d_f s_f^\alpha\right) + \mathrm{div}_\tau(\mathbf{q}_f^\alpha) - [\![\mathbf{q}_m^\alpha]\!] = h_f^\alpha$$

$$\mathbf{q}_f^\alpha = -\eta_f^\alpha(\bar s_f^\alpha)\left(\frac{1}{12}\bar d_f^3\right)\nabla_\tau \gamma \bar p^\alpha$$

(Normal jump: $[\![\mathbf{q}]\!] = \mathbf{q}^+ \cdot \mathbf{n}^+ + \mathbf{q}^- \cdot \mathbf{n}^-$)

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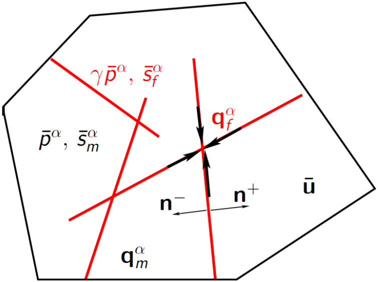
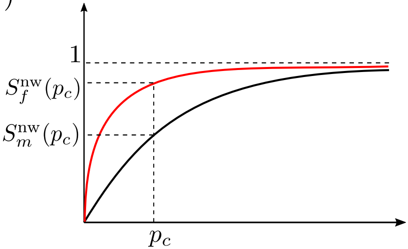


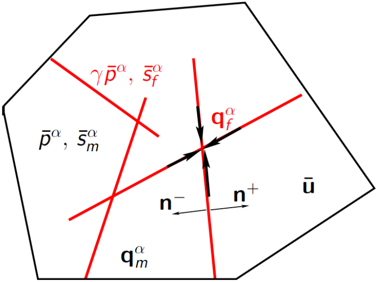
$$\bar s_m^\alpha = S_m^\alpha(\bar p_c), \quad \bar s_f^\alpha = S_f^\alpha(\gamma \bar p_c)$$



Poromechanics

Poromechanical model

$$\begin{cases} -\mathrm{div}\Big(\sigma(\bar{\mathbf{u}}) - b\ \bar{p}_m^E \mathbb{I}\Big) = \mathbf{f} \\ \sigma(\bar{\mathbf{u}}) = 2\mu\ \epsilon(\bar{\mathbf{u}}) + \lambda\ \mathrm{div}(\bar{\mathbf{u}})\ \mathbb{I} \\ (\sigma(\bar{\mathbf{u}}) - b\ \bar{p}_m^E \mathbb{I})\mathbf{n}^{\pm} = -\bar{p}_f^E \mathbf{n}^{\pm} \end{cases}$$



Closure laws

$$\begin{cases} \partial_t \bar{\phi}_m = b\ \mathrm{div}\partial_t \bar{\mathbf{u}} + \dfrac{1}{M}\partial_t \bar{p}_m^E \\ \bar{d}_f = -[\![\bar{\mathbf{u}}]\!] \end{cases}$$

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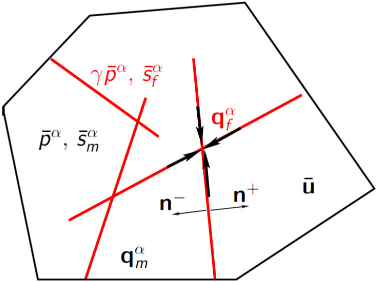

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Let $U_{\mathrm{rt}}(\bar{p}_c) = \int_0^{\bar{p}_c} z\,(S_{\mathrm{rt}}^{\mathrm{nw}})'\,(z)\,\mathrm{d}z$, the equivalent pressures are

$$\bar{p}_m^E = \sum_{\alpha\in\{\mathrm{nw},\mathrm{w}\}} \bar{p}^\alpha\ \bar{s}_m^\alpha - U_m(\bar{p}_c), \quad \bar{p}_f^E = \sum_{\alpha\in\{\mathrm{nw},\mathrm{w}\}} \gamma\bar{p}^\alpha\ \bar{s}_f^\alpha - U_f(\gamma\bar{p}_c)$$

