A fully discrete exact de Rham sequence, with application to magnetostatics

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Australian Government
Australian Research Council
Discrete Functional Analysis: bridging pure and numerical mathematics

Plan

1. Exact sequence of differential operators
2. Principles of discrete exact sequence
3. Fully discrete de Rham sequence
4. Application to magnetostatics

Plan

Spaces and operators

- Ω: open simply connected set in $\mathbb{R}^3$ with connected boundary.

Gradient:

$$\begin{aligned} H^1(\Omega) &= \{u \in L^2(\Omega)\ :\ \operatorname{grad} u \in L^2(\Omega)^3\},\\ \operatorname{grad} &: H^1(\Omega) \to L^2(\Omega)^3. \end{aligned}$$

Curl:

$$\begin{aligned} \boldsymbol{H}(\operatorname{curl};\Omega) &= \{\boldsymbol{u} \in L^2(\Omega)^3\ :\ \operatorname{curl} \boldsymbol{u} \in L^2(\Omega)^3\},\\ \operatorname{curl} &: \boldsymbol{H}(\operatorname{curl};\Omega) \to L^2(\Omega)^3. \end{aligned}$$

Divergence:

$$\begin{aligned} \boldsymbol{H}(\operatorname{div};\Omega) &= \{\boldsymbol{u} \in L^2(\Omega)^3\ :\ \operatorname{div} \boldsymbol{u} \in L^2(\Omega)\},\\ \operatorname{div} &: \boldsymbol{H}(\operatorname{div};\Omega) \to L^2(\Omega). \end{aligned}$$

Exact sequence

- $i_\Omega : \mathbb{R} \to H^1(\Omega)$ natural embedding.

Theorem (Exactness of de Rham sequence)

The following sequence is exact:

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}(\text{curl};\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\},$$

which means that, if $\mathfrak{D}_i$ and $\mathfrak{D}_{i+1}$ are two consecutive operators in the sequence, then

$$\text{Im}\,\mathfrak{D}_i = \text{Ker}\,\mathfrak{D}_{i+1}.$$

Why is this exactness important?

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}(\text{curl};\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\},$$

Stokes problem

$$\left\{\begin{array}{ll} -\Delta \boldsymbol{u} + \text{grad}\, p = f & \text{in } \Omega, \\ \text{div}\, \boldsymbol{u} = 0 & \text{in } \Omega, \\ +\text{BC} & \end{array}\right.$$

- Inf-sup condition: for all $q \in L^2(\Omega)$,

$$\sup_{\boldsymbol{v} \in \boldsymbol{H}(\text{div};\Omega)} \frac{(\text{div}\, \boldsymbol{v}, q)_{L^2}}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\text{div})}} \geq \beta \|q\|_{L^2}.$$

Proof: Fix $q \in L^2(\Omega)$*, and let* $\boldsymbol{v} \in \boldsymbol{H}(\text{div};\Omega)$ *such that* $\text{div}\, \boldsymbol{v} = q$*...*

Why is this exactness important?

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}(\text{curl};\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\},$$

Magnetostatic problem

$$\begin{cases} \boldsymbol{\sigma} - \text{curl}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \text{curl}\,\boldsymbol{\sigma} = \boldsymbol{f} & \text{in } \Omega, \\ \text{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \partial\Omega. \end{cases}$$

- Inf-sup condition: for all $(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{H}(\text{curl};\Omega) \times \boldsymbol{H}(\text{div};\Omega)$,

$$\sup_{(\boldsymbol{\mu},\boldsymbol{w}) \in \boldsymbol{H}(\text{curl}) \times \boldsymbol{H}(\text{div})} \frac{\mathcal{A}((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\mu}, \boldsymbol{w}))}{\|(\boldsymbol{\mu}, \boldsymbol{w})\|_{\boldsymbol{H}(\text{curl}) \times \boldsymbol{H}(\text{div})}} \geq \beta \|(\boldsymbol{\mu}, \boldsymbol{v})\|_{\boldsymbol{H}(\text{curl}) \times \boldsymbol{H}(\text{div})},$$

with bilinear form

$$\mathcal{A}((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\mu}, \boldsymbol{w})) = (\boldsymbol{\tau}, \boldsymbol{\mu})_{L^2} - (\boldsymbol{v}, \text{curl}\,\boldsymbol{\mu})_{L^2} + (\boldsymbol{w}, \text{curl}\,\boldsymbol{\tau})_{L^2} + (\text{div}\,\boldsymbol{v}, \text{div}\,\boldsymbol{w})_{L^2}.$$

Why is this exactness important?

