An arbitrary-order robust polygonal scheme for the Reissner-Mindlin plate problem

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Plan

Reissner–Mindlin plate model

$$\begin{aligned} \boldsymbol{\gamma} + \mathbf{div}(\mathsf{C}\,\mathbf{grad}_{\mathrm{s}}\,\boldsymbol{\theta}) &= \mathbf{0} && \text{in } \Omega, \\ -\operatorname{div}\boldsymbol{\gamma} &= f && \text{in } \Omega, \\ \boldsymbol{\gamma} &= \frac{\kappa}{t^2}(\mathbf{grad}\,u - \boldsymbol{\theta}) && \text{in } \Omega, \\ \boldsymbol{\theta} = \mathbf{0}, \quad u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

- Ω polygonal domain.
- $\boldsymbol{\gamma} : \Omega \to \mathbb{R}^2$: shear strain; $\boldsymbol{\theta} : \Omega \to \mathbb{R}^2$: fibers rotations; $u : \Omega \to \mathbb{R}$: transverse displacement.
- $f : \Omega \to \mathbb{R}$: transverse load; C: linear elasticity tensor; κ: shear modulus.

t: plate thickness.

Weak formulation

Find $(\boldsymbol{\theta}, u) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) + \frac{\kappa}{t^2}(\boldsymbol{\theta} - \mathbf{grad}\, u, \boldsymbol{\eta} - \mathbf{grad}\, v) = \int_\Omega fv$$

$$\forall (\boldsymbol{\eta}, v) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega)$$

with

$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) := \int_\Omega \mathsf{C}\, \mathbf{grad}_{\mathrm{s}}\, \boldsymbol{\theta} : \mathbf{grad}_{\mathrm{s}}\, \boldsymbol{\eta}.$$

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Expected estimates (uniform w.r.t. t):

$$\|\boldsymbol{\theta}\|_{H^2} + \|\boldsymbol{\gamma}\|_{L^2} + t|\boldsymbol{\gamma}|_{H^1} \lesssim 1.$$

General discrete form

Find $(\boldsymbol{\theta}_h, u_h) \in \boldsymbol{\Theta}_h \times U_h$ s.t.

$$a_h(\boldsymbol{\tau}_h, \boldsymbol{\eta}_h) + \frac{\kappa}{t^2}(\boldsymbol{\theta}_h - \mathbf{grad}_h\, u_h, \boldsymbol{\eta}_h - \mathbf{grad}_h\, v_h)_{\boldsymbol{\Theta},h} = \int_\Omega f\; \Pi_h v_h \qquad \forall(\boldsymbol{\eta}, v) \in \boldsymbol{\Theta}_h \times U_h.$$

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First challenge with robustness: take $I_{\boldsymbol{\Theta},h}\boldsymbol{\theta} \in \boldsymbol{\Theta}_h$ and $I_{U_h} u \in U_h$ interpolates of the continuous solution. Assume

$$\mathbf{grad}_h\, I_{U_h} u = I_{\boldsymbol{\Theta},h}(\mathbf{grad}\, u) + \epsilon_h.$$

Then

$$\frac{\kappa}{t^2}(\boldsymbol{\theta}_h - \mathbf{grad}_h\, u_h) = I_{\boldsymbol{\Theta},h}\boldsymbol{\gamma} + \frac{\kappa}{t^2}\epsilon_h.$$

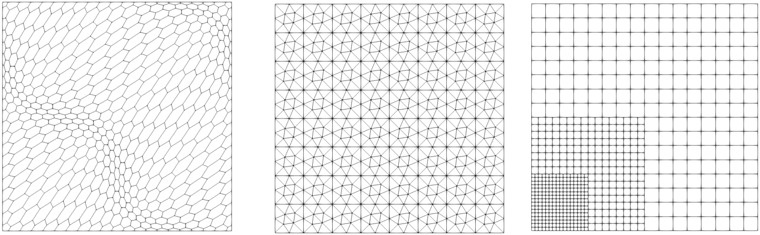
We need $\epsilon_h = 0$ to avoid a t^{-2} factor in the error estimates!