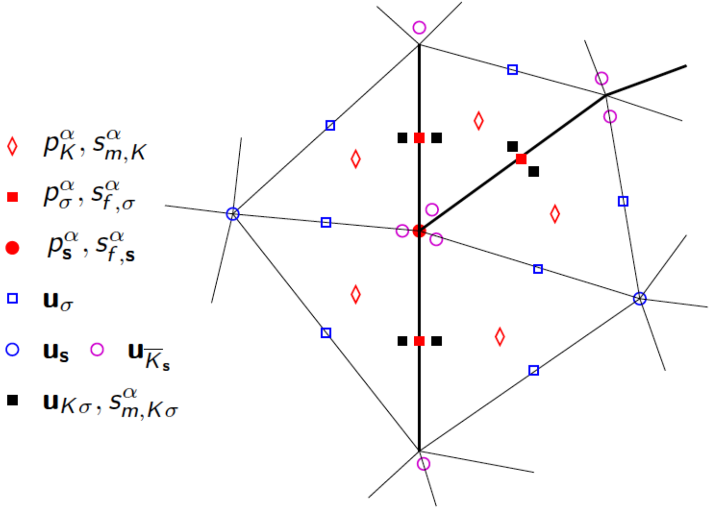
Weak solution: $\bar{p}^\alpha$ and $\bar{\mathbf{u}}$ such that for all smooth test functions $\bar{\varphi}^\alpha$ and $\bar{\mathbf{v}}$:

$$\int_0^T \int_\Omega \Big(-\bar{\phi}_m \bar{s}_m^\alpha \partial_t \bar{\varphi}^\alpha + \eta_m^\alpha(\bar{s}_m^\alpha) \mathbb{K}_m \nabla \bar{p}^\alpha \cdot \nabla \bar{\varphi}^\alpha \Big) \mathrm{d}\mathbf{x}\mathrm{d}t$$
$$+ \int_0^T \int_\Gamma \Big(-\bar{d}_f \bar{s}_f^\alpha \partial_t \gamma \bar{\varphi}^\alpha + \eta_f^\alpha(\bar{s}_f^\alpha) \frac{\bar{d}_f^3}{12} \nabla_\tau \gamma \bar{p}^\alpha \cdot \nabla_\tau \gamma \bar{\varphi}^\alpha \Big) \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t$$
$$- \int_\Omega \bar{\phi}_m^0 \bar{s}_m^{\alpha,0} \bar{\varphi}^\alpha(0,\cdot)\mathrm{d}\mathbf{x} - \int_\Gamma \bar{d}_f^0 \bar{s}_f^{\alpha,0} \gamma \bar{\varphi}^\alpha(0,\cdot)\mathrm{d}\sigma(\mathbf{x})$$
$$= \int_0^T \int_\Omega h_m^\alpha \bar{\varphi}^\alpha \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Gamma h_f^\alpha \gamma \bar{\varphi}^\alpha \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t, \quad \alpha \in \{\mathrm{w}, \mathrm{nw}\},$$

$$\int_0^T \int_\Omega \Big(\mathbb{\sigma}(\bar{\mathbf{u}}) : \mathbb{\varepsilon}(\bar{\mathbf{v}}) - b \bar{p}_m^E \mathrm{div}(\bar{\mathbf{v}}) \Big) \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Gamma \bar{p}_f^E \, [\![\bar{\mathbf{v}}]\!] \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t$$
$$= \int_0^T \int_\Omega \mathbf{f} \cdot \bar{\mathbf{v}} \mathrm{d}\mathbf{x}\mathrm{d}t,$$

Example of discretization

- Two-Point Flux Approximation (TPFA) scheme for Darcy
- $\mathbb{P}_2$ elements for mechanics



Gradient discretization method: motivation

- Abstract discretization framework accounting for a large class of conforming and non conforming discretizations

- Generic stability and convergence analysis under general properties such as
  - Coercivity (discrete Poincaré inequality)
  - Consistency
  - Limit Conformity (for non-conforming methods)
  - Compactness