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}(\text{curl};\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\},$$

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- bilinear form

$$\mathcal{A}((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\mu}, \boldsymbol{w})) = (\boldsymbol{\tau}, \boldsymbol{\mu})_{L^2} - (\boldsymbol{v}, \operatorname{curl} \boldsymbol{\mu})_{L^2} + (\boldsymbol{w}, \operatorname{curl} \boldsymbol{\tau})_{L^2} + (\operatorname{div} \boldsymbol{v}, \operatorname{div} \boldsymbol{w})_{L^2}.$$

Proof: requires two exactness properties in the sequence, to estimate each component of $\boldsymbol{v}$ on $(\operatorname{Ker} \operatorname{div})^\perp$ and $\operatorname{Ker} \operatorname{div}$.

Plan

Objective

- Mimic exact sequence with discrete spaces and operators.

$\rightsquigarrow$ *To be used to design stable numerical schemes.*

- Local construction (element by element), as in standard FE.

- Arbitrary order, based on polynomial spaces of degree $k \geq 0$.

Objective

Local discrete spaces and operators: for T mesh element,

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

Objective

Local discrete spaces and operators: for T mesh element,

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

- Finite Element approach:
  - Finite Element Exterior Calculus (FEEC).
  - Requires elements of certain shapes (tetrahedras, hexahedras...) as in usual FE.
  - Designed in very generic setting, with exterior derivatives etc.

Objective

Local discrete spaces and operators: for T mesh element,

$$\mathbb{R} \xrightarrow{\underline{I}_{\mathrm{grad},T}^k} \underline{X}_{\mathrm{grad},T}^k \xrightarrow{\underline{\boldsymbol{G}}_T^k} \underline{\boldsymbol{X}}_{\mathrm{curl},T}^k \xrightarrow{\underline{\boldsymbol{C}}_T^k} \underline{\boldsymbol{X}}_{\mathrm{div},T}^k \xrightarrow{D_T^k} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

- Virtual Element approach:
  - Applicable on generic meshes with polyhedral elements.
  - Degree decreases by one at each application of differential operator.
  - Functions not fully known, only certain moments or values are accessible.
  - Exactness not usable in a scheme due to the variational crime in VEM.

Plan

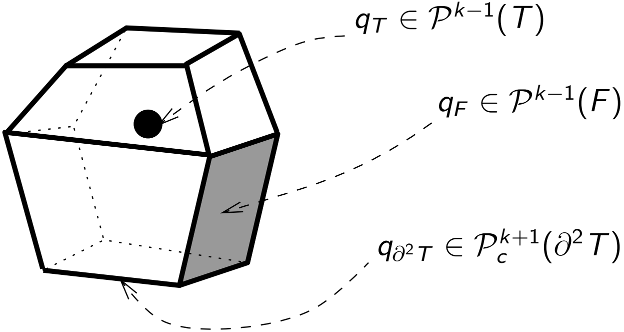
Features

- Applicable on polyhedral elements.
- Arbitrary order of exactness.
- Same order of accuracy along the entire sequence.
- Based on explicit spaces and reconstructed differential operators, exactness holding for these objects.

Principles

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

Gradient unknowns: $\underline{q}_T = (q_T, (q_F)_{F\in\mathcal{F}_T}, q_{\partial^2 T})$.