From the literature...

- Brezzi–Fortin 86: additional unknowns (decompose shear strain).
- Arnold–Falk 89: non-conforming FE and bubble for rotation.
- Brezzi ea. 91, Durán–Liberman 92, Arnold–Falk 97, Lamichhane–Meylan 17: reduced integration/projections.
- Arnold–Brezzi 93: mixed formulation with strain as unknown.
- Arnold ea. 05: discontinuous Galerkin (but robustness only for continuous FE on rotation and non-conforming FE on displacement).
- Lovadina 05, Chinosi ea. 06: non-conforming FE.
- Beirão da Veiga–Mora 11: mimetic finite elements.
- Beirão da Veiga ea. 15, 19: low order virtual elements on re-formulation with shear strain instead of rotation.
- Gallistl–Schedensack 21: Taylor–Hood elements.

Plan

Why polygonal meshes?



(a) Hexagonal mesh (b) Triangular mesh (c) Locally refined mesh

- Increased flexibility for meshing complex geometries (mesh with fine triangles, then agglomerate).
- Easy local refinement to capture locally steep solutions.

2D de Rham complex

Setting $\text{rot}(v_1, v_2) = \partial_1 v_2 - \partial_2 v_1$:

$$\mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\textbf{grad}} \boldsymbol{H}(\text{rot};\Omega) \xrightarrow{\text{rot}} L^2(\Omega) \xrightarrow{0} 0.$$

- Complex: range of an operator included in kernel of the next.
- Exact complex (if Ω topologically trivial): equality.

Reproduction at discrete level has many essential advantages, in particular around stability of schemes for complicated models.

2D Discrete De Rham (DDR) complex

Given a mesh $\mathcal{T}_h$ made of general polygonal elements:

$$\mathbb{R} \hookrightarrow \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{G}^k_h} \underline{X}^k_{\mathrm{rot},h} \xrightarrow{\underline{R}^k_h} \underline{X}^k_{0,h} \xrightarrow{0} 0.$$

Features

- Spaces $\underline{X}^k_{\bullet,h}$ fully discrete, made of vectors of polynomial functions attached to vertces, edges and elements.
- Complex, which is exact if Ω is topologically trivial.
- Arbitrary degree of exactness k.
- Consistent interpolators, discrete operators and inner products.

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Commutating diagrams: in particular,

$$\begin{array}{ccc} C^1(\overline{\Omega}) & \xrightarrow{\mathbf{grad}} & C^0(\overline{\Omega}) \\ \downarrow \underline{I}_{\mathbf{grad},h} & & \downarrow \underline{I}_{\mathrm{rot},h} \\ \underline{X}^k_{\mathbf{grad},h} & \xrightarrow{\underline{G}_h} & \underline{X}^k_{\mathrm{rot},h} \end{array}$$

$$\underline{G}_h(\underline{I}_{\mathbf{grad},h} u) = \underline{I}_{\mathrm{rot},h}(\mathbf{grad}\, u).$$

- This suggests $U_h = \underline{X}^k_{\mathbf{grad},h}$ (then $\epsilon_h = 0$), and thus $\mathbf{\Theta}_h = \underline{X}^k_{\mathrm{rot},h}$.

Extension of $\underline{X}_{\mathrm{rot},h}^{k}$

- Issue: $\underline{X}_{\mathrm{rot},h}^{k}$ only encodes tangential components along the mesh edges (as is the case for the standard Nédélec space).

  $\rightsquigarrow$ Not enough information to reconstruct a consistent robust $\mathbf{grad}_{\mathrm{s}}$.

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- Solution: extend $\underline{X}^k_{\mathrm{rot},h}$ into a space $\mathbf{\Theta}_h$ that includes normal components to the edges.
  On $\mathbf{\Theta}_h$, which has element and edge vector polynomial unknowns, construct a full discrete gradient $\mathsf{G}_h^k : \mathbf{\Theta}_h \to \mathcal{P}^k(\mathcal{T}_h)^{2\times 2}$ using Hybrid High-Order technology.