Gradient discretization of the poromechanical model

Two-phase flow

$X^0_{\mathcal{D}_p}$ = space of discrete unknowns. Reconstruction operators:

- gradient operators on matrix and fracture network

$$\nabla^m_{\mathcal{D}_p} : X^0_{\mathcal{D}_p} \to L^\infty(\Omega)^d, \qquad \nabla^f_{\mathcal{D}_p} : X^0_{\mathcal{D}_p} \to L^\infty(\Gamma)^{d-1};$$

- **piecewise-constant** function operators on matrix and fracture network

$$\Pi^m_{\mathcal{D}_p} : X^0_{\mathcal{D}_p} \to L^\infty(\Omega), \qquad \Pi^f_{\mathcal{D}_p} : X^0_{\mathcal{D}_p} \to L^\infty(\Gamma).$$

Assume $\|v\|_{\mathcal{D}_p} := \|\nabla^m_{\mathcal{D}_p} v\|_{L^2(\Omega)^d} + \|d_0^{3/2}\nabla^f_{\mathcal{D}_p} v\|_{L^2(\Gamma)^{d-1}}$ is a norm on $X^0_{\mathcal{D}_p}$.

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Poromechanics

$X^0_{\mathcal{D}_\mathbf{u}}$ = space of discrete unknowns. Reconstruction operators:

- symmetric gradient operator $\mathbb{e}_{\mathcal{D}_\mathbf{u}} : X^0_{\mathcal{D}_\mathbf{u}} \to L^2(\Omega, \mathcal{S}_d(\mathbb{R}))$,
- displacement function operator $\Pi_{\mathcal{D}_\mathbf{u}} : X^0_{\mathcal{D}_\mathbf{u}} \to L^2(\Omega)^d$,
- normal jump function operator $[\![\cdot]\!]_{\mathcal{D}_\mathbf{u}} : X^0_{\mathcal{D}_\mathbf{u}} \to L^4(\Gamma)$.

Assume $\|\mathbf{v}\|_{\mathcal{D}_\mathbf{u}} := \|\mathbb{e}_{\mathcal{D}_\mathbf{u}}(\mathbf{v})\|_{L^2(\Omega)}$ is a norm on $X^0_{\mathcal{D}_\mathbf{u}}$.

Gradient scheme

Find $p^\alpha \in (X^0_{\mathcal{D}_p})^{N+1}$ and $\mathbf{u} \in (X^0_{\mathcal{D}_\mathbf{u}})^{N+1}$ such that, for all $\varphi^\alpha \in (X^0_{\mathcal{D}_p})^{N+1}$, $\mathbf{v} \in (X^0_{\mathcal{D}_\mathbf{u}})^{N+1}$:

$$\int_0^T \int_\Omega \Big(\delta_t \Big(\phi_\mathcal{D} \Pi^m_{\mathcal{D}_p} s^\alpha_m \Big) \Pi^m_{\mathcal{D}_p} \varphi^\alpha + \eta^\alpha_m (\Pi^m_{\mathcal{D}_p} s^\alpha_m) \mathbb{K}_m \nabla^m_{\mathcal{D}_p} p^\alpha \cdot \nabla^m_{\mathcal{D}_p} \varphi^\alpha \Big) \mathrm{d}\mathbf{x}\mathrm{d}t$$
$$+ \int_0^T \int_\Gamma \delta_t \Big(d_{f,\mathcal{D}_\mathbf{u}} \Pi^f_{\mathcal{D}_p} s^\alpha_f \Big) \Pi^f_{\mathcal{D}_p} \varphi^\alpha \mathrm{d}\sigma(\mathbf{x})$$
$$+ \int_0^T \int_\Gamma \eta^\alpha_f (\Pi^f_{\mathcal{D}_p} s^\alpha_f) \frac{d^3_{f,\mathcal{D}_\mathbf{u}}}{12} \nabla^f_{\mathcal{D}_p} p^\alpha \cdot \nabla^f_{\mathcal{D}_p} \varphi^\alpha \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t$$
$$= \int_0^T \int_\Omega h^\alpha_m \Pi^m_{\mathcal{D}_p} \varphi^\alpha \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Gamma h^\alpha_f \Pi^f_{\mathcal{D}_p} \varphi^\alpha \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t, \quad \alpha \in \{\mathrm{w}, \mathrm{nw}\},$$