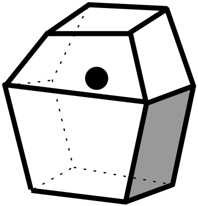
Principles

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

Gradient operator:

$$\underline{\boldsymbol{G}}^k_T \underline{q}_T = (\underbrace{\boldsymbol{G}^k_T \underline{q}_T}_{\in \mathcal{P}^k(T)^3}, (\underbrace{\boldsymbol{G}^k_F(q_F, q_{\partial^2 T})}_{\in \mathcal{P}^k(F)^2})_{F\in\mathcal{F}_T}, (\underbrace{G_E q_E}_{\in \mathcal{P}^k(E)})_{E\in\mathcal{E}_T}).$$

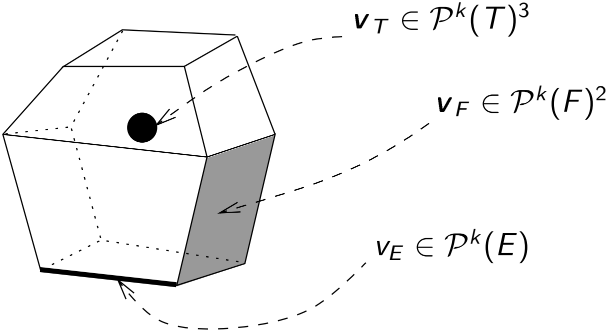
- G_E: derivative along edge.
- $\boldsymbol{G}^k_F$ ($\approx \mathrm{grad}_{|F}$): reconstruction from face and edge, based on formal IBP (divergence formula),
- $\boldsymbol{G}^k_T$ ($\approx$ grad): reconstruction based on formal IBP & face potentials (divergence formula).



Principles

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

Curl unknowns: $\underline{\boldsymbol{v}}_T = (\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F\in\mathcal{F}_T}, (v_E)_{E\in\mathcal{E}_T})$.



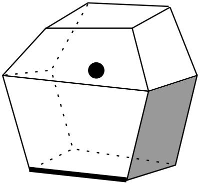
Principles

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

Curl operator:

$$\underline{\boldsymbol{C}}^k_T \underline{\boldsymbol{v}}_T = (\underbrace{\boldsymbol{C}^k_T \underline{\boldsymbol{v}}_T}_{\in \mathcal{P}^k(T)^3}, (\underbrace{C^k_F(\boldsymbol{v}_F, (v_E)_{E\in\mathcal{E}_F})}_{\in \mathcal{P}^k(F))})_{F\in\mathcal{F}_T}).$$

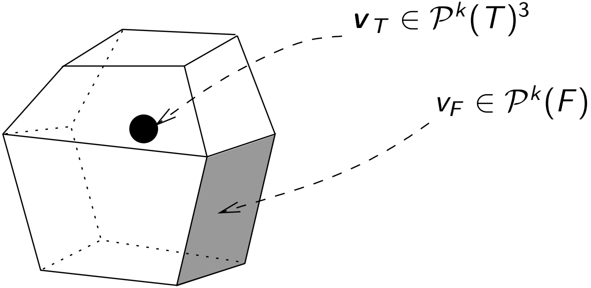
- C^k_F ($\approx$ curl $\cdot \boldsymbol{n}_F$): reconstruction from face and edge, based on formal IBP (rot formula in 2D),
- $\boldsymbol{C}^k_T$ ($\approx$ curl): reconstruction based on formal IBP & face tangential potentials (curl formula).



Principles

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

Divergence unknowns: $\underline{\boldsymbol{v}}_T = (\boldsymbol{v}_T, (v_F)_{F\in\mathcal{F}_T})$.

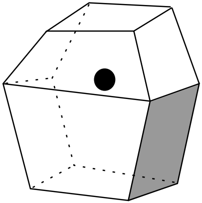


$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

Divergence operator:

$D^k_T \underline{\boldsymbol{v}}_T$ ($\approx$ div) reconstructed in $\mathcal{P}^k(T)$ from divergence formula.

$$\int_T (D^k_T \underline{\boldsymbol{v}}_T) q_T = -\int_T \boldsymbol{v}_T \cdot \mathrm{grad}\, q_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F v_F q_T \quad \forall q_T \in \mathcal{P}^k(T).$$



There's a catch...

$$\mathbb{R} \xrightarrow{\underline{I}_{\mathrm{grad},T}^k} \underline{X}_{\mathrm{grad},T}^k \xrightarrow{\underline{\boldsymbol{G}}_T^k} \underline{\boldsymbol{X}}_{\mathrm{curl},T}^k \xrightarrow{\underline{\boldsymbol{C}}_T^k} \underline{\boldsymbol{X}}_{\mathrm{div},T}^k \xrightarrow{D_T^k} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

- The previous sequence is not exact!