Plan

Arbitrary-order error estimates

Theorem (Error estimate for arbitrary k)

Assume that the exact solution $(\boldsymbol{\eta}, u)$ satisfies $u \in C^1(\overline{\Omega}) \cap H^{k+2}(\Omega)$ and $\boldsymbol{\theta} \in H^1(\Omega)^2 \cap H^{k+2}(\Omega)^2$. Then,

$$\|(\underline{\boldsymbol{\theta}}_h - \underline{I}_{\boldsymbol{\Theta},h}\boldsymbol{\theta}, \underline{u}_h - \underline{I}_{\mathbf{grad},h}u)\|_{\boldsymbol{\Theta}\times U,h} \lesssim h^{k+1}\left(|\boldsymbol{\theta}|_{H^{k+2}} + |\boldsymbol{\gamma}|_{H^{k+1}}\right).$$

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- $\|\cdot\|_{\boldsymbol{\Theta}\times U,h}$ mimics L^2-norm on $\boldsymbol{\Theta}_h \times U_h$.
- Optimal rate of convergence, but not robust w.r.t. t (even for $k = 0$).
- Actually, no method of order $k \geq 1$ is known to be robust w.r.t. t.

Robust low-order error estimates

Theorem (Locking-free error estimate for $k = 0$)

Under the previous assumptions and $k = 0$, it holds

$$\|(\underline{\boldsymbol{\theta}}_h - \underline{I}_{\boldsymbol{\Theta},h}\boldsymbol{\theta}, \underline{u}_h - \underline{I}_{\mathbf{grad},h}u)\|_{\boldsymbol{\Theta}\times U,h} \lesssim h\left(|\boldsymbol{\theta}|_{H^2(\mathcal{T}_h)^2} + t|\boldsymbol{\gamma}|_{H^1(\mathcal{T}_h)^2} + \|\boldsymbol{\gamma}\|_{L^2(\Omega)^2} + \|f\|_{L^2(\Omega)}\right).$$

- Fully robust w.r.t. t: $|\boldsymbol{\theta}|_{H^2(\mathcal{T}_h)^2} + t|\boldsymbol{\gamma}|_{H^1(\mathcal{T}_h)^2} + \|\boldsymbol{\gamma}\|_{L^2(\Omega)^2} \lesssim 1$.

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- Fully robust w.r.t. t: $|\boldsymbol{\theta}|_{H^2(\mathcal{T}_h)^2} + t|\boldsymbol{\gamma}|_{H^1(\mathcal{T}_h)^2} + \|\boldsymbol{\gamma}\|_{L^2(\Omega)^2} \lesssim 1$.
- Proof relies on:
  - Commutation property $\underline{G}_h(\underline{I}_{\mathbf{grad},h}u) = \underline{I}_{\mathrm{rot},h}(\mathbf{grad}\, u)$,
  - Fine lifting of U_h in a conforming space, and a piecewise-constant lifting on $\boldsymbol{\Theta}_h$ based on a local discrete Hodge decomposition.