$$\int_0^T \int_\Omega \Big(\mathbb{\sigma}_{\mathcal{D}_u}(\mathbf{u}) : \mathbb{\epsilon}_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) - b(\Pi^m_{\mathcal{D}_p} p^E_m) \mathrm{div}_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) \Big) \mathrm{d}\mathbf{x}\mathrm{d}t$$
$$+ \int_0^T \int_\Gamma (\Pi^f_{\mathcal{D}_p} p^E_f) [\![\mathbf{v}]\!]_{\mathcal{D}_\mathbf{u}} \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t = \int_0^T \int_\Omega \mathbf{f} \cdot \Pi_{\mathcal{D}_\mathbf{u}} \mathbf{v} \mathrm{d}\mathbf{x}\mathrm{d}t$$

Convergence analysis: main assumptions

- Mobility function $\eta_{\rm rt}^{\alpha}$ continuous, non-decreasing, s.t.

$$0 < \eta_{\rm rt,min}^{\alpha} \leqslant \eta_{\rm rt}^{\alpha}(s) \leqslant \eta_{\rm rt,max}^{\alpha} < +\infty \quad \forall s \in [0,1]$$

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- The sequences $(\mathcal{D}_p^l)_{l\in\mathbb{N}}$, $(\mathcal{D}_{\mathbf{u}}^l)_{l\in\mathbb{N}}$, $\{(t_n^l)_{n=0}^{N^l}\}_{l\in\mathbb{N}}$ of space time Gradient Discretizations satisfy **coercivity, consistency, limit-conformity and compactness** properties

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- There exist a solution $p_l^{\alpha} \in (X_{\mathcal{D}_p^l}^0)^{N^l+1}$, $\mathbf{u}^l \in (X_{\mathcal{D}_{\mathbf{u}}^l}^0)^{N^l+1}$ s.t.
  - (i) $\phi_{\mathcal{D}^l}(t,\mathbf{x}) \geqslant \phi_{m,\min} > 0$,
  - (ii) $d_{f,\mathcal{D}_{\mathbf{u}}^l}(t,\mathbf{x}) \geqslant d_0(\mathbf{x})$, where $d_0 \geqslant 0$ is continuous and vanishes only at the fracture tips

Main convergence result

There are $\bar{p}^\alpha \in L^2(\mathbb{T}; V^0)$ and $\bar{\mathbf{u}} \in L^\infty(\mathbb{T}; \mathbf{U}^0)$ satisfying the weak formulation s.t.

$\Pi^m_{\mathcal{D}^l_p} p^\alpha_l \rightharpoonup \bar{p}^\alpha$	weakly in $L^2(\mathbb{T}; L^2(\Omega))$,
$\Pi^f_{\mathcal{D}^l_p} p^\alpha_l \rightharpoonup \gamma\bar{p}^\alpha$	weakly in $L^2(\mathbb{T}; L^2(\Gamma))$,
$\Pi_{\mathcal{D}^l_{\mathbf{u}}} \mathbf{u}^l \rightharpoonup \bar{\mathbf{u}}$	weakly-$\star$ in $L^\infty(\mathbb{T}; L^2(\Omega)^d)$,
$\phi_{\mathcal{D}^l} \rightharpoonup \bar{\phi}_m$	weakly-$\star$ in $L^\infty(\mathbb{T}; L^2(\Omega))$,
$d_{f,\mathcal{D}^l_{\mathbf{u}}} \to \bar{d}_f$	strongly in $L^\infty(\mathbb{T}; L^p(\Gamma))$ for $2 \leqslant p < 4$,
$\Pi^m_{D^l_p} S^\alpha_m(p^l_c) \to S^\alpha_m(\bar{p}_c)$	strongly in $L^2(\mathbb{T}; L^2(\Omega))$,
$\Pi^f_{D^l_p} S^\alpha_f(p^l_c) \to S^\alpha_f(\gamma\bar{p}_c)$	strongly in $L^2(\mathbb{T}; L^2(\Gamma))$,