There's a catch...

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

- The previous sequence is not exact!

For $X = F, T$ of dimension $d = 2, 3$ let:

- $\boldsymbol{\mathcal{R}}^k(X) = \mathrm{curl}(\mathcal{P}^{k+1}(X)^d)$, $\boldsymbol{\mathcal{R}}^{\mathrm{c},k}(X)$ complement in $\mathcal{P}^k(X)^d$.
- $\boldsymbol{\mathcal{G}}^k(X) = \mathrm{grad}(\mathcal{P}^{k+1}(X)^d)$, $\boldsymbol{\mathcal{G}}^{\mathrm{c},k}(X)$ complement in $\mathcal{P}^k(X)^d$.

There's a catch...

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},T}} \underline{X}^k_{\mathrm{grad},T} \xrightarrow{\underline{\boldsymbol{G}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} \xrightarrow{\underline{\boldsymbol{C}}^k_T} \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$

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- $\boldsymbol{\mathcal{G}}^k(X) = \mathrm{grad}(\mathcal{P}^{k+1}(X)^d)$, $\boldsymbol{\mathcal{G}}^{\mathrm{c},k}(X)$ complement in $\mathcal{P}^k(X)^d$.

Trimmed spaces: face/cell gradients and curls have to be projected on trimmed spaces.

- Gradients in $\mathcal{P}^k(X)^d$ projected on $\boldsymbol{\mathcal{R}}^{k-1}(X) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},k}(X)$.
- Curls in $\mathcal{P}^k(X)^d$ projected on $\boldsymbol{\mathcal{G}}^{k-1}(X) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},k}(X)$.

Commutative diagram

- $\mathcal{N}^k(T) = \mathcal{P}^k(T)^3 + \boldsymbol{x} \times \mathcal{P}^k(T)^3$ (Nédélec space),
- $\mathcal{RT}^k(T) = \mathcal{P}^k(T)^3 + \boldsymbol{x}\mathcal{P}^k(T)$ (Raviart-Thomas space).

► The following diagram is commutative:

$$\begin{array}{ccccccc}
\mathcal{P}^{k+1}(T) & \xrightarrow{\text{grad}} & \mathcal{N}^k(T) & \xrightarrow{\text{curl}} & \mathcal{RT}^k(T) & \xrightarrow{\text{div}} & \mathcal{P}^k(T) \\
\downarrow \underline{I}^k_{\text{grad},T} & & \downarrow \underline{\boldsymbol{I}}^k_{\text{curl},T} & & \downarrow \underline{\boldsymbol{I}}^k_{\text{div},T} & & \downarrow i_T \\
\underline{X}^k_{\text{grad},T} & \xrightarrow{\underline{\boldsymbol{G}}^k_T} & \underline{\boldsymbol{X}}^k_{\text{curl},T} & \xrightarrow{\underline{\boldsymbol{C}}^k_T} & \underline{\boldsymbol{X}}^k_{\text{div},T} & \xrightarrow{D^k_T} & \mathcal{P}^k(T)
\end{array}$$

► Ensures polynomial consistency up to degree k of the discrete sequence.

Other features

Commutative diagram

- $\boldsymbol{\mathcal{N}}^k(T) = \mathcal{P}^k(T)^3 + \boldsymbol{x} \times \mathcal{P}^k(T)^3$ (Nédélec space),
- $\boldsymbol{\mathcal{RT}}^k(T) = \mathcal{P}^k(T)^3 + \boldsymbol{x}\mathcal{P}^k(T)$ (Raviart-Thomas space).