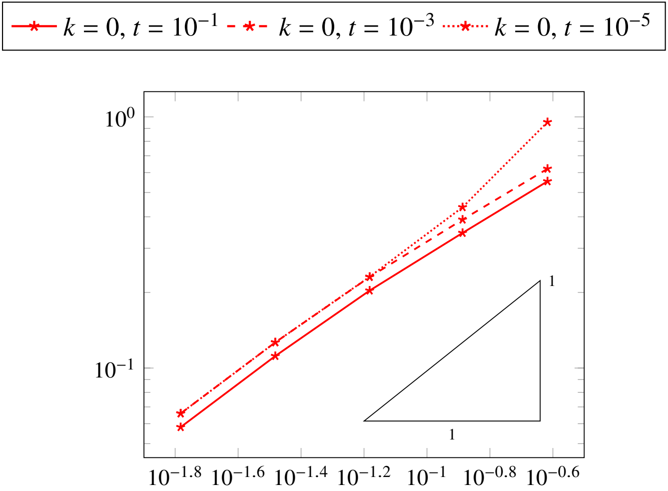
Plan

Smooth polynomial solution

$$
\begin{aligned}
u(\boldsymbol{x}) = {} & \frac{1}{3}x_1^3(1-x_1^3)x_2^3(1-x_2)^3 \\
& - \frac{2t^2}{5(1-\nu)}\Big[x_2^3(x_2-1)^3x_1(x_1-1)(5x_1^2-5x_1+1) \\
& \qquad + x_1^3(x_1-1)^3x_2(x_2-1)(5x_2^2-5x_2+1)\Big], \\
\boldsymbol{\theta}(\boldsymbol{x}) = {} & \begin{bmatrix} x_2^3(x_2-1)^3x_1^2(x_1-1)^2(2x_1-1) \\ x_1^3(x_1-1)^3x_2^2(x_2-1)^2(3x_2-1) \end{bmatrix}.
\end{aligned}
$$

- All $|\boldsymbol{\theta}|_{H^s}$ and $|\boldsymbol{\gamma}|_{H^s}$ uniformly bounded w.r.t. t!

Smooth polynomial solution: hexagonal meshes



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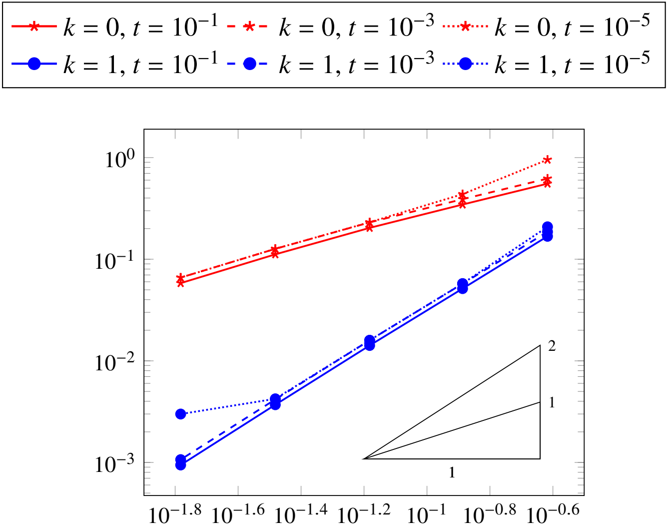