where $\bar{\phi}_m = \bar{\phi}^0_m + b\,\mathrm{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}^0) + \dfrac{1}{M}(\bar{p}^E_m - \bar{p}^{E,0}_m)$ and $\bar{d}_f = -[\![\bar{\mathbf{u}}]\!]$

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$\Pi^f_{D^l_p} S^\alpha_f(p^l_c) \to S^\alpha_f(\gamma\bar{p}_c)$	strongly in $L^2(\mathbb{T}; L^2(\Gamma))$,

- **Previous convergence (Girault et al., '15): single-phase flow, linear case, d_f^3 frozen**

Main convergence result – Steps

1. Energy estimates by taking the phase pressures as test functions in Darcy, the discrete time derivative of the displacement in elasticity.

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2. Weak estimates on time translates

$$\Big|\langle[\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p}s^\alpha_m](\tau)-[\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p}s^\alpha_m](\tau'),\Pi^m_{\mathcal{D}_p}\varphi\rangle_{L^2(\Omega)}$$
$$+\langle[d_{f,\mathcal{D}_u}\Pi^f_{\mathcal{D}_p}s^\alpha_f](\tau)-[d_{f,\mathcal{D}_u}\Pi^f_{\mathcal{D}_p}s^\alpha_f](\tau),\Pi^f_{\mathcal{D}_p}\varphi\rangle_{L^2(\Gamma)}\Big|\leqslant\cdots$$

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3. Strong convergence of s^α_m
   - i. Separate matrix from fractures above using cut-off functions
     $\rightsquigarrow$ strong local time translate estimates

$$\int_0^T \|\Pi^m_{\mathcal{D}_p} s^\alpha_m(\cdot+\tau,\cdot) - \Pi^m_{\mathcal{D}_p} s^\alpha_m\|^2_{L^2(K)}\,dt \leqslant \cdots$$

Main convergence result – Steps

1. Energy estimates by taking the phase pressures as test functions in Darcy, the discrete time derivative of the displacement in elasticity.
2. Weak estimates on time translates
3. Strong convergence of s_m^α
   i. Separate matrix from fractures above using cut-off functions
      $\rightsquigarrow$ strong local time translate estimates
   ii. Space translate estimates from compactness of $\mathcal{D}_p$
   iii. Recover compactness on the full domain from $s_m^\alpha \in [0,1]$

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   1. Separate matrix from fractures above using cut-off functions $\rightsquigarrow$ strong local time translate estimates
   2. Space translate estimates from compactness of $\mathcal{D}_p$
   3. Recover compactness on the full domain from $s^\alpha_m \in [0,1]$
4. Strong convergences of $\phi_{\mathcal{D}}s^\alpha_m$, s^α_m, $d_f s^\alpha_f$, d^α_f and s^α_f
   1. Uniform in time weak in space convergence.
      - Uses discrete Ascoli–Arzelà theorem,
      - First for matrix (using cut-off), then for fracture (using the convergence of matrix variables)