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\mathcal{P}^{k+1}(T) & \xrightarrow{\mathrm{grad}} & \boldsymbol{\mathcal{N}}^k(T) & \xrightarrow{\mathrm{curl}} & \boldsymbol{\mathcal{RT}}^k(T) & \xrightarrow{\mathrm{div}} & \mathcal{P}^k(T) \\
\downarrow \underline{I}^k_{\mathrm{grad},T} & & \downarrow \underline{\boldsymbol{I}}^k_{\mathrm{curl},T} & & \downarrow \underline{\boldsymbol{I}}^k_{\mathrm{div},T} & & \downarrow i_T \\
\underline{X}^k_{\mathrm{grad},T} & \xrightarrow{\underline{\boldsymbol{G}}^k_T} & \underline{\boldsymbol{X}}^k_{\mathrm{curl},T} & \xrightarrow{\underline{\boldsymbol{C}}^k_T} & \underline{\boldsymbol{X}}^k_{\mathrm{div},T} & \xrightarrow{D^k_T} & \mathcal{P}^k(T)
\end{array}$$

- Ensures polynomial consistency up to degree k of the discrete sequence.

Reconstruction of potentials: scalar for gradient, vectorial for curl; enable definition of L^2-inner products used to write schemes in weak formulations.

Going global

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},\Omega}} \underline{X}^k_{\mathrm{grad},\Omega} \xrightarrow{\underline{\boldsymbol{G}}_{\Omega}} \underline{\boldsymbol{X}}^k_{\mathrm{curl},\Omega} \xrightarrow{\underline{\boldsymbol{C}}^k_{\Omega}} \underline{\boldsymbol{X}}^k_{\mathrm{div},\Omega} \xrightarrow{D^k_{\Omega}} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}.$$

Global spaces/operators: by patching local spaces/operators.

- Additional challenges:
  - Global exactness, especially $\mathrm{Ker}\, D^k_\Omega \subset \mathrm{Im}\, \underline{\boldsymbol{C}}^k_\Omega$.
  - Poincaré inequalities (for stability), e.g.

$$\|\underline{\boldsymbol{v}}_\Omega\|_{\underline{\boldsymbol{X}}^k_{\mathrm{curl},\Omega}} \le M \|\underline{\boldsymbol{C}}^k_\Omega \underline{\boldsymbol{v}}_\Omega\|_{\underline{\boldsymbol{X}}^k_{\mathrm{div},\Omega}} \quad \forall v_\Omega \in (\underline{\boldsymbol{X}}^k_{\mathrm{curl},\Omega})^\perp.$$

Plan

Model and exact solution

$$\begin{cases} \boldsymbol{\sigma} - \operatorname{curl} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \operatorname{curl} \boldsymbol{\sigma} = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \partial\Omega. \end{cases}$$

on $\Omega = (0,1)^3$, with exact solution

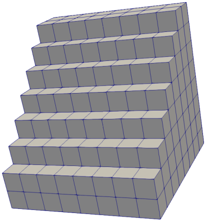
$$\boldsymbol{\sigma}(\boldsymbol{x}) = 3\pi \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \\ 0 \\ -\cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \end{pmatrix},$$

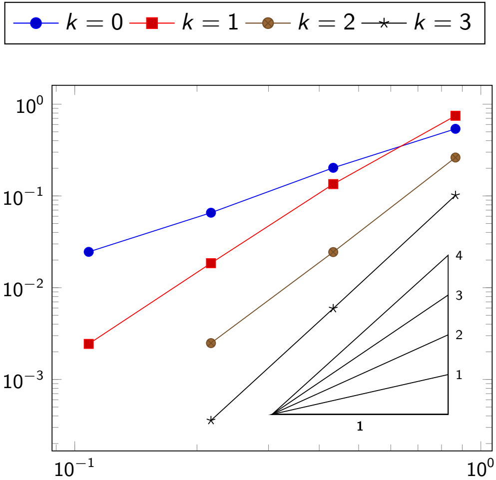
$$\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} \cos(\pi x_1)\sin(\pi x_2)\sin(\pi x_3) \\ -2\sin(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \\ \sin(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \end{pmatrix}.$$

Implementation

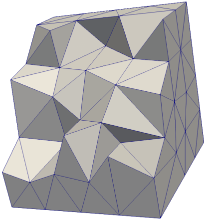
- All spaces and operators entirely implemented in the HArD::Core3D library (`https://github.com/jdroniou/HArDCore`).
  - Open source C++ code for numerical schemes on polyhedral meshes.
  - Based on Eigen linear algebra library (`http://eigen.tuxfamily.org`).
  - Complete and intuitive description of mesh.
  - Routines for handling polynomial spaces (on edges, faces and cells), for quadrature rules, for Gram-like matrices (mass, stiffness), etc.