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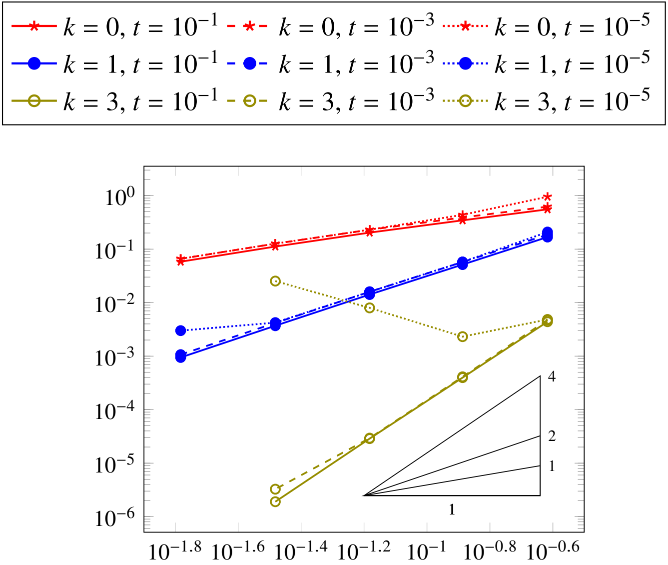
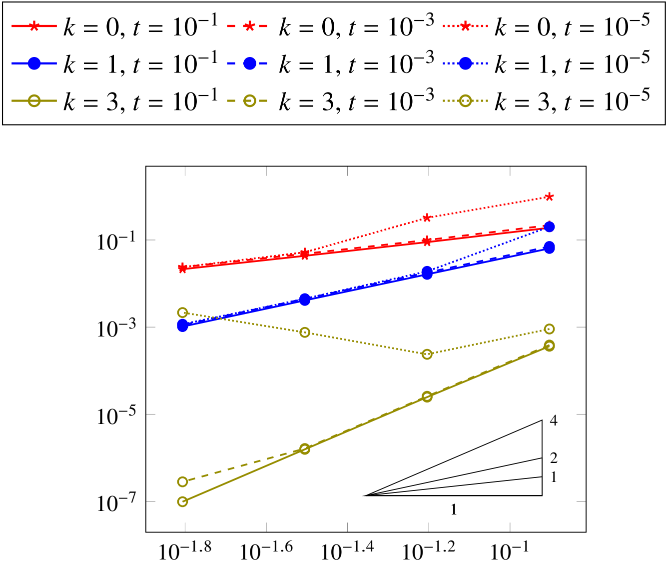
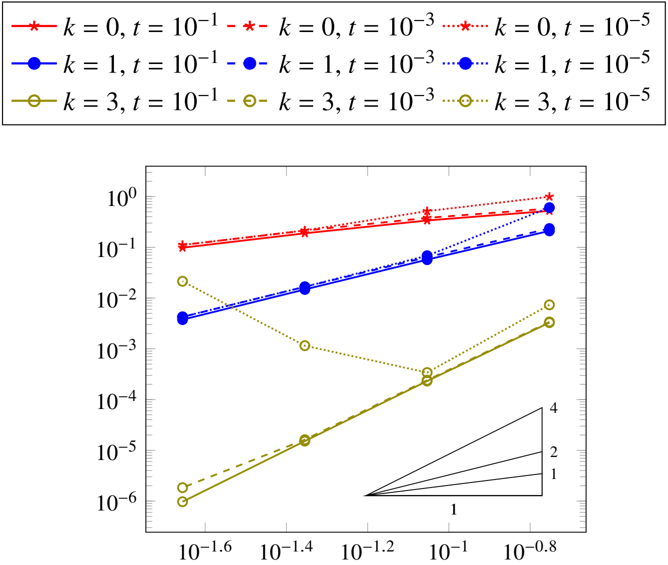


Smooth polynomial solution: triangular meshes



Smooth polynomial solution: locally refined meshes



What is happening here?

Rounding errors.

- For high k, local quantities (e.g. local b_h) have precision $\sim 10^{-14}$.
- For $t = 10^{-5}$, the t^{-2} scaling factor brings precision to $\sim 10^{-4}$.
- Many elements cumulates these...

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- Many elements cumulates these...

No other tests in the literature went to this order and number of elements.
Beirão da Veiga et al. used IGA with $k = 3$, $t = 10^{-3}$ and 300 elements max.

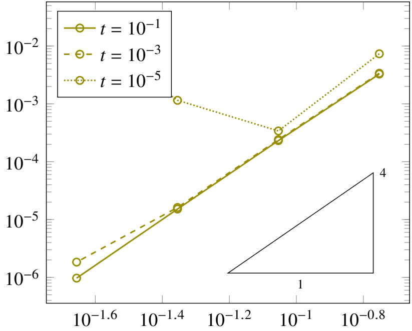


Figure: Locally refined meshes, $k = 3$

Analytical solution with expected behaviour

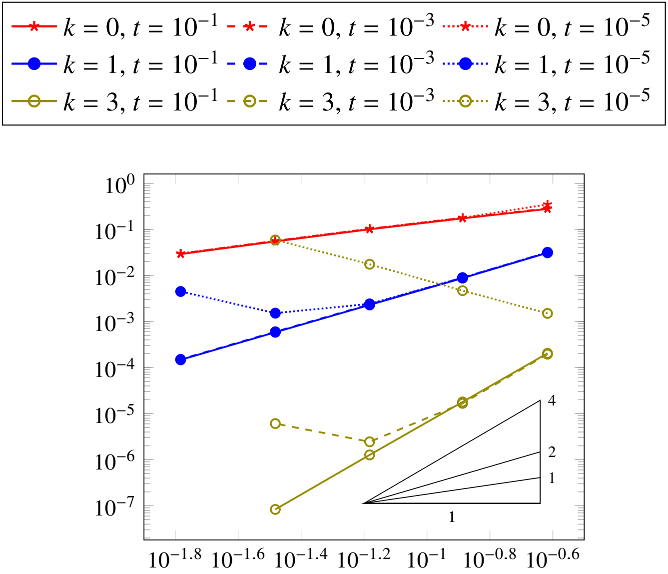
Smooth polynomial solution

- As $t \longrightarrow 0$, $|\boldsymbol{\theta}|_{H^s} \sim 1$ and $|\boldsymbol{\gamma}|_{H^s} \sim 1$ for all s!