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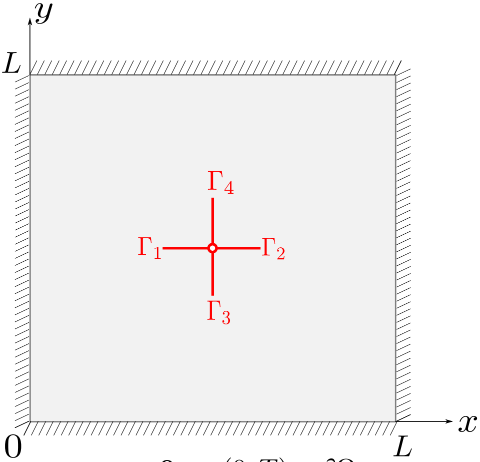
1. Energy estimates by taking the phase pressures as test functions in Darcy, the discrete time derivative of the displacement in elasticity.
2. Weak estimates on time translates
3. Strong convergence of s_m^α
   1. Separate matrix from fractures above using cut-off functions
      $\rightsquigarrow$ strong local time translate estimates
   2. Space translate estimates from compactness of $\mathcal{D}_p$
   3. Recover compactness on the full domain from $s_m^\alpha \in [0,1]$
4. Strong convergences of $\phi_{\mathcal{D}} s_m^\alpha$, s_m^α, $d_f s_f^\alpha$, d_f^α and s_f^α
   1. Uniform in time weak in space convergence.
   2. Strong compactness: combining uniform-in-time weak-in-space, and compactness property in space.
      Need to isolate fracture tips for compactness in space of s_f

Main convergence result – Steps

1. Energy estimates by taking the phase pressures as test functions in Darcy, the discrete time derivative of the displacement in elasticity.
2. Weak estimates on time translates
3. Strong convergence of s_m^α
   - i. Separate matrix from fractures above using cut-off functions $\rightsquigarrow$ strong local time translate estimates
   - ii. Space translate estimates from compactness of $\mathcal{D}_p$
   - iii. Recover compactness on the full domain from $s_m^\alpha \in [0,1]$
4. Strong convergences of $\phi_{\mathcal{D}} s_m^\alpha$, s_m^α, $d_f s_f^\alpha$, d_f^α and s_f^α
   - i. Uniform in time weak in space convergence.
   - ii. Strong compactness: combining uniform-in-time weak-in-space, and compactness property in space.
     Need to isolate fracture tips for compactness in space of s_f
5. Identification of the limit fields and weak solution

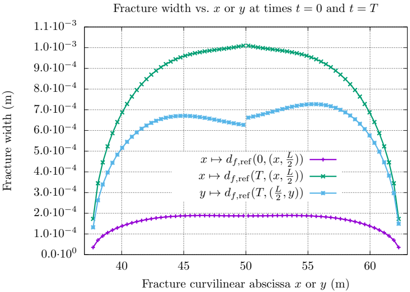
Numerical experiment: convergence test

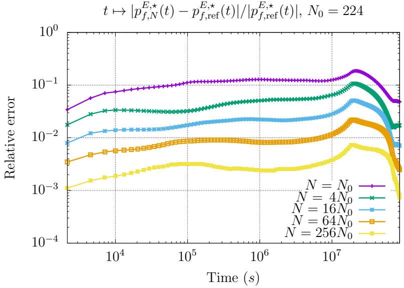
- TPFA scheme for flows
- $\mathbb{P}_2$ elements for mechanics

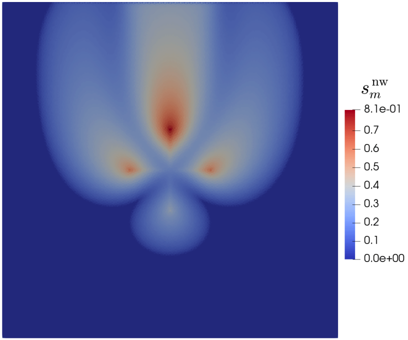


$\mathbf{u} = \mathbf{0}$ on $(0, T) \times \partial\Omega$

Numerical experiment: convergence test







Conclusions & perspectives

Conclusions

- Analysis of a GD for a two-phase flow in deformable fractured porous media
- Linear elastic mechanical behavior, open fractures
- Porosity bounded from below and above by strictly positive constants
- Fracture width larger than a nonnegative function vanishing only at the tips
- Nonlinear flow/mechanics coupling, fracture conductivity $d_f^3/12$ not frozen
- Convergence validation ($\mathbb{P}_2$ elements for mechanics, TPFA for flows)

Perspectives

- Discontinuous pressure models (just submitted)
- Contact, slip, friction between fracture surfaces
- More advanced discretizations (polytopal DG, etc.)

Thanks for your attention