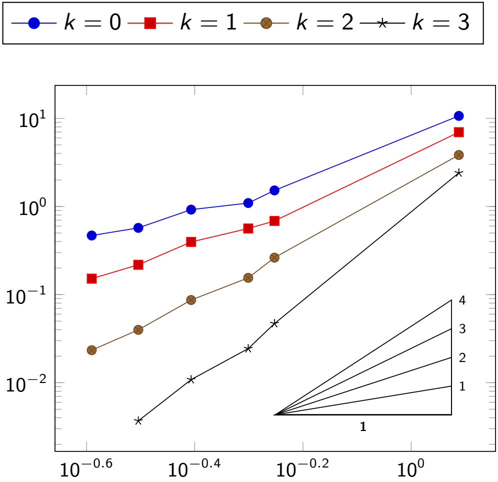
Convergence graphs in energy norm: cubic cells



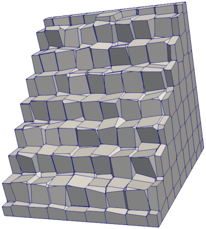


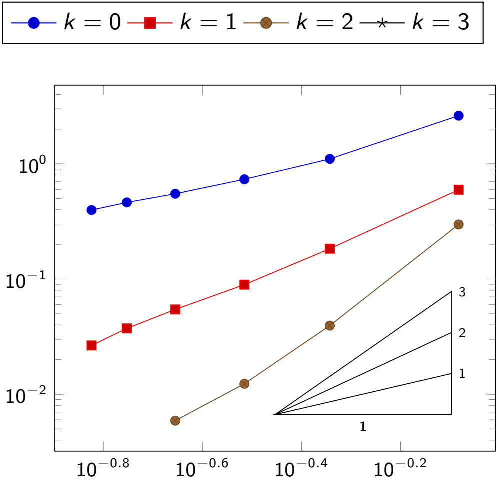
Convergence graphs in energy norm: tetrahedral cells



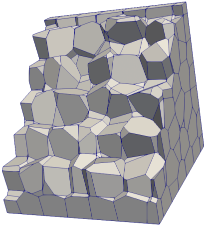


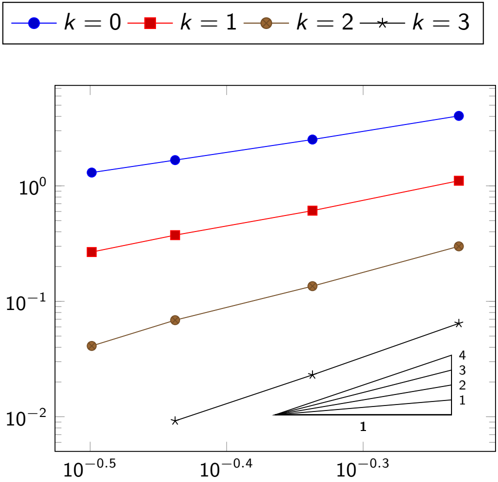
Convergence graphs in energy norm: Voronoi cells 1





Convergence graphs in energy norm: Voronoi cells 2





Conclusion

- Design of a fully discrete (local and global) exact de Rham sequence.
- Purely based on explicit polynomial spaces.
- Applicable on generic polyhedral meshes, and of arbitrary accuracy order.
- Proofs of local and global exactness, and Poincaré inequalities.
- Automatically yields stable discretisations of PDEs.

Bibliography

Main papers:

- *Fully discrete polynomial de Rham sequences of arbitrary degree on polygons and polyhedra*. D. Di Pietro, J. Droniou, and F. Rapetti. Math. Models Methods Appl. Sci. 44p, 2020. `https://arxiv.org/abs/1911.03616`.
- *An arbitrary-order method for magnetostatics on polyhedral meshes based on a discrete de Rham sequence*. D. A. Di Pietro and J. Droniou, 31p, submitted, 2020. `https://arxiv.org/abs/2005.06890`.

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Thanks.