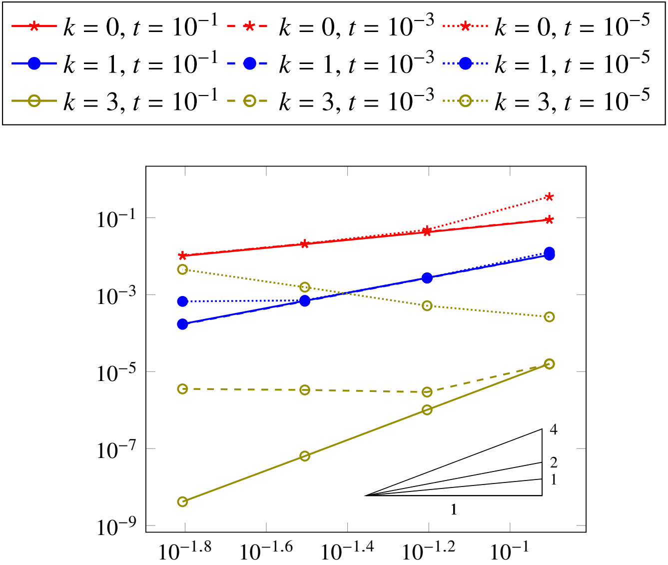
New analytical solution

- As $t \longrightarrow 0$, $|\boldsymbol{\theta}|_{H^2} \sim 1$ and $|\boldsymbol{\gamma}|_{H^s} \sim t^{-s+\frac{1}{2}}$.

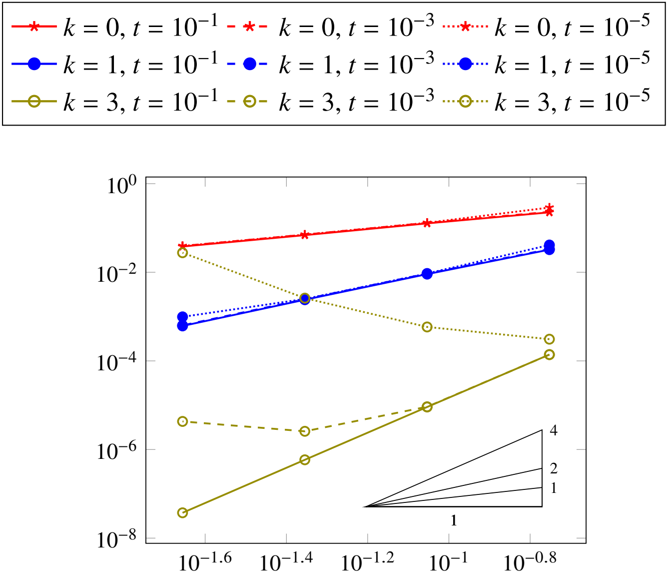
New analytical solution: hexagonal meshes



New analytical solution: triangular meshes



New analytical solution: locally refined meshes



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- Completely robust w.r.t. t for $k = 0$, as expected!
- Increased degradation of convergence sooner (smaller $k \geq 1$, larger t) than for the polynomial solution.
- Much less dependence w.r.t. t than the error estimates lead us to believe.

  E.g.: $|\boldsymbol{\gamma}|_{H^3} \sim t^{-3.5}$ *so for* $k = 3$, *increase between* $t = 10^{-1}$ *and* $t = 10^{-3}$*:*
  - *expected* $\sim 10^7$.
  - *actual: at most* 10^3 *on these meshes.*

Thanks!

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